A Smooth Approximation to the Alternating Gradient Orbit Equations

Keith R. Symon, Wayne University, July 1, 1954.

Introduction

We propose to develop an approximate solution to the equation

$$\frac{d^2x}{ds^2} = f(x, s) \tag{1}$$

where $f(x, s)$ is periodic in s with period S. The approximation will be based upon the assumption that x changes relatively little within a sector length S, and will be expected to be valid when the betatron wavelength is large in comparison with the sector length S. Under these conditions, we will attempt to replace Eq. (1) by its "smooth" approximation

$$\frac{d^2x}{ds^2} = F(x, x') \tag{2}$$

where in nearly all case of interest, the function F will not depend on $x^1 = dx/ds$. The solution of Eq. (2) will indicate the gross features of the betatron oscillations, but will not be expected to reveal details of the motion within a sector, or resonance effects which depend on the sector length.

2. General Theory

We choose a standard reference point within a sector, which we take for convenience to be the point $s = 0(\pm nS)$. Let the solution of Eq. (1) for $x = x_o$, $x^1 = x_o^1$ at $s = 0$ take the values x_S, x_{-S} at reference points in the two adjacent sectors. Then we define

$$F(x, x^1) = \frac{x_S - 2x_o + x_{-S}}{S^2} \tag{3}$$

The right member is a function of x_o, x_o^1, and these are to be replaced by x, x^1 defined by

$$x = x_o,$$
$$x^1 = \frac{x_S - x_{-S}}{2S} \tag{4}$$

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Whenever possible, we will choose the reference point within the sector so as to eliminate the dependence on x_o^1 and hence on x^1. It appears that this can always be done, at least in first approximation. Indeed, in this one dimensional case, we should almost insist on eliminating the dependence on x^1, since otherwise Liouville's theorem, which holds for Eq. (1) in the x, x^1 phase plane, would not hold for Eq. (2).

The approximation defined by Eqs. (2) and (3) will be valid when the mean slope $(x_S - x_o)/S$ does not change rapidly from one sector to the next. Under these circumstances it will usually, though perhaps not always, be reasonable to assume that x does not change appreciably within a sector. We therefore develop $f(x,s)$ in a Taylor series:

$$f(x,s) = f(x_o,s) + f_x(x_o,s)\ (x-x_o) + \tfrac{1}{2}f_{xx}(x_o,s)\ (x-x_o)^2 + \ \cdots \tag{5}$$

The velocity within the two sectors $-S \leq x \leq S$, is then approximately

$$x' = x_o' + \int_o^s f(x_o,s)ds + \int_o^s fx(x_o,s)\ (x-x_o)ds + \ \cdots \tag{6}$$

and the position is

$$x = x_o + x_o's + \int_o^s \int_o^s f(x_o,s)\ dsds + \int_o^s \int_o^s f_x(x_o,s)\ (x-x_o)dsds + \cdots \tag{7}$$

In first approximation

$$x - x_o = x_o's + f_2(x_o,s), \tag{8}$$

where

$$f_2(x_o,s) = \int_o^s \int_o^s f(x_o,s)dsds. \tag{9}$$

We substitute Eq. (8) in Eq. (7) to obtain in second approximation

$$x = x_o + x_o's + f_2(x_o,s) + x_o' \int_o^s \int_o^s sf_x(s_o,s)dsds$$

$$+ \int_o^s \int_o^s f_x(x_o,s)f_2(x_o,s)dsds. \tag{10}$$

Equation (3) now becomes

$$F(x,x') = \frac{f_2(x_o,S) + f_2(x_o,-S)}{S^2} + \frac{1}{S^2}\left[\int_o^S \quad \int_o^{-S}\right]\left[\int_o^s f_x(x_os).\right.$$

$$\left. f_2(x_o,s)ds\right]ds + \frac{x_o'}{S^2}\left[\int_o^S + \int_o^{-S}\right]\int_o^s sf_x(x_o,s)dsds. \tag{11}$$

It seems fairly clear that by choosing the reference point $s = 0$ at an appropriate point within the sector, the last term can be made to vanish, and F will depend only on $x = x_0$. In particular, if $s = 0$ can be so chosen that $f(x,s)$ is symmetrical:

$$f(x, -s) = f(x,s) \tag{12}$$

then Eq. (11) becomes

$$F(x) = \frac{2}{S^2} f_2(x, S) + \frac{2}{S^2} \int_0^S \int_0^s f_x(x,s) f_2(x,s)\, ds\, ds. \tag{13}$$

If $f(x, s)$ has the form

$$f(x,s) = f(x)\ g(s), \tag{14}$$

then in second approximation

$$F(x) = \alpha f(x) + \beta f(x) f_x(x), \tag{15}$$

with
$$\alpha = \frac{2}{S^2} g_2(S), \tag{16}$$

$$\beta = \frac{2}{S^2} \int_0^S \int_0^s g(s)\, g_2(s)\, ds\, ds, \tag{17}$$

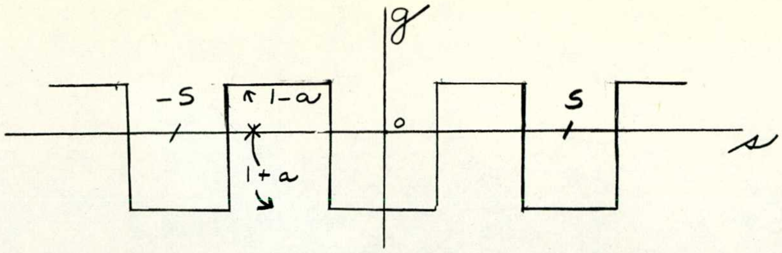
$$g_2(s) = \int_0^s \int_0^s g(s)\, ds\, ds, \tag{18}$$

provided $g(-s) = g(s)$.

