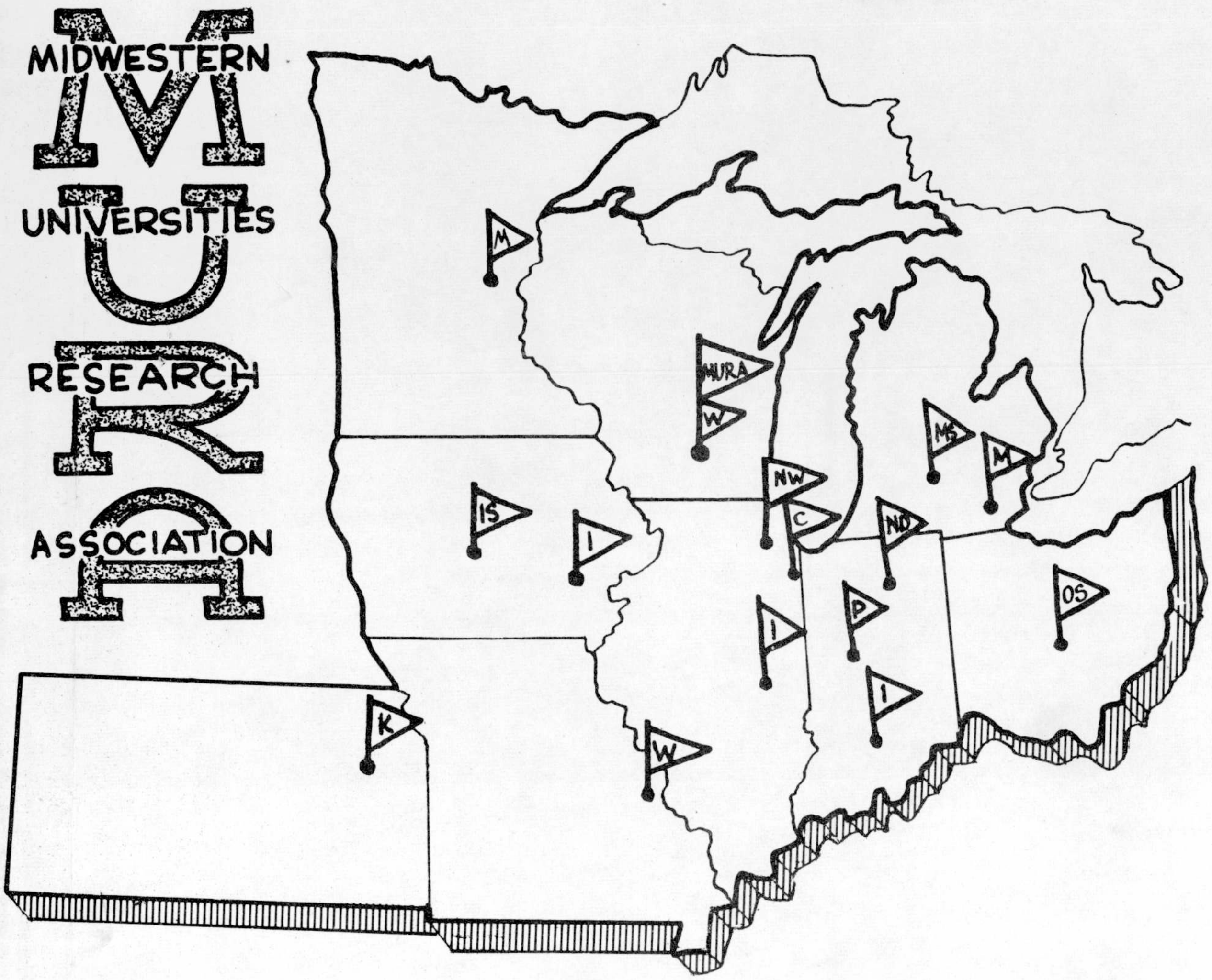

THEORY OF ACCELERATORS WITH A GENERAL MAGNETIC FIELD

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ABSTRACT

This report investigates the particle motion in an accelerator having a general magnetic field. For a wide class of machines, which includes the A.G. synchrotron, FFAG machines, and the fixed frequency cyclotron, a relatively simple solution of the equations of motion has been found. General expressions for the equilibrium orbit, and for the linear motion and tune are obtained. It is expected that the results should be good within an error of about 15% for presently considered machines.

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I. INTRODUCTION

In this report we will investigate the motion of a particle in an accelerator having an arbitrary magnetic field. We will find expressions for the equilibrium orbit, the linear tune and the linear motion. In a series of previous reports[1] we carried through this investigation for the general spiral sector magnetic field which is a scaling machine (the tune remains the same for all energies and orbit radii of the particle). It was stated in these reports that the same methods could be applied with little change to an arbitrary magnetic field. This report will apply the methods of the previous reports to an arbitrary magnetic field.

The method we will use employs two approximations which considerably simplify the equations. Machines presently being considered are such that the use of these two approximations gives an accuracy of 10 to 20% in the results.

