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PHYSICS AND MATHEMATICS

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TIME DEPENDENCE OF THE SLOWING DOWN OF NEUTRONS WITH CONSTANT SCATTERING MEAN FREE PATH

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ABSTRACT

Simple expressions are derived for the time dependence of the slowing down of neutrons in simple media. In Section I, a method suggested by age theory is used, while in Section II, a two parameter scattering function is employed. Results are compared with exact calculations.

I.

We consider the slowing down of neutrons in an infinite, homogenous medium having a constant scattering cross section and either no capture or "1/v" capture. Under these conditions we derive a simple formula for the time behavior of the slowed down neutrons. The formula has the wrong asymptotic behavior for large t but apparently describes the slowing down of most neutrons rather well. It is most useful for light elements.

We use the notation of Marshak, R.M.P. 19, 185 (1947). Let us first consider the case of no capture and let ℓ_0 be the scattering mean free path, v the neutron velocity, and $u = \ln E_0/E$. Then $\psi_0(u,t)$, the collision density, satisfies the equation

$$\frac{\ell_0}{v}\frac{\partial \psi_0(u,t)}{\partial t} + \psi_0(u,t) = \int_0^u du'\ \psi_0(u',t) f_0(u-u') + \delta(u)\ \delta(t). \qquad (1)$$

Introducing the Laplace transform of the collision density with respect to time,

$$\Phi_0(u,s) = \int_0^\infty e^{-st}\ \psi_0(u,t)dt,$$

we find that $\Phi_0(u,s)$ must satisfy the equation

$$\left(1+\frac{s\ell_0}{v}\right)\Phi_0(u,s) = \int_0^u du'\ \Phi(u',s) f_0(u-u') + \delta(u). \qquad (2)$$

If we are interested in energies well below the source energy, the influence of the $\delta(u)$ is negligible, and when we introduce

$$\phi(u,s) = \left(1+\frac{s\ell_o}{v}\right)\Phi_o(u,s),$$

our equation takes the form

$$\Phi(u,s) = \int_o^u \frac{du'}{1+\frac{s\ell_o}{v'}}\,\phi(u',s)f_o(u-u') \qquad (3)$$

Thus $\phi(u.s)$ satisfies just the same equation as would the collision density in a medium with a capture mean free path v/s. It is well known, therefore, that for sufficiently small $s\ell_o/v$, equation (3) has the solution

$$\phi(u,s) = \exp\left\{-\frac{1}{\xi}\int_o^u \frac{s\ell_o/v'}{1+\frac{s\ell_o}{v'}}\,du'\right\}.$$

Here ξ is the average logarithmic energy loss per collision. This equation is surely valid for $\frac{s\ell_o}{v} \ll 1$, and in some cases is in error by only a few percent for $\frac{s\ell_o}{v} = 1$.

Evaluating the integral we see that

$$\Phi_o(u,s) = C\left(1+\frac{s\ell_o}{v}\right)^{-1-\frac{2}{\xi}} \qquad (4)$$

where terms of order $\frac{s\ell_o}{v_o}$ have been neglected in comparison with unity.

This expression is readily inverted and gives

$$\Psi_o(u,t) = \frac{v}{\ell_o} e^{-x} x^{2/\xi} \Big/ 2\Gamma(\tfrac{2}{\xi}) \qquad (5)$$

where $x = \frac{vt}{\ell_o}$.

We note that $\Psi_o(u,t)$ has the following properties:

a) It is maximum when $x = \frac{2}{\xi}$, corresponding to the result of age theory in which $\Psi_o(u,t) \sim \delta(x - \frac{2}{\xi})$.

b) It is normalized so that

$$\int_o^\infty \frac{\ell_o}{v} \Psi_o(u,t)du = 1; \int_o^\infty \Psi_o(u,t)dt = \frac{1}{\xi}$$

c) $$\langle x^k \rangle = \frac{\int_o^\infty x^k \Psi_o(u,x)dx}{\int_o^\infty \Psi_o(u,x)dx} = \frac{\Gamma(k+\frac{2}{\xi}+1)}{\Gamma(\frac{2}{\xi}+1)} \qquad (6)$$

Now it is known that for large x Ψ_o behaves like $x^{+\frac{2}{1-r^2}} e^{-x}$, where $r = \frac{M-1}{M+1}$. (viz. Marshak, page 196) Thus it is clear that our equation (5) is generally poor for large x. For hydrogen, $\xi = 1$, $r = 0$, and the elementary solution is exact. If $M \gg 1$, $x \gg 1$, $\xi \approx \frac{2}{M}$

behaves as $x^M e^{-x}$, while the exact solution behaves as $x^{\frac{M}{2}} e^{-x}$. To see how good it is in the vicinity of $x = \frac{2}{\xi}$ one may compare the moments of the true distribution (calculated by Placzek and given by Marshak, equation 47) with those given by our equation (6). Exactly $\langle x^n \rangle = n! \prod_{k=1}^{n} (1-\lambda_k)^{-1}$, where

$$\lambda_k = \frac{2}{k+2} \frac{1-r^{k+2}}{1-r^2}$$

The comparison follows for the first ten moments with M = 2 and 15; R = ratio of approximate to exact moment. R_G is the corresponding ratio with Goertzel's solution of Part II.