3. Examples

We consider the case where $f(x, s)$ is given by Eq. (14) with

$$g(s) = \begin{cases} -(1+a), & 0 < s < \frac{1}{4}S, \\ 1-a, & \frac{1}{4}S < s < \frac{3}{4}S, \\ -(1+a), & \frac{3}{4}S < s < S. \end{cases} \tag{19}$$



This is a unit square wave displaced to a mean value -a. (We choose the reference point in the middle of a focusing half-sector. The results would be the same if we chose the middle of a defocussing half-sector.) The coefficients α and β, calculated from Eq. (16) and (17) are in this case

$$\alpha = -a \tag{20}$$

$$\beta = -\frac{S^2}{48}(1-4a^2). \tag{21}$$

We may compare the case

$$g(s) = -a - b\cos\frac{2\pi s}{S}. \tag{22}$$

We now calculate:

$$\alpha = -a, \tag{23}$$

$$\beta = -\frac{S^2}{48}\left(\frac{6b^2}{\pi^2} - 4a^2\right). \tag{24}$$

Evidently in smooth approximation the sinusoidal and square wave solutions are identifical if we take

$$b = \frac{\pi}{\sqrt{6}} = 1.27 \tag{25}$$

Let us consider first the linear case, with

$$f(x) = n x, \tag{26}$$

and with g(s) given by Eq. (19). The equation of motion (2) for the smooth approximation is, if we use Eqs. (15), (20), (21),

$$\begin{aligned}\frac{d^2x}{ds^2} &= \alpha n x + \beta n^2 x \\ &= -\left(a n + \frac{S^2 n^2}{48} - \frac{S^2 n^2 a^2}{12}\right)x\end{aligned} \tag{27}$$

If we set

$$k^2 = an + \frac{s^2 n^2}{48} - \frac{s^2 n^2 a^2}{12}, \tag{28}$$

then the motion is stable if $k^2 > 0$, and is given by

$$x = A\cos(kx - \theta_0). \tag{29}$$

The betatron wavelength is $2\pi/k$. In the linear theory of Courant and Snyder (EDC / HSS - 1), the betatron wavelength is $2\pi S/\sigma$. Hence the smooth approximation for the parameter σ is

$$\sigma = kS = \sqrt{anS^2 + \frac{n^2 S^4}{48} - \frac{a^2 n^2 S^4}{12}} \tag{30}$$

Courant and Snyder show that in the special case a = 0 (p. 19), σ is given by

$$\cos\sigma = \cos\psi \cosh\psi, \tag{31}$$

with

$$\psi = \tfrac{1}{2} S\sqrt{n}. \tag{32}$$

Formula (30) gives, for this case,

$$\sigma = \frac{1}{\sqrt{3}}\psi^2. \tag{33}$$

If we expand Eq. (31) in a power series in σ and ψ, we obtain

$$\sigma = \frac{1}{\sqrt{3}}\psi^2\left(1 + \frac{17}{420}\psi^4 + \ldots\right) \tag{34}$$

A plot of σ s ψ as given by Eqs. (31) and (33) is given in Fig. 1.

If we set

$$n_1 = n(1+a),$$

$$n_2 = -n(1-a),$$

(35)

$$a = \frac{n_1 + n_2}{n_1 - n_2},$$

$$n = \frac{n_1 - n_2}{2},$$

(36)

then the stability limits for Eq. (27) are determined by the condition

$$\frac{n_1 + n_2}{2} + \frac{s^2(n_1+n_2)^2}{192} - \frac{s^2(n_1+n_2)^2}{48} = 0,$$

(37)

or

$$3\psi_1^4 + 10\psi_1^2\psi_2^2 + 3\psi_2^4 - 24(\psi_1^2 + \psi_2^2) = 0 \qquad (38)$$

with

$$\psi_1 = \frac{1}{2}s\sqrt{n_1}, \quad \psi_2 = \frac{1}{2}s\sqrt{n_2}. \qquad (39)$$

The exact formula for the stability limits, as given by Courant and Snyder is

$$\cos\sigma = \cos\psi_1\cos\psi_2 - \frac{\psi_1^2 + \psi_2^2}{2\psi_1\psi_2}\sin\psi_1\sin\psi_2 = \pm 1. \qquad (40)$$

Eqs. (38) and (40) are plotted in Fig. 2. If we expand Eq. (40) in powers of ψ_1 and ψ_2, then, with the plus sign on the right, Eq. (40) agrees to fourth degree terms with Eq. (38). The minus sign on the right gives the limits at the bottom of the necktie ($|n|$ large) and the smooth approximation is not valid there.