The first assumption we will make is that on the equilibrium orbit, $r = r(\theta)$

$$\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2 << 1 \tag{1.1}$$

Equation (1.1) says that the slope of the equilibrium orbit is small, or that the transverse component of the particle's velocity is small compared with the longitudinal component of the velocity.

The second assumption we will make is that

$$\frac{\Delta H_z}{H_z} << 1 \tag{1.2}$$

where ΔH_z is the change in the magnetic field over the radial extent of the equilibrium orbit, and H_z is the maximum value of the magnetic field on the equilibrium orbit.

The percentage variation in H_z over the radial extent of the equilibrium orbit seems to be about 10% for presently considered machines. The error in our calculations is usually of order $(\Delta H_z / H_z)^2$. Eq. (1.1) is best fulfilled for machines with a large N, but even for machines having N = 4, we usually find that $(1/r)^2 (dr/d\theta)^2$ is of the order of 10%.

The above two assumptions are the only restrictions we will put on the magnetic field. We will assume that the magnetic field is given in the median plane (z = 0) and we will write it as

$$H_z = - \sum_{n=-\infty}^{\infty} G_n(r)\, e^{i\omega_n \theta} , \tag{1.3}$$

where $\omega_n = nN$. It is clear that the form of Eq. (1.3) is entirely general. The functions $G_n(r)$ will in general be complex. We can write

$$G_n(r) = H_n(r)\, e^{-i\beta_n(r)} , \tag{1.4}$$

where $H_n(r)$ and $\beta_n(r)$ are real. $\beta_n(r)$ gives the spiralling of the magnets.

II. SUMMARY OF RESULTS

In this section we will summarize the more important results obtained.

The magnetic field in the median plane is written as

$$H_z = - \sum_{n} G_n(r)\, e^{i\omega_n \theta} , \tag{2.1a}$$

where $\omega_n = nN$ and

$$G_n(r) = H_n(r)\, e^{-i\beta_n(r)} \tag{2.1b}$$

$H_n(r)$ and $\beta_n(r)$ being real.

The average radius, R, of the equilibrium orbit depends on the momentum p of the particle. It is easier to find p as a function R and this relation is

$$p = \frac{eR}{2c}\left[H_0(R) + \sqrt{H_0^2(R) + 4\alpha}\right], \tag{2.2a}$$

$$\alpha = 2\sum_{n\geq 1} \frac{1}{\omega_n^2}\left[R H_n'(R) H_n(R) + 2 H_n^2(R)\right]. \tag{2.2b}$$

Equations (2.2) give the relation between the average radius of the equilibrium orbit, R, and the particle momentum, p. The equilibrium orbit $r = r_s(\theta)$ is given by

$$x_s(\theta) = \frac{eR}{pc}\sum_{n\neq 0}\frac{1}{\omega_n^2} G_n(R)\, e^{i\omega_n\theta} \tag{2.3a}$$

where x_s is defined by

$$r_s(\theta) = R(1 + x_s). \tag{2.3b}$$

The r-tune ν_r will depend on the momentum, p, or it may be considered as a function of R, the average radius of the equilibrium orbit. $\nu_r(R)$ is given by

$$\begin{aligned}\nu_r^2 = {} & \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 \\ & + 2\left(\frac{eR}{pc}\right)^2 \sum_{n\geq 1}\frac{1}{\omega_n^2}\Big\{R^2 H_n'' H_n + R^2 H_n'^2 \\ & \qquad + 8R H_n' H_n + 6 H_n^2\Big\}.\end{aligned} \tag{2.4}$$

In Eq. (2.4), the H_n, H_n', H_n'' are to be evaluated at $r = R$. Note also that p depends on R according to Eq. (2.2).

The z - tune ν_z is given by

$$\nu_z^2 = 2\left(\frac{eR}{pc}\right)^2 \sum_{n\geq 1} H_n^2 - \frac{eR}{pc} R H_0'$$
$$+ 2\left(\frac{eR}{pc}\right)^2 \sum_{n\geq 1} \frac{1}{\omega_n^2}\left\{2R^2 H_n^2 \beta_n'^2 - 2R H_n' H_n + R^2 H_n'^2 - R^2 H_n'' H_n\right\}. \qquad (2.5)$$

We may note that both the r-tune, ν_r and the average radius of the equilibrium orbit, R, do not depend on the $\beta_n(r)$ which give the spiralling of the magnets.

In Section V, the above equations are applied to the special case of FFAG scaling machines. Separate reports are planned which will apply the above general results to the A.G. synchrotron and to the constant frequency cyclotron.

III. THE EQUILIBRIUM ORBIT

The procedure we will follow in finding the equilibrium orbit for an accelerator having an arbitrary magnetic field is very similar to that used in the report MURA-258, where the equilibruim orbit for the general spiral sector magnetic field was found.

