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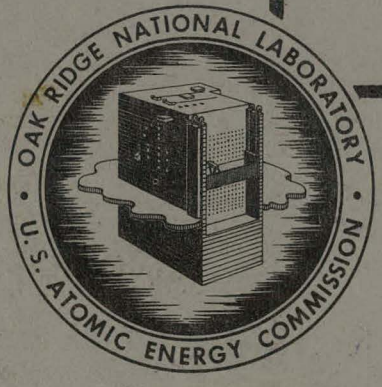
THEORY OF NEUTRON CHAIN REACTIONS

Volume II, Part I

HOMOGENEOUS NUCLEAR CHAIN REACTIONS



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THEORY OF NEUTRON CHAIN REACTIONS

Volume II, Part I

HOMOGENEOUS NUCLEAR CHAIN REACTIONS

Chapter V. NEUTRON CHAIN REACTIONS

Chapter VI. PILE EQUATIONS

Chapter VII. THEORY OF REFLECTORS AND THE METHOD OF GROUPS

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August 10, 1951

THEORY OF NEUTRON CHAIN REACTIONS

The following notes comprise Part I, Volume II, of a book on neutron chain reactions which is largely an outgrowth of a course given in the original Clinton Laboratories Training School. A first draft of some of these notes has already appeared. Because of the demands of the reactor program, we have considered it best to put out a second preliminary edition even though there are inadequacies in the text.

Part II, Volume II, is still in preparation as is Volume III entitled "Lattice Theory".

VOLUME II, PART I

HOMOGENEOUS NUCLEAR CHAIN REACTIONS

INTRODUCTION

The previous section of this book deals with the general problem of neutron diffusion. In the sequel we shall apply the results obtained already to the theory of slow neutron chain reacting systems.

A neutron chain reacting system or "pile" is a device containing fissionable material (such as U^{235}, Pu^{239}, U^{233}) which produces neutrons as the result of neutron induced fission in the fissionable material. It is not necessary that the reaction be divergent or potentially divergent for it to be called a chain reaction: a small piece of ordinary uranium is a chain reacting system (although a very inefficient one) since, if neutrons are introduced into the system, more neutrons will be produced from neutrons which are absorbed in the U^{235} and induce fission. The word "chain" thus refers to the fact that the agency which causes fission, namely, the neutrons, are themselves produced as a result of fission. In this way every fission event gives rise to a chain of subsequent fissions; every neutron gives rise to a sequence of daughter neutrons.

A chain reaction can be divergent, convergent, or critical. If more neutrons are produced by neutron induced fission(1) than are absorbed in the system or leak out, the system is said to be <u>divergent</u>. If fewer neutrons are produced than are absorbed or leak out, the system is <u>convergent</u>.

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(1) Neutrons produced by an extraneous source, such as Ra-Be, are not counted here.

And if the number of neutrons produced is just equal to the total number which are absorbed or which leak out, the system is critical.

Evidently any chain reacting system in which more neutrons are produced than are absorbed can be made divergent by building the system large enough. For, as the size of the system is increased, the relative importance of neutron leakage becomes less; in an infinite system no neutrons are lost by leakage. The size of a chain reacting system at which the neutron production from neutron induced fission is just balanced by the leakage and the absorption is called the critical size. Of course the critical size will depend on the shape of the system -- a critical sphere requires less material than say a cylinder, since, because of its smaller surface to volume ratio the leakage of neutrons out of a sphere is smaller than out of a cylinder of the same volume. For a given shaped system the critical size depends both on the number of neutrons produced per neutron absorbed inside the system -- that is, on the multiplication constant (denoted by k_∞ or simply k) -- and also on a length, the migration length (denoted by M), which characterizes the scale of the neutron diffusion processes.

The multiplication constant and the migration length depend on the nuclear properties, such as capture cross sections, mean free paths, number of neutrons produced per fission, etc., of the materials contained in the chain reacting system. They also depend on the configuration of the materials contained in the system -- whether the fissionable material is lumped or spread out, the mass ratio of fissionable to non-fissionable material, etc. The calculation of the multiplication constant and migration length comprises "microscopic" pile theory -- "microscopic" because the multiplication constant and migration length are intensive properties of a pile which are character-

istic of the small scale structure of the pile and are nearly independent of the gross size of the system.

From the microscopic properties of a chain reacting pile it is possible to derive various extensive quantities -- for example, the critical size of the system or the neutron distribution. All of these quantities depend upon the size of the chain reacting system; they can be computed without reference to the detailed structure of the pile once the multiplication constant and migration length are known. For this reason their calculation is said to comprise "macroscopic" pile theory. In the present section we shall give an exposition of macroscopic pile theory. Microscopic pile theory will be treated in the final section.

CHAPTER V

NEUTRON CHAIN REACTIONS

Nuclear Reactions in Fissionable Materials

The possibility of establishing a neutron chain reaction depends upon the fact that certain heavy isotopes, notably those with an odd number of neutrons, such as U^{235} and Pu^{239}, undergo fission when they absorb neutrons. The fission act is accompanied by the emission of several fast neutrons; the energy of these fast neutrons is greater than the neutron energy required to induce fission.

It is important that the energy of the fission neutrons exceed the fission threshold; otherwise the produced neutrons could not induce further fission. It may be mentioned that the idea of a neutron chain reaction based on the $Be^9(n,2n)Be^8$ reaction was proposed by L. Szilard in about 1933, at which time the reaction was erroneously believed to be exothermic. Since then all n-2n reactions have been shown to be endothermic.

For fissionable species containing an odd number of neutrons, the fission threshold is below thermal energy ($\sim$0.025 ev), and so these nuclei fission with thermal neutrons. Probably all the known even neutron fissionable nuclei, except some isotopes of elements 94 or higher, have fission thresholds many kilovolts above thermal energy. This difference between the even and odd neutron isotopes of the same element is attributed by Bohr and Wheeler to the higher binding energy per neutron in an even neutron nucleus compared with an odd neutron nucleus. An odd neutron heavy nucleus gains an excitation energy of about 6 Mev when it captures a neutron while an even neutron nucleus gains only 5 Mev excitation energy. This difference in exci-

tation energy accounts for the fact that odd neutron heavy nuclei are thermally fissionable, while even neutron nuclei have finite fission thresholds.

The fission threshold of a heavy nucleus depends, according to the drop model, on the ratio of the Coulomb energy to the surface tension energy; that is, upon the ratio Z^2/A where Z is the atomic number and A is the atomic weight. Thus, of two isotopes, both having even neutron numbers or both having odd neutron numbers, the lighter isotopes will have the lower fission threshold and will probably be more readily fissionable.

In the following table we list the thermal neutron fission cross sections, σ_f, of all the known thermally fissionable nuclei and the fission excitation threholds, E_f, of those isotopes which undergo fission only with fast neutrons. See Table on page V-3.

Table V-I

Thermal Fission Cross-Sections and Fission Thresholds of Heavy Isotopes

Element	Z	A	$\sigma_f(\times 10^{24})$ cm^2	E_f(Mev)**	Reference Vol.	Reference Paper
Ra	88	223	<100		14B	19.5*
Ra(ThX)	88	224				
Ra	88	226	$\leqslant 1.1 \times 10^4$		14B	19.6
Ra(MsTh$_1$)	88	228	<2		14B	19.5
Ac	89	227	<20		14B	19.5
Ac(MsTh$_2$)	89	228				
Th	90	227	600 ± 200		14B	19.5
Th(RdTh)	90	228	$\leqslant .1$		17B	9.10
Th	90	229	40 ± 10		17B	9.1
Th(I$_o$)	90	230	$\leqslant .001$		17B	9.12
Th(UY)	90	231				
Th	90	232	$\leqslant 2 \times 10^{-5}$	6.40	17B	9.12
Th(UX$_1$)	90	234				
Pa	91	230	15,000 - 20,000		17B	9.15
Pa	91	231	$.010 \pm .005$	5.89	17B	9.12
Pa	91	232	700 ± 200		17B	9.15
Pa	91	233	<0.1		17B	9.10
Pa(UZ)	91	234				
Pa(UX$_2$)	91	234				
U	92	232	70 ± 10		17B	9.11
U	92	233	$\overline{505}$		LA-140A	
U(U$_{II}$)	92	234	$\leqslant 2.1$		LA-140A	
U(AcU)	92	235	550		LA-140A	
U	92	237				
U(U$_I$)	92	238		6.02		
U	92	239				
Np	93	237	$.018 \pm .005$	5.55	14B	22.26
Np	93	238	1000 - 1500		14B	22.28
Np	93	239				
Np	93	240				
Pu	94	238	$\leqslant 18$		14B	22.24
Pu	94	239	765		LA-140A	
Pu	94	240	$\leqslant 180$		14B	22.10
Am	95	241	2.9 ± 0.5		14B	22.11
Cm	96	242	$\sim\overline{1500}$		14B	22.25

* This and similar numbers are PPR reference numbers. The data was obtained privately from W. N. Manning and represents work of A. Ghiorso, F. T. Hageman, M. Studier, A. VanWinkle, S. Peterson, E. Westman, P. W. Osborn and others. The data in this table were taken directly from the draft of K. Way's report "The Fission Process".

** E_f = Neutron Threshold Energy (measured) + Neutron Binding Energy (calculated)

At the time (1939) Bohr and Wheeler wrote their papers on the theory of fission it was believed that fission was a very much faster process than γ-ray emission (lifetime against fission $\sim 10^{-16}$ sec compared to lifetime against γ-emission $\sim 10^{-13}$ sec). Thus the fission width even at thermal energy was believed to be so great that narrow fission resonances were out of the question. In 1943, McDaniel et al observed low lying resonances in the fission of U^{235}; subsequently such resonances were found in all the fissionable isotopes. The total widths of these fission resonances, ~ 0.1 ev, were not particularly different from the usual γ-ray widths found in elements at the heavy end of the periodic chart, which would imply that radiative capture could compete successfully with fission at thermal energies. This proved to be the case: measurements of the thermal fission cross-sections and the total capture cross-section in fissionable isotopes showed that the two were appreciably different. The capture cross-section, which includes both fission and radiative capture, was as much as 40% higher than the fission cross-section, in some cases.

In the following table we give the ratio of thermal radiative capture to fission cross-sections for the three important fissionable species, U^{233}, U^{235}, and Pu^{239}.

Table V-II

Ratio of Thermal Radiative Capture (σ_r) to Fission (σ_f) Cross-sections

	$\alpha = \sigma_r/\sigma_f$	Reference
U^{233}	0.12	MP-126
U^{235}	0.18	LA-158, LA-140A
Pu^{239}	0.48	LA-140A, LA-91

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Production of Neutrons in Fissionable Materials

The fission process is accompanied by the emission of several fast neutrons. We denote the number of neutrons produced per fission by ν. This number is greater than η, the number of neutrons produced per neutron absorbed in the fissionable material, because not all the neutrons absorbed in a fissionable isotope lead to fission. Evidently ν is related to η by the relation

$$\eta = \frac{\sigma_f}{\sigma_r + \sigma_f}\nu = \frac{\nu}{1+\alpha} \qquad (5.1)$$

where $\alpha = \sigma_r/\sigma_f$.

The process of neutron emission during fission is rather obscure, and so predictions of the energy dependence of ν and η are apt to be misleading. The quantity ν is usually believed to be rather constant at low neutron energies ($< 10{,}000$ ev). On the other hand there is experimental evidence that η is very strongly energy dependent even at energies of a few volts. At very high energies processes can be imagined in which the incident neutron is scattered inelastically but leaves the nucleus so highly excited that it can still undergo fission. Such a "fission-n" reaction would lead to a rather higher ν than ordinary fission. Actually there is some evidence that ν for very fast neutrons is a little greater than ν for slow neutrons.

At high energies other processes, such as n-2n, will compete with fission. Since η from true fission is larger than 2, a strong n-2n process would tend to reduce η as observed experimentally.

If the fissionable isotope is mixed with a non-fissionable isotope of the same element, as for example natural U, in which the isotopic ratio of U^{235} to U^{238} is about 0.7%, the quantity η_R is defined as the number of neutrons produced

per neutron absorbed in the mixture. If the isotopic ratio $U^{235}:U^{238}$ is R, then

$$\eta_R = \frac{R\sigma_f(U^{235})\nu}{R\sigma_f(U^{235}) + R\sigma_\gamma(U^{235}) + \sigma_\gamma(U^{238})} \quad . \tag{5.2}$$

For ordinary uranium this quantity is about 1.33. The values of η and ν for thermal fission in various fissionable species are given in the accompanying table.

Table V-III

	η	ν	Reference
U^{233}	2.37	2.67	LA-140, CP-2297
U^{235}	2.09	2.47	LA-140
Pu^{239}	1.97	2.91	LA-140

The Fission Spectrum

It is generally believed that the neutrons emitted during fission are "boiled off" from the moving fission fragments. The energy of the fission neutrons according to this picture should be distributed according to an evaporation formula suitably modified to take account of the motion of the fission fragments.

Several measurements of the energy spectrum of the fission neutrons (briefly, the "fission spectrum") have been made. The most satisfactory measurements -- those of H. T. Richards (LA-84, LA-200), and of P. Demers, (MP-215) -- were performed by irradiating a photographic plate with fission neutrons from U^{235} and observing the recoil proton tracks. The results of

these experiments are given in Figure 5-I. The fission spectrum rises to a maximum at about 2 Mev and then trails off; the average neutron energy is about 2.3 Mev. The maximum neutron energy observed in any of these experiments is about 10 Mev; probably there are neutrons of higher energy than this. According to D. Hughes, the tail of the fission spectrum is an exponentially decreasing function $e^{-E/1.6}$, where E is measured in Mev.

There is some evidence that the fission spectra from slow and fast neutron fissions are not identical (LA-200). This is not surprising since essentially new processes which result in neutron emission (inelastic scattering + fission, n-2n) probably occur at high energy.

Delayed Neutrons

Most of the neutrons produced during fission are emitted practically instantaneously. According to measurements of R. R. Wilson the delay between separation of the fission fragments and emission of the neutrons is less than 10^{-9} sec. Besides these prompt neutrons a few neutrons are emitted only after very much longer time delays. These delayed neutrons comprise about 1% or less of the instantaneous neutrons. The exact number of delayed neutrons depends strongly on the fissionable species and probably, to a much lesser extent, upon the energy of the neutrons inducing fission.

The origin of the delayed neutrons was elucidated by Bohr and Wheeler (Phys. Rev. 56, 426, 1939). They pointed out that during the successive β-decays of the fission fragments it is probable that some nuclei are formed in very highly excited states. If the excitation energy exceeds the binding energy of a neutron, then the nucleus will decay by neutron emission instead of β particle emission. The neutron emission follows immediately upon the formation of the highly excited nucleus; but this nucleus, being a daughter of a chain of previous β-decays, is

formed rather long after fission. Consequently the neutrons emitted by it follow fission after a time determined by the half-lives of the previous β-emitters.

Identification of the fission fragments which emit two of the delayed neutrons was accomplished by Snell et al (CP-1967). He found that the delayed neutrons of half life 56 sec were emitted by Br^{87} and the delayed neutrons of the half life 22 sec were emitted by I^{137}. Recently Sugarman has identified the 6.5 sec emitter to be a Br isotope with mass number between 87 and 90 (CP-2621).

The delayed neutrons are emitted with definite mean lives, τ_i, as to be expected considering the fact that they follow β-decay of certain fission products. Various observers have established the existences of at least five delayed neutron periods. These are given along with the abundances, β_i, relative to the total number of prompt neutrons per fission, in the following table:

Table V-IV

Delayed Neutron Mean Lives and Relative Abundances

τ_i = mean life; β_i = relative abundance

U^{233}		U^{235}		Pu^{239}	
τ_i(sec)	β_i	τ_i(sec)	β_i	τ_i(sec)	β_i
80.2	1.83×10^{-4}	79.8	2.60×10^{-4}	79.5	1.16×10^{-4}
31.74	5.82×10^{-4}	32.1	17.26×10^{-4}	32.4	9.35×10^{-4}
6.51	8.57×10^{-4}	6.46	22.81×10^{-4}	7.0	11.18×10^{-4}
2.19	6.22×10^{-4}	1.9	24.45×10^{-4}	1.55	10.51×10^{-4}
0.61	1.82×10^{-4}	0.61	8.50×10^{-4}		

The energy of the delayed neutrons is, according to Wolland Burgy, considerably lower on the average than the energy of the prompt neutrons.

The mean delayed neutron energy is about 0.7 Mev; this may be compared with the average prompt neutron energy which is about 2.3 Mev.

The prompt neutrons probably come from virgin fission fragments before they lose any of their energy by β-decay. It is therefore not too surprising that their energy should be higher than that of neutrons emitted from nuclei farther down in a fission chain.

The existence of the delayed neutrons is of paramount importance for the practical operation of a chain reaction. If the multiplication constant is so low that the delayed neutrons are necessary for maintaining the reaction divergent, the delayed neutrons will act as "pacemakers" for the reaction. The time scale of any temporal change in the reaction will be largely determined by the delayed neutron periods. Since these periods are of the order of seconds, while the lifetime of a neutron in a slow neutron chain reaction is about 10^{-3} seconds, fluctuations in neutron density which, in the absence of the delayed neutrons, would occur in times of the order of milliseconds actually are slowed down to times of the order of several seconds. This makes the control of a chain reacting pile relatively simple.

Other Neutron Producing Nuclear Reactions in a Pile

Nuclear reactions other than heavy isotope fission produce neutrons in certain types of piles. The most important of these, particularly in piles employing heavy water or beryllium as moderator, are γ-n and n-2n reactions.

The threshold for the reaction $D^2(\gamma,n)H^1$ is 2.2 Mev; for the reaction $Be^9(\gamma\text{-}n)Be^8$ it is 1.65 Mev. Since there are several γ-rays with energy greater

than this among the fission product radiations, photoneutrons are present in any chain reacting system containing D or Be. Such neutrons have been observed in the Argonne heavy water pile.

Since the photoneutrons are produced from γ-rays emitted by the fission products, they are delayed with respect to the prompt neutrons. Their intensities and periods depend to some extent upon the time during which the fissionable material has been exposed to neutrons. This is because the long-lived fission products saturate only after long irradiations; the composition of the fission products present immediately after an irradiation, and consequently the number of γ-n neutrons produced by fission product γ-rays, therefore, depends upon the total time of irradiation.

Available measurements on the periods and relative abundances of photoneutrons produced in heavy water by fission products do not agree very well. In the following table we give the relative abundances of heavy water photoneutrons following very long neutron irradiation of U^{235}. The data of Bernstein and Preston is probably more reliable than that of Hughes, Spaatz and Cahn. Both sets of data indicate that the photoneutrons from heavy water do not amount to more than 1/3 of the delayed neutrons.

Table V-V

Mean-Lives, τ_i, and Abundances, β_i, (Relative to Total Prompt Fission Neutrons) of D photoneutrons following long irradiation of U^{235}

Bernstein and Preston		Hughes, Spaatz and Cahn	
β_i	τ_i	β_i	τ_i
0.0122×10^{-4}	76 hr	0.0054×10^{-4}	35 hr
0.0387×10^{-4}	6.3 hr	0.15×10^{-4}	2.9 hr
0.284×10^{-4}	2.4 hr	0.20×10^{-4}	46 min
0.250×10^{-4}	39.3 min	0.35×10^{-4}	9.4 min
0.400×10^{-4}	11.1 min	1.20×10^{-4}	130 sec
1.32×10^{-4}	2.7 min	7.61×10^{-4}	43.3 sec
2.14×10^{-4}	47 sec	10.91×10^{-4}	9.65 sec
7.32×10^{-4}	35 sec		

The n-2n reactions in Be and D have the same thresholds as the γ-n reactions, 1.65 Mev and 2.2 Mev. Some of the prompt neutrons have greater energy than this; and in fact, evidence for a fairly strong n-2n reaction in Be moderated piles has been found. We shall consider this matter in greater detail in the section on microscopic pile theory.

Elementary Characteristics of a Chain Reaction

Slow Neutron Chain Reactions

The fission cross sections of the fissionable nuclei are remarkably large at thermal energy (Table V-I): the ratio of cross-sections of U^{235} and U^{238} is about 250 at thermal energy as against an average ratio of about 2 over the resonance region. Thus, in ordinary U, where the isotopic contents of U^{235} is 0.7%, only about 1 out of every 75 neutrons absorbed above the Cd cutoff (0.4 volts) causes fission, while roughly one out of every two neutrons absorbed thermally causes fission. For this reason, it is advantageous to reduce the energy of the fission neutrons in a chain reacting system to thermal energy and thus insure that most of the neutrons are absorbed when they are very slow.

The slowing down of neutrons in a pile is accomplished by mixing the uranium with a moderator of low atomic weight, such as H_2O, D_2O, Be, BeO, C, etc. All moderators (except He, which is impractical because it is a gas) absorb some neutrons. Consequently the amount of moderator which can be used in a slow neutron chain reacting system is limited by the requirement that not too many neutrons be lost in the moderator. The best moderator from a theoretical viewpoint is the one for which the ratio of slowing down power ($\sigma_s \xi$) to capture cross-section (σ_a) is a maximum, for, the higher this ratio, the fewer neutrons will be absorbed parasitically by the moderator for a given amount of moderation.

This ratio for several moderators is given in the following table:

Moderator	H_2O	D_2O	He	Be	C	O
$\sigma_s \xi/\sigma_a$	63	1600	∞	126	169	1000

Of course the choice of moderator in any given practical case depends not only on this ratio but also on such quantities as density, availability, structural properties, etc.

While a fission neutron is being slowed in a medium containing uranium and a moderator it may be absorbed in one of the resonances of U^{238}. If too many neutrons are lost in U^{238} resonances a chain reaction is impossible; hence methods must be devised to reduce the resonance absorption in U^{238}.

The most obvious method is to use separated U^{235} or Pu^{239} in the chain reaction. Unfortunately this method is expensive. The discovery of alternative, much cheaper methods to establish a chain reaction was one of the main achievements of the Plutonium Project.

The use of a moderator in itself reduces the resonance absorption in U^{238}. According to equation (4.28) the probability that a neutron escape resonance capture in a homogeneous mixture of uranium and moderator is

$$p = \exp - \frac{N_u}{N_m \sigma_m \xi_m} \left(\int \sigma_{au} \frac{dE}{E} \right)_{eff}$$

where N_u is the number of atoms of U^{238}/c.c., $N_m \sigma_m \xi_m$ is the slowing down power of the moderator per c.c., and $\left(\int \sigma_{au} \frac{dE}{E} \right)_{eff}$ is the effective resonance absorption integral per atom of U^{238}. As the slowing down power of the moderator per atom of U is increased -- and this is most easily accomplished

by increasing the amount of moderator -- the fraction of neutrons which escape resonance capture increases. If increase of p were the only consideration, it would be advantageous to increase the amount of moderator without limit. However, as has been pointed out, all practical moderating substances capture some thermal neutrons. This sets a limit to the amount of moderator which can be used. Because of this competition between two opposing requirements, namely, large amount of moderator to reduce U^{238} resonance capture, and small amount of moderator to reduce thermal neutron capture in the moderator, there is an optimum moderator to U ratio for which the multiplication constant of a chain reacting system is highest. One of the problems of microscopic pile theory is to calculate this optimum.

Heterogeneous versus Homogeneous Arrangements

There is another way to reduce the resonance absorption in U^{238}. This is to dispose the uranium heterogeneously as a lattice of lumps throughout the moderator. The advantage of such an arrangement is that, because the resonance absorption is largely confined to sharp, very deep levels, the uranium atoms inside the lump are shielded from resonance neutrons by the atoms on the surface. The effectiveness of this self-shielding in reducing the resonance absorption has been calculated in Chapter IV.

It was found that the value of

$$\left(\int \sigma_{au} \frac{dE}{E}\right)_{eff}$$

can be reduced by lumping from 240×10^{-24} cm^2, the value for finely divided uranium to perhaps 15×10^{-24} cm^2, depending on the exact dimensions of the lump (Eq.4.51).

This decrease in the resonance absorption of uranium caused by lumping is crucial for the establishment of a chain reaction in a system containing unenriched uranium. In fact it was the idea of lumping, proposed independently by Szilard in this country and by Harteck in Germany, which made it possible to build a successful chain reaction out of graphite and ordinary uranium.

The main reason that lumping improves the multiplication constant is that, because of the discrete character of the resonance absorption, levels, (and their very great height) the value of $\left(\int \sigma_{au} \frac{dE}{E}\right)_{eff}$ is less in a lump than in a homogeneous mixture. There is a second effect of lumping, namely, that moderation of neutrons through the resonance absorption energy region takes place away from the uranium. This also reduces, though rather slightly, the resonance absorption. However, this effect is more than balanced by the fact that, since thermal neutrons are absorbed in the uranium lump, the thermal neutron density is depressed there relative to the density in the moderator. Consequently the absorption in the moderator is increased by an amount which, in lattices of practical interest, more than offsets the improvement in resonance absorption caused by the fact that the neutrons slow down far from the lump.

The total effect on the multiplication constant caused by lumping is the sum of three effects, two of which are advantageous, and one disadvantageous:

Advantages of Lumping

1) Reduction in $\left(\int \sigma_{au} \frac{dE}{E}\right)_{eff}$

2) Reduction in number of resonance neutrons near lump.

Disadvantage of Lumping

1) Increase in thermal absorption by moderator because of increase in relative number of thermal neutrons in moderator.

The net effect of lumping is advantageous since advantage (1) much more than balances the difference between disadvantage (1) and advantage (2). Since advantage (1) accrues only because the absorption is concentrated in very deep lines, there would be a net loss instead of a net gain from lumping if the resonance absorption in U^{238} were not confined to discrete levels. It may be mentioned that in many semi-popular discussions of the chain reaction, the only advantage claimed for lumping is (2); this is evidently erroneous.

In the calculation of the critical size and other macroscopic properties of piles the lattice structure complicates the analysis very considerably. We shall in this section ignore the lattice structure, and shall consider the actual pile to be replaced by a homogeneous one whose microscopic properties, on the average, are the same as those of the lattice pile. This procedure is approximately correct as long as the dimensions of a cell are small compared to the size of the pile. In Chapter the error which is involved in this simplification will be calculated.

The Multiplication Constant

Uranium chain reactions of the type we consider, i.e., slow neutron chain reactions may be described by tracing the life history of a neutron from the time of its birth as a fast neutron to the time of its death, by capture, as a slow neutron. For the moment we shall not be concerned with neutrons which might leak out; we therefore imagine a chain reacting system having a certain definite arrangement of uranium and moderator and whose overall size is infinite. In such an infinite system neutrons can be lost only by capture.