Evidently our Ψ_o is useful near $x = \frac{2}{\xi}$ and is off by approximately a factor 2 when $x = \frac{2}{\xi} + 10$. Ψ_o is clearly most useful for light scattering elements in which Ψ_o is down by a large value from its maximum when $x = \frac{2}{\xi} + 10$.

If "1/v" capture is present a similar equation may be easily derived. Let ℓ_o be the scattering mean free path, and ℓ the total mean free path. Then the collision density, $\Psi(u,t)$, is given by the approximate expression:

$$\Psi(u,t) = \frac{v}{\ell} e^{-x} (x_o)^{2/\xi} / 2\Gamma(\tfrac{2}{\xi}) \qquad (7)$$

where $x = \frac{vt}{\ell}$, $x_o = \frac{vt}{\ell_o}$. This expression is so normalized that

$$\int_0^{\infty} \frac{\ell}{v} \Psi(u,t)du = e^{-\alpha t},$$

TABLE I

n	R(M=2)	R(M=15)	R_G(M=2)	R_G(M=15)
1	1.0441	1.0197	.9899	.99985
2	1.1031	1.0565	.961	.9983
3	1.168	1.1086	.915	.9947
4	1.235	1.1786	.856	.9887
5	1.300	1.2658	.7882	.9799
6	1.364	1.3716	.717	.9679
7	1.426	1.4972	.645	.9526
8	1.486	1.6449	.575	.9338
9	1.544	1.8170	.509	.9116
10	1.600	2.0164	.447	.8860

where $\ell_{capture} = \frac{v}{\alpha}$. We note that in equation (7), the build-up factor, $x_o^{2/\xi}$, is determined by the scattering properties of the medium; while the attenuation factor, e^{-x}, is determined by the total m.f.p..

II. TIME BEHAVIOR WITH GOERTZEL'S SOLUTION

G. Goertzel and E. Greuling have developed a method of modifying the scattering cross section and scattering function, $f_o(u-u')$, so as to obtain a good approximation to the flux. They use the scattering function

$$f_o(u-u') = \frac{\xi}{\gamma^2} e^{-(u-u')/\gamma}$$

with a scattering cross section equal to $\frac{\xi}{\gamma}$ times the true one. Here ξ is the usual logarithmic energy loss and γ is related to the second moment of the slowing down function, namely

$$\gamma = \frac{1}{(1-r^2)\xi} \left[1-r^2 \left\{ 1- \ln r^2 + \frac{(\ln r^2)^2}{2} \right\} \right] .$$

In this method the flux, $\psi(u,t)$, will satisfy the equation:

$$\frac{\xi}{\gamma} \frac{1}{\ell_o} \psi(u,t) + \frac{1}{v} \frac{\partial \psi(u,t)}{\partial t} = \int_0^u \psi(u',t) \frac{1}{\ell_o} \frac{\xi}{\gamma^2} e^{-(u-u')/\gamma} du' + \delta(u)\, \delta(t), \quad (8)$$

where ℓ_o is the scattering m.f.p. and no capture has been assumed.

Introducing the Laplace transform of ψ with respect to time;

$$\phi(u,s) = \int_0^\infty e^{-st}\psi(u,t)dt,$$

we find that $\phi(u,s)$ satisfies the equation

$$\left(\frac{\xi}{\gamma}\frac{1}{\ell_o} + \frac{s}{v}\right)\phi(u,s) = \int_0^u \phi(u',s)\frac{\xi}{\ell_o\gamma^2}e^{-(u-u')/\gamma}du' + \delta(u). \qquad (9)$$

If we now let

$$\phi(u,s) = X(u,s) + \frac{\delta(u)}{\left(\frac{\xi}{\gamma}\frac{1}{\ell_o} + \frac{s}{v}\right)}$$

and introduce $\ell_o' = \ell_o \frac{\gamma}{\xi}$, we see that $X(u,s)$ must satisfy

$$\left(1 + \frac{s\ell_o'}{v}\right)X(u,s) = \int_0^u X(u',s)\frac{1}{\gamma}e^{-(u-u')/\gamma}du' + \frac{\frac{1}{\gamma}e^{-u/\gamma}}{\left(\frac{1}{\ell_o'} + \frac{s}{v_o}\right)} \qquad (10)$$

This is equivalent to the differential equation

$$\frac{\partial X(u,s)}{\partial u} = -X(u,s)\frac{\frac{s}{v}\left(\frac{1}{\gamma} + \frac{1}{2}\right)}{\frac{1}{\ell_o'} + \frac{s}{v}} \qquad (11)$$

plus the boundary condition

$$X(o,s) = \ell_o'/\gamma \quad (1+ \frac{s\ell_o'}{v_o})^{-2} \; .$$

The solution of this equation is

$$X(u,s) = \frac{\ell_o'}{\xi} \frac{(1+ \frac{s\ell_o'}{v_o})^{2/\gamma -1}}{(1+ \frac{s\ell_o'}{v})^{2/\gamma +1}} \tag{12}$$

We have now only to invert this expression to find the flux. This can be done in terms of a confluent hypergeometric function - viz. Doetsch Laplace Transformation, page 310.