We will write the magnetic field in the median plane as

$$H_z = - \sum_n G_n(r)\, e^{i\omega_n \theta}, \qquad (3.1)$$

where $\omega_n = nN$.

To find the equilibrium orbit, we must solve the equation

$$\frac{d}{d\theta}\left\{\frac{r'}{\sqrt{r^2+r'^2}}\right\} - \frac{r}{\sqrt{r^2+r'^2}} = \frac{e}{pc}\, r H_z . \tag{3.2}$$

Let us expand the motion about a circle of radius R, where R is chosen to make the motion about this circle as small as possible. The method of choosing R is given later on. It will turn out that R is essentially the average radius of the equilibrium orbit.

Let us then write

$$r = R(1+x) \tag{3.3}$$

Our two assumptions given by Eqs. (1.1) and (1.2) can then be written as

$$\left(\frac{dx}{d\theta}\right)^2 \ll 1 , \tag{3.4a}$$

$$\frac{r G_n'}{G_n}\, x \ll 1 , \tag{3.4b}$$

where all quantities in Eqs. (3.4) are to be evaluated on the equilibrium orbit.

Using Eqs. (3.3) and (3.4a), we can rewrite Eq. (3.2) in terms of x as

$$x'' - (1+x) = \frac{eR}{pc}(1+x)^2 H_z . \tag{3.5}$$

To solve Eq. (3.5) for the equilibrium orbit, we make use of the assumption that the magnetic field does not vary very much over the radial extent of the orbit. We can then expand H_z in powers of x and we will keep up to linear terms in x.

$$H_z(r,\theta) = H_z(R,\theta) + H_{z,r}(R,\theta)\, Rx + \ldots\ldots \tag{3.6}$$

where $H_{z,r}$ means $\partial H_z/\partial r$.

Putting this expansion for H_z back into Eq. (3.5) gives

$$x'' - (1+x) = \frac{eR}{pc}(1+x)^2 (H_z + H_{z,r} R x + \dots)$$

$$= \frac{eR}{pc}\left\{H_z + x\,(R H_{z,r} + 2H_z) + \dots\right\},$$

$$x'' + x\left[-\frac{eR}{pc}(R H_{z,r} + 2H_z) - 1\right] = 1 + \frac{eR}{pc} H_z + \dots \; . \tag{3.7}$$

In Eq. (3.7), H_z, $H_{z,r}$, are to be evaluated at $r = R$.

Eq. (3.7) can also be written as

$$x'' + (E_s - g(\theta)) = f(\theta), \tag{3.8a}$$

where

$$E_s - g(\theta) = -\frac{eR}{pc}(R H_{z,r} + 2H_z) - 1, \tag{3.8b}$$

$$f(\theta) = 1 - \frac{eR}{pc} H_z, \tag{3.8c}$$

and the Fourier components of $g(\theta)$ and $f(\theta)$ are given by

$$g_n = -\frac{eR}{pc}(R G_n' + 2G_n), \tag{3.9a}$$

$$E_s = +\frac{eR}{pc}(R G_0' + 2G_0) - 1, \tag{3.9b}$$

$$f_n = \frac{eR}{pc} G_n, \tag{3.9c}$$

$$f_0 = 1 + \frac{eR}{pc} G_0. \tag{3.9d}$$

Note that the definition of E_s is to make $g_0 = 0$, and the g_n and f_n are defined by $g(\theta) = \sum_n g_n \exp(i w_n \theta)$, $f_n = \sum_n f_n \exp(i w_n \theta)$. Note also that the G_n in Eqs. (3.9) are to be evaluated at r = R.

Now, following the procedure of MURA-258, we solve Eq. (3.8) by expanding x in the Flocquet solutions, $u_\nu(\theta)$ of the equation

$$u'' + (E - g(\theta))\,u = 0 \tag{3.10}$$

which is just the left side of Eq. (3. 8), except that E_s is replaced by E , and E may take on all values for which Eq. (3.10) has stable solutions.

Thus, as in MURA-258, we find the solution

$$\chi = \sum_n u_n(\theta) \frac{1}{E_s - E_n} \int d\theta'\, u_n^*(\theta')\, f(\theta'), \tag{3.11}$$

where $u_n(\theta)$ is the solution of Eq. (3.10) corresponding to the E -value E_n and having the tune $\nu_n = nN,\ n = 0, \pm 1, \pm 2 \ldots$

We now choose R so that the term in the expansion Eq. (3.11) which corresponds to $\nu_n = 0$ vanishes. This means that R is determined by the condition

$$\int d\theta\, u_0^*(\theta)\, f(\theta) = 0, \tag{3.12}$$

where $u_0(\theta)$ is the Flocquet solution of Eq. (3.10) having the tune $\nu = 0$.