Four distinct steps can be distinguished in the life cycle of a pile neutron; these are

a. Birth as a fast neutron as the result of fission.

b. Absorption above the fission threshold in U^{238} and production of new neutrons as a result of fast fission.

c. Radiative capture by U^{238} during moderation (so-called resonance capture), or resonance capture by U^{235} followed by fission.

d. Radiative capture as a slow neutron in the moderator or in impurities (so-called parasitic capture), or capture in fissionable material with production of fast neutrons from fission.

Evidently any given neutron cannot experience all phases of this cycle. A few neutrons in the slow neutron chain reactions of primary interest will be multiplied as a result of fast neutron fission in U^{238} or other fast neutron fissionable material. Some will be captured at resonance energy and therefore will not be available for further fission. The majority of neutrons will survive to thermal energies ($\sim$ .025 ev) and will there be captured parasitically or will produce new neutrons by thermal fission.

Fast and Slow Neutron Chain Reactions

The fraction of neutrons which survive any particular part of the cycle will depend upon the composition of the pile. If there is little moderator present and if the fissionable material ("metal") is lumped in large lumps, more neutrons will cause fast fission than if the lumps are small or the metal to moderator ratio is small. Again if there is a large amount of moderator present, most of the neutrons will be slowed to thermal energy, and the bulk of the fissions will be produced by thermal neutrons.

Thus a uranium chain reaction can be characterized by the energy of the neutrons which cause the fissions. In a bomb practically all the fissions are caused by neutrons whose energy is above 1/2 Mev; such a chain reaction is

referred to as a "fast neutron reaction", or simply "fast" reaction. In a chain reaction which operates on unenriched uranium only a few percent of the fissions are caused by fast or resonance neutrons; almost all the fissions are caused by thermal neutrons, and such a reaction is therefore called a "slow" reaction.

The Generation Time

Basic to the discussion of neutron chain reactions is the notion of the generation time. We define this quantity for a slow neutron reaction in the following manner. Each slow neutron fission in a pile is induced by a neutron which was born as a direct result of slow fission occurring at a time ℓ_i before, or it is induced by a neutron which is a direct fast fission or resonance fission descendant of neutrons produced by slow fission at time ℓ_i earlier. The time ℓ_i may be called the effective lifetime of the neutron causing fission. Because of statistical fluctuations in the neutron capture process, and because some neutrons do have fast and resonance fission descendants while others do not, the time intervals ℓ_i between successive directly related slow neutron fissions will not all be the same. Let $N(\ell_i)$ denote the number of slow neutron fissions per second per c.c. induced by neutrons whose effective lifetime is ℓ_i. Then we define the average generation time (ℓ) as

$$\ell = \frac{\sum_i \ell_i \, N(\ell_i)}{\sum_i N(\ell_i)} , \qquad (5.3)$$

the summation being over all neutron lifetimes.

The generation time is a measure of the average time between successive generations of slow neutron fissions. In a pile whose macroscopic composition

is not uniform the generation time will vary from place to place. Where there is a large amount of moderator the generation time will be larger than where there is a small amount of moderator.

If the pile is so rich in fissionable material that no thermal fission takes place, this definition of the generation time evidently becomes inappropriate. In that case it is more relevant to define a generation time as the average time interval between successive fissions -- not successive thermal fissions. However, for slow neutron piles it is more convenient to define the generation time as the average time between successive slow neutron fissions.

During one generation time a certain number of slow neutrons are created and another number of slow neutrons are captured in the infinite system. The ratio of the number of neutrons which become slow (i.e., are "produced") in one generation time to the number of slow neutrons captured in one generation is called the multiplication constant of the system. It is denoted by k_∞ since it is defined for an infinitely large system.

The multiplication constant in a slow neutron system is evidently the product of four factors, corresponding to the four phases in the neutron cycle. Suppose a single slow neutron is captured in the system. How many slow neutron, on the average, will be produced one generation time later? The slow neutron may be captured either in the moderator or other inert material present in the system, in which case it does not produce fission, or it may be captured in the fissionable material. (In this connection we call any isotopic mixture of U^{235} and U^{238} "fissionable material"). The number of fast neutrons which immediately result from the capture of one slow neutron is therefore the product

$$f\eta$$

where f is the probability that the slow neutron is captured in the uranium,

and η is the number of neutrons produced per slow neutron captured in the fissionable material. The quantity f is called the thermal utilization.

Some of the fast neutrons produced by slow neutron fission induce fast neutron fission in the fissionable material. Most of these fast neutron fissions occur in the U^{238} in a system in which unenriched U is used. The total number of fast neutrons produced in one generation as the result of the capture of a slow neutron is, thus, greater than $f\eta$ by a factor ϵ (> 1), the fast multiplication constant. This factor is defined as the ratio of the number of neutrons produced by all fissions in one generation to the number of neutrons produced by slow neutron fission in one generation.

During the slowing down process the fast neutrons which leave the uranium have only a probability p (resonance escape probability) of escaping resonance capture in U^{238}. The total number of slow neutrons created in one generation per thermal neutron absorbed is therefore

$$k_\infty = f\eta\epsilon p. \qquad (5.4)$$

For every slow neutron absorbed in one generation time, there must have been $\frac{1}{p} - 1$ neutrons absorbed in the resonance absorption energy range. The total number of neutrons absorbed (both as slow and as resonance) per thermal neutron absorbed per generation time is therefore $1/p$. Similarly the total number of neutrons which are produced in one generation time per thermal neutron absorbed -- this includes both those neutrons which are destined to be captured at resonance and those which are destined to be captured thermally -- is k_∞/p. The ratio is k_∞. Hence an alternative definition of the multiplication constant is the total number of neutrons produced, on the average, in one generation time,

divided by the total number of neutrons absorbed, on the average, in one generation time.

The Elementary Pile Equation

We now consider a chain reacting system which is of finite size. The neutron density in this system evidently will remain stationary if the average number of neutrons produced in one generation is equal to the average number lost in one generation. In an infinite system neutrons can be lost only by capture. In a finite system neutrons can be lost by leakage out of the system as well as by capture. Hence it is customary to define an "effective" multiplication constant, k_{eff}, defined as the number of neutrons produced on the average in one generation time, divided by the total number of neutrons which are absorbed or leak out on the average in one generation time. The quantity k_{eff} is to be contrasted with k, defined as the total number of neutrons produced on the average in one generation time, divided by the total number of neutrons absorbed in one generation. The condition that the neutron density in a chain reaction be stationary is evidently $k_{eff} = 1$. For any finite system k_{eff} is less than k. The difference between k and k_{eff} is less than k. The difference between k and k_{eff} depends on the size of the system; as the chain reacting pile becomes infinite the value of k_{eff} approaches k.

If the energy spectrum of the neutrons in the finite system is the same as the spectrum in the infinite system having the same uranium-moderator configuration, then k is the same as k_{∞}. The neutron spectra in the finite pile and in its infinite prototype will be nearly the same provided the dimensions of the pile are large compared to the range of fission neutrons. We shall be interested in unenriched systems which are always large enough for this to be true. It is for this reason we find it appropriate to introduce the quantity

k_∞, and in fact, to divide pile theory into microscopic and macroscopic parts. If the pile is so small that the neutron energy spectrum is very different from that in the infinite prototype, the quantity k is still a useful one, but it is no longer necessarily equal to k_∞. In such a case the introduction of k_∞ as defined here,* is rather irrelevant, and, in fact, the distinction between microscopic and macroscopic pile theory is blurred.

Near the boundaries of even a large system the neutron energy spectrum will deviate from that characteristic of an infinite pile. This is usually not too important, as will be shown later, since the macroscopic properties of a pile are hardly affected by its microscopic properties near the boundaries. There are some cases, however -- notably the calculation of the effect of reflectors -- where the change in energy spectrum near a boundary is important. In this case the distinction between macroscopic and microscopic theory is not very useful.

With these preliminaries in mind, we proceed to derive an equation describing the neutron distribution in a slow neutron chain reacting pile. We consider a homogeneous and isotropic pile which contains no extraneous neutron sources (e.g. Ra-Be). Our task will be to compute the critical size of such a pile, and to determine the distribution of neutrons in it. Two assumptions will be made in most of the work of this chapter. First, we assume the pile is homogeneous. Since most piles utilizing unenriched uranium have a lattice structure, this assumption apparently eliminates unenriched piles from consideration. In order to include such lattice piles in the theory we use the stratagem mentioned previously: namely, we replace the lattice pile by a homogeneous pile having the same microscopic properties as the lattice. The neutron distribution calculated for the homogeneous model will then reproduce

* It is still possible, however, as pointed out by F. L. Friedman, to introduce a multiplication constant which is explicitly a function of the neutron energy spectrum.

the large scale features of the distribution in the lattice pile but will not include the local irregularities introduced by the lattice structure. These local irregularities are perturbations superposed on the large scale neutron distribution; they will be dealt with in Part 3 of this volume.

A second assumption which will be made in much of what follows is that the multiplication constant, k, of the pile is close to unity. This is almost always the case for piles using unenriched uranium. The requirement $k \approx 1$ is equivalent to the assumption that the pile is large compared to the mean distance that a neutron travels from birth to death.

In order to write down an equation which describes the steady state neutron distribution, we use the fact that in a steady state the number of neutrons which become slow per second in a cubic centimeter must equal the net number which diffuse out of, plus the number captured in, this cubic centimeter per second. Let $\Phi_s(\underline{r})$ denote the thermal flux (i.e., number times velocity of thermal neutrons per c.c.) at the point whose position vector is $\underline{r}$. Since the angular distribution is not used in this section, no confusion will arise from our use of Φ_s instead of Φ_{o_s} for the total slow flux. We assume the system is so large that elementary diffusion theory is applicable; then the net diffusion of slow neutrons out of a unit volume in one second is

$$\nabla \cdot D_o \nabla \Phi_s(\underline{r})$$

where D_o is the diffusion coefficient for slow neutrons. The diffusion coefficient D_o is related to other constants by the equation

$$D_o = \frac{N\sigma_a}{\lambda^2} \qquad (5.5)$$

where $N\sigma_a$ is the average macroscopic absorption cross-section of the pile, (evaluated for the average velocity), and $1/\mathcal{K}(=L)$ is the diffusion length in the pile.

Let $q(E_s,\underline{r})$ denote the number of neutrons which become thermal per c.c. per second at $\underline{r}$. In the steady state

$$\nabla \cdot D_o \nabla \Phi_s(\underline{r}) - N\sigma_a \Phi_s(\underline{r}) + q(E_s,\underline{r}) = 0 . \qquad (5.6)$$

The number of neutrons $q(E_s,\underline{r})$ which become thermal per c.c. per sec., that is, the "thermal slowing down density", evidently comprises all those neutrons which were produced by fission throughout the pile and which, in diffusing away from their points of origin, happen to become thermal at $\underline{r}$. Let $P(E_s,\underline{r},\underline{r}')$ be the probability that a fission neutron, created in unit volume at $\underline{r}'$ inside the pile, becomes a slow neutron in unit volume at $\underline{r}$. We call the function $P(E_s,\underline{r},\underline{r}')$ the "finite slowing down kernel" -- "finite" since it describes the slowing down probability in a finite pile. Since the number of fission neutrons produced per c.c. per second at r' is $\frac{k}{p} N\sigma_a \Phi_s(\underline{r}')$, the total number becoming thermal per c.c. per second at $\underline{r}$ is

$$q(E_s,\underline{r}) = \int_{\text{pile}} \frac{k}{p} N\sigma_a \Phi_s(\underline{r}')\, P(E_s,\underline{r},\underline{r}')d\underline{r}', \qquad (5.7)$$

the integral being extended over the pile.

Upon substituting (5.7) into (5.6), we obtain

$$\nabla \cdot D_o \nabla \Phi_s(\underline{r}) - N\sigma_a \Phi_s(\underline{r}) + \int_{\text{pile}} \frac{k}{p} N\sigma_a \Phi_s(\underline{r}')\, P(E_s,\underline{r},\underline{r}')d\underline{r}' = 0, \qquad (5.8)$$

an integro-differential equation whose solution determines the slow neutron density in a slow neutron chain reaction. We call (5.8) the "pile equation".

If the pile is uniform, the pile equation may be written

$$L^2 \Delta \bar{\Phi}_s(\underline{r}) + \frac{k}{p} \int_{\text{pile}} \bar{\Phi}_s(\underline{r}') \, P(E_s,\underline{r},\underline{r}')d\underline{r}' - \bar{\Phi}_s(\underline{r}) = 0, \qquad (5.9)$$

since the square of the pile diffusion length, L^2, is just

$$L^2 = \frac{1}{\mathcal{X}^2} = D_o/N\sigma_a \, .$$

The Fast Flux and the Boundary Conditions

Before the thermal neutron density or critical size can be determined from the pile equation, it is necessary to first specify the distribution of the non-thermal neutrons, and second to impose boundary conditions on the neutron fluxes of all energies.

The slowing down density at energy E, that is, the number of neutrons which cross energy E per c.c. per second is denoted, as in Chapter III, by $q(E,\underline{r})$. If $P(E,\underline{r},\underline{r}')$ is the probability that a fission neutron created in unit volume at $\underline{r}'$ crosses the energy E in a unit volume at $\underline{r}$, then evidently

$$q(E,\underline{r}) = \int_{\text{pile}} \frac{k}{p} N\sigma_a \bar{\Phi}_s(\underline{r}') \, P(E,\underline{r},\underline{r}') \, d\underline{r}' \, ; \qquad (5.10)$$

when $E = E_s$, $q(E,\underline{r})$ is just the thermal slowing down density, $q(E_s,\underline{r})$ which appears in the pile equation. If $\bar{\Phi}_s(\underline{r})$ and $P(E,\underline{r},\underline{r}')$ are known, (5.10) determines the slowing down density, and therefore the fast flux, at every energy.

If the chain reacting system contained only slow neutrons, the boundary condition would be essentially the one stated in Chapter I; namely, $\bar{\Phi}_s(\underline{r})$ extrapolates to zero at $0.71\,\lambda_t$ beyond the physical boundary, λ_t being the

thermal neutron transport mean free path.*

The boundary condition on the fast neutron flux -- that is, on $q(E,\underline{r})$ -- will in general differ from the boundary condition on $\Phi_s(\underline{r})$, since the mean free path is always energy dependent. The fact that the extrapolation distances for $q(E,\underline{r})$ and $\Phi_s(\underline{r})$ may differ complicates the theory tremendously. To simplify matters it is usually customary to assume the two distances are the same. This is not at all correct for chain reactions whose critical size is comparable to a mean free path; however, in slow neutron piles the extrapolation distance is always so small compared to the pile dimension that the error introduced by assuming the extrapolation distance to be energy independent is unimportant. The size of this error is discussed in Chapter VII .

There are two slightly different ways in which the extrapolation distance boundary condition can be formulated. Either the condition can be taken as

$$\Phi_s(\underline{r}),\ q(E,\underline{r}) \qquad \text{vanish on extrapolated boundary}$$

or, it can be taken as

$$\frac{1}{\Phi_s}\frac{\partial \Phi_s}{\partial \nu} = \left[-.71\,\lambda_t(E_s)\right]^{-1}$$

$$\frac{1}{q}\frac{\partial q}{\partial \nu} = \left[-.71\,\lambda_t(E)\right]^{-1} \qquad \text{on physical boundary} \qquad (5.11)$$

where $\frac{\partial}{\partial \nu}$ is the derivative with respect to the outward normal, ν .

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* The exact numerical coefficient of λ_t in the extrapolation distance formula depends on the curvature of the bounding surface, and on the ratio of capture to scattering cross-section. The constant 0.71 applies if the curvature is small compared to λ_t, and if $\sigma_a/\sigma_s \ll 1$.

The two ways of writing the boundary conditions are identical if the fluxes inside the pile are linear near the pile boundary. This is always the case if the pile is sufficiently large. The theory of the pile turns out to be more straightforward if the boundary condition is stated in terms of the normal logarithmic derivatives, Eq. (5.11), although actual calculations are easier with the other form of the boundary condition.

For convenience we collect together the pile equation and the boundary conditions:

$$\nabla \cdot D_o \nabla \Phi_s(\underline{r}) - N\sigma_a \Phi_s(\underline{r}) + q(E_s,\underline{r}) = 0 \qquad (5.9)$$

$$q(E,\underline{r}) = \int_{\text{pile}} \frac{k}{p} N\sigma_a \Phi_s(\underline{r}') \, P(E,\underline{r},\underline{r}') d\underline{r}' \qquad (5.10)$$

$$\frac{1}{\Phi_s}\frac{\partial \Phi_s}{\partial \nu} = \left[-.71\lambda_t(E_s)\right]^{-1}, \quad \frac{1}{q}\frac{\partial q}{\partial \nu} = \left[-.71\lambda_t(E)\right]^{-1} \qquad (5.11)$$

on physical boundary

For the remaining discussion we shall assume the pile to be uniform, and the transport mean free path -- that is, the extrapolation distance -- to be energy independent unless otherwise stated.

The Asymptotic Pile Equation

Far from the boundaries of the pile the moderation of neutrons proceeds as though the pile were infinitely large. In the expression for the slowing down density (5.10) it is therefore permissible to write $P_\infty(E,|\underline{r}-\underline{r}'|)$, where $P_\infty(E,|\underline{r}-\underline{r}'|)$ is the slowing down kernel in an infinite system, rather than

$P(E,\underline{r},\underline{r}')$. Thus, the slowing down density far from a boundary, which we denoted by $q_\infty(E,\underline{r})$, is

$$q_\infty(E,\underline{r}) = \int_{\substack{\text{all}\\ \text{space}}} \frac{k}{p} N\sigma_a \Phi_s(\underline{r}')\, P_\infty(E,|\underline{r}-\underline{r}'|)d\underline{r}'. \qquad (5.12)$$

The integration can be extended over all space rather than over the pile provided $P_\infty(E,|\underline{r}-\underline{r}'|)$ falls off so rapidly with $\underline{r}-\underline{r}'$ that there is no appreciable contribution to the slowing down density from points close to the boundary. In all cases of physical importance $P_\infty(E,|\underline{r}-\underline{r}'|)$ falls off sufficiently fast with $\underline{r}-\underline{r}'$ for this to be true.

The pile equation at points far from the boundary is, for a uniform pile,

$$L^2\Delta\Phi_s(\underline{r}) - \Phi_s(\underline{r}) + \frac{k}{p}\int_{\substack{\text{all}\\ \text{space}}} \Phi_s(\underline{r}')\, P_\infty(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' = 0. \qquad (5.13)$$

This equation may be referred to as the asymptotic pile equation, and its solutions, since they are valid in general only far from boundaries, may be called asymptotic solutions of the pile equation.

The asymptotic pile equation is simpler than the exact pile equation on two accounts: first, because the range of integration is over all space; and second, because the slowing down kernel $P_\infty(E,|\underline{r}-\underline{r}'|)$ depends only on $|\underline{r}-\underline{r}'|$; that is, it is a symmetric, displacement kernel.

The finite slowing down kernel, $P(E,\underline{r},\underline{r}')$ will ordinarily satisfy the same linear differential equation as $P_\infty(E,|\underline{r}-\underline{r}'|)$ -- for example, the Fermi

age equation, or the group equation -- depending on the picture used to represent the slowing down process. Thus $P(E,\underline{r},\underline{r}')$ in such cases is expressible as a superposition of $P_\infty(E,|\underline{r}-\underline{r}'|)$. The process of superposing $P_\infty(E,|\underline{r}-\underline{r}'|)$ to yield $P(E,\underline{r},\underline{r}')$ corresponds to distributing a system of images whose effects mutually cancel on the extrapolated boundary and so cause $P(E,\underline{r},\underline{r}')$ to vanish there. Since these images must be placed at various distances from the boundary it is not surprising that the finite kernel, $P(E,\underline{r},\underline{r}')$ is not, in general, a displacement kernel, while the infinite one is. Another way of putting it is that the presence of boundaries introduces the possibility of leakage and therefore makes the probability that a fast neutron created at $\underline{r}'$ will appear at $\underline{r}$ depend not only on $|\underline{r}-\underline{r}'|$, but also on the distance from $\underline{r}$ or $\underline{r}'$ to the pile boundary.

The infinite slowing down kernel is normalized so that the probability that a neutron will cross energy E is the resonance escape probability, p(E):

$$\int P_\infty(E,|\underline{r}-\underline{r}'|)d\underline{r}' = 4\pi \int_0^\infty P_\infty(E,r)r^2dr = p(E). \qquad (5.14)$$

The resonance escape probability to thermal energy $p(E_s)$, is denoted, as usual, simply by p.

As an example which will make the distinction between $P_\infty(E,|\underline{r}-\underline{r}'|)$ and $P(E,\underline{r},\underline{r}')$ clear, consider a semi-infinite slab in which the Fermi age slowing down picture is applicable, and in which $p(E) = 1$. Then $P_\infty(E,|\underline{r}-\underline{r}'|)$ is

$$\frac{1}{\left[4\pi\tau(E)\right]^{1/2}}\left[e^{-\frac{|x-x'|^2}{4\tau(E)}}\right],$$

while the finite kernel, which vanishes on the slab boundary (x = 0) is

$$\frac{1}{\left[4\pi\tau(E)\right]^{1/2}}\left[e^{-\frac{|x-x'|^2}{4\tau(E)}} - e^{-\frac{|x+x'|^2}{4\tau(E)}}\right].$$

The second term represents an image which is placed at -x' and whose effect is to bring the finite kernel to zero at the boundary.

Asymptotic Solution of the Pile Equation; the Fundamental Theorem of Pile Theory

We now prove the following "fundamental theorem of pile theory":

The asymptotic uniform pile equation

$$L^2\Delta\Phi_s(\underline{r}) - \Phi_s(\underline{r}) + \frac{k}{p}\int_{\substack{\text{all}\\ \text{space}}} \Phi_s(\underline{r}')\, P_\infty(E_s|\underline{r}-\underline{r}'|)d\underline{r}' = 0 \qquad (5.13)$$

is satisfied by any solution of

$$\Delta\Phi_s + B^2\Phi_s = 0 \qquad (5.15)$$

provided B^2 is a root of the characteristic equation

$$-L^2B^2 - 1 + \frac{k}{p}\bar{P}_\infty(E_s,B^2) = 0 \qquad (5.16)$$

where $\bar{P}_\infty(E_s,B^2)$ is the three-dimensional Fourier transform of $P_\infty(E_s,|\underline{r}|)$; i.e.,

$$\bar{P}_\infty(E_s,B^2) = \int_{-\infty}^{\infty} e^{i\underline{B}\cdot\underline{r}}\, P_\infty(E_s,|\underline{r}|)d\underline{r} = 4\pi\int_0^\infty \frac{\sin Br}{Br}\, P_\infty(E_s,r)r^2dr. \qquad (5.17)$$

The slowing down density at energy E is

$$q_\infty(E,\underline{r}) = \frac{k}{p} N\sigma_a \, \bar{P}_\infty(E,B^2) \, \bar{\Phi}_s(\underline{r}); \qquad (5.18)$$

that is, $q_\infty(E,r)$ is proportional at every energy to $\bar{\Phi}_s(\underline{r})$.

To prove this theorem we first prove the last statement (5.17). Consider a solution $\bar{\Phi}_s(\underline{r})$ of

$$\Delta \bar{\Phi}_s(\underline{r}) + B^2 \bar{\Phi}_s(\underline{r}) = 0. \qquad (5.15)$$

This function may be expressed as a Fourier integral

$$\bar{\Phi}_s(\underline{r}) = \int A(\underline{\alpha}) \, e^{i\underline{\alpha}\cdot\underline{r}} d\underline{\alpha}$$

where $\underline{\alpha} = \alpha_x \underline{i} + \alpha_y \underline{j} + \alpha_z \underline{k}$ is a vector integration variable. Since $\bar{\Phi}_s$ satisfies (5.15), the function $A(\underline{\alpha})$ must be zero unless $\alpha^2 = B^2$; i.e., it is a δ-function on the surface of a sphere of radius B. From (5.12),

$$q_\infty(E,\underline{r}) = \frac{k}{p} N\sigma_a \int_{\substack{\text{all} \\ \text{space}}} \bar{\Phi}_s(\underline{r}') \, P_\infty(E,|\underline{r}-\underline{r}'|) d\underline{r}'$$

$$= \frac{k}{p} N\sigma_a \iint A(\alpha) \, e^{i\underline{\alpha}\cdot\underline{r}'} \, P_\infty(E,|\underline{r}-\underline{r}'|) d\underline{r}' \, d\underline{\alpha} \, .$$

Introducing the factor $e^{i\underline{\alpha}\cdot(\underline{r}-\underline{r})}$, we obtain

$$q_\infty(E,\underline{r}) = \frac{k}{p} N\sigma_a \int A(\underline{\alpha}) \, e^{i\underline{\alpha}\cdot\underline{r}} \, d\underline{\alpha} \int e^{i\underline{\alpha}\cdot(\underline{r}'-\underline{r})} \, P_\infty(E,|\underline{r}-\underline{r}'|) d|\underline{r}-\underline{r}'| =$$

$$= \frac{k}{p} N\sigma_a \, \bar{P}_\infty(E,B^2) \, \bar{\Phi}_s(\underline{r})$$

where $\bar{P}_\infty(E,B^2)$ is the three dimensional Fourier transform of $P_\infty(E,|\underline{r}-\underline{r}'|)$. The three dimensional transform can be written

$$\bar{P}_\infty(E,B^2) = \int e^{i\underline{B}\cdot\underline{r}} P_\infty(E,|\underline{r}|)d\underline{r}$$

where $B = B_x\underline{i} + B_y\underline{j} + B_z\underline{k}$.

On shifting to polar coordinates r, $\mu = \cos\theta, \varphi$, this becomes

$$\bar{P}_\infty(E,B^2) = 4\pi \int_0^\infty \frac{\sin Br}{Br} P_\infty(E,r)r^2dr.$$

Thus (5.17) is proved.

To prove the remainder of the theorem, we write

$$q_\infty(E_s,\underline{r}) = \frac{k}{p} N\sigma_a \bar{P}_\infty(E_s,B^2)\Phi_s(\underline{r})$$

and substitute into the uniform pile equation. The result is the characteristic equation (5.16):

$$-L^2B^2 - 1 + \frac{k}{p}\bar{P}(E_s,B^2) = 0; \qquad (5.16)$$

thus we have proved that a solution of (5.15) will satisfy the asymptotic pile equation if B^2 is a root of (5.16).

The solution of the asymptotic pile equation is, according to the fundamental theorem, equivalent to the solution of the wave equation (5.15) in which B^2 is a root of the characteristic equation (5.16). The quantity B^2 is called the "buckling" of the pile.* Since it is determined as the root of an equation (5.16) whose parameters $(L^2,k,\bar{P}_\infty,p)$, depend only on the microscopic properties of the pile, the buckling B^2 is also a microscopic property of the pile.