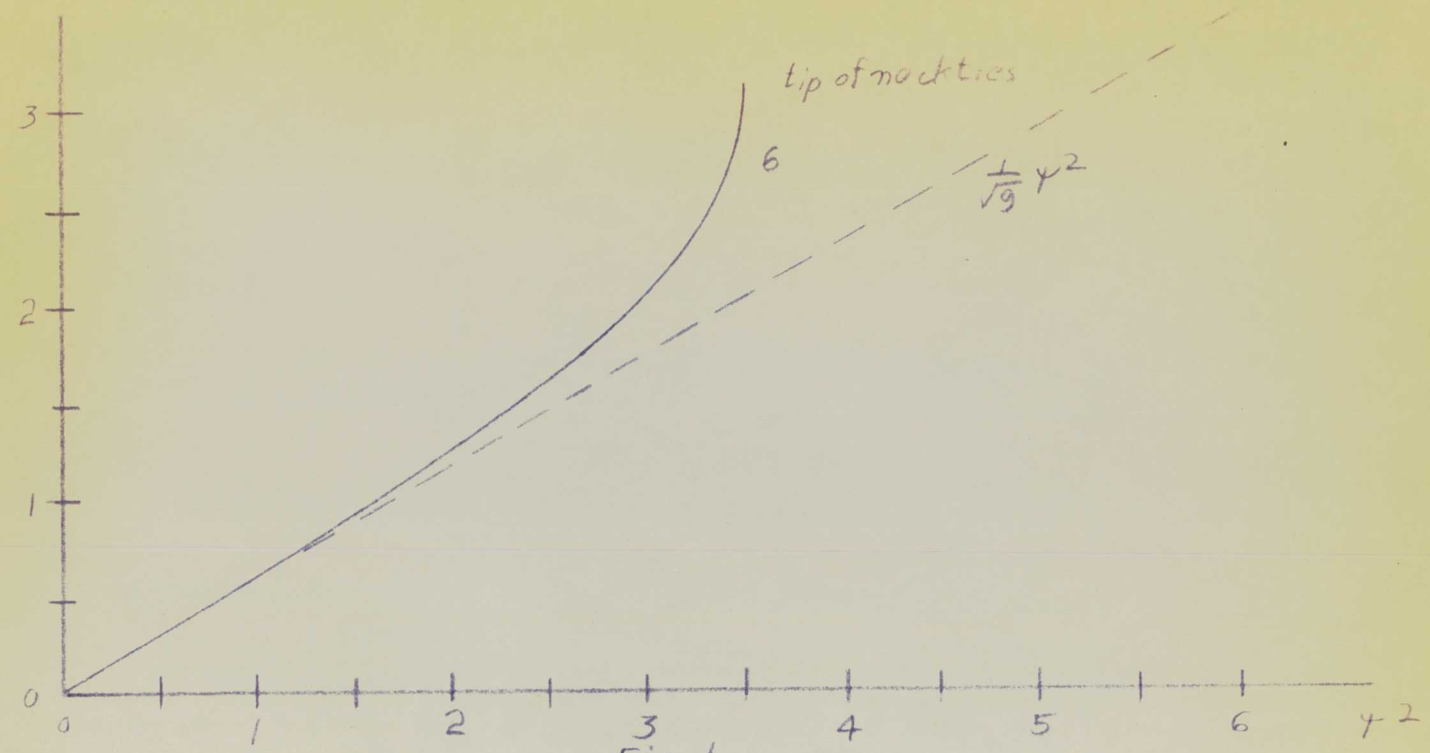

tip of neckties
6
$\frac{1}{\sqrt{9}}\psi^2$
3
2
1
0
0
1
2
3
4
5
6
ψ^2

Fig. 1

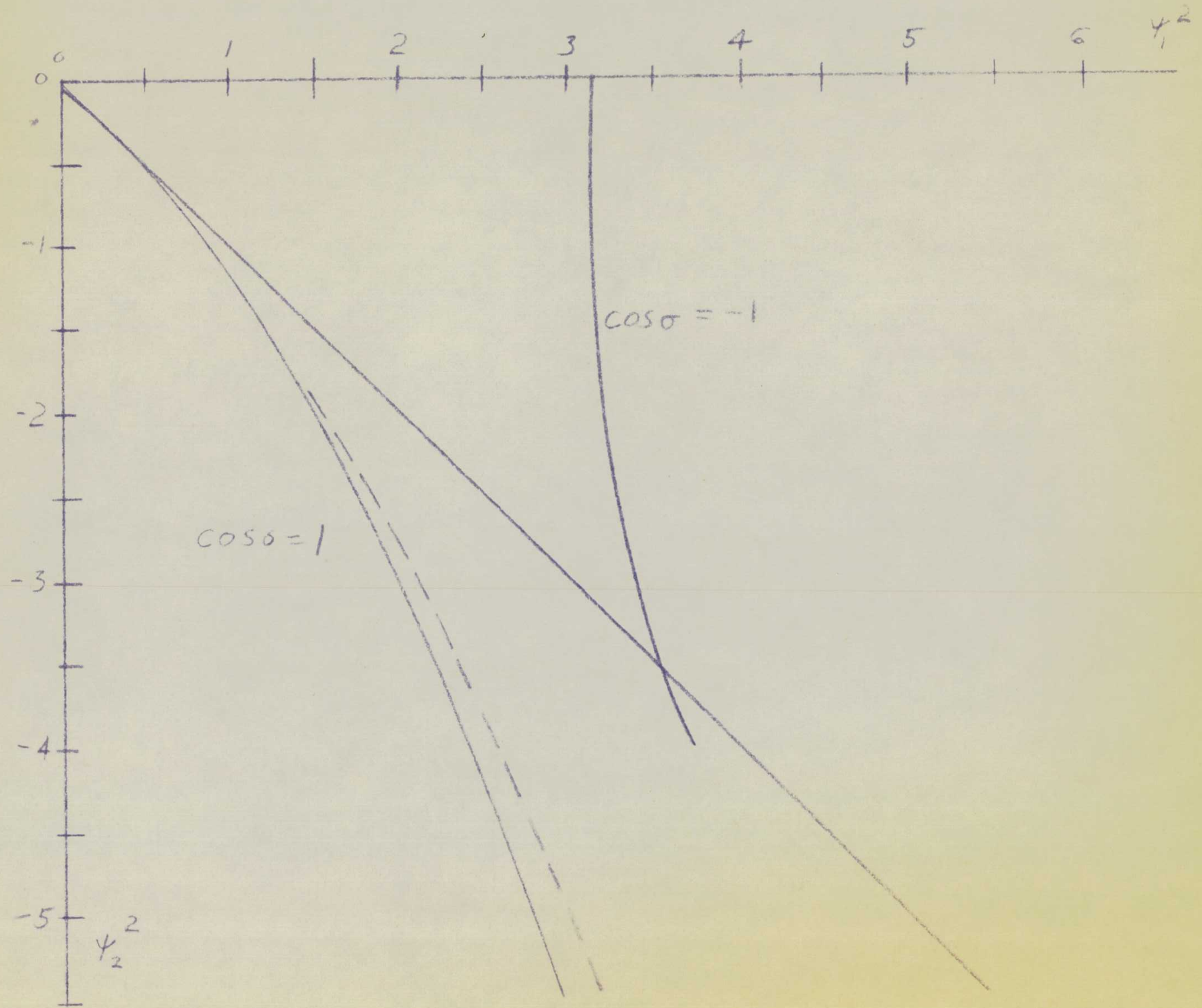

0
1
2
3
4
5
6
ψ_1^2
0
-1
-2
-3
-4
-5
ψ_2^2
$\cos\sigma = -1$
$\cos\sigma = 1$

Fig. 2

We next consider the non-linear case with cubic terms:

$$f(x) = nx - \frac{e}{3}x^3. \tag{41}$$

The equation of motion in smooth approximation is

$$\frac{d^2x}{ds^2} = (\alpha n + \beta n^2)x - \frac{e}{3}(\alpha + 4\beta n)x^3 + \frac{e^2}{3}\beta x^5. \tag{42}$$

If we take g(s) from Eq. (19), with a = 0 (on the necktie diagonal), this becomes

$$\frac{d^2x}{ds^2} = -\frac{S^2 n}{48}\left(nx - \frac{4e}{3}x^3 + \frac{e^2}{3n}x^5\right). \tag{43}$$

We note that the cubic term in Eq. (43) is relatively four times as large in comparison with the linear term as in Eq. (41). If we solve Eq. (43) by the energy method, we have:

$$\frac{1}{2}x'^2 = E - V(x), \tag{44}$$

where E is a constant of the motion and

$$V(x) = \frac{S^2 n^2}{48}\left(\frac{1}{2}x^2 - \frac{1}{3}\frac{e}{n}x^4 + \frac{1}{18}\frac{e^2}{n^2}x^6\right). \tag{45}$$

V(x) is plotted in Fig. 3, together with a phase plot for the motion. There are stable fixed points at $x = 0, \pm\sqrt{\frac{3n}{e}}$, and unstable fixed points at $x = \pm\sqrt{\frac{n}{e}}$.

The general character of the phase plot of Fig. 3 is in agreement with results of Iliac solutions of the orbit equation for e = n and for values of nS^2 less than about 2, and for trajectories within the region of the phase plane shown in Fig. 3.* There is a region of instability in the

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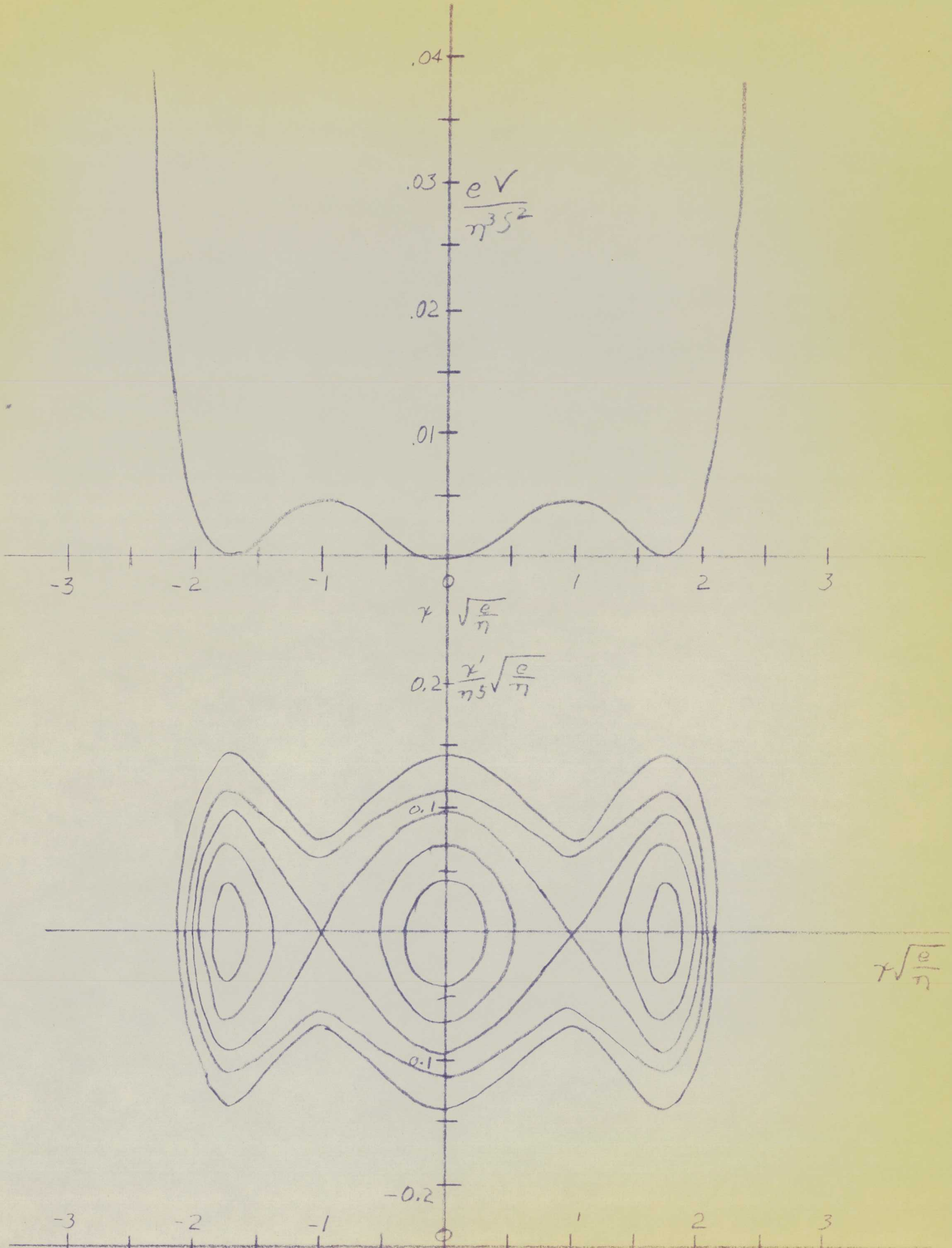
* I am indebted to John Powell for a discussion of the agreement between the Iliac solutions and the smooth approximation.