The Flocquet solution $u_0(\theta)$ is given by[2]

$$u_0(\theta) = 1 - \sum_{n \neq 0} \frac{g_n}{w_n^2} e^{i w_n \theta}, \tag{3.13}$$

where $g(\theta) = \sum_{n \neq 0} g_n \exp(i w_n \theta)$, and the g_n are given by Eq. (3.9) as

$$g_n = -\frac{eR}{pc}\left(RG_n' + 2G_n\right). \tag{3.14}$$

Using this expression for $\mu_0(\theta)$ and Eqs. (3.9) for $f(\theta)$, Eq. (3.12) becomes

$$1 - \frac{eR}{pc}G_0 - \left(\frac{eR}{pc}\right)^2 \sum_{n\neq 0} \frac{1}{\omega_n^2} G_{-n}\left\{RG_n' + 2G_n\right\} = 0 \tag{3.15}$$

Eq. (3.15) determines R, which will turn out to be the average orbit radius. We can also write this equation in terms of H_n, β_n where $G_n = H_n \exp(-i\beta_n)$.

$$G_n' = e^{-i\beta_n}\left\{H_n' - iH_n\beta_n'\right\}, \tag{3.16}$$

and

$$G_{-n}G_n' = H_nH_n' - iH_n^2\beta_n',$$

since $H_{-n} = H_n$ and $\beta_{-n} = -\beta_n$:

Thus the equation for R becomes

$$1 - \frac{eR}{pc}H_0 - 2\left(\frac{eR}{pc}\right)^2 \sum_{n\geq 1} \frac{1}{\omega_n^2}\left\{RH_n'H_n + 2H_n^2\right\} = 0. \tag{3.17}$$

It may be noted that $\beta_n(r)$, which gives the spiralling of the magnets, does not enter into the equation for the average orbit radius, R.

Equation (3.17) can be regarded as an equation which determines the average orbit radius R as a function of the particle momentum, p. The equation, however, can be solved more easily for p as a function of R, and we find that

$$p = \frac{eR}{2c}\left[H_0 + \sqrt{H_0^2 + 4\alpha}\right], \tag{3.18a}$$

where

$$\alpha = 2 \sum_{n \geq 1} \frac{1}{\omega_n^2} \left[R H_n' H_n + 2 H_n^2 \right] . \tag{3.18b}$$

The equilibrium orbit is now found from Eq. (3.11), following MURA-258,

$$x = \frac{eR}{pc} \sum_{n \neq 0} \frac{1}{\omega_n^2 - E_s} G_n(R) e^{i \omega_n \theta} , \tag{3.19a}$$

where

$$E_s = \frac{eR}{pc} (R G_0' + 2 G_0) - 1 , \tag{3.19b}$$

One can usually neglect E_s compared with ω_n^2, if $\nu_r << N$, ν_r being the linear r-tune.

The circumference factor C is given by the maximum value of the function $C(\theta)$ (see MURA-258), where

$$C(\theta) = \frac{eR}{pc} H_z ,$$

$$C(\theta) = \frac{eR}{pc} \sum_n G_n(r) e^{i \omega_n \theta} , \tag{3.20}$$

where in Eq. (3.20), r is evaluated on the equilibrium orbit.

We can then write for the $G_n(r)$ in Eq. (3.20)

$$G_n(r) = G_n(R) + G_n'(R) R x + \cdots , \tag{3.21a}$$

and

$$C(\theta) = \frac{eR}{pc} \left\{ \sum_n \left[G_n(R) + G_n'(R) R x \right] e^{i \omega_n \theta} \right. \tag{3.21b}$$

The function $C(\theta)$ given by Eq. (3.21b) must be maximized to find the circumference factor C. This is difficult to do for a general magnetic field; however, we can get a good estimate of C by assuming that the zeroth and first harmonics dominate in H_z and neglecting the second term in Eq. (3.21b). We find then that

$$C = \frac{eR}{pc} \left\{ H_0(R) + 2 H_1(R) \right\} \tag{3.22}$$

IV. THE LINEAR TUNE AND THE LINEAR MOTION

To treat the linear motion, we will use essentially the same procedure as in MURA-273 where the linear motion was investigated for the general spiral sector magnetic field.

First let us consider the radial oscillation. The equation of motion in the radial direction is given by

$$\frac{d}{d\theta} \frac{r'}{\sqrt{r^2+r'^2+z'^2}} - \frac{r'}{\sqrt{r^2+r'^2+z'^2}} = F_r , \tag{4.1a}$$

where

$$F_r = \frac{e}{pc}\left(r H_z - z' H_\theta \right) . \tag{4.1b}$$

We wish to expand this equation about the equilibrium orbit. Let $r = r_s(\theta)$ represent the equilibrium orbit. $r_s(\theta)$ has been found in Section III. Let $r_s = R(1+x_s)$, then x_s is given by Eq. (3.19) as

$$x_s = \frac{eR}{pc} \sum_{n\neq 0} \frac{1}{\omega_n^2} G_n(R) e^{i\omega_n \theta} . \tag{4.2}$$

Expanding about the equilibrium orbit will be simpler if we first make use of our assumption that $(dr/d\theta)^2/r^2 << 1$ to simplify Eq. (4.1a).