*The word buckling, proposed by J. Wheeler, arises from the fact that $B^2 = -\Delta\Phi_s/\Phi_s$ and is therefore a measure of the curvature of the slow neutron density.

Relation Between Asymptotic and Finite Pile Equations

The asymptotic pile equation and its asymptotic solution are valid in general only at distances from the boundary which are large compared to the slowing down range of the neutrons. Closer to the boundary than this there will generally be non-asymptotic solutions for which the rather remarkable proportionality between the slowing down density and the slow neutron density implied in equation (5.18) no longer holds.

There is one very important case, however, in which the asymptotic solution is rigorously correct to within a distance of a mean free path, rather than a slowing down range, of the boundary and the breakdown of the asymptotic solution then arises only because elementary diffusion theory ceases to be valid so close to the boundary. This important case is stated in the following theorem:

If the extrapolation distance, as defined by (5.11) is independent of neutron energy, and if the finite and infinite slowing down kernels satisfy the same linear equation, then the asymptotic solution holds everywhere in a critical pile, except within a distance of the order of a mean free path from the boundary. The slowing down density in the pile is identical with the infinite slowing down density except within a distance of the order of a mean free path from the boundary.

To prove this theorem we first use the fact that the infinite slowing down density is proportional to the slow neutron flux Φ_s

$$q_\infty(E,\underline{r}) = \frac{k}{p} N\sigma_a \bar{P}_\infty(E,B^2)\, \Phi_s(\underline{r})$$

provided

$$\Delta\Phi_s + B^2\Phi_s = 0.$$

Since $q_\infty(E,\underline{r})$ is proportional to $\underline{\Phi}_s(\underline{r})$, the two functions q_∞ and $\underline{\Phi}_s$ satisfy the same boundary conditions on the pile surface.

The finite slowing down kernel $P(E,\underline{r},\underline{r}')$ and the infinite slowing down kernel $P_\infty(E,|\underline{r}-\underline{r}'|)$ both satisfy the same linear equation by hypothesis. Consequently, the finite slowing down density

$$q(E,\underline{r}) = \frac{k}{p} N\sigma_a \int_{\text{pile}} \underline{\Phi}_s(\underline{r}')\, P(E,\underline{r},\underline{r}')d\underline{r}'$$

and the infinite slowing down density

$$q_\infty(E,\underline{r}) = \frac{k}{p} N\sigma_a \int_{\substack{\text{all}\\ \text{space}}} \underline{\Phi}_s(\underline{r}')\, P_\infty(E,|\underline{r}-\underline{r}'|)d\underline{r}',$$

being linear superpositions of functions which satisfy a linear equation, also satisfy the same equation. The slowing down density $q(E,\underline{r})$ in the pile satisfies certain boundary conditions which, by hypothesis, are identical with the boundary conditions satisfied by $\underline{\Phi}_s$. But we have already shown that $q_\infty(E,\underline{r})$ being proportional everywhere to $\underline{\Phi}_s(\underline{r})$ satisfies the same boundary conditions as $\underline{\Phi}_s(\underline{r})$. Hence $q_\infty(E,\underline{r})$ and $q(E,\underline{r})$ are identical, and the asymptotic solution $\underline{\Phi}_s(\underline{r})$ satisfies the pile equation wherever the pile equation and the extrapolation distance boundary condition are valid, i.e., within a distance of the order of a mean free path of the boundary.

Physically, the sense of this theorem is somewhat as follows: the neutron distribution in a finite pile can be calculated (under the restrictions stated in the hypothesis) by extending the pile out to infinity and finding the asymptotic neutron distribution in this infinite system. This solution oscillates,

positive neutron densities alternating with negative ones ad infinitum. The positive and negative densities are so distributed that on the pile boundary, their superposed effect satisfies the boundary conditions.

The regions in which the neutron density is negative may be called "negative" piles. The positive and negative piles are very similar to the system of images which are commonly used to solve ordinary boundary value problems. Ordinarily the system of images can be constructed only if the bounding surface has sufficient symmetry -- for example, if it is an infinite slab, or a cube, etc. The theorem shows that even for arbitrary shaped surface, an appropriate image system of positive and negative piles can be constructed; in fact, the intensity and distribution of the images is automatically given by the analytic continuation of the asymptotic neutron distribution outside the pile.

This manner of solving the finite pile equation by analytically continuing the pile out to infinity was suggested by Fermi. It is analogous to the Born-von Karman method of calculating the lattice vibrations in a finite crystal by extending the crystal lattice out to infinity.

At first sight the requirement that $P(E,\underline{r},\underline{r}')$ and $P_\infty(E,|\underline{r}-\underline{r}'|)$ satisfy the same linear equation may seem too restrictive. Actually this is not the case: the three slowing down kernels which are used in pile theory, namely the Fermi age kernel, the group picture kernel, and the transport kernel all satisfy linear equations. This holds true, of course, for kernels obtained by convolution of these three.

The requirement that the extrapolation distance is independent of energy is very inaccurate in a hydrogenous mixture, but it is fair in most other cases.

If the pile is very large compared to a mean free path the assumption that the extrapolation distance is energy independent is evidently not very important.

In any case, as we have already remarked, the whole argument breaks down within a mean free path of the boundary because diffusion theory is invalid there. Since the overall size of a fast neutron pile is often of the order of a mean free path, the theorem is not applicable in such a case. Thus the actual neutron distribution in a small fast neutron pile falls off much faster near the boundary than does the asymptotic density. For slow neutron piles, on the other hand, the size is always so large that the deviations from diffusion theory near the boundary are almost always unimportant.

Existence of a Critical Size

According to the fundamental theorem of pile theory, a solution of the wave equation

$$\Delta\Phi_s + B^2\Phi_s = 0 \qquad (5.15)$$

will satisfy the pile equation provided the buckling, B^2, satisfies the characteristic equation

$$-L^2B^2 - 1 + \frac{k}{p}\bar{P}_\infty(E_s,B^2) = 0. \qquad (5.16)$$

The buckling in (5.15) determines the "wave length" of the neutron distribution; i.e., for a given value of B^2 the neutron density has nodes on some surface. Only if the extrapolated pile surface coincides with the surface on which the neutron density has its first node will the solution of (5.15) satisfy both the pile equation and the boundary conditions (neutron density positive inside the pile, neutron density zero on the extrapolated boundary). Thus,

for a pre-assigned set of microscopic parameters, k, L^2, p, $P_\infty(E,|\underline{r}-\underline{r}'|)$, the pile must have very particular physical dimensions in order for the buckling of the solution which satisfies the boundary conditions to simultaneously satisfy the characteristic equation (5.16). This size is called the "critical size" of the pile.

If the pile is critical, B^2 must simultaneously fulfill two conditions: first, B^2 must satisfy the characteristic equation (5.16); and second, B^2 must have such a value that the neutron density determined from (5.15) vanishes on the extrapolated boundary. Two different "bucklings" can therefore be distinguished. On the one hand there is the numerical value

$$B_g^2 = - \frac{\Delta \Phi_s}{\Phi_s} \tag{5.17}$$

which the buckling must have if the neutron density is to vanish on the extrapolated boundary of the pile. This number is called the geometric buckling and is here denoted by B_g^2. For a sphere of radius R_c its numerical value is clearly π^2/R_c^2, since the solution of the wave equation

$$\Delta \Phi_s + \frac{\pi^2}{R_c^2} \Phi_s = 0$$

is positive inside a sphere of radius R_c and vanishes on the surface of the sphere. On the other hand, for any disposition of the fissionable material for which the microscopic quantities L^2, k, p, $P_\infty(E,|\underline{r}-\underline{r}'|)$ are pre-assigned, there is a value of B^2 which satisfies the characteristic equation. Thus B^2, the "material" buckling, may be denoted by B_m^2; it is a microscopic characteristic of a particular disposition of fissionable material in a moderating medium.

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The remarks are summarized in the following theorem:

A chain reacting pile will be critical if its geometric buckling B_g^2 and its material buckling, B_m^2, are equal.

In order for a finite critical size to exist at all the buckling of the neutron density must be positive. For the buckling, being proportional to $-\triangle\Phi_s/\Phi_s$, determines the curvature of the neutron distribution. If the buckling is positive ($B^2 > 0$) the neutron density distribution is concave downward. Conversely, if the buckling is negative the distribution is concave upward. Evidently, if a solution of the pile equation which falls to zero at the boundary is to exist, the curvature of the neutron distribution must be downward. For a given positive value of the material buckling, and a given pile shape, there is one and only one critical pile size for which the neutron density will fall to zero at the pile boundary. If the pile is smaller than this, the neutron density will not reach zero at the boundary; if the pile is larger, the density will fall to zero before the boundary is reached.

On physical grounds there will be a finite critical size -- that is, the buckling will be positive -- only if $k > 1$.

This is readily seen mathematically in the following manner:

The critical equation may be written

$$k = \frac{p(1 + L^2B^2)}{\overline{P}_\infty(E_s,B^2)}$$

where

$$\overline{P}_\infty(E_s,B^2) = 4\pi \int_0^\infty \frac{\sin Br}{Br} P_\infty(E_s,r)r^2dr.$$

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If the buckling vanishes, $B^2 = 0$,

$$\bar{P}_\infty(E_s,0) = 4\pi \int_0^\infty P_\infty(E_s,r)r^2dr = p;$$

hence, when $B^2 = 0$,

$$k = \frac{p}{\bar{P}_\infty(E_s,0)} = 1. \tag{5.18}$$

Since $B^2 = 0$ implies an infinitely large pile, equation (5.1) means that the critical size is infinite if $k = 1$.

If $B^2 > 0$,

$$\bar{P}_\infty(E_s,B^2) = 4\pi \int_0^\infty \frac{\sin Br}{Br} P_\infty(E_s,r)r^2dr < p$$

since $\left|\frac{\sin Br}{Br}\right| < 1$. Hence, when $B^2 > 0$,

$$k = \frac{p(1 + L^2B^2)}{\bar{P}_\infty(E_s,B^2)} > 1;$$

i.e., if the critical size is finite, k must exceed unity.

If $B^2 < 0$,

$$\bar{P}_\infty(E_s,B^2) = 4\pi \int_0^\infty \frac{\sinh |Br|}{|Br|} P_\infty(E_s,r)r^2dr > p$$

since $\frac{\sinh |Br|}{|Br|} > 1$. Hence, when $B^2 < 0$,

$$k < 1;$$

i.e., if k is less than unity there is no finite critical size.

The Critical Equation for a Large Pile

If the pile is very large the general critical equation always reduces to a particularly simple form. Since large critical size means small B^2, the critical equation for a large pile can be obtained by expanding $\overline{P}_\infty(E_s,B^2)$ in powers of B^2. Thus

$$\overline{P}_\infty(E_s,B^2) = 4\pi \int_0^\infty \frac{\sin Br}{Br} P_\infty(E_s,r)r^2dr \approx 4\pi \int_0^\infty P_\infty(E_s,r)r^2dr - \frac{4\pi}{6} B^2 \int_0^\infty P_\infty(E_s,r)r^4dr + \cdots \tag{5.20}$$

The first integral is equal to p because of the way in which $P_\infty(E_s,r)$ has been normalized. The second integral is $p\,\frac{\overline{r^2}}{6}(E_s)$, where $\overline{r^2}(E_s)$ is the mean square distance that a neutron travels from the point of its birth as a fission neutron to the point at which it becomes a slow neutron. Substituting (5.20) into (5.16) we obtain for the characteristic equation

$$B^2 = \frac{k - 1}{L^2 + k\,\frac{\overline{r^2}}{6}(E_s)} \,. \tag{5.21}$$

The approximate form of the characteristic equation serves to verify the statement that B^2 is positive (i.e., critical size is finite) only if $k > 1$.

The expansion of $\overline{P}_\infty(E_s,B^2)$ is valid only if $B^2\overline{r^2}$ is small; i.e., if the reciprocal of the buckling is much larger than the mean square range. The reciprocal of B is always the same order of magnitude as the dimension of a chain reacting system, providing the extrapolation distance is small compared with this dimension. Hence, if the linear dimension of a chain reaction is large compared to the slowing down range $\sqrt{\overline{r^2}(E_s)}$, the simple relation (5.21)

between B^2 and $k - 1$ holds. In particular, this result is independent of the particular form of $\overline{P}_\infty(E_s, |\underline{r}-\underline{r}'|)$.

The simplified characteristic equation can be written

$$B^2 = \frac{k - 1}{L^2 + \frac{\overline{r^2}}{6}(E_s) + (k-1)\frac{\overline{r^2}}{6}(E_s)} \approx \frac{k-1}{M^2}\left[1 - \frac{(k-1)}{6}\frac{\overline{r^2}}{M^2}(E_s)\right]$$

where $M^2 = L^2 + \frac{1}{6}\overline{r^2}(E_s)$ is called the migration area. Since this equation is derived in the first place on the assumption that $k - 1 \ll 1$, it is correct, to the first order in $k - 1$, to write

$$B^2 = \frac{k - 1}{M^2}\ ; \qquad (5.22)$$

this simplest form of the pile equation is applicable in any pile in which unenriched uranium is used, since $k - 1$ is always small in such piles.

The square root of the migration area is called the migration length. Since $L^2 = \frac{1}{6}$ x mean square distance that a slow neutron travels before capture, the migration area is $\frac{1}{6}$ x mean square distance a neutron travels from birth to death. The migration length determines the length scale of a chain reacting system in which k is close to unity.

Physical Significance of the Characteristic Equation; k_{eff} and k_{ex}

The characteristic equation (5.16) which the geometric buckling satisfies if the pile is critical has a simple physical interpretation. This interpretation follows from the fact that the Fourier transform of the slowing down kernel, $\overline{P}_\infty(E,B^2)$, is the average probability that a fission neutron, born in a pile whose geometric buckling is B^2, crosses energy E inside the pile. To

see this we recall from the fundamental theorem that the slowing down density at energy E in a pile whose geometric buckling is B_g^2 is

$$q(E,\underline{r}) = \frac{k}{p}\,\bar{P}_\infty(E,B_g^2)\,N\sigma_a\Phi_s(\underline{r}).$$

Since $q(E,\underline{r})$ is the number of neutrons which cross E per c.c. per sec at $\underline{r}$, the total number Q(E) of neutrons which cross E per second inside the pile is

$$Q(E) = \int_{\text{pile}} q(E,\underline{r})d\underline{r} = \frac{k}{p}\,\bar{P}_\infty(E,B_g^2)N\sigma_a \int_{\text{pile}} \Phi_s(\underline{r})d\underline{r}$$

while the number of fission neutrons created per second in the pile is

$$\frac{k}{p}\,N\sigma_a \int_{\text{pile}} \Phi_s(\underline{r})d\underline{r}.$$

The ratio of these two expressions is $\bar{P}_\infty(E,B_g^2)$, and this is therefore the average probability that a neutron crosses energy E inside the pile.

Suppose we start with N fission neutrons and follow their history through one generation. Of these neutrons

$$N\,\bar{P}_\infty(E_s,B_g^2)$$

become slow inside the pile. Some of these slow neutrons leak out of the pile; the rest are absorbed. The ratio of slow neutron leakage out of the pile to slow neutron absorption is

$$\frac{D_o\int \nabla\Phi_s \cdot d\underline{s}}{N\sigma_a \int_{\text{pile}} \Phi_s\, d\underline{r}} \qquad (5.23)$$

where $\int \nabla \Phi_s \cdot d\underline{s}$ denotes the integral over the pile surface of the normal derivative of the slow neutron flux. By Gauss' theorem, and the fact that Φ_s satisfies the wave equation (5.15), this ratio can be written

$$\frac{\text{slow neutron leakage}}{\text{slow neutron absorption}} = L^2 B_g^2 \tag{5.24}$$

Hence $$\frac{\text{slow neutrons absorbed}}{\text{slow neutrons created}} = \frac{1}{1 + L^2 B_g^2}$$

The total number of neutrons absorbed as slow, if we started with N fission neutrons, is the product of $N \bar{P}_{\infty}(E_s, B^2)$ and (5.24); that is

$$\frac{N \bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2}$$

and therefore the total number of second generation fission neutrons is

$$\frac{k}{p} N \frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2} . \tag{5.25}$$

In a steady state, i.e., if the pile is critical, the number of fission neutrons at the beginning of the second generation must be the same as N, the number at the beginning of the first generation. Hence the critical condition is

$$\frac{k}{p} N \frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2} = N$$

which, upon cancellation of N, is seen to be the characteristic equation (5.16).

Since

$$\frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2}$$

is the total number of slow neutrons absorbed in one generation per neutron either absorbed or leaking, the quantity

$$k_{eff} = \frac{k}{p} \frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2} \tag{5.26}$$

is the number of neutrons created in one generation per neutron removed either by leakage or absorption. According to p. 20 this quantity is the "effective multiplication constant" which was there denoted by k_{eff} Evidently k_{eff} depends on B_g^2, i.e., on the pile size, and is therefore a macroscopic pile property. The critical condition (5.16) is simply

$$k_{eff} = 1. \tag{5.27}$$

If the geometric buckling is reduced -- that is, if the pile size is increased beyond the critical size -- then more neutrons will be produced in one generation than are lost by leakage or absorption. The k_{eff} will be larger than unity by an amount which is called the "excess" multiplication factor, k_{ex};

$$k_{ex} = k_{eff} - 1 = \frac{k}{p} \frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2 B_g^2} - 1. \tag{5.28}$$

The excess multiplication factor determines the rate at which the neutron density grows in a super-critical pile. It is therefore of fundamental importance in the theory of pile kinetics.

Fast and Slow Leakage

According to (5.24) the number of slow neutrons which leak out of the pile per slow neutron absorbed is

$$L^2 B^2.$$

Since every slow neutron which is created in the pile must ultimately leak out or be captured, (5.24) leads to:

$$\frac{\text{slow neutrons leaking out of pile}}{\text{slow neutrons created inside pile}} = \frac{L^2B^2}{1 + L^2B^2} \quad . \tag{5.29}$$

We now compute the fast neutron leakage. The relative fraction of fast neutrons which leak out or are captured in resonances will, of course, depend upon whether the resonance capture occurs predominantly at very high energy, before the neutrons have slowed down, or at low energy after their moderation has been completed. It is fairly realistic, in most slow neutron piles, to assume that all the resonance capture occurs at a single energy just above thermal energy. Then if N fast neutrons start to slow down

$N\,\bar{P}_\infty(E_s,B^2)$ become thermal inside pile

$N\,\frac{\bar{P}_\infty}{p}\,(E_s,B^2)$ reach the resonance line inside pile

$N\left[1 - \frac{\bar{P}_\infty}{p}\,(E_s,B^2)\right]$ leak out of pile while being slowed.

$$\frac{p - \bar{P}_\infty(E_s,B^2)}{p\,\bar{P}_\infty\,(E_s,B^2)} = \frac{\text{fast neutrons leaking out of pile}}{\text{slow neutrons created inside pile}} \quad . \tag{5.30}$$

Hence we can write, by combining (5.29) and (5.30),

$$\frac{\mathcal{L}_f}{\mathcal{L}_s} = \frac{\text{fast neutrons leaking out of pile}}{\text{slow neutrons leaking out of pile}} = \frac{(p - \bar{P}_\infty(E_s,B^2)\,(1 + L^2B^2)}{L^2B^2 p\,\bar{P}_\infty\,(E_s,B^2)} \tag{5.31}$$

where $\mathcal{L}_f$ and $\mathcal{L}_s$ denote fast and slow leakages respectively. Since $\frac{1 + L^2B^2}{\bar{P}_\infty\,(E_s,B^2)} = \frac{k}{p}$, we have in a critical pile

$$\frac{\mathcal{L}_f}{\mathcal{L}_s} = \frac{k(p - \bar{P}_\alpha(E_s, B^2))}{p^2 L^2 B^2}, \tag{5.32}$$

and this becomes, in a large pile ($k \approx 1$),

$$\frac{\mathcal{L}_f}{\mathcal{L}_s} \approx \frac{k}{p} \frac{\overline{r^2(E_s)}}{6L^2}. \tag{5.33}$$

Aside from the factor $\frac{k}{p}$, which is necessarily close to unity in a large pile, the fast and slow leakages are just in proportion to the contributions of the fast and slow neutrons to the migration area.

The ratio $\mathcal{L}_f/\mathcal{L}_s$ given in (5.32) is correct only if the resonance absorption occurs after the slowing down process has been completed. A corresponding formula can easily be derived for the case of resonance absorption before moderation begins. In a slow neutron pile resonance absorption may occur at all energies, although it is usually heaviest a few volts before thermal energy.

Neutron Distributions and Critical Sizes in Bare Piles with Various Geometries

In this section we calculate the neutron distributions and critical sizes for piles of rather simple shape.

The method used is to solve the wave equation (5.15) in which B^2 is the real root of the characteristic equation (5.16). In all cases we consider here piles which are uniform so that the buckling is independent of position. We also suppose that the extrapolation distance is energy independent; the density of slow and fast neutrons are therefore proportional throughout the pile. The mathematical problem, then, is the same as finding the fundamental mode for a vibrating medium of a particular shape and size.

Infinite Slab

The pile equation for a slab which is infinite in the y and z directions is

$$\frac{d^2\Phi}{dx^2} + B^2\Phi = 0 \qquad (5.34)$$

with the solution

$$\Phi = \Phi_o \cos Bx, \qquad (5.35)$$

where Φ is the neutron flux and Φ_o is the arbitrary value of Φ at $x = o$, which we take to be the center plane of the pile. We do not distinguish between slow and fast neutrons since the two are proportional. The boundary condition $\Phi = o$ at $x = \pm a$, where a is the extrapolated half thickness of the pile, can be satisfied only if

$$a = \frac{\pi}{2B}, \qquad (5.36)$$

and this is the relation between the critical half thickness and the buckling of an infinite slab pile. The critical neutron distribution (5.35) is a maximum in the center of the pile and falls to zero at the extrapolated edge. This is characteristic of the neutron distribution in any bare pile which is simply connected and which has a constant buckling everywhere. The average neutron flux, $\overline{\Phi}$, is given by

$$\overline{\Phi} = \frac{\Phi_o}{2a} \int_{-a}^{a} \cos \frac{\pi x}{2a}\, dx = \frac{2}{\pi} \Phi_o ; \qquad (5.37)$$

i.e., the central flux is $\frac{\pi}{2} = 1.57$ times the average flux. The center to average

neutron flux is an important quantity in the design of an actual pile since it determines how much more intense than the average is the heat production at the pile center.

Another average which is important in pile theory is the average of the square of the neutron flux. For a slab pile this is

$$\overline{\Phi^2} = \frac{\Phi_o^2}{2a} \int_{-a}^{+a} \cos^2 \frac{\pi x}{2a}\, dx = \frac{\Phi_o^2}{2}, \tag{5.38}$$

i.e., $\overline{\Phi^2}$ is 1/2 the square of the neutron flux at the center of the pile.

The structure of the critical size formula (5.36) deserves some study since it has the same basic form for all geometries. The critical dimension a is in general proportional to 1/B. If the pile is very large, then the approximate formula (5.22) for B^2 can be used. The critical dimension then becomes

$$a = \frac{\pi M}{2\sqrt{k - 1}} \tag{5.39}$$

i.e., a is proportional to M. Thus for a large pile the migration length determines the scale of a chain reacting system. The dimension a is also inversely proportional to $\sqrt{k - 1}$; only if k exceeds unity is there a finite critical size. If k is just equal to one. The pile is infinitely large, and if k is less than one, a is imaginary; that is, there can be no critical pile for which k is less than one. The constant of proportionality between a and $M/\sqrt{k - 1}$ is $\pi/2$ in the case of a slab; in other geometries this constant has different values.

Sphere

The pile equation is

$$\frac{d^2(\Phi_r)}{dr^2} + \frac{2}{r}\frac{d\Phi_r}{dr} + B^2\Phi_r = 0 \qquad (5.40)$$

with the solution (regular at the origin, r = o)

$$\bar{\Phi} = \bar{\Phi}_o \frac{\sin Br}{Br} , \qquad (5.41)$$

The critical radius R_c is

$$R_c = \frac{\pi}{B} \qquad (5.42)$$

which, for a large pile becomes

$$R_c = \frac{\pi M}{\sqrt{k - 1}} \qquad (5.43)$$

and the critical volume, V_c, is

$$V_c = \frac{4}{3}\pi R_c^3 = \frac{4}{3}\frac{\pi^4}{B^3} = \frac{129.9}{B^3} . \qquad (5.44)$$

The critical volume is inversely proportional to the 3/2 power of the buckling. This is a general result which follows from dimensional considerations.

The value of $\bar{\Phi}$ is $\frac{\bar{\Phi}_o}{3.04}$, and the value of $\bar{\Phi}^2$ is $\frac{\bar{\Phi}_o^2}{6.08}$.

Infinite Right Circular Cylinder

The pile equation is

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} + B^2\Phi = 0 \qquad (5.45)$$

with the solution (regular at the origin)

$$\Phi = \Phi_o J_o(Br) \qquad (5.46)$$

where J_o is the Bessel function of the first kind of zero-th order. The critical radius is

$$R_c = \frac{2.405}{B} \tag{5.46}$$

which becomes, for a large cylinder,

$$R_c = \frac{2.405\ M}{\sqrt{k-1}} . \tag{5.47}$$

The value of $\bar{\Phi}$ is

$$\bar{\Phi} = \frac{2\Phi_o}{R_c^2} \int_0^{R_c} J_o(2.405 \frac{r}{R_c}) r dr = \frac{2J_1(2.405)}{2.405} \Phi_o = \frac{\Phi_o}{2.32} ;$$

and the value of $\bar{\Phi}^2$ is

$$\bar{\Phi}^2 = J_1^2 (2.405)\ \Phi_o^2 = \frac{\Phi_o^2}{3.71} .$$

The integrals used to evaluate these averages are given in Jahnke-Emde, Tables of Functions, p. 146.