.04
.03
$\frac{eV}{\eta^3 s^2}$
.02
.01
-3
-2
-1
0
1
2
3
$x\sqrt{\frac{e}{\eta}}$
0.2
$\frac{x'}{\eta s}\sqrt{\frac{e}{\eta}}$
0.1
0.1
-0.2
-3
-2
-1
0
1
2
3
$x\sqrt{\frac{e}{\eta}}$

Fig. 3

phase plane outside the portion shown in Fig. 3 which corresponds to the instabilities beyond $\sigma = \pi$ in the linear case and which is not predicted by the smooth approximation. As nS^2 increases, this region of instability shrinks, until when nS^2 is about 3, the motion is unstable outside the separatrix through the two unstable fixed points. The fixed points at $x = 0$, $\pm\sqrt{\frac{3n}{\ell}}$ are determined by the zeroes of $f(x)$ and are therefore correctly given by the smooth approximation in all cases. The fixed points at $x = \pm\sqrt{\frac{n}{\ell}}$ are located correctly by the smooth approximation within 10% for $nS^2 < 6$.

4. The Region of Validity of the Smooth Approximation

The basic assumption made in the smooth approximation is represented by Eqs. (2) and (3) and will be expected to be valid when the betatron wavelength is much longer than the sector length. In the linear case,

$$F(x) = -k^2 x, \tag{46}$$

the smooth approximation for the betatron oscillations is

$$x = A \sin k\, s, \tag{47}$$

and the condition of validity will be

$$kS << 2\pi. \tag{48}$$

In the non-linear case, k^2 is not constant and Eq. (47) will not hold, but it seems reasonable to require still that Eq. (48) hold, with k^2 given by Eq. (46):

$$S\sqrt{\left|\frac{F(x)}{x}\right|} << 2\pi, \tag{49}$$

say

$$S\sqrt{\left|\frac{F(x)}{x}\right|} < \frac{2\pi}{10}. \tag{50}$$

In the linear case, on the necktie diagnonal, k is given by Eq. (28):

$$k = \frac{nS}{4\sqrt{3}} \tag{51}$$

and Eq. (50) becomes

$$\frac{nS^2}{4\sqrt{3}} < \frac{2\pi}{10}, \tag{52}$$

or, approximately

$$\psi^2 < \frac{\pi\sqrt{3}}{5} = 1.06 \tag{53}$$

According to Fig. 1, is given accurately by the smooth approximation somewhat beyond this limit.

It is interesting to note that if we set

$$kS = \pi, \tag{54}$$

which is the condition for the $\sigma = \pi$ resonance, we obtain

$$\psi^2 = \pi\sqrt{3} = 5.3 \tag{55}$$

The correct value of ψ^2 at $\sigma = \pi$ is 3.5, so that is appears that an order of magnitude estimate of the location of the $\sigma = \pi$ stability boundary may be made by using the smooth approximation, although the smooth approximation does not of course predict instability beyond $\sigma = \pi$. One may expect even in the non-linear case that the stability limits are given in order of magnitude at any rate by the condition that the betatron wavelength in smooth approximation be 2S.

If we apply Eq. (50) to the non-linear example (Eq. (43)) in the preceding action, we obtain the restriction

$$\frac{nS^2}{4\sqrt{3}} \sqrt{\left| 1 - \frac{4}{3}\frac{e}{n}x^2 + \frac{1}{3}\frac{e^2}{n^2}x^4 \right|} < \frac{2\pi}{10}. \tag{56}$$

For $x << \sqrt{\frac{n}{e}}$, this is the same restriction as Eq. (52) for the linear case. For large values of x, the restriction on allowable values of nS is more stringent. Near the fixed point $x = \sqrt{\frac{n}{e}}$, the square root in Eq. (56) is of the order 1. (In applying Eq. (56), one should presumably ignore cancellations which an reduce the argument of the square root to zero at fixed points.) Hence we should expect the approximation to be valid for

$$nS^2 < \frac{4\pi\sqrt{3}}{5} = 4.3 \tag{57}$$

In fact, the approximation locates this fixed point within 10 percent for $nS^2 < 6$. We may try to estimate the stability boundaries by applying the ideas of the preceding paragraph and setting

$$\frac{nS^2}{4\sqrt{3}} \sqrt{\left|1 - \frac{4}{3}\frac{e}{n}x^2 + \frac{1}{3}\frac{e^2}{n^2}x^4\right|} = \pi \tag{58}$$