Eq. (4.1a) can then be written as

$$r'' - r = r F_r \tag{4.3}$$

Now let us introduce the variable ρ which will give the radial displacement relative to the equilibrium orbit. ρ is defined by

$$r = r_s(\theta) + \rho. \tag{4.4}$$

We now expand Eq. (4.3) in powers of ρ keeping only up to terms which are linear in ρ,

$$\begin{aligned} & r_s'' - r_s + \rho'' - \rho \\ & \quad = \frac{e}{pc}(r_s + \rho)^2 \left(H_z + H_{z,r}\,\rho + \dots \right), \end{aligned} \tag{4.5}$$

or

$$\rho'' + \rho \left[- \frac{e r_s}{pc} (r_s H_{z,r} + 2 H_z) - 1 \right] = 0 .$$

In Eq. (4.5), H_z and $H_{z,r}$ are to be evaluated at $r = r_s(\theta)$. In obtaining Eq. (4.5) we used the result that $r_s(\theta)$ satisfies the equation

$$r_s'' - r_s = \frac{e}{pc} r_s^2 H_z . \tag{4.6}$$

We introduce the dimensionless variable μ,

$$\mu = \frac{1}{R} \rho, \tag{4.7}$$

where R is the average radius of the equilibrium orbit, and write Eq. (4.5) as

$$\mu'' + n_x(\theta) \mu = 0, \tag{4.8a}$$

where

$$n_x(\theta) = - \frac{e r_s}{pc} \left(r_s H_{z,r} + 2 H_z \right) - 1 \tag{4.8b}$$

In the expression for $n_x(\theta)$, H_z and $H_{z,r}$ are to be evaluated at $r = r_s(\theta)$. To do this we write $r_s = R(1+x_s)$ and then expand H_z and $H_{z,r}$ about r = R. Thus

$$H_z(r_s,\theta) = H_z(R,\theta) + H_{z,r}(R,\theta)\, R x_s + \dots, \tag{4.9a}$$

$$H_{z,r}(r_s,\theta) = H_{z,r}(R,\theta) + H_{z,rr}(R,\theta)\, R x_s + \dots, \tag{4.9b}$$

and we can write $n_x(\theta)$ as

$$\begin{aligned} n_x(\theta) = -\frac{eR}{pc}(1+x_s)\Big[& R(1+x_s)(H_{z,r} + H_{z,rr} R x_s + \dots) \\ & + 2(H_z + H_{z,r} R x_s \dots)\Big] - 1, \end{aligned} \tag{4.10a}$$

$$\begin{aligned} n_x(\theta) = -\frac{eR}{pc}\Big\{ & R H_{z,r} + 2H_z \\ & + x_s\left[R^2 H_{z,rr} + 4R H_{z,r} + 2H_z\right]\Big\} - 1, \end{aligned} \tag{4.10b}$$

where in Eq. (4.10b), H_z and $H_{z,r}$ are now to be evaluated at r = R.

For finding the tune, it is more convenient to write Eq. (4.8) as

$$u'' + (E_0 - g(\theta))\, u = 0, \tag{4.11a}$$

where

$$\begin{aligned} E_0 = {} & \frac{eR}{pc}(RG_0' + 2G_0) \\ & + \left(\frac{eR}{pc}\right)^2 \sum_{m\neq 0} \frac{G_{-m}}{\omega_m^2}\left(R^2 G_m'' + 4G_m' + 2G_m\right) - 1, \end{aligned} \tag{4.11b}$$

or

$$E_0 = \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 + 2\left(\frac{eR}{pc}\right)^2 \sum_{m\geq 1} \frac{1}{\omega_m^2}\left\{R^2 H_m H_m'' - R^2 H_m^2 \beta_m'^2 + 4RH_mH_m' + 2H_m^2\right\},$$

$$g_n = -\left(\frac{eR}{pc}\right)\left(RG_n' + 2G_n\right) - \left(\frac{eR}{pc}\right)^2 \sum_{m\neq n} \frac{G_{n-m}}{\omega_{n-m}^2}\left(R^2 G_m'' + 4RG_m' + 2G_m\right). \qquad (4.11c)$$

The g_n are the coefficients in the expansion $g(\theta) = \sum_n g_n \exp(i\omega_n\theta)$, and E_0 is so defined that $g_0 = 0$. In Eqs. (4.11), the G_n are to be evaluated at r = R.

Eqs. (4.11) are the equations giving the linear radial motion. They can be solved to give the r-motion and the r-tune. The second term in Eq. (4.11c) for g_n can usually be neglected.

Let us find the tune of the radial motion, ν_r. The r-tune ν_r depends on E_0 and g_n according to the relation

$$\nu_r^2 = E_0 + \sum_{n\neq 0} \frac{|g_n|^2}{\omega_n^2}, \qquad (4.12)$$

if $g_n/N^2 \ll 1$ and $(\nu/N)^2 \ll 1$.