Finite Right Circular Cylinder of Height H_c, Radius R_c

The pile equation is

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} + \frac{d^2\Phi}{dz^2} + B^2\Phi = 0 \tag{5.48}$$

with the solution which vanishes at $(R_c, \pm \frac{1}{2} H_c)$

$$\Phi = \Phi_o J_o(2.405 \frac{r}{R_c}) \cos \frac{\pi z}{H_c} \tag{5.49}$$

provided

$$\frac{\pi^2}{H_c^2} + \frac{2.405^2}{R_c^2} = B^2. \tag{5.50}$$

This equation determines the critical radius in terms of the pile height and buckling. The minimum radius of a critical cylinder is found when $H_c = \infty$; this leads to the critical equation (5.47) for an infinite cylinder. The minimum height is found when $R = \infty$ and is just the value found for the infinite slab in (5.42). In both these extreme cases the pile volume is infinite. There must therefore be an optimum radius to height ratio for which the pile volume, for a given buckling, is a minimum. To find this minimum volume, V_{min}, we write, from (5.50)

$$V_c = \pi R_c^2 H_c = \frac{\pi \times 2.405^2 H_c^3}{H_c^2 B^2 - \pi^2}$$

and the minimum occurs when

$$\frac{dV_c}{dH_c} = 0.$$

Solving this equation we find

$$V_{min} = \frac{1.5 \times \sqrt{3} \times 2.405^2 \times \pi^2}{B^3} = \frac{148.3}{B^3} . \qquad (5.51)$$

The diameter of the minimum volume critical cylinder is 1.08 times its height.

The distribution (5.49) in the finite cylinder is the product of two distributions, the one characteristic of an infinite cylinder, the other characteristic of an infinite slab. Corresponding to this factorability of the solution is the fact that the critical radius of the finite cyclinder, which is determined from

$$\frac{2.405^2}{R_c^2} = B^2 - \frac{\pi^2}{H_c^2} ,$$

is the same as that of an infinite cylinder whose buckling is less than B^2 by

an amount π^2/H_c^2. The quantity, $B^2 - \pi^2/H_c^2$, which is denoted by B_r^2, is called the "radial" buckling. Similarly, the critical height is given by

$$\frac{\pi^2}{H_c^2} = B_H^2$$

where $B_H^2 = B^2 - \frac{2.405^2}{R_c^2}$ is called the longitudinal buckling. The total geometric buckling is thus the sum of the bucklings in each direction. This example illustrates the general principle that whenever the pile equation (5.15) is separable, the critical dimension in any direction is calculated as though the pile were infinite in all other directions but with a buckling which is reduced from B^2 by the partial bucklings which are "assigned" to the other directions.

Because the finite cylinder solution is factorable into infinite cylinder and finite slab solutions, the density averages are also products of the averages for the two infinite shapes. Thus

$$\overline{\Phi} = \frac{2J_1(2.405)}{2.405} \times \frac{2}{\pi} \Phi_o = \frac{\Phi_o}{3.64}$$

and

$$\overline{\Phi^2} = J_1^2(2.405) \times \frac{1}{2} \Phi_o^2 = \frac{\Phi_o^2}{7.42} .$$

Rectangular Parallelopiped of Sides $2a_1$, $2a_2$, $2a_3$

The pile equation is

$$\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2} + \frac{d^2\Phi}{dz^2} + B^2\Phi = 0 \qquad (5.52)$$

with the solution which vanishes on the pile boundary ($x = \pm a_1$, $y = \pm a_2$, $z = \pm a_3$)

$$\Phi = \Phi_o \cos \frac{\pi x}{2a_1} \cos \frac{\pi y}{2a_2} \cos \frac{\pi z}{2a_3} \qquad (5.53)$$

provided

$$\frac{\pi^2}{4a_1^2} + \frac{\pi^2}{4a_2^2} + \frac{\pi^2}{4a_3^2} = B^2 \ . \tag{5.54}$$

The solution is the product of three infinite slab solutions, one for each of the coordinate directions. The critical condition (5.54) determines one of the critical dimensions in terms of the other two, which are arbitrary so long as they exceed a certain minimum value. This minimum value is found by putting $a_1 = a_2 = \infty$. Equation (5.54) then reduces to the formula already found for the critical infinite slab.

For a cube, $a_1 = a_2 = a_3$, the critical dimension is

$$a_1 = \frac{\pi\sqrt{3}}{2B}$$

and the critical volume, V_c, is

$$V_c = 8a_1^3 = \frac{\pi^3 \cdot 3^{3/2}}{B^3} = \frac{161.1}{B^3} \ . \tag{5.55}$$

Comparing the critical volumes for a sphere (5.44), an optimum cylinder (5.51), and a cube all with the same buckling,

$$V_{c_{sphere}} : V_{c_{cylinder}} : V_{c_{cube}} = 1 : 1.142 : 1.240 \ ;$$

i.e., the figure with the smallest surface to volume ratio requires the smallest critical volume for a given buckling. This result is not surprising since the smaller the surface to volume ratio, the smaller is the probability that a neutron will escape from the system by leakage.

The averages for a parallelopiped are again the product of three slab averages since the neutron distribution (5.53) is factorable. These averages are:

$$\overline{\Phi} = \left(\frac{2}{\pi}\right)^3 \Phi_o = \frac{\Phi_o}{3.88}$$

$$\overline{\Phi^2} = \left(\frac{1}{2}\right)^3 \Phi_o^2 = \frac{\Phi_o^2}{8} .$$

More Complicated Shapes

Various more complicated shapes have been solved analytically; essentially any shapes for which the fundamental vibrational mode is known is a solvable pile shape. We list some of these shapes below:

Sector of a Sphere:

The function

$$\Phi = \Phi_o \frac{1}{\sqrt{\mu_\ell \, r/R_c}} \, J_{\ell+\frac{1}{2}}(\mu_\ell \, r/R_c) \, P_\ell(\cos\theta) \qquad (5.56)$$

is a solution of the pile equation which satisfies the boundary condition $\Phi = o$ on the boundary of a spherical sector (Fig.5-II), provided the half angle α of the sector is a zero of

$$P_\ell(\cos\alpha) = 0, \qquad (5.57)$$

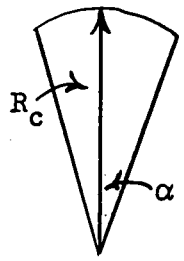

Figure 5-II

μ_ℓ is a zero of

$$J_{\ell+\frac{1}{2}}(\mu_\ell) = 0 \qquad (5.58)$$

and

$$\frac{\mu_\ell^2}{R_c^2} = B^2, \tag{5.59}$$

where R_c is the radius of the sphere from which the sector has been cut. Thus (5.56) gives, for different values of ℓ, the neutron distribution in a discrete set of sectors whose apex half-angles are given by (5.57). For a hemisphere, $\ell = 1$, $\alpha = \pi/2$, and the neutron flux is

$$\Phi = \frac{\Phi_o}{\sqrt{\mu_1 \, r/R_c}} \, J_{3/2}(\mu_1 \, r/R_c) \cos\theta \tag{5.60}$$

where

$$\mu_1 = 4.49, \quad R_c = 4.49/B, \tag{5.61}$$

and the critical volume of the hemisphere is

$$V_c = 190/B^3. \tag{5.62}$$

The critical volume of a hemisphere is 1.46 times as large as that of a sphere having the same buckling. This is another illustration of the fact that for a given buckling the figure with the smaller surface to volume ratio requires the smaller critical volume. To find the critical volumes of sectors whose half-angles are not zeros of the Legendre Polynomials, interpolation between the values of V_c found for sectors whose angles do satisfy (5.57) can be used.

Sector of a Cylinder:

To solve a pile which is shaped like a cylindrical sector we proceed in a similar manner. The function

$$\Phi = \Phi_o \cos\frac{\pi z}{H_c} \, J_\ell \, (\mu_\ell \, r/R_c) \cos\ell\theta \tag{5.63}$$

is a solution of the pile equation which satisfies the boundary condition $\Phi = 0$ on the boundary of a cylindrical sector pile whose height is H_c, provided the half apex angle, $\alpha = \pi/2$;

$$J_\ell(\mu_\ell) = 0$$

and

$$\frac{\mu_\ell^2}{R_c^2} + \frac{\pi^2}{H_c^2} = B^2. \qquad (5.64)$$

Again (5.63) gives the neutron distribution in a discrete set of cylindrical sector piles whose half-angles are $\alpha = \pi/2\ell$ $(\ell \neq 0)$. The critical size for sectors whose half-angles are not $\pi/2\ell$ can be estimated by interpolation just as in the previous example.

Elliptic Cylinder:

The pile equation in circular coordinates is

$$\frac{d^2\Phi}{dr^2} + \frac{1}{r}\frac{d\Phi}{dr} + \frac{1}{r^2}\frac{d^2\Phi}{d\theta^2} + B^2\Phi = 0 \qquad (5.65)$$

where r and θ are the circular coordinates of a point measured from the center of the ellipse. If the cylinder is of finite height the total buckling B^2 must be replaced by the radial buckling; otherwise the equation for the radial distribution is just (5.65). The solutions of this equation appropriate to an ellipse are the Mathieu functions; however, since these are not tabulated very extensively it is more convenient to write the solution as

$$\Phi(r,\theta) = \sum A_{2m} J_{2m}(Br) \cos 2m\theta \, . \qquad (5.66)$$

(This amounts to expanding the Mathieu functions in a series of Bessel functions; cf. Whittaker and Watson, Modern Analysis (1927) p. 427). The even multiples of m are used because, on an ellipse, $r(\theta) = r(180-\theta)$. The problem is to

determine the A_{2m} and the value of B so that $\Phi(r,\theta)$ vanishes at every point on the boundary of the ellipse.

A simple way to satisfy the boundary conditions approximately is to first cut off the series at, say, m = 3. This places at our disposal B and three A_{2m}'s; they can be so adjusted that Φ vanishes at three points on the ellipse in the first quadrant (12 points in all quadrants). The value of θ_i at which to make $\Phi(r,\theta_i)$ vanish are rather arbitrary. A choice which gives excellent convergence, and which is suggested by the necessity for $\int\Phi(r,\theta)ds$ to vanish along the boundary of the ellipse, is to take the Φ_i to be the roots of

$$P_{2m}(\cos\theta) = 0. \tag{5.57}$$

This choice ensures, according to Gauss' integral approximation formula (Whittaker and Robinson, The Calculus of Observations p. 159), that the integral of Φ along the boundary will be small at each stage in the approximation.

For m = 3, the roots of (5.57) are $\theta_1 = 76^\circ\ 12'$, $\theta_2 = 48^\circ\ 37'$, $\theta_3 = 21^\circ\ 10'$. Corresponding to these three angles are the three radii r_1, r_2, and r_3. The approximate boundary condition is therefore satisfied if

$$A_oJ_o(Br_i) + A_2J_2(Br_i)\cos 2\theta_i + A_4J_4(Br_i)\cos 4\theta_i = 0 \quad (i = 1,2,3),$$

and this set of homogeneous linear equations in A_{2m} is solvable if the determinant of coefficients vanishes:

$$\begin{vmatrix} J_o(Br_1) & \left[J_2(Br_1)\cos 2\theta_1\right] & \left[J_4(Br_1)\cos 4\theta_1\right] \\ J_o(Br_2) & \left[J_2(Br_2)\cos 2\theta_2\right] & \left[J_4(Br_2)\cos 4\theta_2\right] \\ J_o(Br_3) & \left[J_2(Br_2)\cos 2\theta_3\right] & \left[J_4(Br_3)\cos 4\theta_3\right] \end{vmatrix} = 0. \tag{5.68}$$

This is a transcendental equation in B which can be solved numerically for any particular ellipse. We list below some values which have been obtained essentially by this method by N. Morehouse:

Critical Dimensions of an Ellipse

Eccentricity	Semi-Major Axis	B
0 (circle)	1	2.405
0.80	1	2.51
0.85	1	2.63
0.90	1	2.84
0.94	1	3.12

The Variation Method

In many cases the pile shape is so complicated that an exact solution of the pile equation is out of the question. It is then often appropriate, in order to find the critical size and the neutron density, to apply a variation method, e.g. the Ritz method.

We review the method very briefly. The solution of the pile equation

$$\triangle \Phi + B^2 \Phi = 0 \tag{5.69}$$

is that function $\Phi(x,y,z)$ which, among all functions which satisfy the pile boundary conditions (Φ everywhere non-negative, Φ vanishes on extrapolated pile boundary), makes the integral over the pile volume

$$I = \int \left[(\nabla \Phi)^2 - B^2 \Phi^2 \right] dv \tag{5.70}$$

an extremum. The pile equation is the Euler equation corresponding to this variation problem.

The idea of the Ritz method is to assume some arbitrary form of Φ depending on parameters α_ν, substitute into (5.70), and then determine α_ν so that I is a relative minimum. Thus, if we consider a set of functions $f_\nu(x,y,z)$ which satisfy the boundary conditions, we may set

$$\Phi = \sum_{\nu=1}^{\infty} \alpha_\nu f_\nu(x,y,z); \tag{5.71}$$

we then seek the values of α_ν which make the functional I an extremum among all functions representable as a linear combination of ν functions of the form (5.71). If the f_ν are chosen to be members of a complete orthogonal set which satisfy the boundary conditions, then the infinite set of linear equations in α_ν obtained by requiring that I be an extremum will indeed determine the α_ν and B exactly. Usually it is simpler to choose the f_ν as a sequence of polynomials because in that case the evaluation of the integral is simplified.

As an example we solve the sphere problem by the variation method. We consider the sequence of functions (which vanish at $r = R_c$)

$$f_1 = (r - R_c)^2$$

$$f_2 = (r - R_c)^4$$

and set

$$\Phi = \alpha_1(r - R_c)^2 + \alpha_2(r - R_c)^4.$$

Substituting this assumed form of Φ into (5.70), integrating, and then setting the derivatives of I with respect to α_1 and α_2 equal to zero, we obtain

$$(.2667 - .0190\, B^2R_c^2)\alpha_1 + (.1524 - .0079\, B^2R_c^2)\alpha_2 R_c^2 = 0$$

$$(.1524 - .0079\, B^2R_c^2)\alpha_1 + (.1270 - .0040\, B^2R_c^2)\alpha_2 R_c^2 = 0$$

These homogeneous equations have a solution only if the determinant of their coefficients vanish; this will be the case if $B^2R_c^2 = 11.6$. The exact value of $B^2R_c^2$ is $\pi^2 = 9.87$.

Numerical Methods

When analytic methods are unfeasible, it is always possible to integrate the pile equation numerically. If the pile is two-dimensional the solution is relatively simple. When the pile is three-dimensional the numerical work becomes great. It is usually more convenient in setting up the problem to assume the pile size in advance and then to determine the characteristic value B^2 for the equation

$$\Delta\Phi + B^2\Phi = 0.$$

The available numerical methods are too numerous to mention; an excellent summary is given by H. W. Emmons, Quarterly of Applied Mathematics, 2, 175, (1944).

Critical Size of Anisotropic Pile

The critical size and the neutron distribution in a pile which is anisotropic is found by an easy generalization of the previous results, provided the pile is large -- i.e., $B_m^2 \approx k\text{-}1/M^2$. Consider a pile which is so constructed that its diffusion properties are different in the three coordinate directions. A pile which is traversed by a large number of parallel cylindrical

air cooling channels would be an example, since the neutrons diffuse more easily in the direction of the channels than perpendicular to them. The age of a neutron is now different in the three directions, and instead of one age, τ, three ages, τ_1, τ_2, τ_3, (where subscripts refer to x,y,z) are required to characterize the fast neutron diffusion. Similarly three diffusion lengths, L_1^2, L_2^2, L_3^2, which are respectively 1/6 the mean square distances that a slow neutron travels in each of the three coordinate directions, are needed to characterize the slow neutron diffusion. The pile equation in the simplest approximation must therefore be replaced by

$$M_1^2 \frac{\partial^2 \Phi}{\partial x^2} + M_2^2 \frac{\partial^2 \Phi}{\partial y^2} + M_3^2 \frac{\partial^2 \Phi}{\partial z^2} + (k-1)\Phi = 0 \qquad (5.72)$$

where $M_i^2 = \tau_i + L_i^2$ is the migration area in the i^{th} direction. The solution for a rectangular parallelopiped of sides $2a_1$, $2a_2$, $2a_3$ is

$$\Phi = \Phi_o \cos \frac{\pi x}{2a_1} \cos \frac{\pi y}{2a_2} \cos \frac{\pi z}{2a_3} \qquad (5.73)$$

provided

$$M_1^2 \frac{\pi^2}{4a_1^2} + M_2^2 \frac{\pi^2}{4a_2^2} + M_3^2 \frac{\pi^2}{4a_3^2} = k - 1. \qquad (5.74)$$

This equation reduces to (5.22) when the pile is isotropic ($M_1 = M_2 = M_3$).

The ratio of the parallel to the perpendicular migration lengths when the asymmetry is caused by an array of parallel empty channels is given essentially in Eq. (1.122). Thus the diffusion length ratio is

$$\frac{L_{11}^2}{L_\perp^2} = \frac{1 + f(1 + 6\gamma\bar{\ell}/\lambda)}{1 + f(1 + 3\gamma\bar{\ell}/\lambda)} \qquad (5.75)$$

where f is the fraction of the total volume occupied by the channels, $\bar{\ell}$ is the hydraulic radius of the channel, γ is a form factor depending on the shape of the channel, and λ is the mean free path for thermal neutrons. The ratio of τ in the parallel and perpendicular directions is given by a similar formula, but λ must be averaged over the fast neutron energy spectrum.

Effect of Density on Critical Size

Each critical dimension of a pile is inversely proportional to the square root of the material buckling, B_m^{-1}. Since B_m^{-1} has the same dimensions as the mean free path, it must depend on the density, d, of the pile material in the same way as the mean free path, namely $B_m^{-1} \sim d^{-1}$. This is strictly true if the pile is homogeneous. If the pile is heterogeneous with lumps of fissionable material scattered throughout the moderator, the proportionality to 1/d is in general not quite exact. In this case a change in density will change the details of the thermal neutron distribution, and this will change the relative number of neutrons absorbed in the moderator and in the fissionable material. As a result, L, and hence B_m^{-1} will not be strictly proportional to 1/d.

Since each critical linear dimension is always proportional to B_m^{-1}, which in turn is proportional to d^{-1}, the critical volume of a homogeneous pile must be proportional to d^{-3},

$$V_c \sim d^{-3},$$

provided the multiplication factor k is independent of density. The critical mass, m_c, is $V_c d$. Hence

$$m_c \sim d^{-2}; \qquad (5.76)$$

i.e., the critical mass of a homogeneous chain reacting system is inversely proportional to the square of the density of the system.

No such simple result can be quoted for a heterogeneous system because in this case B_m^{-1} is not strictly proportional to d^{-1} and k is not strictly independent of d. However, in most examples of practical interest, the relation $m_c \sim d^{-2}$ is sufficiently accurate to be a useful approximation even for heterogeneous systems.

The Approach to Critical

The effective multiplication constant, k_{eff}, in a pile which is sub-critical is less than unity. A source of neutrons placed in such a pile will be multiplied in strength a finite number of times. It is easy to calculate the total multiplication of the source provided the source distribution is the same as the steady state critical distribution, and the energy of the source neutrons is the same as the energy of the fission neutrons.

We consider a sub-critical pile whose geometric buckling is B_g^2. Suppose a source of fission neutrons is distributed over the pile according to a function $Q(\underline{r})$ which satisfies the wave equation

$$\triangle Q(\underline{r}) + B_g^2 Q(\underline{r}) = 0, \qquad (5.77)$$

and extrapolates to zero on the extrapolated boundary. We call such a distribution the "fundamental" or "zero-th harmonic" distribution. In a slab pile of thickness 2a, for example, this means that the source strength is distributed like

$$Q(x) = Q_o \cos \frac{\pi x}{2a} . \qquad (5.78)$$

A source neutron which originates close to the pile boundary evidently will produce, on the average, fewer than k_{eff} neutrons in one generation, while a source neutron which originates at the center of the pile will produce

on the average more than k_{eff} neutrons in one generation. However, if the source distribution is the same as the fundamental distribution then the number of neutrons produced in one generation by a source neutron will on the average be just k_{eff}.

It is a simple matter to compute the total number of daughter neutrons produced in all successive generations by a fundamental distribution of primary source neutrons. We start with one average neutron produced at the beginning of the first generation. In the first generation this neutron has the probability $\overline{P}_{\infty}(E_s,B_g^2)$ of becoming slow inside the pile. Each neutron which slows down produces k_{eff} slow neutrons in the first generations, k_{eff}^2 slow neutrons in the second generation ..... k_{eff}^n slow neutrons in the n^{th} generation. The total number of slow neutrons produced in all generations is clearly

$$\overline{P}_{\infty}(E_s,B^2)(1 + k_{eff} + k_{eff}^2 + ... k_{eff}^n + ...) = \frac{\overline{P}_{\infty}(E_s,B^2)}{1 - k_{eff}} . \qquad (5.79)$$

Since for every slow neutron produced, $\frac{1}{\overline{P}_{\infty}(E_s,B^2)}$ fast neutrons must have started a slowing down act, the total number of neutrons produced per primary source neutron is $\frac{1}{1 - k_{eff}}$. Thus the primary source distribution, which in the case of the slab pile was

$$Q(\underline{r}) = Q_o \cos \frac{\pi x}{2a} ,$$

is increased by multiplication in successive generations until it becomes

$$\frac{Q_o}{1 - k_{eff}} \cos \frac{\pi x}{2a} = - \frac{Q_o}{k_{ex}} \cos \frac{\pi x}{2a} \qquad (5.80)$$

where the excess multiplication constant, k_{ex}, is $k_{eff} - 1$.

The excess multiplication constant is, according to (5.28),

$$k_{ex} = k_{eff} - 1 = \frac{k\,\bar{P}_{\infty}(E_s, B_g^2)}{p\left[1 + L^2B_g^2\right]} - 1. \qquad (5.28)$$

As the pile is made larger $\frac{\bar{P}_{\infty}(E_s, B_g^2)}{1 + L^2B_g^2}$ increases and therefore k_{eff} approaches unity. Thus the effective source strength in a sub-critical pile, according to (5.80) increases toward ∞ as the size of the sub-critical pile is made to approach the critical size.

This fact is of great practical significance since it is the basis of the experimental method for determining how near critical a sub-critical pile is. The ratio of the primary source intensity to the multiplied source intensity is - k_{ex}. A counter placed near a chain reacting system containing a fission source distributed (in a slab) like $\cos \frac{\pi x}{2a}$, will record a number which is proportional to the multiplied source intensity. The ratio of this reading to the counter reading when the chain reacting system is removed but the source remains, is - k_{ex}. In assembling a chain reacting system it is customary to record this ratio as a function of the volume of the system. The volume at which this ratio extrapolates to zero is the critical volume of the system, since at this point k_{ex} is zero. Thus it is unnecessary to build a chain reacting system all the way to critical in order to estimate very reliably the critical volume.

From k_{ex} is is possible to determine k if $\frac{\bar{P}_{\infty}(E_s, B_g^2)}{p\,(1 + L^2B_g^2)}$ is known (Eq. 5.28). This quantity can often be estimated from a knowledge of the moderation and thermal diffusion properties of the medium. Measurement of the multiplication

rate in a finite system can thus be used to estimate k, the multiplication constant in an infinite system. This is the method which was used extensively at Los Alamos and on the German atomic energy project.

To assume that a neutron source will have the same spatial dependence as the fundamental distribution or the same energy dependence as the fission neutrons is of course unrealistic. Multiplication experiments are ordinarily done with point Ra-Be sources, or with neutrons from spontaneous fission. In neither case does the space distribution of the source satisfy

$$\triangle Q + B_g^2 Q = 0;$$

in addition, the energy of the Ra-Be neutrons differs from the energy of the fission neutrons. Because of this energy difference, the slowing down function for a source neutron $P_{\infty_S}(E,|\underline{r}-\underline{r}'|)$ will in general differ from the corresponding function for a fission neutron. We therefore must generalize the foregoing considerations to the case of a source distribution which is arbitrary in both space and energy.

We write the pile equation for a bare, uniform pile in which the extrapolation distance is energy independent (or failing this in which we consider only asymptotic solutions) as

$$D_o \triangle \phi_s - N\sigma_a \phi_s + q(E_s,\underline{r}) = 0. \qquad (5.81)$$

In this equation, $q(E_s,\underline{r})$, the number of slow neutrons produced per c.c. and second at $\underline{r}$, consists of two parts. One part arises from those neutrons which were produced by neutron induced fission; the other arises from neutrons which were emitted by the source. We can therefore write

$$q(E_s,\underline{r}) = \frac{kN\sigma_a}{p} \int \phi_s(\underline{r}')\, P_\infty(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' + \int Q(\underline{r}')\, P_{\infty_S}(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' \qquad (5.82)$$

where $Q(\underline{r}')$ is the number of source neutrons emitted per c.c. and second at $\underline{r}'$. For a point source $Q(\underline{r}')$ is a δ-function.

In order to solve Eq. (5.82) we shall expand $\Phi_s(\underline{r})$ and $Q(\underline{r})$ in a series of the characteristic functions $Z_n(\underline{r})$ which satisfy the wave equation

$$\Delta Z_n(\underline{r}) + B_n^2 Z_n(\underline{r}) = 0 \tag{5.83}$$

and the boundary condition $Z_n(\underline{r}) = 0$ on the extrapolated pile boundary. The numbers B_n^2 are the characteristic numbers for this problem; they form a monotonically increasing sequence

$$B_o^2 < B_1^2 < B_2^2, \text{ etc.}$$

The lowest characteristic number B_o^2 is identical with B_g^2, the geometric buckling of the pile.