Let us suppose we expect difficulty only for large x, and keep only the la st term under the square root:

$$\frac{nS^2}{12}\left(\frac{e}{n}x^2\right) = \pi \tag{59}$$

If we take $\sqrt{\frac{e}{n}}x = 2$, which is the maximum value of x on the separatrix in Fig. 3, we obtain

$$nS^2 = 3\pi = 9.4 \tag{60}$$

In fact, the instability boundary reaches the separatrix for $nS^2 = 3$, so that we have rather overestimated nS^2, which is not surprising in view of the several rather crude approximations which went into the result (Eq. 60). If we apply Eq. (50) to estimate the region in the phase plane where the smooth approximation should be valid, and if we keep only the highest order terms in x, we obtain

$$\frac{nS^2}{12}\frac{e}{n}x^2 < \frac{2\pi}{10}, \tag{61}$$

or

$$\left|\sqrt{\frac{e}{n}}\,x\right| < \sqrt{\frac{12\pi}{5nS^2}} \qquad (62)$$

With $nS^2 = 3$, this gives

$$\left|\sqrt{\frac{e}{n}}\;x\right| < 1.6 \qquad (63)$$

Orbits meeting this requirement should be given fairly accurately by the smooth approximation, a conclusion which is supported by the example in the previous section.

Another approximation involved in the development of the general theory in Section 1 was the solution of Eq. (1) within a sector by successive approximations. This is not essential to the method, but is probably the only convenient general way of carrying out the smooth approximation procedure. It will be valid provided that the force $f(x,s)$ does not differ greatly from $f(x_0,s)$ within a sector, say perhaps

$$|x'S|\left|\frac{\partial F}{\partial x}\right| << |F|_{max} \qquad (64)$$

where f_{max} is the maximum value of f within the sector. In most cases, this condition reduces to

$$\frac{x'S}{x_{max}} << \delta, \qquad (65)$$

where δ is a number which is 1 in the linear case and 1/3 in the extreme cubic case. This condition will usually be essentially equivalent to the condition (49) that the betatron wavelength be much longer than the sector length.

5. Comparison with the Method of Sigurgeirsson

Sigurgeirsson* has given a perturbation treatment of the strong focussing problem with very nearly the same results as the method of approximation discussed here, although the approach is different. Sigurgeirsson considers the case where $f(x,s)$ is given by our Eq. (14) (he treats the two dimensional motion), and arrives at an approximate equation of motion of the same form as Eq. (15), with, in our notation,

*T. Sigurgeirsson, "Focussing in a synchrotron with periodic field, perturbation treatment." CERN/T/TS-3, May, 1953.

$$\alpha = 0, \quad \text{(He assumes } \int_0^S g(s)\,ds = 0). \tag{66}$$

$$\beta = \frac{1}{S}\int_0^S g(s)\,g_2(s)\,ds = \overline{g(s)\,g_2(s)}, \tag{67}$$

where $g_2(s)$ is given by our Eq. (18). Sigurgeirsson writes the integrals in Eq. (18) as indefinite integrals and states that the integration constants are to chosen so that $g_2(s)$ is a (presumably periodic) function with mean value zero. In the case $g(s) = 0$ treated by Sigurgeirsson (on the necktie diagonal), Eq. (67) gives the same result for β if our Eq. (18) is used for $g_2(s)$, with the proviso that the reference point $s = 0$ be so chosen that $\int_0^s g(s)\,ds$ have zero mean. Equation (67) can be transformed by integration by parts into a more convenient form:

$$\beta = -\frac{1}{S}\int_0^S \left[\int_0^s g(s)\,ds\right]^2 ds = \overline{\left[\int_0^s g(s)\,ds\right]^2} \tag{68}$$

It can be shown that our formula (17) for β gives the same values as Sigurgeirsson's formula (67) (when $\overline{g(s)} = 0$). Formula (68), due to Sigurgeirsson, is far easier to compute than our formula (17).