If we substitute for E_0 and g_n the expressions given in Eqs. (4.11), we find for the r-tune

$$\nu_r^2 = \frac{eR}{pc}\left(RG_0' + 2G_0\right) - 1 + \left(\frac{eR}{pc}\right)^2 \sum_{m\neq 0} \frac{1}{\omega_m^2}\left\{G_{-m}\left[R^2 G_m'' + 4RG_m' + 2G_m\right] + \left|RG_m' + 2G_m\right|^2\right\}. \qquad (4.13)$$

We can write Eq. (4.13) in terms of H_n and β_n, by using

$$G_n = H_n e^{-i\beta_n},$$

$$G_n' = e^{-i\beta_n}\left(H_n' - iH_n\beta_n'\right),$$

$$G_n'' = e^{-i\beta_n}\left(H_n'' - i2H_n'\beta_n' - iH_n\beta_n - H_n\beta_n^2\right), \tag{4.14}$$

and $H_n = H_{-n}$, $\beta_n = -\beta_{-n}$.

We find that

$$\begin{aligned} \nu_r^2 = {} & \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 \\ & + 2\left(\frac{eR}{pc}\right)^2 \sum_{m\geq 1} \frac{1}{\omega_m^2}\Big\{R^2 H_m H_m'' - R^2 H_m^2 \beta_m'^2 \\ & \qquad + 4RH_m H_m' + 2H_m^2 \\ & \qquad + \left(RH_m' + 2H_m\right)^2 + R^2\beta_m'^2 H_m^2\Big\}, \end{aligned} \tag{4.15}$$

or

$$\begin{aligned} \nu_r^2 = {} & \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 \\ & + 2\left(\frac{eR}{pc}\right)^2 \sum_{m\geq 1} \frac{1}{\omega_m^2}\Big\{R^2 H_m H_m'' + R^2 H_m'^2 \\ & \qquad + 8RH_m H_m' + 6H_m^2\Big\}. \end{aligned} \tag{4.16}$$

It may be noticed that the magnet spiralling which is given by $\beta_n(r)$ does not contribute to the r-tune.

To complete the treatment of the linear r-motion, we should find the Flocquet solutions of Eq. (4.11). The r-motion is given by a linear combination of U_ν and U_ν^*, and U_ν is given by (see MURA-273)

$$u_\nu = e^{i\nu\theta} \left\{ 1 - \sum_{n \neq 0} \frac{g_n}{\omega_n^2} \frac{1}{1 + 2\frac{\nu}{\omega_n}} e^{i\omega_n \theta} \right\}, \tag{4.16a}$$

where the g_n are given by Eq. (4.11c).

The linear z -motion can be treated in the same way. The equation of motion in the z -direction is

$$\frac{d}{d\theta} \frac{z'}{\sqrt{r^2 + r'^2 + z'^2}} = F_z , \tag{4.17a}$$

where

$$F_z = \frac{e}{pc} \left(r' H_\theta - r H_r \right). \tag{4.17b}$$

Before expanding Eq. (4.17) in powers of z , we use our assumption, that $(dz/d\theta)^2/r^2 << 1$, to simplify the equation, and we get

$$z'' = r F_z \tag{4.18}$$

The expansion for F_z up to linear terms is (see MURA-273)

$$F_z = \frac{e}{pc} \left(\frac{r_s'}{r_s} H_{z,\theta} - r_s H_{z,r} \right) z + \ldots , \tag{4.19}$$

where $H_{z,\theta}$ and $H_{z,r}$ are to be evaluated on the equilibrium orbit where $r = r_s(\theta)$, $z = 0$.

The expansion for $r F_z$ is then

$$r F_z = (r_s + \rho) F_z$$

$$r F_z = \frac{e r_s}{pc} \left(\frac{r_s'}{r_s} H_{z,\theta} - r_s H_{z,r} \right) z + \ldots \tag{4.20}$$

Eq. (4.18) expanded up to linear terms in z is then given by

$$z'' - z\left[\frac{e r_s}{pc}\left(\frac{r_s'}{r_s} H_{z,\theta} - r_s H_{z,r}\right)\right] = 0. \tag{4.21}$$

We introduce the variable y,

$$y = \frac{1}{R} z \tag{4.22}$$

and write Eq. (4.21) as

$$y'' + n_y(\theta)\, y = 0, \tag{4.23a}$$

$$n_y = \frac{-e r_s}{pc}\left(\frac{r_s'}{r_s} H_{z,\theta} - r_s H_{z,r}\right). \tag{4.23b}$$