We expand $Q(\underline{r})$ and $\Phi_s(\underline{r})$ in $Z_n(\underline{r})$:

$$\begin{aligned} Q(\underline{r}) &= \sum Q_n Z_n(\underline{r}) \\ \Phi_s(\underline{r}) &= \sum \Phi_n Z_n(\underline{r}) \end{aligned} \tag{5.84}$$

where Q_n and Φ_n are coefficients in the expansion, and substitute into the pile equation (5.82). Since each $Z_n(\underline{r})$ satisfies a wave equation, then by the same argument that led to the fundamental theorem of pile theory it follows that

$$\int Z_n(\underline{r}') \, P(E_s, |r-r'|) dr' = \overline{P}(E_s, B_n^2) Z_n(\underline{r}). \tag{5.85}$$

Consequently $q(\underline{r})$ can be written

$$q(E_s, r) = \frac{k}{p} N\sigma_a \sum \overline{P}_\infty(E_s, B_n^2) \, \Phi_n Z_n(\underline{r}) + \sum \overline{P}_{\infty_s}(E_s, B_n^2) \, Q_n Z_n(\underline{r}). \tag{5.86}$$

The pile equation reduces to

$$\sum \left\{ -B_n^2 L^2 \Phi_n Z_n(\underline{r}) - \Phi_n Z_n(\underline{r}) + \frac{k}{p} \bar{P}_\infty (E_s, B_n^2) \Phi_n Z_n(\underline{r}) + \frac{Q_n}{N\sigma_a} \bar{P}_{\infty_s} (E_s, B_n^2) Z_n(\underline{r}) \right\} = 0$$

from which

$$\Phi_n = - \frac{Q_n \bar{P}_{\infty_s} (E_s, B_n^2)}{N\sigma_a \left(\frac{k}{p} \bar{P}_\infty (E_s, B_n^2) - 1 - L^2 B_n^2 \right)} . \tag{5.87}$$

The slow neutron density can therefore be written

$$\Phi_s(\underline{r}) = - \sum \frac{Q_n \bar{v} \ell_n \bar{P}_{\infty_s} (E_s, B_n^2) Z_n(\underline{r})}{k_{ex}(B_n^2)} \tag{5.88}$$

where

$$k_{ex}(B_n^2) = \frac{k \bar{P}_\infty (E_s, B_n^2)}{p(1 + L^2 B_n^2)} - 1 \tag{5.89}$$

and

$$\bar{v} \ell_n = \frac{1}{N\sigma_a (1 + L^2 B_n^2)} . \tag{5.90}$$

According to (5.88) the slow neutron density consists of a sum of harmonics each of which has the factor $k_{ex}(B_n^2)$ in the denominator. According to the definition of $k_{ex}(B_n^2)$, this quantity can be interpreted as the excess multiplication constant in a pile whose geometric buckling is B_n^2. Since the B_n^2 form a monotone increasing sequence, the successive $k_{ex}(B_n^2)$ remain negative. Thus, as the pile approaches critical, the first term in the series (5.88) grows indefinitely -- i.e., the neutron intensity increases, and the distribution acquires more and more nearly the shape of the fundamental. Close to critical then, the slow neutron density approaches

$$\Phi_s(\underline{r}) \approx - \frac{Q_0 \bar{v} \ell_0 \bar{P}_{\infty_s} (E_s, B_0^2) Z_0(\underline{r})}{k_{ex}(B_0^2)} . \tag{5.91}$$

Now the lifetime multiplied by $\bar{v}$ of a slow neutron in an infinite pile is $\frac{1}{N\sigma_a}$; in a finite pile of geometric buckling B_n^2, it is shorter because of the leakage -- in fact, it is just

$$\bar{v}\ell_n = \frac{1}{N\sigma_a(1 + L^2B_n^2)} . \tag{5.92}$$

The quantity

$$\frac{\bar{\Phi}_s(\underline{r})}{\bar{v}\ell_n}$$

therefore must be the number of slow neutrons produced per second per c.c. at $\underline{r}$ in a pile whose geometric buckling is B_n^2. The zero-th harmonic of the original source strength is $Q_oZ_o(\underline{r})$; the fraction of those which become slow is evidently

$$Q_oP_{\infty_s}(E_s,B_o^2)Z_o(\underline{r}).$$

Hence the ratio of slow neutrons produced per second to the zero-th harmonic of the primary slow neutron source strength approaches

$$-\frac{1}{k_{ex}(B_o^2)}$$

as the pile approaches critical.

To compute the ratio of the multiplied source strength of fast neutrons to the original source strength, we need only calculate the total number $q_o(\underline{r})$ of neutrons born either by fission or by emission from the primary source per second per c.c. Evidently

$$q_o(\underline{r}) = Q(\underline{r}) + \frac{k\,N\sigma_a}{p}\bar{\Phi}_s(\underline{r}) \tag{5.93}$$

and from (5.88) this can be transformed to

$$q_o(\underline{r}) = \sum \left[Q_n - \frac{Q_n \frac{k}{p}\bar{P}_{\infty_s}(E_s,B_n^2)}{\frac{k}{p}\bar{P}_\infty(E_s,B_n^2)-1-L^2B_n^2} \right] Z_n(r) \qquad (5.94)$$

$$= \sum \frac{Q_n}{k_{ex}(B_n^2)} \left[\frac{\frac{k}{p}\left[\bar{P}_\infty(E_s,B_n^2)-\bar{P}_{\infty_s}(E_s,B_n^2)\right]}{1+L^2B_n^2} - 1 \right] Z_n(r) \ .$$

Close to critical only the zero-th harmonic appears. Hence the multiplication of the zero-th harmonic approaches

$$\frac{q_o(r)}{Q_oZ_o(r)} \rightarrow \frac{-1}{k_{ex}(B_o^2)} \left[1 - \frac{\frac{k}{p}\left[\bar{P}_\infty(E_s,B_o^2) - \bar{P}_{\infty_s}(E_s,B_o^2)\right]}{1+L^2B_o^2} \right] \ . \qquad (5.95)$$

If the primary source neutrons have fission energy, then $\bar{P}_\infty(E_s,B_n^2) = P_{\infty_s}(E_s,B_n^2)$, and the multiplication of the zero-th harmonic of the source strength becomes simply

$$\frac{-1}{k_{ex}(B_o^2)} \ .$$

This is identical with the result (5.80) obtained by counting up the generation by generation daughters from a primary fission source whose spatial distribution included only the zero-th harmonic.

Elementary Kinetics of a Slow Neutron Chain Reaction

Thus far in this chapter we have considered only stationary chain reacting systems -- that is, systems in which the neutron density is constant in time. In this section we shall develop the simplest properties of chain reactors in which the neutron density is time dependent.

If a localized transient disturbance of the neutron density occurs in a critical pile, this disturbance will be propagated over the pile (in the manner described in Chapter I) until it dissipates itself and the neutron intensity returns to its original distribution. The propagation of a neutron intensity wave can be described by a decomposition of the instantaneous neutron distribution into a superposition of many spatial harmonics, each of which decays with a different decay time. The details of how the neutron density reestablishes itself following a localized disturbance will be described in Chapter X . In this section we consider the simpler problem of how the neutron density as a whole -- i.e., the fundamental -- rises or falls in a pile which is super-critical or sub-critical.

A simplifying assumption will be made throughout the discussion. We shall assume that the time required for a fast neutron born at $\underline{r}'$ to appear as a slow neutron at $\underline{r}$ is negligible compared to a generation time. In a graphite pile the average slowing down time is only about 1/50 of the slow neutron lifetime, so that here, as in fact in all slow neutron piles, the slowing down time is really a small part of the generation time. According to this assumption, the production of slow neutrons at time t may be taken to be proportional to the fast neutron production at time t rather than at some earlier, "retarded" time, determined by how long it takes for a fast neutron to become slow.

The time dependent behavior of a chain reaction is complicated by the fact that not all the neutrons are produced instantaneously after fission. In addition to the delayed neutrons from fission, there may be delayed photoneutrons produced by fission product γ-rays interacting with atoms of the

moderator. The presence of delayed neutrons, generally speaking, expands the time scale of a chain reaction. Instead of the fundamental time being the generation time, it is a much longer time determined by the delayed neutron periods and the importance of the delayed neutrons to the chain reaction.

We now set up the time dependent equations for a slow neutron chain reaction. Let $\Phi_s(\underline{r},t)$ be the slow neutron flux and let $C_i(\underline{r},t)$ be the density of radioactive nuclei which will emit, with a mean life τ_i, a delayed neutron. The yield of the i^{th} group of delayed neutrons we denote by β_i; that is, per prompt neutron produced, β_i delayed neutrons of period τ_i are emitted. The ratio of all delayed to prompt neutrons in one fission is $\beta = \sum \beta_i$; the ratio of all delayed to all neutrons is therefore $\frac{\beta}{1+\beta}$.

Since each fission neutron is accompanied by $\frac{\beta_i}{1+\beta}$ i^{th} type delayed neutrons, and each such delayed neutron originated from a radioactive nucleus, the production rate of nuclei $C_i(\underline{r},t)$ is

$$\frac{k}{p}\frac{\beta_i}{1+\beta} N\sigma_a \Phi_s(\underline{r},t)$$

while the rate of destruction is

$$\frac{C_i(\underline{r},t)}{\tau_i}.$$

The time rate of change of C_i is therefore

$$\frac{\partial C_i}{\partial t}(\underline{r},t) = \frac{k}{p}\frac{\beta_i}{1+\beta} N\sigma_a \Phi_s(\underline{r},t) - \frac{C_i(\underline{r},t)}{\tau_i}. \qquad (5.96)$$

For the slow neutron flux we have

$$\frac{1}{v}\frac{\partial \Phi_s}{\partial t}(\underline{r},t) = D_o \Delta \Phi_s(\underline{r},t) - N\sigma_a \Phi_s(\underline{r},t) + q(E_s,\underline{r},t) \qquad (5.97)$$

where $q(E_s,\underline{r},t)$ is the production rate of slow neutrons at $\underline{r}$. The appropriate velocity $\bar{v}$ in (5.97) is clearly the average velocity; for the term

$$\frac{1}{\bar{v}}\frac{\partial \Phi_s}{\partial t}$$

is just the time rate of change of n, the total slow neutron density. Since

$$\frac{1}{\bar{v}}\frac{\partial \Phi_s}{\partial t} = \frac{\partial n}{\partial t},$$

$$\bar{v} = \frac{\Phi_s(\underline{r},t)}{n(\underline{r},t)} = \frac{\int vn(E,\underline{r},t)dE}{\int n(E,\underline{r},t)dE}, \tag{5.98}$$

the integration extending over all "thermal" neutron energies.

We make the usual assumption about the pile being uniform, and the extrapolation distance being energy independent. The production of slow neutrons is the sum of production by slowing of prompt neutrons and of delayed neutrons:

$$q(E_s,\underline{r},t) = \frac{k}{p} N\sigma_a \int_{\substack{\text{all}\\ \text{space}}} \Phi_s(\underline{r}',t)\left(1-\frac{\beta}{1+\beta}\right)P_\infty^p(E_s,|\underline{r}-\underline{r}'|)d\underline{r}'$$

$$+ \sum_i \int_{\substack{\text{all}\\ \text{space}}} \frac{C_i(\underline{r},t)}{\tau_i} P_\infty^i(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' \tag{5.99}$$

where we distinguish between $P_\infty^p(E_s,|\underline{r}-\underline{r}'|)$, the slowing down kernel for the prompt neutrons, and $P_\infty^i(E_s,|\underline{r}-\underline{r}'|)$, the slowing down kernel for the i^{th} delayed neutron. The kernels P^p and P^i are not necessarily the same because the energy of the delayed neutrons is lower than the energy of the prompt neutrons.

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The use of the infinite kernels and the extension of the integration over all space is justified (theorem - page 32) by the assumption that the pile is uniform with energy independent extrapolation distance.

In the steady state, $\frac{\partial \Phi_s}{\partial t} = \frac{\partial C_i}{\partial t} = 0$, $\frac{C_i}{\tau_i} = \frac{k}{p} N\sigma_a \frac{\beta_i}{1+\beta} \Phi_s$, and therefore, from (5.99)

$$q(E_s,\underline{r}) = \frac{k}{p} N\sigma_a \int \Phi_s(\underline{r}') \left[(1 - \frac{\beta}{1+\beta}) P_\infty^p(E_s,|\underline{r}-\underline{r}'|) + \sum_i \frac{\beta_i}{1+\beta} P_\infty^i(E_s,|\underline{r}-\underline{r}'|) \right] d\underline{r}' . \tag{5.100}$$

The expression for $q(E_s,\underline{r})$ in (5.100) is the usual one for the steady state slowing down density, since the quantity in the brackets is the weighted sum of the slowing down kernels for the prompt and the delayed neutrons and this sum is equal to the overall slowing down kernel previously denoted by $P_\infty(E_s,|\underline{r}-\underline{r}'|)$, i.e.,

$$P_\infty(E_s,|\underline{r}-\underline{r}'|) = \frac{1}{1+\beta} P_\infty^p(E_s,|\underline{r}-\underline{r}'|) + \sum_i \frac{\beta_i}{1+\beta} P_\infty^i(E_s,|\underline{r}-\underline{r}'|). \tag{5.101}$$

We suppose that the geometric buckling of the pile is B_g^2. Then the assumption

$$\Phi_s(\underline{r},t) = \Phi_{s_o}(\underline{r})\, e^{t/T}$$

$$C_i(\underline{r},t) = C_{i_o}(\underline{r})\, e^{t/T}$$

$$q(\underline{r},t) = q_o(\underline{r})\, e^{t/T}$$

where T is called the "period" of the pile, reduces the pile equations

(5.96), (5.97), and (5.99) to

$$D_o \Delta \bar{\Phi}_{s_o} - (N\sigma_a + \frac{1}{\bar{v}\,T})\,\bar{\Phi}_{s_o} + q_o = 0 \tag{5.102}$$

$$\frac{k}{p}\frac{\beta_i}{1+\beta} N\sigma_a \bar{\Phi}_{s_o} - \left(\frac{1}{\tau_i} + \frac{1}{T}\right) C_{i_o} = 0 \tag{5.103}$$

$$q_o(E_s,\underline{r}) = \frac{k}{p} N\sigma_a \int \bar{\Phi}_{s_o}(\underline{r}')(1-\frac{\beta}{1+\beta})P_\infty^p(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' + \int \sum_i \frac{C_{i_o}}{\tau_i}(\underline{r}')P_\infty^i(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' . \tag{5.104}$$

It should be noticed in the first equation that the term arising from the time dependence behaves formally like an additional absorption.

To solve (5.102) we suppose that $\bar{\Phi}_{s_o}$ and C_{i_o} satisfy the wave equation with buckling B_g^2, and that these functions satisfy the pile boundary conditions. Then by the fundamental theorem of pile theory applied to (5.104)

$$q_o(E_s,\underline{r}) = \frac{k}{p} N\sigma_a (1-\frac{\beta}{1+\beta})\overline{P_\infty^p}(E_s,B_g^2)\bar{\Phi}_{s_o}(\underline{r}) + \sum_i \frac{C_{i_o}}{\tau_i}(\underline{r})\,\overline{P_\infty^i}(E_s,B_g^2). \tag{5.105}$$

If $\frac{C_{i_o}}{\tau_i}$ in (5.105) is replaced by its value from Eq. (5.103) and the result is substituted into (5.102), there finally results the following characteristic equation:

$$\frac{k\left[\overline{P_\infty^p}(E_s,B_g^2) + \sum_i \beta_i\,\overline{P_\infty^i}(E_s,B_g^2)\right]}{p(1+\beta)\,(1+L^2B_g^2)} - 1 = \frac{k\sum_i \frac{\beta_i \tau_i}{T+\tau_i}\overline{P_\infty^i}(E_s,B_g^2)}{p(1+\beta)\,(1+L^2B_g^2)} + \frac{\ell}{T(1+L^2B_g^2)} \tag{5.106}$$

where $\ell = \frac{1}{N\sigma_a \bar{v}}$ is the lifetime of a thermal neutron in an infinite pile.

The quantity

$$\frac{\overline{P_\infty^p}(E_s,B_g^2) + \sum_i \beta_i \overline{P_\infty^i}(E_s,B_g^2)}{(1+\beta)}$$

is the Fourier transform of the slowing down kernel for all neutrons, (both prompt and delayed) produced by fission; we have been denoting it by $\overline{P}_\infty(E_s,B_g^2)$ with no superscript. Also, the quantity $\frac{\ell}{1+L^2B_g^2}$ is the lifetime of a neutron in a finite pile of geometric buckling B_g^2; we denote it by ℓ^*.

Hence, Eq. (5.106) can be written

$$\frac{k\,\overline{P}_\infty(E_s,B_g^2)}{p(1+L^2B_g^2)} - 1 = \frac{\ell^*}{T} + \frac{k}{p(1+\beta)(1+L^2B_g^2)} \sum_i \frac{\beta_i \tau_i}{T+\tau_i} \overline{P_\infty^i}(E_s,B_g^2). \qquad (5.107)$$

The quantity on the left hand side is the excess multiplication constant -- i.e., $k_{ex}(B_g^2)$. If $T = \infty$, then from Eq. (5.107), $k_{ex} = 0$, and the pile is critical.

The equation

$$k_{ex}(B_g^2) = \frac{\ell^*}{T} + \frac{k}{p(1+\beta)(1+L^2B_g^2)} \sum_i \frac{\beta_i \tau_i}{T+\tau_i} \overline{P_\infty^i}(E_s,B_g^2) \qquad (5.108)$$

is a relation between the period T, the geometric buckling and the microscopic multiplication properties of a non-critical pile. It is a generalization of the characteristic equation (5.16) which relates the multiplication properties to the buckling in a critical pile; (5.108) reduces to this characteristic equation when $T = \infty$.

If there are no delayed neutrons, (5.108) may be written

$$T = \frac{\ell^*}{k_{ex}(B_g^2)} ; \tag{5.109}$$

i.e., the period of the pile is the lifetime divided by $k_{ex}(B_g^2)$. This result is intuitive; for $k_{ex}(B_g^2)$ is the rate at which neutrons are multiplied per generation, and ℓ^* is (because we neglect the time required for a neutron to become slow) the generation time. Hence k_{ex}/ℓ^* is the multiplication rate per second, and the reciprocal is therefore the time required for the neutron density to multiply by a factor e. According to (5.109), if $k_{ex} > 0$, the pile is super-critical, T is positive, and the neutron intensity rises exponentially.

The presence of the delayed neutrons complicates the preceding considerations somewhat, since T is then the solution of (5.108) which is a high degree algebraic equation. However, if the pile period is very long compared to the longest delayed neutron period, then T can be factored out from the sum in (5.108). The resulting formula for the period is

$$T \approx \frac{1}{k_{ex}}\left[\ell^* + \frac{k}{p(1+\beta)} \sum_i \frac{\beta_i \tau_i \overline{P_\infty^i}(E_s, B_g^2)}{1 + L^2 B_g^2}\right] . \tag{5.110}$$

Since ℓ^*/k_{ex} is the period if there were no delayed neutrons, the presence of the delayed neutrons increases the pile period by a factor

$$1 + \frac{k}{p(1+\beta)\ell^*} \sum_i \frac{\beta_i \tau_i \overline{P_\infty^i}(E_s, B_g^2)}{1 + L^2 B_g^2} . \tag{5.111}$$

The quantity $\frac{\overline{P^i_\infty}(E_s,B_g^2)}{1 + L^2B_g^2}$ is the probability that an i^{th} delayed neutron is absorbed while slow inside the pile, while $\frac{k\,\beta_i}{p(1+\beta)}$ is the number of i^{th} delayed neutrons produced per slow neutron absorbed. Hence, $\frac{k\,\beta_i}{p(1+\beta)}\,\frac{\overline{P^i}(E_s,B_g^2)}{1 + L^2B_g^2}$ is the ratio of the number of i^{th} delayed neutrons which are absorbed as slow neutrons to the number of all slow neutrons absorbed. Since only a delayed neutron which is absorbed as a slow neutron inside the pile is effective in lengthening the pile period, it is natural that the lifetimes τ_i in (5.111) are weighted with the number of delayed neutrons which are absorbed as slow neutrons. Thus the pile period is increased, because of the delayed neutrons, by the ratio of the weighted mean lives of those delayed neutrons which take part in the multiplication, to the generation time. The quantity $\frac{1}{1+\beta}\sum_i \beta_i \tau_i$ is of order 10^{-1} sec; $\frac{k}{p}\frac{\overline{P^i_\infty}(E_s,B_g^2)}{1 + L^2B_g^2} \approx 1$, and $\ell^* \sim 10^{-3}$ sec in a graphite pile. The presence of the delayed neutrons in this case therefore lengthens the pile period by a factor of about 100.

Since (5.108) is an algebraic equation of high degree, there are several other roots depending on the number of delayed neutron periods. These roots are all negative, regardless of the value of k_{ex}; they therefore give rise to transients in the neutron density. A detailed discussion of these transients will be deferred until later.

The Reactivity of a Pile; The Inhour Formula

Let us divide (5.108) through by k_{eff}:

$$\frac{k_{ex}(B_g^2)}{k_{eff}(B_g^2)} = \frac{\ell^*}{T\,k_{eff}} + \sum_i \frac{\beta_i\,\tau_i}{T+\tau_i}\,\frac{\overline{P^i_\infty}(E_s,B_g^2)}{\overline{P}_\infty(E_s,B_g^2)} \quad . \qquad (5.112)$$

The quantity $\frac{k_{ex}(B_g^2)}{k_{eff}(B_g^2)}$ is called the reactivity of the pile, and we denote it by $\rho(B_g^2)$:

$$\text{Pile reactivity} = \rho(B_g^2) = \frac{k_{ex}(B_g^2)}{k_{eff}(B_g^2)} . \tag{5.113}$$

The excess multiplication constant is the difference between the effective multiplication constant and unity; i.e., the difference between the actual effective multiplication constant and the value which this constant must have if the pile is just critical. For this reason k_{ex} is sometimes denoted by $\delta k_{eff}(B_g^2)$. In this notation

$$\rho(B_g^2) = \frac{\delta k_{eff}(B_g^2)}{k_{eff}(B_g^2)} .$$

In terms of the reactivity, the period equation (5.112) is

$$\rho(B_g^2) = \frac{\ell^*}{T\, k_{eff}} + \sum_i \frac{\beta_i \tau_i}{T + \tau_i} \frac{\overline{P_\infty^i}(E_s, B_g^2)}{\overline{P}_\infty(E_s, B_g^2)} . \tag{5.114}$$

The reactivity is a pure number. It has become conventional to express it in a unit called the inhour (inverse hour). An inhour is defined as that value of $\rho(B_g^2)$ for which the pile period is one hour. Thus

$$\text{Reactivity in inhours} = \frac{\frac{\ell^*}{k_{eff} T} + \sum_i \frac{\beta_i \tau_i}{T + \tau_i} \frac{\overline{P_\infty^i}(E_s, B_g^2)}{\overline{P}_\infty(E_s, B_g^2)}}{\frac{\ell^*}{3600\, k_{eff}} + \sum_i \frac{\beta_i \tau_i}{3600 + \tau_i} \frac{\overline{P_\infty^i}(E_s, B_g^2)}{\overline{P}_\infty(E_s, B_g^2)}} . \tag{5.115}$$

For long periods the pile period in hours is equal to the reciprocal of the reactivity expressed in inhours.

It is customary in most cases to ignore the difference between $\overline{P^i_\infty}(E_s, B_g^2)$ and $\overline{P}_\infty(E_s, B_g^2)$. Then the reactivity is

$$\rho(B_g^2) = \frac{\ell^*}{T\, k_{eff}} + \sum_i \frac{\beta_i \tau_i}{T + \tau_i} . \qquad (5.116)$$

If T is large compared to ℓ^* the second term in (5.116) is much larger than the first. Moreover, in this term only constants β_i, τ_i, appear. These constants are the same for all piles in which the fissionable material and the photoneutron production is the same. Thus for long periods the relation between the reactivity and the period is independent of the size of the pile or of the disposition of the fissionable material. This is not true of the relation (5.108) between k_{ex} and T, and it is for this reason -- viz., the convenience of having a universal formula independent of a particular pile -- that the reactivity $\rho(B_g^2)$ instead of $k_{ex}(B_g^2)$ is used to describe the degree of criticality of a pile.

Relation between Various Definitions of k_{eff}, k_{ex} and Reactivity

There has been considerable confusion in project literature concerning the appropriate definition of the effective multiplication constant, the excess multiplication constant and the reactivity. The definitions used here are self-consistent, although they are not exactly the same as the ones usually used. It is worthwhile at this point to examine the various definitions.

We define k_{eff} as the number of neutrons created in one generation per neutron removed either by leakage or by absorption. Since the lifetime against absorption or leakage is

$$\ell^* = \frac{\ell}{1 + L^2 B_g^2} ,$$

the kinetic equation for slow neutrons with no delayed production is

$$\frac{d\Phi_s}{dt} = \frac{k_{eff}-1}{\ell^*}\,\Phi_s \tag{5.117}$$

$$= \left[\frac{k\,\bar{P}(E_s,B_g^2)}{p} - L^2B_g^2 - 1\right]\frac{\Phi_s}{\ell} \tag{5.118}$$

$$= \left[k - (1 + L^2B_g^2)\,\frac{p}{\bar{P}(E_s,B_g^2)}\right]\frac{\Phi_s\,\bar{P}(E_s,B_g^2)}{\ell p}\,. \tag{5.119}$$

In our notation,

$$k_{ex} = k_{eff}-1,$$

and the pile period is ℓ^*/k_{ex}.

In terms of the infinite pile lifetime, ℓ, the pile period is (from (5.118)

$$\frac{\ell}{\frac{k}{p}\,\bar{P}(E_s,B_g^2) - L^2B_g^2 - 1}$$

and in much of the project literature the quantity in the denominator is defined as k_{ex}. The effective multiplication constant, according to this definition, is

$$\frac{k}{p}\,\bar{P}(E_s,B_g^2) - L^2B_g^2\,.$$

Finally, Soodak (see lecture notes, p. 49) uses the quantity $\dfrac{\ell p}{\bar{P}(E_s,B_g^2)}$ as the fundamental lifetime, and he defines k_{ex} as

$$k - (1 + L^2B_g^2)\,\frac{p}{\bar{P}(E_s,B_g^2)}\,.$$

For a very large pile, $k \approx 1$, all these definitions are identical to first order in $k - 1$, namely

$$k_{eff} = k - M^2 B_g^2 \ . \tag{5.120}$$

Our definition of the reactivity

$$\rho(B_g^2) = \frac{k_{ex}}{k_{eff}} = \frac{\dfrac{k\,\bar{P}(E_s,B_g^2)}{p(1 + L^2B_g^2)} - 1}{\dfrac{k\,\bar{P}(E_s,B_g^2)}{p\,(1 + L^2B_g^2)}} = \frac{k - \dfrac{p(1 + L^2B_g^2)}{\bar{P}(E_s,B_g^2)}}{k} \tag{5.121}$$

is $1/k$ times Soodak's reactivity (denoted by ρ_S).