Making use of the formula

$$\int_o^\infty e^{-st} \frac{t^{m-\frac{1}{2}} M_{k,m}(t)dt}{\Gamma(2m+1)} = \frac{(s-\frac{1}{2})^{k-m-\frac{1}{2}}}{(s+\frac{1}{2})^{k+m+\frac{1}{2}}}$$

where

$$M_{k,m}(t) = t^{m+\frac{1}{2}} e^{-\frac{t}{2}} {}_1F_1(m+\frac{1}{2}-k,\; 2m+1;\; t)$$

and ${}_1F_1$ is the usual hypergeometric function,

$${}_1F_1(a,\zeta,t) = 1+ \frac{a}{1!\zeta} t + \frac{a(a+1)}{2!\zeta(\zeta+1)} t^2 + \ldots$$

we find that

$$\psi(u,t) = \frac{1}{\xi \ell_o'} v^{(1+\frac{2}{\gamma})} v_o^{(1-\frac{2}{\gamma})} t \, e^{-\frac{vt}{\ell_o'}} {}_1F_1(1-\frac{2}{\gamma}, 2; \frac{(v-v_o)t}{\ell_o'}) + $$

$$+ v_o \, \delta(u) e^{-v_o t/\ell_o}. \qquad (13)$$

The moments of $\psi(u,t)$ with respect to t, or of $\psi(u,x)$ with respect to x may be computed with the help of equation (12). Thus we may write

$$\langle x^k \rangle = \frac{\int_0^\infty x^k \psi(u,x)dx}{\int_0^\infty \psi(u,x)dx} = (\frac{v}{\ell_o})^k \frac{\xi}{\ell_o'} \frac{\partial^k}{\partial s^k} \left\{ X(u,s) \right\}_{s=0} \qquad (14)$$

which gives

$$\langle x \rangle = \frac{\gamma}{\xi} \left[\frac{2}{\gamma} \frac{v_o - v}{v_o} + \frac{v_o + v}{v_o} \right]$$

$$\langle x^2 \rangle = (\frac{\gamma}{\xi})^2 \left[(\frac{2}{\gamma} \frac{v_o - v}{v_o} + \frac{v_o + v}{v_o})(\frac{2}{\gamma} \frac{v_o - v}{v_o} + 2 \frac{v_o + v}{v_o}) - 2 \frac{v}{v_o} \right].$$

For $v << v_o$, and $\frac{v_o t}{\ell_o'} >> (\frac{2}{\gamma})^2$ $\psi(u,t)$ has the asymptotic form

$$\psi(u,t) \sim \frac{1}{\xi} \frac{\ell_o'}{t} (\frac{vt}{\ell_o'})^{2/\gamma + 1} e^{-\frac{v}{\ell_o'} t_o} \Big/ \Gamma(\frac{2}{\gamma} + 1), \qquad (15)$$

and it follows that the moments will be given by

$$\langle x^k \rangle = \left(\frac{\gamma}{\xi}\right)^k \frac{\Gamma\left(\frac{2}{\gamma}+1+k\right)}{\Gamma\left(\frac{2}{\gamma}+1\right)} \tag{16}$$

It is clear that this asymptotic behavior is not correct, but again the function $\psi(u,t)$ may not be a bad approximation in the region where it is relatively large.

When $M >> 1$, $\gamma \approx \frac{4}{3M}$, so that the ψ of the equation behaves as $x^{\frac{3}{2M}} e^{-\frac{3}{2}x}$. For very large x, it is clear that this solution behaves badly because it has the wrong factor in the exponential, in fact, it is worse than the age solution.

The moments (16) have been compared with the exact moments of Placzek and the results are tabulated for M = 2 and M = 15 in Table I. It is seen that for M = 15, the Goertzel solution is vastly superior to the simple solution. The Goertzel solution is probably superior for $M \geq 3$, save perhaps at very large x where the correct asymptotic build-up factor should be used, anyway.

If "1/v" capture is present, then the Goertzel solution for the flux takes the form, for $v_o >> v$, $\frac{v_o t}{\ell_o'} > \left(\frac{2}{\gamma}\right)^2$

$$\psi(u,t) = \frac{\ell_o'}{\xi t}\left(\frac{vt}{\ell_o'}\right)^{2/\gamma+1} e^{-\left(\frac{v}{\ell_o'}+\alpha\right)t} \Big/ \left(\frac{2}{\gamma}+1\right) \tag{17}$$

where the capture mean free path has been taken to be $\frac{v}{\alpha_m}$.