In the expression for $n_y(\theta)$, $H_{z,\theta}$ and $H_{z,r}$ are to be evaluated at $r = r_s(\theta)$. To do this, we write $r_s = R(1 + x_s)$ and then expand $H_{z,r}$ and $H_{z,\theta}$ about r = R. Thus

$$H_{z,\theta} = H_{z,\theta}(R,\theta) + H_{z,\theta r}(R,\theta)\, R x_s + \cdots, \tag{4.24a}$$

$$H_{z,r} = H_{z,r}(R,\theta) + H_{z,rr}(R,\theta)\, R x_s + \ldots, \tag{4.24b}$$

and we can write $n_y(\theta)$ as

$$n_y(\theta) = -\frac{eR(1+x_s)}{pc}\Big[x_s' H_{z,\theta} - R(1+x_s)\left(H_{z,r} + H_{z,rr} R x_s + ..\right)\Big] \tag{4.25a}$$

$$n_y(\theta) = -\frac{eR}{pc}\Big\{x_s' H_{z,\theta} - R H_{z,r} - x_s\left[R^2 H_{z,rr} + 2R H_{z,r}\right]\Big\}, \tag{4.25b}$$

where in Eq. (4.25b), $H_{z,\theta}$, $H_{z,r}$ and $H_{z,rr}$ are now to be evaluated at r = R.

It is more convenient to write Eq. (4.23) as

$$y'' + (E_0' - f(\theta))\, y = 0, \tag{4.26a}$$

where

$$E_0' = \left(\frac{eR}{pc}\right)^2 \sum_{m \neq 0} |G_m|^2 - \left(\frac{eR}{pc}\right) R\, G_0'$$

$$- \left(\frac{eR}{pc}\right)^2 \sum_{m \neq 0} \frac{G_{-m}}{w_m^2} \left\{ R^2 G_m'' + 2 R G_m' \right\}, \tag{4.26b}$$

or

$$E_0' = 2\left(\frac{eR}{pc}\right)^2 \sum_{m \geq 1} |H_m|^2 - \left(\frac{eR}{pc}\right) R\, H_0'$$

$$- 2\left(\frac{eR}{pc}\right)^2 \sum_{m \geq 1} \frac{1}{w_m^2} \left\{ R^2 H_m H_m'' - R^2 H_m^2 \beta_m'^2 + 2 R H_m' H_m \right\},$$

$$f_m = - \frac{eR}{pc} R\, G_m' \tag{4.26c}$$

The f_m are the coefficients in the expansion $f(\theta) = \sum_m f_m \exp(i w_m \theta)$, and E_0' is so defined that $f_0 = 0$. In Eqs. (4.26), the G_n are evaluated at r = R, and in finding the f_m we have dropped the terms which are linear in x_s in Eq. (4.25b).

Eqs. (4.26) are the equations giving the linear z-motion. They can be solved to give the z-motion and the z-tune.

The z-tune depends on E_0' and f_m according to the relation

$$\nu_z^2 = E_0' + \sum_{n \neq 0} \frac{1}{w_n^2} |f_n|^2, \tag{4.27}$$

if $f_n / N^2 \ll 1$ and $(\nu_z / N)^2 \ll 1$.

If we substitute for E_0' and f_n the expressions given in Eqs. (4.26), we find for the z -tune

$$\nu_z^2 = \left(\frac{eR}{pc}\right)^2 \sum_{m \neq 0} |G_m|^2 - \frac{eR}{pc} R G_0' + \left(\frac{eR}{pc}\right)^2 \sum_{m \neq 0} \frac{1}{\omega_m^2} \Big\{ |R G_m'|^2 - G_{-m} \left[R^2 G_m'' + 2 R G_m' \right] \Big\} . \tag{4.28}$$

We can write Eq. (4.28) in terms of H_n and β_n as

$$\nu_z^2 = 2 \left(\frac{eR}{pc}\right)^2 \sum_{m \geq 1} H_m^2 - \left(\frac{eR}{pc}\right) R H_0' + 2 \left(\frac{eR}{pc}\right)^2 \sum_{m \geq 1} \frac{1}{\omega_m^2} \Big\{ R^2 H_m'^2 + R^2 H_m^2 \beta_m'^2 - \left(R^2 H_m H_m'' - R^2 H_m^2 \beta_m'^2 + 2 R H_m H_m' \right) \Big\} , \tag{4.29}$$

or

$$\nu_z^2 = 2 \left(\frac{eR}{pc}\right)^2 \sum_{m \geq 1} H_m^2 - \left(\frac{eR}{pc}\right) R H_0' + 2 \left(\frac{eR}{pc}\right)^2 \sum_{m \geq 1} \frac{1}{\omega_m^2} \Big\{ 2 H_m^2 R^2 \beta_m'^2 + R^2 H_m'^2 - R^2 H_m H_m'' - 2 R H_m H_m' \Big\} . \tag{4.30}$$

The Flocquet solutions of Eqs. (4.26) may be found from Eq. (4.16a) by simply replacing the g_n by the f_n given in Eq. (4.26c).

V. APPLICATION TO FFAG MACHINES

In this section we will illustrate the general results derived in SectionIII and IV by applying them to a particular machine. We will apply them to the scaling FFAG machine. The results for the scaling FFAG machine have been extensively checked by numerical calculations and have been found to be good within an error of about 15%.