Because of the difference between our k_{ex} and Soodak's k_{ex}, the relation between the definitions of reactivity is

$$\rho(B_g^2) = \frac{k_{ex}}{k_{eff}} = \frac{\rho_S}{k} = \frac{k_{exS}}{k} .$$

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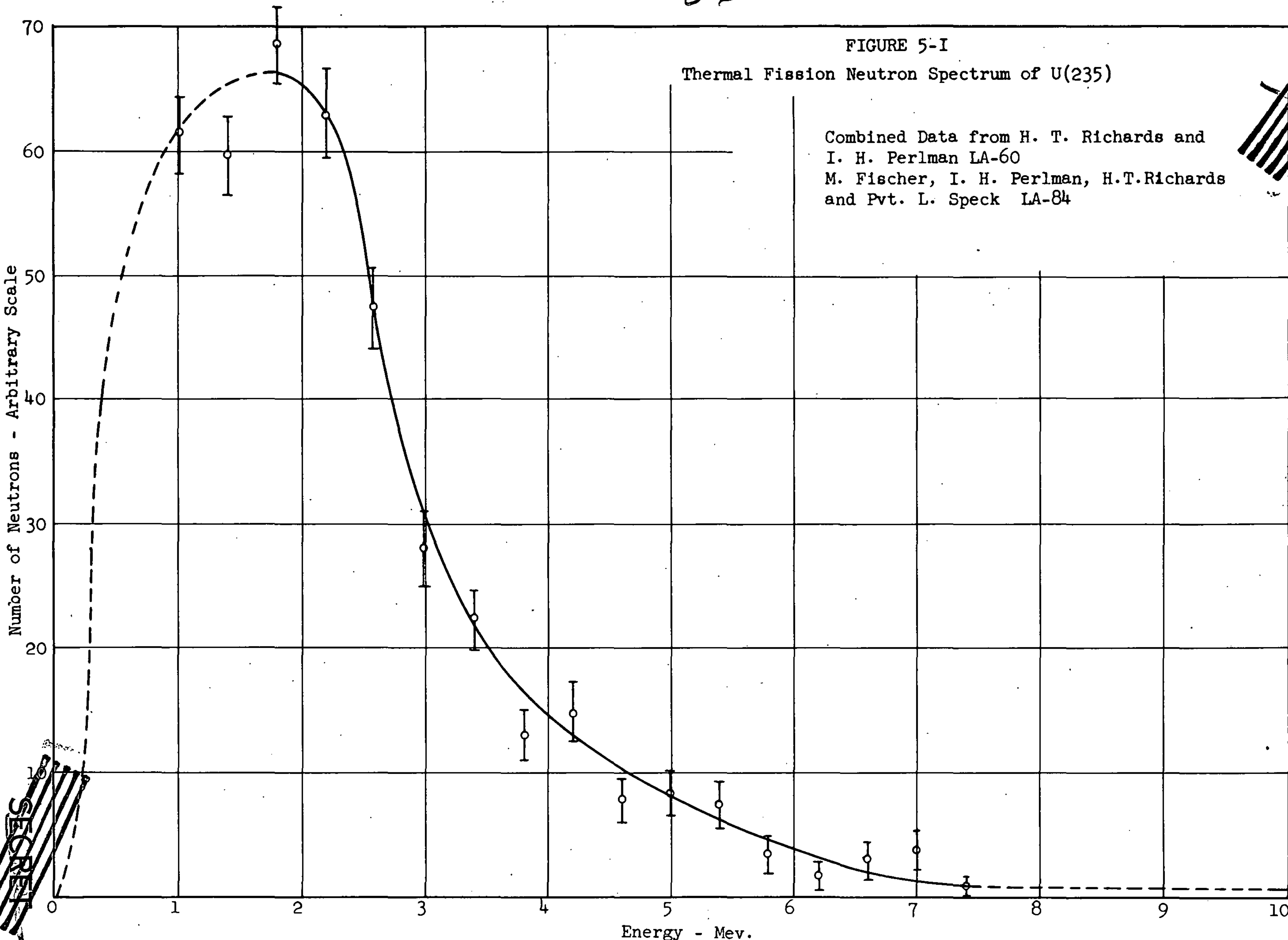

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CHAPTER VI

PILE EQUATIONS

In the previous chapter the asymptotic theory of a slow neutron chain reactor in which the slowing down kernel is an arbitrary function of $|\underline{r}-\underline{r}'|$ was developed. The whole theory was seen to rest upon the formula for $k_{eff}(B_g^2)$, viz.,

$$k_{eff}(B_g^2) = \frac{k}{p} \frac{\overline{P}_\infty(E_s, B_g^2)}{1 + L^2 B_g^2} . \tag{6.1}$$

This formula was shown to be valid in a bare pile provided the pile is large compared to a mean free path, and the extrapolation distance is energy independent. The relation $k_{eff}(B_g^2) = 1$ is the characteristic equation which determines the buckling of a critical pile.

In this chapter we shall discuss the various special forms of $P_\infty(E, |\underline{r}-\underline{r}'|)$ which are used in actual pile calculations, and we shall derive the particular expressions for $k_{eff}(B_g^2)$ appropriate for each slowing down kernel.

In principle there is no ambiguity about the proper kernel to be used in a particular pile: the correct $P_\infty(E, |\underline{r}-\underline{r}'|)$ is the solution of the Boltzmann equation with energy loss. However, the accurate solution of this equation is exceedingly complicated. In practice it is necessary to use either an experimentally determined kernel, or to use some rather simple analytic approximation to the experimental kernel.

The Moment Form of the Characteristic Equation

Before discussing particular analytic forms of $P_\infty(E, |\underline{r}-\underline{r}'|)$ we shall point out how the value of k_{eff} of a bare pile, and therefore the critical

size, is determined by the moments of the slowing down distribution. From its definition,

$$\bar{P}_\infty(E_s,B_g^2) = 4\pi \int_0^\infty P_\infty(E_s,r) r^2 \frac{\sin B_g r}{B_g r}\, dr. \tag{6.2}$$

This becomes, on expanding $\frac{\sin B_g r}{B_g r}$ in Taylor's series and integrating term by term

$$\bar{P}_\infty(E_s,B_g^2) = 4\pi \sum_{n=o}^{\infty} \frac{(-)^n}{(2n+1)!} B_g^{2n} \int_0^\infty r^{2n} P_\infty(E_s,r) r^2 dr. \tag{6.3}$$

Now the $2n^{th}$ moment of the slowing down distribution is

$$\overline{r^{2n}}(E_s) = \frac{4\pi \int_0^\infty r^{2n} P_\infty(E_s,r) r^2 dr}{4\pi \int_0^\infty P_\infty(E_s,r) r^2 dr} = \frac{4\pi}{p} \int_0^\infty r^{2n} P_\infty(E_s,r) r^2 dr;$$

hence (6.3) can be written

$$\bar{P}_\infty(E_s,B_g^2) = p \sum_{n=o}^{\infty} \frac{(-)^n}{(2n+1)!} B_g^{2n} \overline{r^{2n}(E_s)}$$

and $k_{eff}(B_g^2)$ is

$$k_{eff}(B_g^2) = \frac{k\, \bar{P}_\infty(E_s,B_g^2)}{p(1 + L^2B_g^2)} = \frac{k}{1 + L^2B_g^2} \sum_{n=o}^{\infty} \frac{(-)^n}{(2n+1)!} B_g^{2n} \overline{r^{2n}(E_s)}. \tag{6.4}$$

The critical equation is

$$k_{eff}(B_g^2) = 1 = \frac{k}{1 + L^2B_g^2} \sum_{n=o}^{\infty} \frac{(-)^n}{(2n+1)!} B_g^{2n} \overline{r^{2n}(E_s)}. \tag{6.5}$$

If the buckling is small compared to $\overline{r^2}^{-1}$ -- and this implies that the pile dimension is large compared to $\sqrt{\overline{r^2}}$ -- then only the first terms in the expansion need be retained. The critical equation becomes simply

$$B_g^2 = \frac{k - 1}{M^2}$$

which was obtained already in (5.22).

The importance of (6.5) is that it expresses the critical buckling of a bare pile in terms of the moments of the slowing down distribution, and these moments can in principle be determined experimentally. The number of moments required to determine the critical size depends on the pile dimensions; in the limit of a very large pile, only the second moment is sufficient, while for a smaller, higher moments are needed.

From an experimentally measured distribution of indium resonance neutrons around a point source of fission neutrons, it is possible to compute the slowing down moments. The measurement of the slowing down moments is not easy, for several reasons:

1) To determine a high order moment it is necessary to measure neutrons at a large distance from the course. This invariably leads to intensity difficulties.

2) The required slowing down moments are from fission energy to thermal energy while the neutrons measured in the usual way with Cd-covered In foils have energy about 1.4 ev. Increments must therefore be applied to the observed In resonance moments to yield the thermal energy moments, and these increments are rather uncertain.

3) Strictly, the slowing down distribution must be measured in a medium which has the same resonance absorption as the pile. If a slowing down distribution is measured in a sub-critical pile the observed In resonance distribution

will be falsified because of neutrons produced by thermal fissions throughout the medium. This difficulty can be avoided by heavily poisoning the subcritical pile with Cd; this suppresses the thermal fission without affecting the resonance absorption very much.

Analytic Approximations to the Slowing Down Kernel

A number of analytic forms for the slowing down kernel have been used in critical calculations. The particular analytic form for $P_\infty(E_s,r)$ is always a compromise between the actual experimental kernel and a manageable mathematical approximation to the experimental data.

We list below the various slowing down kernels and the corresponding formulas for k_{eff}.

1) Simple Gaussian (Fermi) Kernel

If the fission spectrum were monochromatic, and the moderator atomic weight large, then the simple Gaussian kernel is appropriate:

$$P_\infty(E,r) = p(E) \frac{e^{-r^2/4\tau(E)}}{\left[4\pi\tau(E)\right]^{3/2}} \qquad (6.6)$$

where $\tau(E)$ is the age from fission energy ($\tau = o$) to energy E. The quantity p(E) is the fraction of neutrons which survive to energy E without undergoing resonance capture; in this notation $p(E_s) = p$. The Gaussian kernel was introduced in this country by Fermi; it has been used in most elementary pile calculations in which the moderator is graphite, Be, BeO, or D_2O.

The Fourier transform of $P_\infty(E,r)$ is

$$\bar{P}_\infty(E,r^2) = 4\pi \int_0^\infty p(E) \frac{e^{-r^2/4\tau(E)}}{\left[4\pi\tau(E)\right]^{3/2}} \frac{\sin Br}{Br} r^2 dr = p(E)\, e^{-B^2\tau(E)}. \qquad (6.7)$$

Thus the slowing down density at energy E is

$$q(E) = \frac{k}{p} N\sigma_a \Phi_s \bar{P}_\infty(E, B_g^2) = \frac{k}{p} p(E) N\sigma_a \Phi_s e^{-B_g^2 \tau(E)} \tag{6.8}$$

and the expression for $k_{eff}(B_g^2)$ is

$$k_{eff}(B_g^2) = \frac{k e^{-B_g^2 \tau(E_s)}}{1 + L^2 B_g^2} . \tag{6.9}$$

The critical relation is

$$k_{eff}(B_g^2) = 1 = \frac{k e^{-B_g^2 \tau(E_s)}}{1 + L^2 B_g^2} ; \tag{6.10}$$

this equation was first derived by Fermi and is sometimes called the Fermi equation.

The solution of (6.10) -- i.e., the value of B_g^2 which satisfies (6.10) for pre-assigned k, $\tau(E_s)$, and L^2 -- can best be found graphically. A plot of (6.10) for various values of the parameters has been prepared by E. C. Campbell. We reproduce it in Fig. (5-I). By means of this graph it is a simple matter to calculate the critical buckling of a pile in which the simple Gaussian slowing down picture is applicable.

2) Gaussian Kernel with Fission Spectrum Spread

Wigner has pointed out that a more realistic slowing down kernel in a moderately heavy moderator is a superposition of Gaussian kernels integrated over the fission spectrum. The fission spectrum can be represented by a function

$$f(E')dE'$$

where $f(E')dE'$ is the fraction of neutrons produced between energy E' and E' + dE' per fission. The age of neutrons of energy E which have originated at energy E' is

$$\tau(E,E').$$

According to the simplest Gaussian slowing down picture, in which the first and last collisions are neglected

$$\tau(E,E') = \int_E^{E'} \frac{1}{N\sigma_s \, N\sigma_{tr} \xi} \frac{dE}{E}, \tag{6.11}$$

so that, in this simplest approximation, the age is additive, i.e.,

$$\tau(E,E') = \quad (E,\infty) - \quad (E',\infty).$$

The contribution to the slowing down distribution at energy E from fission neutrons between E' and E' + dE' is

$$p(E) \frac{e^{-r^2/4\tau(E,E')}}{\left[4\pi\tau(E,E')\right]^{3/2}} f(E')dE'$$

and therefore the overall slowing down kernel is

$$P(E,r) = p(E) \int_0^{\infty} \frac{e^{-r^2/4\tau(E,E')}}{\left[4\pi\tau(E,E')\right]^{3/2}} f(E')dE'.$$

Since the Fourier transform of this is

$$\overline{P}_{\infty}(E,B^2) = p(E) \int_0^{\infty} e^{-B^2\tau(E,E')} f(E')dE', \tag{6.12}$$

the equation for $k_{eff}(B_g^2)$ is

$$k_{eff}(B_g^2) = \frac{k}{1 + L^2B_g^2} \int_0^\infty e^{-B_g^2\tau(E_s,E')} f(E')dE' \tag{6.13}$$

while the Fermi equation, modified for the fission spectrum spread, is

$$\frac{k}{1 + L^2B_g^2} \int_0^\infty e^{-B_g^2\tau(E_s,E')} f(E')dE' = 1. \tag{6.14}$$

This reduces to the monochromatic Fermi equation (6.10) when $f(E')$ is a δ-function.

If a monochromatic fission spectrum picture is used, there is the problem of what value of $\tau(E_s)$ must be used in order that the critical buckling predicted by (6.10) be the same as the critical buckling predicted by (6.14). By comparing (6.10) and (6.14) it is seen that if

$$e^{-B_g^2\tau(E_s)} = \int_0^\infty e^{-B_g^2\tau(E_s,E')} f(E')dE' \tag{6.15}$$

then the monochromatic equation (6.10) and the more nearly correct (6.14) will give the same critical buckling. Since, according to (6.15)

$$\tau(E_s) = -\frac{1}{B_g^2} \ln \int_0^\infty e^{-B_g^2\tau(E_s,E')} f(E')dE'$$

the appropriate average $\tau(E_s)$ depends on B_g^2; i.e., it is different for every different sized pile.

If the pile is large, the moment expansion (6.5) of the characteristic equation converges well. The moment form of (6.14) is found by expanding

the exponential and integrating term by term:

$$\int_0^\infty e^{-B_g^2 \tau(E_s,E')} f(E')dE' = 1 - B_g^2 \int_0^\infty \tau(E_s,E')\, f(E')dE' + \frac{B_g^4}{2!} \int_0^\infty \tau^2(E_s,E') f(E')dE' \ldots$$

The first integral in this expansion is the arithmetic mean of $\tau(E_s,E')$ averaged over the fission spectrum. Only this term enters if the pile is very large. The migration area, which measures the length scale in a large pile, is therefore

$$M^2 = L^2 + \int_0^\infty \tau(E_s,E')\, f(E')dE'; \qquad (6.16)$$

i.e., it is the <u>arithmetic</u> mean of the age (in addition to L^2) which determines the migration area.

If the pile is not large, then the higher moments in (6.5) must be taken into account in computing the critical size. These moments have been measured in graphite by Hill and Roberts. The results are summarized in the following table:

Observed and Calculated Slowing Down Moments
to In Resonance in Graphite

	Calculated	Observed by Hill and Roberts
$\overline{\tau} = \int \tau(E_s,E) f(E)dE$	341 cm^2	329 cm^2
$\overline{\tau^2} = \int \tau^2(E_s,E) f(E)dE$	12.3×10^4 cm^4	14.8×10^4 cm^4
$\overline{\tau^3} = \int \tau^3(E_s,E) f(E)dE$	4.8×10^7 cm^6	13.3×10^7 cm^6

The measured moments beyond $\overline{r^2}$ are much higher than the calculated values. The reason for the discrepancy probably lies in two facts; first, that the calculations are based on a simple age theory while the actual slowing down function has a longer tail than simple age theory would predict; and second, that the fission spectrum may have a fairly strong high energy tail which is not included in the calculations.

3) The All-Thermal Pile

The simplest of all pile pictures is one in which the fission neutrons are assumed to be produced with thermal energy. The slowing down kernel in this case degenerates to a δ-function,

$$P_\infty (E_s, r) = p\, \delta(\underline{r})$$

since, if the neutrons are produced as thermals, they "become" slow at their point of origin. The normalization is unity since there is no resonance capture, p = 1. The Fourier transform is simply

$$\overline{P}_\infty (E_s, B^2) = 4\pi \int_0^\infty \delta(\underline{r})\, \frac{\sin Br}{Br}\, r^2 = 1,$$

and the characteristic equation is

$$\frac{k}{1 + L^2 B_g^2} = 1;$$

i.e.,

$$B_g^2 = \frac{k - 1}{L^2}\ .$$

This "one-group" pile picture has been applied to very large fast neutron chain reactors, in which all the neutrons can be considered to be roughly monoenergetic.

4) The "Fast" Pile

The general formalism can be adopted to the fast pile -- i.e., a pile in which there is no moderation at all -- provided the symbols are properly interpreted. We re-define the generation time as the average time between successive collisions. The multiplication factor, k, is just the number of neutrons produced per collision; since a collision can result in fission, absorption, or scattering,

$$k = \nu \frac{N\sigma_f}{N\sigma} . \tag{6.18}$$

Furthermore, since the generation is taken to be the time between successive collisions, the "slowing down kernel" must be replaced by the probability that a neutron produced at $\underline{r}'$ will suffer a collision at $\underline{r}$. This is just the transport kernel

$$\frac{N\sigma\, e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|^2} ,$$

if isotropic scattering is assumed. This transport kernel is normalized to unity -- i.e., $p = 1$. Finally, since there is no moderation, there are no thermal neutrons: this can be fitted into our formalism by setting $N\sigma_a = \infty$, which implies $L^2 = 0$.

The Fourier transform of the transport kernel is

$$\bar{P}_\infty (E_s, B^2) = 4\pi\, N\sigma \int_0^\infty \frac{e^{-N\sigma r}}{4\pi r^2} \frac{\sin Br}{Br} r^2 dr = N\sigma \int_0^\infty e^{-N\sigma r} \frac{\sin Br}{Br} dr =$$

$$= \frac{N\sigma}{B} \int_{N\sigma}^\infty \int_0^\infty e^{-yr} \sin Br\, dr dy = \frac{N\sigma}{B} \tan^{-1} \frac{B}{N\sigma}. \tag{6.19}$$

Substituting this into the general expression for $k_{eff}(B_g^2)$ and using (6.18) and $L^2 = 0$, we obtain

$$k_{eff}(B_g^2) = \frac{k\,N\sigma}{B_g} \tan^{-1} \frac{B_g}{N\sigma} = \frac{\nu\, N\sigma_f}{B_g} \tan^{-1} \frac{B_g}{N\sigma} \tag{6.20}$$

and the characteristic equation for the critical buckling is

$$\frac{\nu\, N\sigma_f}{B_g} \tan^{-1} \frac{B_g}{N\sigma} = 1. \tag{6.21}$$

There is a striking analogy between this critical equation and the characteristic equation which determines the thermal diffusion length from the thermal absorption and scattering cross-sections. If we set

$$B_g = i\,\mathcal{K},$$

then the critical equation becomes

$$\frac{k\,N\sigma}{\mathcal{K}} \tanh^{-1} \frac{\mathcal{K}}{N\sigma} = 1. \tag{6.22}$$

This is identical with the characteristic equation (1.20) for the thermal diffusion length, $\mathcal{K}$, if k is still interpreted as the number of neutrons resulting from one collision. In an absorbing medium this is

$$\frac{N\sigma_s}{N\sigma},$$

and therefore (6.22) can be written as

$$\frac{N\sigma_s}{\mathcal{K}} \tanh^{-1} \frac{\mathcal{K}}{N\sigma} = 1.$$

It is, of course, not surprising that the characteristic equation for the asymptotic solution in an absorbing medium is formally the same as

the characteristic equation for the asymptotic solution in a producing medium, provided the neutron energy does not change. In both the producing and absorbing case the asymptotic solutions satisfy wave equations:

$$\Delta\underline{\Phi} + B^2\underline{\Phi} = 0 \qquad \text{producing medium}$$

$$\Delta\underline{\Phi} - \varkappa^2\underline{\Phi} = 0 \qquad \text{absorbing medium}$$

as well as satisfying the transport equation without energy loss:

$$\underline{\Phi}(\underline{r}) = k\int N\sigma\underline{\Phi}(\underline{r}')\,\frac{e^{-N\sigma|r-r'|}}{4\pi|\underline{r}-\underline{r}'|^2}\,d\underline{r}' \qquad \text{producing medium}$$

$$\underline{\Phi}(\underline{r}) = \frac{N\sigma_s}{N\sigma}\int N\sigma\underline{\Phi}(\underline{r}')\,\frac{e^{-N\sigma|\underline{r}-\underline{r}'|}}{4\pi|\underline{r}-\underline{r}'|^2}\,d\underline{r}' \qquad \text{absorbing medium.}$$

The two sets of equations are identical in form, and therefore the characteristic equations are, except for the sign of B^2 and $\varkappa^2$, identical.

The critical buckling calculated from the characteristic equation (6.21), is the buckling of the asymptotic solution. Since fast piles are usually not large compared to a mean free path, the extrapolation distance is a fair fraction of the wave length of the asymptotic solution. An error in the value of the extrapolation can therefore lead to serious error in the critical size of a fast pile. The simple formula for the extrapolation distance, ℓ,

$$\ell = 0.71\,\lambda_{tr}$$

which holds at a plane surface of a non-capturing medium, is not generally valid. The exact expression for the extrapolation distance is fairly complicated and to derive it would lead us too far afield; it is discussed in Frankel and Nelson's report on "Displacement Integral Equations" -- LA-53.

5) The Two-Group Picture

One of the most generally useful slowing down kernels is the "two-group" picture, discussed in Chapter III. In this model, neutrons are either "fast" or "slow" and moderation is described as a diffusion process. A fast neutron is assumed to diffuse without energy loss until it has travelled a mean square distance equal to its actual slowing down length, at which time it disappears by becoming a slow neutron. The slowing down kernel is

$$P_\infty(E_s,r) = \frac{p\, e^{-r/L_1}}{4\pi r\, L_1^2}$$

where $L_1^2 = \frac{\overline{r^2}}{6}(E_s)$. The Fourier transform of this "group" kernel is

$$\overline{P}_\infty(E_s,B^2) = \frac{p}{L_1^2}\int_0^\infty \frac{e^{-r/L_1}}{r}\,\frac{\sin Br}{Br}\,r^2dr = \frac{p}{1+L_1^2B^2}\,; \tag{6.23}$$

the corresponding $k_{eff}(B_g^2)$ is

$$k_{eff}(B_g^2) = \frac{k}{(1+L^2B_g^2)(1+L_1^2B_g^2)} \tag{6.24}$$

and the critical equation is

$$\frac{k}{(1+L^2B_g^2)(1+L_1^2B_g^2)} = 1. \tag{6.25}$$

The two-group characteristic equation is a second degree algebraic equation in B_g^2:

$$L^2L_1^2(B_g^4) + (L^2+L_1^2)B_g^2 + 1 - k = 0,$$

the roots of which are

$$B_g^2 = \frac{-(L^2+L_1^2) \pm \sqrt{(L^2+L_1^2)^2 + 4L^2L_1^2(k-1)}}{2L^2L_1^2}\,. \tag{6.26}$$

Of the two roots one is positive, the other negative (provided $k > 1$). If $k - 1$ is small (large pile) then the two roots are approximately

$$B_{g1}^2 \approx \frac{k-1}{L^2 + L_1^2} = \frac{k-1}{M^2} \tag{6.27}$$

$$B_{g2}^2 \approx -\left[\frac{1}{L^2} + \frac{1}{L_1^2} + \frac{k-1}{M^2}\right] . \tag{6.28}$$

The positive root B_{g1}^2 is the usual expression for the buckling of the asymptotic solution in a large pile. The negative root gives rise to non-asymptotic solutions which behave like exponentials rather than like oscillating functions. These non-asymptotic solutions are not needed to satisfy the boundary conditions in a bare pile in which the extrapolation distance is energy independent. However, if the pile is not bare, or if the dependence of the extrapolation distance on energy is taken into account, then these non-asymptotic solutions must be included in order to satisfy the boundary conditions.

The occurrence of non-asymptotic solutions is, of course, entirely general and does not depend on the particular slowing down model. In the Gaussian case, for example, the characteristic equation is transcendental, and there are therefore an infinite number of non-asymptotic solutions in addition to the usual asymptotic solution.

Convolution Kernels

Slowing down kernels which are convolutions of other kernels occupy an important place in pile theory. A two-fold convolution kernel is one which can be written as

$$P(|\underline{r}-\underline{r}'|) = \int P_1(|\underline{r}-\underline{r}''|)\, P_2(|\underline{r}''-\underline{r}'|)\, d\underline{r}'' . \tag{6.29}$$

A three-fold convolution kernel is

$$P(|\underline{r}-\underline{r}'|) = \iint P_1(|\underline{r}-\underline{r}''|)\, P_2(|\underline{r}'''-\underline{r}''|)\, P_3(|\underline{r}''-\underline{r}'|)d\underline{r}'''d\underline{r}'' \ .$$

The generalization to n-fold convolution kernels is immediate.

The Fourier transform of a convolution of functions is the product of the Fourier transforms of the separate functions. This is easily seen, since if $P(|\underline{r}-\underline{r}'|)$ is a two-fold convolution, then

$$\bar{P}(B^2) = \int e^{i\underline{B}\cdot\underline{r}}\, P(|\underline{r}|)d\underline{r}$$

$$= \iint e^{i\underline{B}\cdot\underline{r}}\, P_1(|\underline{r}-\underline{r}'|)\, P_2(|\underline{r}'|)d\underline{r}d\underline{r}'$$

$$= \iint e^{i\underline{B}\cdot(\underline{r}-\underline{r}')}\, P_1(|\underline{r}-\underline{r}'|)\, e^{i\underline{B}\cdot\underline{r}}\, P_2(|\underline{r}'|)d\underline{r}d\underline{r}'$$

$$= \bar{P}_1(B^2)\, \bar{P}_2(B^2).$$

For an n-fold convolution, the Fourier transform is

$$\bar{P}(B^2) = \bar{P}_1(B^2)\, \bar{P}_2(B^2)\, \bar{P}_3(B^2) \ldots\ldots \bar{P}_n(B^2) \ . \qquad (6.30)$$

The general expression for $k_{eff}(B^2)$ becomes, for a convolution kernel,

$$k_{eff}(B_g^2) = \frac{k}{p} \frac{\bar{P}_1(E_s,B_g^2)\, \bar{P}_2(E_s,B_g^2) \ \ldots\ \bar{P}_n(E_s,B_g^2)}{1 + L^2B_g^2} \ . \qquad (6.31)$$

From the three fundamental kernels -- the Gaussian, the transport, and the diffusion -- three two-fold convolution kernels can be constructed. These are:

a) The Christy-Wheeler Kernel (Convolution of Gaussian and Transport)

Christy and Wheeler (CF-400) suggested that the slowing down in hydrogenous media could be analyzed into a long first flight, during which the neutron suffers no collision, followed by a Gaussian diffusion away from the point of first collision. Such a picture is rather appropriate in hydrogen-containing moderators because of the way in which the proton scattering cross-section varies. At high energy the cross-section is small; it rises rapidly below 100 kev. The first collision therefore occurs only after a relatively long flight. Once a collision has been suffered, there is a good chance that the energy of the neutron will be reduced enough to make its further diffusion a sequence of short zig-zag paths. These give rise to the Gaussian. While this is by no means an accurate description of the slowing down in H_2O, it is sufficiently good to make possible rather accurate predictions of the critical size of H_2O chain reacting systems.