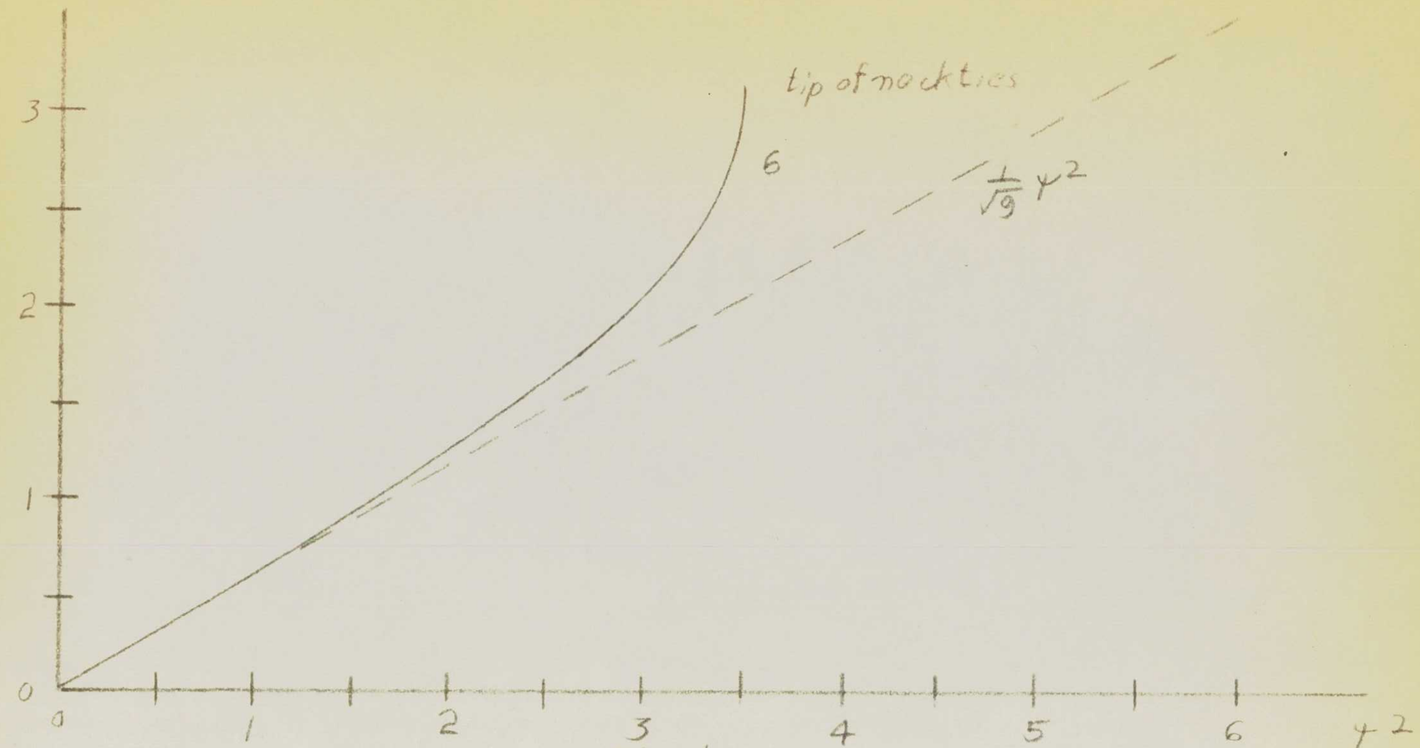
It is interesting that one can arrive directly at Sigurgeirsson's result by the method of Section 1 if we replace Eq. (3) by

$$F(x, x') = \frac{x'_s - x'_0}{s}, \tag{69}$$

a procedure which seems reasonable, but which gives difficulties off the necktie diagonal (a case not treated by Sigurgeirsson), and to which there are certain theoretical objections. I have no doubt one can easily modify Sigurgeirsson's procedure to give results equivalent to ours off the diagonal where $\overline{g(s)} \neq 0$.

6. The Ripple

Once Eq. (2) is solves for the smoothed motion, one can determine the details of the motion within a sector approximately by using Eq. (8) or (10) for the motion within a sector, where x_0, x'_0 are the values for the smooth solution at a nearby reference point. One can thus determine approximately maximum excursions, path lengths, etc. This fact is pointed out by Sigurgeirsson (loc. cit.).



tip of neckties
6
$\frac{1}{\sqrt{9}}\psi^2$
0
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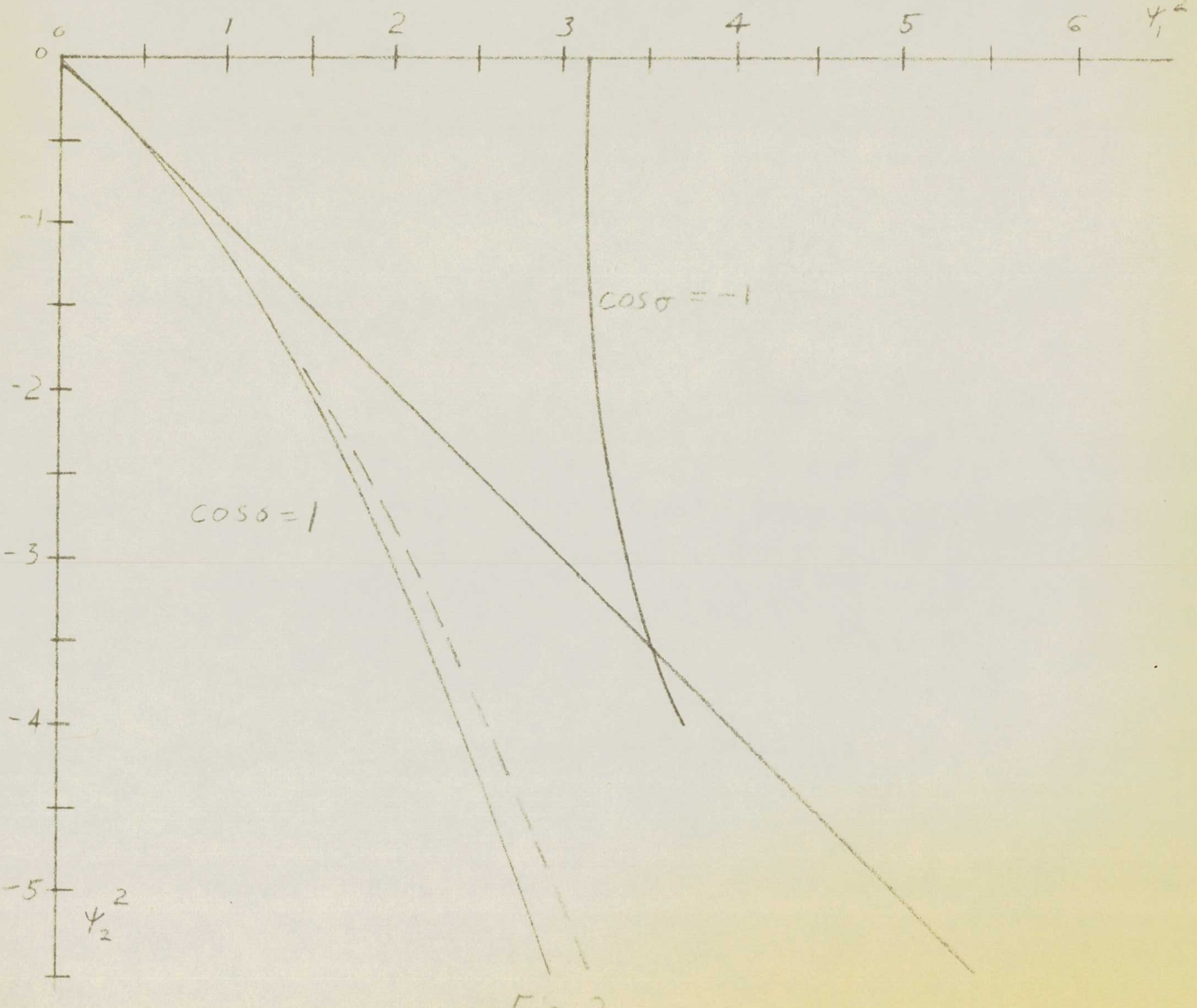
Fig. 1