Scaling FFAG Machine

For this machine, the magnetic field in the median plane is given by

$$H_z = -H\left(\frac{r}{r_1}\right)^K \sum_n h_n e^{i w_n \varphi} , \tag{5.1a}$$

where

$$\varphi = \theta - \frac{1}{wN} \ln \frac{r}{r_1} . \tag{5.1b}$$

Thus for this field the $G_n(r)$ are

$$G_n(r) = -H\left(\frac{r}{r_1}\right)^K h_n e^{-i\frac{n}{w} \ln(r/r_1)} , \tag{5.2a}$$

and

$$H_n(r) = -H\left(\frac{r}{r_1}\right)^K |h_n| , \tag{5.2b}$$

$$\beta_n(r) = -\frac{n}{w} \ln \frac{r}{r_1} + \alpha_n , \tag{5.2c}$$

where

$$h_n = |h_n| e^{i\alpha_n} . \tag{5.2d}$$

We will now find the tune ν_r and ν_z for the machine using Eqs. (4.16) and (4.30). The first step is to find $p(R)$, the connection between the momentum p and the average radius of the equilibrium orbit, R. This is given by Eq. (3.18),

$$p = \frac{eR}{2c}\left[H_0 + \sqrt{H_0^2 + 4\alpha}\right], \tag{5.3a}$$

$$\alpha = 2\sum_{n\geq 1}\frac{1}{\omega_n^2}\left[R H_n' H_n + 2H_n^2\right]. \tag{5.3b}$$

Thus we find

$$\alpha = 2H^2\left(\frac{R}{r_1}\right)^{2K}\sum_{n\geq 1}\frac{1}{\omega_n^2}|h_n|^2(K+2), \tag{5.4a}$$

and

$$p = \frac{eHR}{2c}\left(\frac{R}{r_1}\right)^K\left[h_0 + \sqrt{h_0^2 + 8\sum_{n\geq 1}\frac{1}{\omega_n^2}|h_n|^2(K+2)}\right]. \tag{5.4b}$$

The tune ν_r is given by

$$\begin{aligned}\nu_r^2 = {} & \frac{eR}{pc}\left(RH_0' + 2H_0\right) - 1 \\ & + 2\left(\frac{eR}{pc}\right)^2\sum_{n\geq 1}\frac{1}{\omega_n^2}\left\{R^2 H_n H_n'' + R^2 H_n'^2 \right. \\ & \left. + 8RH_nH_n' + 6H_n^2\right\}.\end{aligned} \tag{5.5}$$

We note that

$$R H_n' = K H_n, \tag{5.6a}$$

$$R^2 H_n'' = K(K-1)H_n \tag{5.6b}$$

and that

$$\frac{eRH}{pc} = b\left(\frac{R}{r_1}\right)^{-K} , \tag{5.7a}$$

where

$$b = \frac{2}{h_0 + \sqrt{h_0^2 + 8\sum\limits_{n\geq 1}\frac{1}{\omega_n^2}|h_n|^2(K+2)}} . \tag{5.7b}$$

We find then for ν_r

$$\nu_r^2 = b\,h_0\,(K+2) - 1 + 2b^2\sum_{n\geq 1}\frac{1}{\omega_n^2}|h_n|^2\left[K(K-1) + K^2 + 8K + 6\right] , \tag{5.8a}$$

or

$$\nu_r^2 = bh_0(K+2) - 1 + 2b^2\sum_{n\geq 1}\frac{1}{\omega_n^2}|h_n|^2\left[2K^2 + 7K + 6\right] . \tag{5.8b}$$

The tune ν_z is given by

$$\nu_z^2 = 2\left(\frac{eR}{pc}\right)^2\sum_{n\geq 1} H_n^2 - \frac{eR}{pc}RH_0' + 2\left(\frac{eR}{pc}\right)^2\sum_{n\geq 1}\frac{1}{\omega_n^2}\left\{2R^2H_n^2\beta_n'^2 - 2RH_nH_n' + R^2H_n'^2 - R^2H_nH_n''\right\} . \tag{5.9}$$

Thus

$$\nu_z^2 = 2b^2 \sum_{n \geq 1} |h_n|^2 - b h_0 K$$

$$+ 2b^2 \sum_{n \geq 1} \frac{1}{\omega_n^2} |h_n|^2 \left\{ 2 \frac{n^2}{w^{-2}} - 2K + K^2 - K(K-1) \right\},$$

(5.10a)

or

$$\nu_z^2 = 2b^2 \sum_{n \geq 1} |h_n|^2 \left\{ 1 + \frac{2}{w^{-2} N^2} - \frac{K}{\omega_n^2} \right\} - b h_0 K$$

(5.10b)

The above results are essentially the same results found in reports MURA-258 and MURA-273 for the scaling FFAG machine.

REFERENCES

1. G. Parzen, MURA-258 and MURA-273
2. G. Parzen, MURA-200