The Christy-Wheeler kernel is

$$\bar{P}_\infty(E_s,|\underline{r}-\underline{r}'|) = p\int \frac{e^{-\frac{|\underline{r}-\underline{r}''|^2}{4\tau}}}{(4\pi\tau)^{3/2}} \frac{N\sigma e^{-N\sigma|\underline{r}''-\underline{r}'|}}{4\pi|\underline{r}''-\underline{r}'|^2}\, d\underline{r}''. \tag{6.32}$$

The mean free path $1/N\sigma$ and the age τ are to be considered as arbitrary parameters to be chosen so as to best fit the observed slowing distribution.

The Fourier transform of the Gaussian function is given in (6.7), the transform of the transport kernel is given in (6.19). Hence the Fourier transform of the Christy-Wheeler kernel is the product of (6.7) and (6.19):

$$\bar{\bar{P}}_\infty(E_s,B^2) = p\, e^{-\tau B^2} \frac{N\sigma}{B} \tan^{-1}\frac{B}{N\sigma}. \tag{6.33}$$

The expression for $k_{eff}(B_g^2)$ is

$$k_{eff}(B_g^2) = \frac{k\, e^{-\tau B_g^2}\, N\sigma B_g^{-1} \tan^{-1} \frac{B_g}{N\sigma}}{1 + L^2 B_g^2} \tag{6.34}$$

and the characteristic equation is

$$\frac{k\, e^{-\tau B_g^2}\, N\sigma B_g^{-1}}{1 + L^2 B_g^2} \tan^{-1} \frac{B_g}{N\sigma} = 1. \tag{6.35}$$

This characteristic equation was used to estimate the critical buckling of the Los Alamos water-boiler -- the first chain reactor in which enriched fissionable material was used.

b) The Heavy Water Kernel (Convolution of Diffusion and Gaussian)

The slowing down function in heavy water is well represented by a convolution of a diffusion and a Gaussian kernel:

$$\bar{P}_\infty(E_s, B^2) = p \int \frac{e^{-|\underline{r}-\underline{r}''|^2/4\tau}}{(4\pi\tau)^{3/2}} \frac{e^{-|r''-r'|/L_1}}{4\pi|r''-r'|L_1^2}\, d\underline{r}''. \tag{6.36}$$

According to Friedman and Wattenberg (CP-3453), the best agreement with experiment results if $\tau = 58$ cm^2 and $L_1 = 48$ cm^2. According to (6.7) and (6.23) the Fourier transform of this kernel is

$$\bar{P}_\infty(E_s, B^2) = \frac{p\, e^{-\tau B^2}}{1 + L_1^2 B^2}, \tag{6.37}$$

the effective multiplication is

$$k_{eff}(B_g^2) = \frac{k\, e^{-\tau B^2}}{(1 + L_1^2 B_g^2)(1 + L^2 B_g^2)} \tag{6.38}$$

and the critical equation is

$$\frac{k\, e^{-\tau B_g^2}}{(1 + L_1^2 B_g^2)(1 + L^2 B_g^2)} = 1. \qquad (6.39)$$

It may be noted that the expression for $k_{eff}(B_g^2)$ in any slow neutron reactor, viz.,

$$\frac{k}{p} \frac{\bar{P}_\infty(B_g^2)}{1 + L^2 B_g^2}$$

contains the factor $1 + L^2 B_g^2$ in the denominator, regardless of the slowing down kernel. Since $(1 + L^2 B_g^2)^{-1}$ is the Fourier transform of the diffusion kernel, the product

$$(1 + L^2 B_g^2)^{-1} \bar{P}_\infty (B_g^2)$$

is the Fourier transform of a two-fold convolution kernel whose component kernels are $P_\infty (\underline{r}-\underline{r}'')$ and $\frac{e^{-|\underline{r}-\underline{r}'|/L}}{4\pi |\underline{r}'-\underline{r}| L^2}$. This convolution kernel represents the probability that a fission neutron at $\underline{r}''$ actually is <u>absorbed</u> as a slow neutron at $\underline{r}$, rather than becomes a slow neutron at $\underline{r}$. The splitting off of the thermal diffusion part of this composite kernel is really a matter of notation which is done simply to conform to the usual practice in pile theory literature.

c) <u>Convolution of Diffusion and Transport Kernel</u>

This combination, which leads to the characteristic equation

$$\frac{k}{(1 + L^2 B_g^2)(1 + L_1^2 B_g^2)} \frac{N\sigma}{B_g} \tan^{-1} \frac{B_g}{N\sigma} = 1. \qquad (6.40)$$

has not been used in pile calculations.

d) The Multigroup Picture

The kernel which is formed by the convolution of several diffusion kernels leads to the so-called "multigroup" picture; the two-group picture previously discussed is a special case. The Fourier transform of an n-group slowing down kernel in which the diffusion length in each group is L_i is

$$\frac{p}{(1 + L_1^2B^2)(1 + L_2^2B^2).....(1 + L_n^2B^2)}$$

and therefore the expression for $k_{eff}(B_g^2)$ is

$$k_{eff}(B_g^2) = \frac{k}{(1 + L_1^2B_g^2)(1 + L_2^2B_g^2)....(1 + L_n^2B_g^2)(1 + L^2B_g^2)}, \qquad (6.41)$$

and the characteristic equation is

$$\frac{k}{(1 + L^2B_g^2)\prod_{i=1}^{n}(1 + L_i^2B_g^2)} = 1. \qquad (6.42)$$

The characteristic equation in an n-group theory is algebraic of degree n + 1 in B_g^2 (n + 1 because the thermal group is counted separately from the n fast groups). If $k > 1$, there are always n complex or negative roots in addition to the one positive, real B_g^2. The positive B_g^2 is associated with the asymptotic solution; the other roots are associated with the non-asymptotic ones.

If the number of groups becomes large, but the sum of the squares of the "diffusion" lengths is held constant ($\sum L_i^2 = \tau$), the group picture passes into the Gaussian picture. This was already pointed out in Chapter II. That the characteristic equation (6.42) passes into (6.10) is readily seen. In the limit, (6.42) is

$$\frac{k}{(1 + L^2B_g^2)} \left[\lim_{n \to \infty} \prod_{i=1}^{n} (1 + L_i^2 B_g^2) \right]^{-1} = 1, \quad \Sigma L_i^2 = \tau .$$

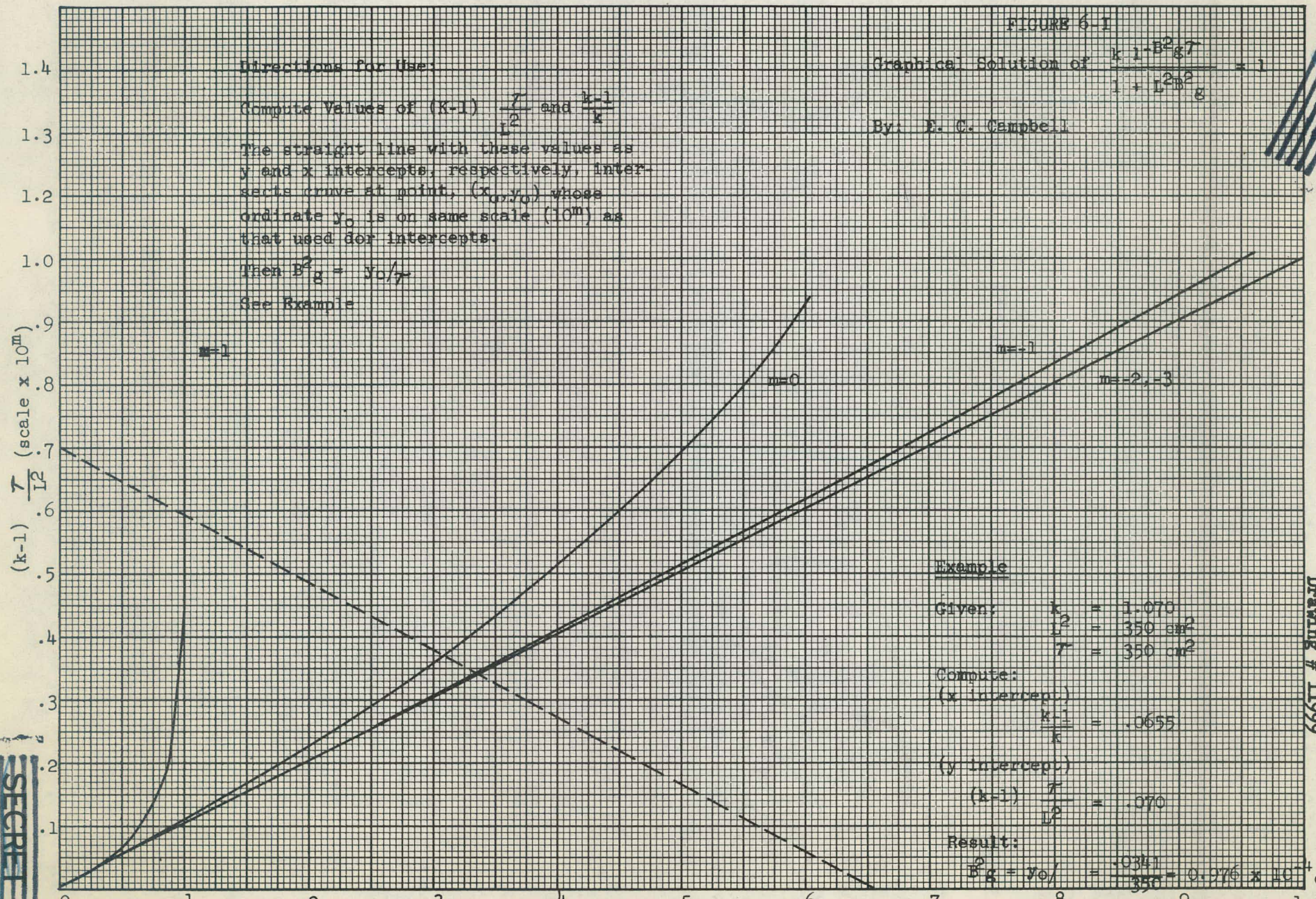
Now

$$\lim_{n \to \infty} \prod_{i=1}^{n} (1 + L_i^2 B_g^2) = e^{\tau B_g^2} , \quad \Sigma L_i^2 = \tau$$

as can be verified by taking logarithms of both sides. Hence (6.42) becomes

$$\frac{k\, e^{-\tau B_g^2}}{(1 + L^2 B_g^2)} = 1,$$

which is the Fermi equation.



CHAPTER VII

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THEORY OF REFLECTORS AND THE METHOD OF GROUPS

The critical mass of a chain reacting system can be reduced by surrounding the system by a scattering material, such as graphite. The function of such a "reflector" or "tamper" is to reflect back into the chain reactor neutrons which would otherwise have leaked out of the system. Thus a reflector reduces the neutron leakage; if placed around a just critical, non-reflected pile, a reflector would make the pile super-critical, or if placed around a slightly sub-critical reactor, a reflector of proper size might make an otherwise sub-critical reactor just critical.

The study of reflected reactors is considerably more complicated than the study of non-reflected reactors with energy independent extrapolation distance. The reason is that in the latter case the neutron energy spectrum is the same throughout the pile, while in the reflected pile the neutron spectrum changes in the neighborhood of the reflector. This complication arises from the fact that the multiplying and slowing down properties in the reflector are not the same as in the active portion of the reactor. In the region of the interface between reactor and reflector the neutron energy spectrum will therefore undergo a change from the shape characteristic of the reactor to that characteristic of the reflector. The asymptotic solution of the pile equation still holds far from the reflector-pile interface and there the energy spectrum is independent of position. However, close to the interface non-asymptotic transient solutions are excited which, on superposition, serve to describe the change in neutron energy spectrum as the interface is approached.

In this chapter it will be our purpose to outline the procedures which have been used with success in calculating the critical size and neutron distributions of reflected chain reactors. It will be seen that the problem can

be solved rigorously if, as in the practically important case of a graphite moderated and reflected pile, the slowing down properties of the reflector and the pile are the same. In the general case where the reflector and reactor have different slowing down properties only rather special approximate techniques, such as the multi-group methods, are available.

Simple One-Group Reflector Theory

In many cases of practical importance, the chain reactor is so large compared to a slowing down length that the transient solutions referred to above do not distort the neutron distribution in the bulk of the reactor. In such cases a one-group pile model is fairly adequate. The appropriate one-group pile equation in a large pile can be obtained by starting with the pile equation (5.13)

$$L^2 \Delta \phi_s(\underline{r}) - \phi_s(\underline{r}) + \frac{k}{p} \int_{\substack{\text{all} \\ \text{space}}} \phi_s(\underline{r}') \, P_{oo}(E_s, |\underline{r}-\underline{r}'|) d\underline{r}' = 0 \qquad (5.13)$$

and expanding $\phi_s(\underline{r}')$ around the point $\underline{r}$ under the integral:

$$\phi_s(\underline{r}') = \phi_s(\underline{r}) + (\underline{r}'-\underline{r}) \cdot \nabla \phi_s(\underline{r}) + \frac{1}{2!} \left[(\underline{r}'-\underline{r}) \cdot \nabla\right]^2 \phi_s(\underline{r}) + \ldots\ldots \qquad (7.1)$$

If the pile is large, the neutron buckling is small, and therefore the Taylor's expansion can be broken off after the first two terms. Upon substituting (7.1) into the integral in (5.13), and evaluating the resulting integrals in polar coordinates, we obtain

$$\Delta \phi_s(\underline{r}) + \frac{k-1}{M^2} \phi_s(\underline{r}) = 0 \qquad (7.2)$$

where M^2 is the migration area $= L^2 + \frac{k\overline{r^2}(E_s)}{6}$. The equation (7.2) is formally the same as the one-group thermal pile equation with M playing the role of the diffusion length L.

Consider now a large reactor which is surrounded by a reflector in which the diffusion length is L_r. We shall denote quantities referring to the reactor by subscript p, and quantities referring to the reflector by subscript r. Then, dropping the subscript s, we may write the following pair of equations for the

system pile and reflector:

$$\Delta\Phi_p + B_p^2 \Phi_p = 0 \qquad \text{in pile} \qquad (7.3a)$$

$$\Delta\Phi_r - \frac{1}{L_r^2} \Phi_r = 0 \qquad \text{in reflector} \qquad (7.3b)$$

with the boundary conditions

$$\Phi_p = \Phi_r \qquad (7.4a)$$

$$D_p \frac{\partial \Phi_p}{\partial n} = D_r \frac{\partial \Phi_r}{\partial n} \qquad \text{on pile reflector interface} \qquad (7.4b)$$

where $B_p^2 = \frac{k-1}{M^2}$, and $\frac{\partial}{\partial n}$ is the normal derivative. In addition to (7.4) there is of course the requirement

$$\Phi_r = 0 \qquad \text{on extrapolated boundary of reflector.} \qquad (7.5)$$

The critical equation is obtained by finding that solution of equations (7.3) which satisfies the boundary conditions (7.4) and (7.5). Before solving the reflected pile equation we note that the slope of the neutron distribution at the pile-reflector interface will be less steep than it would be at the physical boundary of an unreflected pile (Fig. 7-I). In other words the presence of the reflector increases the effective extrapolation distance in the reactor, and therefore reduces the critical

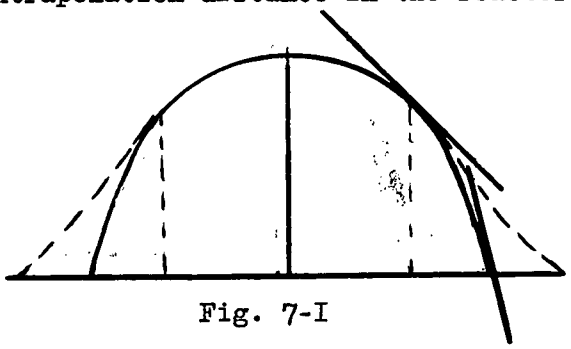
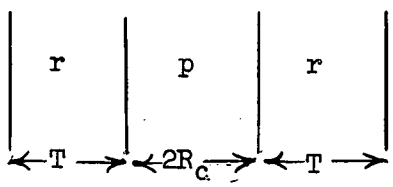


Fig. 7-I

dimensions of a chain reactor. This of course is just a somewhat more mathematical way of saying that a reflector reduces the neutron leakage.

We now calculate the critical dimension of an infinitely wide slab pile with a reflector of (extrapolated) thickness T on each side.



As has already been found, the critical dimension of the slab, if unreflected, is

$$R_{c_o} = \frac{\pi}{2B_p} \tag{7.6}$$

$2R_{c_o}$ being the width of the unreflected pile, and the neutron distribution is

$$\phi_p = \phi_{po} \cos B_p x \ . \tag{7.7}$$

In the reflected case, the solution of (7.3a) is

$$\phi_p = \phi_{po} \cos B_p x \ , \tag{7.8}$$

while the solution of (7.3b) in the right hand reflector, which satisfies (7.5), is

$$\phi_r = A \sinh \varkappa_r (T + R_c - |x|), \tag{7.9}$$

where $\varkappa_r = \frac{1}{L_r}$ and R_c is the half width of the reflected pile. The quantities A and R_c must be determined so that the continuity conditions (7.4) are satisfied. Substituting (7.8) and (7.9) into (7.4) we obtain

$$\phi_{po} \cos B_p R_c = A \sinh \varkappa_r T$$

$$B_p D_p \phi_{po} \sin B_p R_c = D_r A \varkappa_r \cosh \varkappa_r T$$

that is,

$$B_p D_p \tan B_p R_c = D_r \mathscr{k}_r \coth \mathscr{k}_r T \ . \qquad (7.10)$$

The transcendental equation (7.10) is the critical equation for the reflected pile. If T = 0, the pile is unreflected and the critical equation becomes

$$\tan B_p R_{c_o} = \infty \ ,$$

which, of course, leads to (7.6), i.e., $R_{c_o} = \frac{\pi}{2B_p}$.

If the pile and reflector are of finite height and breadth, then the transverse buckling must be subtracted from B_p^2 . Thus, for a system of square cross-section with extrapolated height H, the distribution is

$$\Phi_p = \Phi_{po} \cos\sqrt{B_p^2 - \frac{2\pi^2}{H^2}}\, x \cos\frac{\pi y}{H} \cos\frac{\pi z}{H} \qquad (7.8a)$$

$$\Phi_r = A \sinh\sqrt{\mathscr{k}_r^2 + \frac{2\pi^2}{H^2}}\,(T + R_c - |x|) \cos\frac{\pi y}{H} \cos\frac{\pi z}{H} \ . \qquad (7.9a)$$

The critical equation has the same form as 7.10 except that B_p is replaced by $\left(B_p^2 - \frac{2\pi^2}{H^2}\right)^{1/2}$ and $\mathscr{k}_r$ by $\left(\mathscr{k}_r^2 + \frac{2\pi^2}{H^2}\right)^{1/2}$.

The reduction in critical size effected by the reflector, the so-called reflector saving, δ, is easily computed from (7.10). We introduce into (7.10) the reflector saving,

$$\delta = R_{c_o} - R_c \ , \qquad (7.11)$$

so that (7.10) becomes, by virtue of the unreflected critical condition,

$$B_p D_p \tan\left(\frac{\pi}{2} - B_p \delta\right) = D_r \mathscr{k}_r \coth \mathscr{k}_r T$$

or

$$\tan B_p\delta \doteq \frac{B_p D_p}{D_r \mathcal{K}_r} \tanh \mathcal{K}_r T. \qquad (7.12)$$

The reflector saving is small compared to R_c, since it was assumed to begin with that the reactor is large; hence $B_p\delta \ll 1$, and it is permissible to replace $\tan B_p\delta$ by $B_p\delta$. This corresponds to ignoring the curvature in the neutron distribution close to the reflector interface, a procedure which is valid if the pile is large and δ is small. In this approximation the reflector saving turns out to be

$$\delta \approx \frac{D_p}{D_r} L_r \tanh \frac{T}{L_r} . \qquad (7.13)$$

If $T \ll L_r$, this becomes

$$\delta \approx \frac{D_p}{D_r} T$$

while, in case $D_p = D_r$ and the reflector is infinitely thick,

$$\delta \approx L_r; \qquad (7.14)$$

i.e., the reflector savings in a large pile reflected by an infinite reflector having the same diffusion coefficient as the pile is equal to the diffusion length of the reflector.

Explicit expressions for the one-group reflector saving in a spherical or cylindrical pile can be derived by solving the pile equation in the appropriate coordinate system. Thus in a spherical pile surrounded by a spherical reflector of thickness T, the neutron distribution in the pile is

$$\Phi_p = \Phi_{po} \frac{\sin B_p r}{B_p r} \qquad (7.15)$$

and in the reflector it is

$$\Phi_r = A \frac{\sinh \mathcal{L}_r(R_c + T - r)}{\mathcal{L}_r r} \quad . \tag{7.16}$$

So far the sphere and the plane problems are identical except that $r\Phi$ in the sphere replaces Φ in the plane. However, the boundary conditions at R_c

$$\Phi_{po} \frac{\sin B_p R_c}{B_p R_c} = A \frac{\sinh \mathcal{L}_r T}{\mathcal{L}_r R_c} \tag{7.17}$$

$$\Phi_{po} D_p \left[\frac{\cos B_p R_c}{R_c} - \frac{\sin B_p R_c}{B_p R_c^2} \right] = - D_r A \left[\frac{\cosh \mathcal{L}_r T}{R_c} + \frac{\sinh \mathcal{L}_r T}{\mathcal{L}_r R_c^2} \right] \tag{7.18}$$

differ from the plane boundary conditions in that an additional term is present in (7.18) which vanishes only if the two diffusion constants are the same.

By solving (7.17) and (7.18) we find the reflector saving $\delta = R_{c_o} - R_c$ in the case $D_p = D_r$ to be given again by

$$\delta \approx L_r \tanh \frac{T}{L_r} \tag{7.19}$$

while if $D_p \neq D_r$, δ is given by the formula (again valid for $B_p\delta \ll 1$)

$$\delta \approx \frac{1}{2} \left[\frac{D_r}{D_p} \delta_o + R_{c_o} - \sqrt{\left(R_{c_o} + \frac{D_r}{D_p} \delta_o \right)^2 - 4 R_{c_o} \delta_o} \right] \tag{7.20}$$

where

$$\delta_o = \frac{D_p}{D_r} L_r \tanh T/L_r .$$

If $D_r = D_p$, or if $R_{c_o} \to \infty$, the reflector saving (7.20) reduces to the plane reflector saving, (7.13). If $D_r < D_p$, it can be shown that the spherical

reflector saving is greater than the plane reflector saving, while if $D_r > D_p$, the reverse is true.

The neutron distribution in a reflected pile as computed with the one-group theory is compared to the distribution in an unreflected pile in Fig. 7-I. Evidently, as is seen from the figure, the ratio of central flux to average flux in a reflected pile is lower than in an unreflected one having the same buckling.

Two-Group Reflector Theory

In the one-group theory, the fast neutron flux is assumed to be proportional to the slow flux even at the reflector-pile interface. This is not correct near the reflector-pile interface, since, in the reflector, the ratios of slowing down power to thermal absorption cross-section and to multiplication rate usually have values quite different from the values of these ratios in the pile. If the pile is very large, the region in which the fast and slow fluxes are not proportional to each other occupies a rather small fraction of the pile, and therefore no great error is introduced by using the one-group picture. However, in a small reactor the region of non-proportionality may extend throughout the pile, and it is therefore necessary to take account more accurately of the fast neutrons. A simple method of taking the fast neutrons into account is to use the two-group pile model, and to require continuity of flux and current of both fast and slow neutrons.

The two-group pile equations will be discussed first for the problem of an unreflected pile with energy independent extrapolation distance. The appropriate equations are derived, as in Chapter III, by considering the neutron balance for fast and slow neutrons separately. We denote the flux of fast neutrons by Φ_1 and the flux of slow neutrons by Φ_2. The number of fast neutrons produced per cubic centimeter per second is evidently

$$\frac{k}{p} N\sigma_2 \Phi_2$$

where p is the resonance escape probability and σ_2 is the slow neutron absorption cross-section. The fact that these fast neutrons are produced at fission energy, and this energy bears little relation to the average energy of the fast neutron group is ignored in the group picture. Taking $\frac{k}{p} N\sigma_2 \Phi_2$ as the source of fast neutrons, we obtain for the fast neutrons

$$D_1 \Delta \Phi_1 - N\sigma_1 \Phi_1 + \frac{k}{p} N\sigma_2 \Phi_2 = 0 \qquad (7.21a)$$

where $N\sigma_1$ is the "absorption", i.e. slowing down cross-section, for the fast neutrons (cf. equation (3-106) of Chapter III). It will be recalled from Chapter III that $N\sigma_1$ and D_1 are chosen so that $\left(\frac{D_1}{N\sigma_1}\right)^{1/2}$ is the experimentally observed slowing down length. The slow neutrons in the pile satisfy the equation

$$D_2 \Delta \Phi_2 - N\sigma_2 \Phi_2 + pN\sigma_1 \Phi_1 = 0 \; ; \qquad (7.21b)$$

the slow neutron source term, $pN\sigma_1\Phi_1$, represents those fast neutrons which slow down and escape resonance capture. Because of the schematic manner in which the fast neutrons are lumped together, it is impossible in the two-group method to take account properly of the fact that the resonance capture occurs continuously over the whole of the fast neutron energy range. All that is achieved is to make the net resonance escape probability equal to p regardless of the energy distribution of the resonance capture. For the boundary condition we take

$$\Phi_1(\underline{R}_c) = \Phi_2(\underline{R}_c) = 0 \; , \qquad (7.23)$$

$\underline{R}_c$ being a radius vector which defines the extrapolated boundary of the pile. We assume that the extrapolation distance is the same for both fast and slow neutrons.