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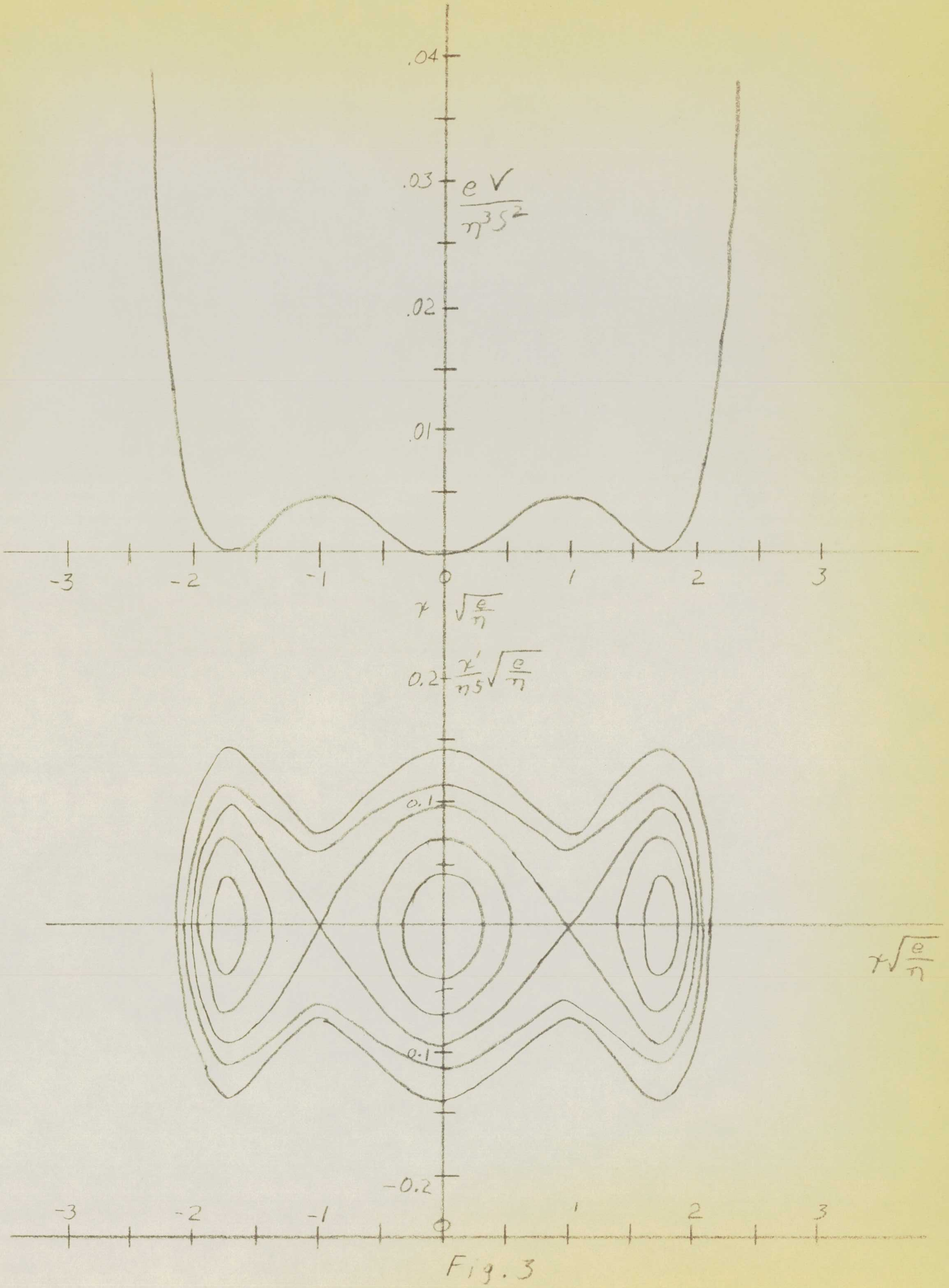
ψ_2^2
$\cos\sigma = -1$
$\cos\sigma = 1$

Fig 2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$x\sqrt{\frac{e}{\eta}}$
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Fig. 3