The pair of pile equations (7.21) can be solved in terms of the solutions of the wave equation

$$\Delta Z_i(B_i r) + B_i^2 Z_i(B_i r) = 0 \tag{7.24}$$

where B_i is to be determined later. Thus the general solution of (7.21) can be written

$$\Phi_1 = Z_1(B_1 r) + bZ_2(B_2 r) \tag{7.25a}$$

$$\Phi_2 = \frac{D_1 B_1^2 + N\sigma_1}{\frac{k}{p} N\sigma_2} Z_1(B_1 r) + b \frac{D_1 B_2^2 + N\sigma_1}{\frac{k}{p} N\sigma_2} Z_2(B_2 r) \tag{7.25b}$$

where b is an arbitrary constant. That (7.25) does satisfy (7.21a) is verified by direct substitution. In order to satisfy (7.21b), Φ_2 must also be given by

$$\Phi_2 = \frac{pN\sigma_1}{N\sigma_2 + D_2 B_1^2} Z_1(B_1 r) + \frac{bpN\sigma_1}{N\sigma_2 + D_2 B_2^2} Z_2(B_2 r). \tag{7.26}$$

The two expressions for Φ_2 are compatible if and only if

$$\frac{D_1 B_i^2 + N\sigma_1}{\frac{k}{p} N\sigma_2} = \frac{p\, N\sigma_1}{N\sigma_2 + D_2 B_i^2} ; \tag{7.27}$$

i.e., if B_i^2 satisfies the characteristic equation

$$\frac{k}{(1 + L_1^2 B_i^2)\,(1 + L_2^2 B_i^2)} = 1 \tag{7.28}$$

where

$$L_1 = \left(\frac{D_1}{N\sigma_1}\right)^{1/2}, \quad L_2 = \left(\frac{D_2}{N\sigma_2}\right)^{1/2} \tag{7.29}$$

are the slowing down length and diffusion length, respectively. Equation (7.28) is recognized immediately as the two-group critical equation (6.25) which was derived in Chapter VI. In an unreflected pile with energy independent extrapolation distance, the critical size is determined by (7.28) provided B_1^2 is the geometric buckling.

The two roots of (7.28) are

$$B_1^2 = \frac{1}{2}\left[-\left(\frac{1}{L_1^2} + \frac{1}{L_2^2}\right) + \sqrt{\left(\frac{1}{L_1^2} + \frac{1}{L_2^2}\right)^2 + \frac{4(k-1)}{L_1^2 L_2^2}} \right] \tag{7.30a}$$

and

$$B_2^2 = \frac{1}{2}\left[-\left(\frac{1}{L_1^2} + \frac{1}{L_2^2}\right) - \sqrt{\left(\frac{1}{L_1^2} + \frac{1}{L_2^2}\right)^2 + \frac{4(k-1)}{L_1^2 L_2^2}} \right] \tag{7.30b}$$

If the pile is very large, $k-1 \ll 1$ and the radicals can be expanded. For a large, pile, therefore,

$$B_1^2 \approx \frac{k-1}{M^2} \tag{7.31a}$$

$$B_2^2 \approx -\left[\frac{1}{L_1^2} + \frac{1}{L_2^2} + B_1^2\right] \tag{7.31b}$$

where

$$M^2 = L_1^2 + L_2^2$$

is the pile migration area.

We digress at this point to establish an important property of the two solutions $Z_1(B_1 r)$ and $Z_2(B_2 r)$. By multiplying (7.24) by $Z_i(B_i r)$, integrating over the volume v enclosed by the surface S, and applying Green's theorem, we find

$$B_i^2 = \frac{\int_v (\nabla Z_i)^2 dv}{\int_v Z_i^2 dv} - \frac{\int_S Z_i \frac{\partial Z_i}{\partial n} dS}{\int_v Z_i^2 dv} \tag{7.32}$$

Now, according to (7.32), if either Z_i or $\frac{\partial Z_i}{\partial n}$ vanishes on the surface S, B_i^2, being the quotient of the two positive quantities, must also be positive. If B_i^2 is negative, neither Z_i nor $\frac{\partial Z_i}{\partial n}$ can vanish over any closed surface S.* Thus the two solutions of (7.24) can be classified according to whether B_i^2 is positive or negative. Since, according to (7.30), $B_1^2 > 0$ and $B_2^2 < 0$, $Z_1(B_1\underline{r})$ is a function which may pass through zero on certain surfaces, while $Z_2(B_2\underline{r})$ can never vanish on a closed surface. The function Z_1 is analogous to a cosine, while Z_2 is analogous to a hyperbolic cosine.

We now apply the boundary condition that the fast and slow fluxes vanish on the extrapolated boundary:

$$\Phi_1(R_c) = Z_1(B_1\underline{R}_c) + bZ_2(B_2\underline{R}_c) = 0 \tag{7.33a}$$

$$\Phi_2(\underline{R}_c) = \frac{D_1B_1^2 + N\sigma_1}{\frac{k}{p}N\sigma_2} Z_1(B_1\underline{R}_c) + b\frac{D_1B_2^2 + N\sigma_1}{\frac{k}{p}N\sigma_2} Z_2(B_2\underline{R}_c) = 0 . \tag{7.33b}$$

This pair of equations can be satisfied only if

$$Z_1(B_1\underline{R}_c) = 0, \quad bZ_2(B_2\underline{R}_c) = 0,$$

and, since $Z_2(B_2\underline{R}_c)$ can never vanish on the boundary, b = 0. Thus the only solution of (7.21) which satisfies the boundary conditions in an unreflected pile is

$$\Phi_1(\underline{r}) = Z_1(B_1\underline{r}) \tag{7.34a}$$

$$\Phi_2(\underline{r}) = \frac{D_1B_1^2 + N\sigma_1}{\frac{k}{p}N\sigma_2} Z_1(B_1\underline{r}). \tag{7.34b}$$

In other words, according to the two-group theory, the fast and slow neutron densities are everywhere proportional in an unreflected pile with energy

* The function Z_i is of course continuous along with its first and second derivatives over the region v.

independent extrapolation distance. This is a special case of the fundamental theorem of pile theory and could have been predicted from the general theory of Chapter V.

It is seen (7.25) that the neutron distribution in the two-group theory is, in general, made up of two parts. First, there is the $Z_1(B_1 r)$ part, which is oscillatory outside the pile, and, since it appears in the unreflected pile, is to be identified with the asymptotic solution. In addition there is the non-asymptotic, non-oscillatory, solution, $Z_2(B_2 r)$, which occurs whenever $b \neq 0$; i.e., in a pile with reflector. If both solutions are admissible, then, since the coefficients before Z_1 and Z_2 in (7.33b) are different, the fast and slow neutron densities are not proportional throughout the pile. It is in this manner -- by the "excitation" of additional non-asymptotic solutions -- that the variation in neutron spectrum close to a boundary is taken into account by the mathematical apparatus of the group theories. Since every additional group adds another non-asymptotic solution, the more groups there are the more nearly the details of the variation of the neutron energy spectrum over the pile can be described.

We now consider, by two-group theory, the same infinite slab problem which was solved in the previous section by the one-group method. The slab of finite height and breadth can of course be reduced to the infinite slab by assuming a cosine variation in the transverse direction. In the pile the fast and slow fluxes satisfy

$$D_{1p} \frac{d^2\Phi_{1p}}{dx^2} - N_p\sigma_{1p}\Phi_{1p} + \frac{k}{p} N_p\sigma_{2p}\Phi_{2p} = 0 \qquad (7.35a)$$

$$D_{2p} \frac{d^2\Phi_{2p}}{dx^2} - N_p\sigma_{2p}\Phi_{2p} + pN_p\sigma_{1p}\Phi_{1p} = 0 \qquad (7.35b)$$

while in the reflector, where no fast neutrons are created, the fluxes satisfy

$$D_{1r}\frac{d^2\Phi_{1r}}{dx^2} - N_r\sigma_{1r}\Phi_{1r} = 0 \tag{7.36a}$$

$$D_{2r}\frac{d^2\Phi_{2r}}{dx^2} - N_r\sigma_{2r}\Phi_{2r} + N_r\sigma_{1r}\Phi_{1r} = 0 \tag{7.36b}$$

where subscripts p and r refer to pile and reflector, respectively. The boundary conditions are

$$D_{1p}\frac{d\Phi_{1p}}{dx} = D_{1r}\frac{d\Phi_{1r}}{dx}\ , \quad D_{2p}\frac{d\Phi_{2p}}{dx} = D_{2r}\frac{d\Phi_{2p}}{dx}$$

$$\Phi_{1p} = \Phi_{1r}\ , \quad \Phi_{2p} = \Phi_{2r} \tag{7.37}$$

at $x = \pm R_c$, the reflector-pile interface; and

$$\Phi_{1r} = \Phi_{2r} = 0 \tag{7.38}$$

at $x = \pm (R_c + T)$, the extrapolated boundary of the reflector; the extrapolation distance being assumed the same for fast and slow neutrons. The solution of the reflector equation which vanishes on the outer boundary is

$$\Phi_{1r}(x) = \alpha \sinh B_{1r}(R_c + T - |x|) \tag{7.39a}$$

$$\Phi_{2r}(x) = \frac{-\alpha N_r\sigma_{1r}}{D_{2r}B_{1r}^2 - N_r\sigma_{2r}} \sinh B_{1r}(R_c + T - |x|) + \beta \sinh B_{2r}(R_c + T - |x|) \tag{7.39b}$$

where

$$B_{1r}^2 = \frac{1}{L_{1r}^2}\ , \quad B_{2r}^2 = \frac{1}{L_{2r}^2}\ . \tag{7.40}$$

In the pile,

$$\phi_{1p}(x) = a\cos B_{1p}x + b\cosh B'_{2p}x \tag{7.41a}$$

$$\phi_{2p}(x) = \frac{D_{1p}B_{1p}^2 + N_p\sigma_{1p}}{\frac{k}{p}N_p\sigma_{2p}} a\cos B_{1p}x + b\,\frac{D_{1p}B_{2p}^2 + N_p\sigma_{1p}}{\frac{k}{p}N_p\sigma_{2p}} \cosh B'_{2p}x \tag{7.41b}$$

where B_{1p}^2 and B_{2p}^2 are the two roots (7.30a) and (7.30b), and B'_{2p} denotes the positive square root of the absolute value of B_{2p}^2.

The expressions for the fluxes involve the four arbitrary constants α, β, a, b, and the unknown critical half-width, R_c. Since the boundary conditions (7.37) consist of four homogeneous linear equations, the vanishing of the determinant of coefficients is the critical equation for R_c. Three of the remaining constants can be determined in terms of the fourth; in this manner the fast and slow fluxes are determined within an arbitrary multiplicative constant.

The algebraic details of substituting (7.41) and (7.39) into (7.37) and setting the determinant of coefficient equal to zero are straightforward but tedious. The critical equation turns out to be

$$B_{1p}\tan(B_{1p}R_c) = \frac{1}{Y}\left[\frac{N_r\sigma_{1r}B'_{2p}\tanh(B'_{2p}R_c)}{D_{2p}(B_{2r}^2 - B_{1r}^2)}(B_{2r}\coth B_{2r}T - B_{1r}\coth B_{1r}T)\right.$$

$$+\frac{D_{1r}D_{2r}(B_{2p}^2 - B_{1p}^2)}{D_{2p}\frac{k}{p}N_p\sigma_{2p}} B_{1r}B_{2r}\coth B_{1r}T\coth B_{2r}T \tag{7.42}$$

$$\left.+\frac{B'_{2p}\tanh B'_{2p}R_c}{\frac{k}{p}N_p\sigma_{2p}}\left\{(D_{1p}B_{2p}^2 + N_p\sigma_{1p})\frac{D_{1r}}{D_{1p}}B_{1r}\coth B_{1r}T - (D_{1p}B_{1p}^2 + N_p\sigma_{1p})\frac{D_{2r}}{D_{2p}}B_{2r}\coth B_{2r}T\right\}\right]$$

where

$$Y = \frac{N_r\sigma_{1r}}{D_{2p}(B_{2r}^2 - B_{1r}^2)}\left(B_{1r}\coth B_{1r}T - B_{2r}\coth B_{2r}T\right) + \frac{D_{1p}(B_{2p}^2 - B_{1p}^2)}{\frac{k}{p}N_p\sigma_{2p}} B'_{2p}\tanh B'_{2p}R_c$$

$$+ \frac{1}{\frac{k}{p}N_p\sigma_{2p}}\left[\left(D_{1p}B_{2p}^2 + N_p\sigma_{1p}\right)\frac{D_{2r}}{D_{2p}}B_{2r}\coth B_{2r}T - \left(D_{1p}B_{1p}^2 + N_p\sigma_{1p}\right)\frac{D_{1r}}{D_{1p}}B_{1r}\coth B_{1r}T\right]$$

and the expressions for the constants are

$$\frac{b}{a} = -\left[\frac{1}{Y}\frac{\cos B_{1p}R_c}{\cosh B'_{2p}R_c}\right]\left[\frac{N_r\sigma_{1r}}{D_{2p}(B_{2r}^2 - B_{1r}^2)}\left(B_{1r}\coth B_{1r}T - B_{2r}\coth B_{2r}T\right)\right.$$

$$\left. + \frac{D_{1p}B_{1p}^2 + N_p\sigma_{1p}}{\frac{k}{p}N_p\sigma_{2p}}\left(\frac{D_{2r}}{D_{2p}}B_{2r}\coth B_{2r}T - \frac{D_{1r}}{D_{2p}}B_{1r}\coth B_{1r}T\right)\right]$$

$$\frac{c}{a} = \left[\frac{1}{Y}\frac{\cos B_{1p}R_c}{\sinh B_{1r}T}\right]\left[\frac{D_{1p}(B_{2p}^2 - B_{1p}^2)}{\frac{k}{p}N_p\sigma_{2p}}\left(B'_{2p}\tanh B'_{2p}R_c + \frac{D_{2r}}{D_{2p}}B_{2r}\coth B_{2r}T\right)\right]$$

$$\frac{e}{a} = \left[\frac{1}{Y}\frac{\cos B_{1p}R_c}{\sinh B_{2r}T}\right]\left[\frac{D_{1p}(B_{2p}^2 - B_{1p}^2)}{\frac{k}{p}N_p\sigma_{2p}}\right]\left\{\left(B'_{2p}\tanh B'_{2p}R_c + \frac{D_{2r}}{D_{2p}}B_{1r}\coth B_{1r}T\right)\right.$$

$$\left.\left(\frac{D_{1p}B_{1p}^2 + N_p\sigma_{1p}}{\frac{k}{p}N_p\sigma_{2p}}\right) - \frac{N_r\sigma_{1r}}{D_{2r}(B_{2r}^2 - B_{1r}^2)}\left(B'_{2p}\tanh B'_{2p}R_c + \frac{D_{2r}}{D_{2p}}B_{1r}\coth B_{1r}T\right)\right\}.$$

If the pile is large, $B_{1p}\delta \ll 1$ and the reflector saving is

$$\delta = \delta_2\left\{1 + \frac{\left(1+\delta_2 B'_{2p}\tanh B'_{2p}R_c\right)\left[\frac{N_r\sigma_{1r}}{D_{2r}(B_{2r}^2 - B_{2r}^2)}\left(\frac{D_{2r}D_{1p}}{D_{2p}D_{1r}} - \frac{\delta_1}{\delta_2}\right) + \left(\frac{D_{1p}B_{2p}^2 + N_p\sigma_{1p}}{\frac{k}{p}N_p\sigma_{1p}}\right)\left(\frac{\delta_1}{\delta_2} - 1\right)\right]}{\frac{D_{1p}(B_{2p}^2 - B_{1p}^2)}{\frac{k}{p}N_p\sigma_{2p}} + B'_{2p}\tanh B'_{2p}\left[\frac{\delta_2 N_r\sigma_{1r}}{D_{2r}(B_{2r}^2 - B_{1r}^2)}\left(\frac{\delta_1}{\delta_2} - \frac{D_{2r}D_{1p}}{D_{2p}D_{1r}}\right) + \frac{\delta_2(D_{1p}B_{2p}^2 + N_p\sigma_{1p}) - \delta_1(D_{1p}B_{1p}^2 + N_p\sigma_{1p})}{\frac{k}{p}N_p\sigma_{2p}}\right]}\right\} \quad (7.43)$$

where

$$\delta_2 = \frac{D_{2p}}{D_{2r}} \frac{1}{B_{2r}} \tanh B_{2r} T$$

$$\delta_1 = \frac{D_{1p}}{D_{1r}} \frac{1}{B_{1r}} \tanh B_{1r} T \ .$$

According to the one-group theory the reflector saving is just δ_2. Thus the reflector saving, according to the two-group theory, is larger than that predicted by the one-group theory by the factor contained within the braces of Eq. (7.43). Physically the larger reflector saving predicted by the two-group theory can be understood since the mean square distance a neutron travels before capture in the reflector is a migration length, not a diffusion length. Since the migration length is longer than the diffusion length, the chance that a neutron returns to the reactor is actually larger than the one-group theory predicts, and this leads to a larger reflector saving in the two-group theory.

The transient part of the solution in the pile falls off roughly exponentially from the pile-reflector surface, the length constant being $\frac{1}{B'_{2p}}$. In a large pile, this length is, according to (7.31b), roughly

$$\frac{1}{B'_{2p}} \approx \left[\sqrt{\frac{1}{L_{1p}^2} + \frac{1}{L_{2p}^2}} \right]^{-1} \ . \qquad (7.44)$$

If the two lengths L_{1p} and L_{2p} are very different, the transient length will be nearly equal to the shorter of the two.

The neutron distribution in the pile and reflector computed by two-group theory is plotted in Figure 7-II . The most interesting feature of the distribution is that the slow neutron density has a maximum, or at least a point of

inflection, in the reflector a few centimeters away from the interface. This hump arises from the fact that in the reflector slow neutrons are produced by slowing down of fast neutrons, but they are absorbed very much less strongly in the reflector than in the reactor. For the pile whose distribution is computed in Figure 7-II the ratio of slowing down power to slow neutron absorption cross-section is 97 in the reflector and 4.4 in the reactor. Thus the slow neutrons pile up in the reflector, and this manifests itself as a rise in the slow neutron density near the reactor. Farther from the interface the rate of moderation of fast neutrons decreases because of the distance from the original source, and the slow neutron density decays along with the fast neutron density.

Multi-group Theory; Choice of Constants

The two-group method is readily generalized to any number of groups by increasing the number of intervals into which the neutron energy is divided. As long as there is no resonance capture (which may or may not result in fission) there is no difficulty in writing down the multi-group equations, since the flux, Φ_i, in the i^{th} energy interval remains constant throughout the energy interval. If resonance capture occurs in the i^{th} energy interval with probability p_i, then the multi-group equations acquire a certain ambiguity. This ambiguity arises because the group picture allows no change of Φ_i within a given energy interval, whereas the resonance capture in general results in a more or less continuous decrease of the neutron density as the neutrons are moderated.

A consistent scheme for taking account of resonance capture is to view $N\sigma_i\Phi_i$ always as the number of neutrons passing through the i^{th} energy interval per c.c. per second and entering the $(i+1)^{st}$ interval if there were no resonance capture. The quantity σ_i is the slowing down cross-section for all

groups except thermal; for thermal it is the true capture cross-section. Resonance capture reduces the number of neutrons per c.c. per second entering the interval to

$$N\sigma_i p_i \Phi_i ,$$

p_i being the resonance escape probability in the i^{th} interval. Thus the number of i^{th} interval neutrons resonance absorbed per second per c.c. is

$$N\sigma_i \Phi_i - N\sigma_i p_i \Phi_i = N\sigma_i (1 - p_i)\Phi_i .$$

Hence, in the absence of resonance fission, and under the assumption that all fission neutrons enter the fastest group, the multi-group equations in a uniform system are

$$D_i \Delta \Phi_i - N\sigma_i p_i \Phi_i - N\sigma_i (1 - p_i)\Phi_i + N\sigma_{i-1} p_{i-1} \Phi_{i-1} = 0 \qquad (7.45)$$

or simply

$$D_i \Delta \Phi_i - N\sigma_i \Phi_i + N\sigma_{i-1} p_{i-1} \Phi_{i-1} = 0 \qquad (7.46)$$

for $i \neq 1$, the highest energy interval. For the highest energy,

$$D_1 \Delta \Phi_1 - N\sigma_1 \Phi_1 + \frac{k}{p} N\sigma_n \Phi_n = 0, \qquad (7.47)$$

where p, the total resonance escape probability, is the product of all previous p_i:

$$p = p_1 p_2 p_3 \cdots\cdots p_{n-1} . \qquad (7.48)$$

If resonance fission occurs, as in an enriched pile with rather low moderator to U ratio, the equations evidently become

$$D_i \Delta \Phi_i - N\sigma_i \Phi_i + N\sigma_{i-1} p_{i-1} \Phi_{i-1} + \sum_j N\sigma_j (1-p_j) k_{ji} \Phi_j = 0 \qquad (7.49)$$

where k_{ji} is the number of neutrons created in the i^{th} energy interval per neutron absorbed in the j^{th} interval.

The appropriate expression for p_i in terms of the absorption and scattering cross-section of the system follows from Chapter IV. If the i^{th} energy interval extends over the range E_{i-1} to E_i, then

$$p_i = \exp -\left\{ \frac{1}{N\sigma_{si}\xi} \left[\int_{E_{i-1}}^{E_i} N\sigma_{ai} \frac{dE}{E} \right]_{eff} \right\}$$

where the integrand of the effective resonance absorption integral is a complicated expression involving the surface and volume of the fissionable material, as well as the absorption and scattering cross-sections, σ_{ai} and σ_{si}. For a homogeneous system,

$$\left[\int N\sigma_{ai} \frac{dE}{E} \right]_{eff} = \int \frac{N\sigma_{ai}}{1 + \sigma_{ai}/\sigma_{si}} \frac{dE}{E}$$

which, for very small σ_{ai}/σ_{si} becomes

$$\left[\int N\sigma_{ai} \frac{dE}{E} \right]_{eff} \approx \int N\sigma_{ai} \frac{dE}{E} .$$

If the group interval is very small so that the resonance absorption is also small, then

$$1 - p_i \approx \frac{1}{N\sigma_{si}\xi} \left[\int_{E_{i-1}}^{E_i} N\sigma_{ai} \frac{dE}{E} \right]_{eff}$$

$$\approx \frac{N\sigma_{ai}{}_{eff}}{N\sigma_{si}\xi} \ln \frac{E_i}{E_{i-1}}$$

where $N\sigma_{a_{i_{eff}}}$ is an average effective absorption cross-section, so defined that

$$N\sigma_{a_{i_{eff}}} \int_{E_{i-1}}^{E_i} \frac{dE}{E} = \left[\int_{E_{i-1}}^{E_i} N\sigma_{ai} \frac{dE}{E} \right]_{eff} .$$

Now the slowing down cross-section, $N\sigma_i$, is so defined that $\frac{N\sigma_{si}}{N\sigma_i}$ is the average number of collisions in the i^{th} energy interval, i.e.

$$\frac{1}{N\sigma_i} = \frac{\ell n\, E_i/E_{i-1}}{N\sigma_{si}\xi} \quad ; \tag{7.50}$$

hence, in the limit of small group interval, and small absorption,

$$1 - p_i \approx \frac{N\sigma_{a_{i_{eff}}}}{N\sigma_i} \quad . \tag{7.51}$$

The group equations (7.49) therefore may also be written

$$D_i \Delta\Phi_i - N\sigma_i\Phi_i + (N\sigma_{i-1} - N\sigma_{a_{i-1_{eff}}})\Phi_{i-1} + \sum_j N\sigma_{a_{j_{eff}}} k_{ji}\Phi_j = 0; \tag{7.52}$$

for very small absorption, $\sigma_{a_{eff}}$ may be replaced by σ_a.

Equations rather similar to (7.50) have been derived by starting with the age slowing down equation, including absorption, and approximating the differential equation by difference equations. The objection to such a procedure is that the age equation does not hold if there is absorption (cf. p. 27 Chapter III), and therefore group equations, derived from the age equation are not necessarily more valid than equations (7.50) which are also derived from a simplified, but admittedly incorrect slowing down picture.

We consider the group equations in the form (7.46-7). There are n equations in an n-group picture. The solution for Φ_i can be expressed, as in the two-group

theory, as an n-fold linear combination of functions $Z(B_\nu \underline{r})$ each of which satisfies a wave equation

$$\Delta Z_\nu(B_\nu \underline{r}) + B_\nu^2 Z_\nu(B_\nu \underline{r}) = 0.$$

Upon substituting a solution of the form

$$\phi_i = A_i Z_\nu(B_\nu \underline{r})$$

into (7.46-7), there results a set of n linear homogeneous equations in A_i; for these equations to be compatible the determinant

$$\begin{vmatrix} -(D_1 B_\nu^2 + N\sigma_1) & 0 & \cdots & 0 & \frac{k}{p} N\sigma_n \\ N\sigma_1 p_1 & -(D_2 B_\nu^2 + N\sigma_2) & \cdots & 0 & 0 \\ 0 & 0 & \cdots & N\sigma_{n-1} p_{n-1} & -(D_n B_\nu^2 + N\sigma_n) \end{vmatrix} = 0 \qquad (7.53)$$

This determinant is easily expanded, and it is found that each B_ν satisfies the n-group characteristic equation

$$\frac{k}{(1 + L_1^2 B_\nu^2)(1 + L_2^2 B_\nu^2)\cdots(1 + L_n^2 B_\nu^2)} = 1 \qquad (7.54)$$

where

$$L_i^2 = \frac{D_i^2}{N\sigma_i} \quad . \qquad (7.55)$$

Equation (7.54) is recognized as the critical equation (6-1)

$$\frac{k\overline{P_\infty}(B_\nu^2)}{p(1 + L^2 B_\nu^2)} = 1 \qquad (7.56)$$

as applied to the n-group system, since the Fourier transform of the slowing down kernel in each group is $p_i(1 + L_i^2 B_\nu^2)^{-1}$.

The characteristic equation (7.54) is algebraic of the n^{th} order in B_ν^2. There are therefore n values of B_ν^2, and 2n values of B_ν, corresponding to the positive and negative square roots of B_ν^2. The total number of linearly independent $Z_\nu(B_\nu r)$ is 2n ----- which evidently follows from the fact that the n-group equations comprise n equations each of second order.

The roots B_ν^2 fall into two classes -- one root which corresponds to the asymptotic solution, and the remainder which correspond to the non-asymptotic solutions. The asymptotic solution root, which we denote by B_o^2, is given, approximately, by

$$B_o^2 \approx \frac{k-1}{\sum_i L_i^2} \tag{7.57}$$

provided $k \approx 1$, i.e., the reactor is large. This result has already been shown to hold for any slowing down picture (Ch.VI); it can be derived directly from (7.54) by neglecting terms of order B_ν^4 or higher in equation (7.54).

The remaining roots are, in general, complex. The non-asymptotic solutions which correspond to them behave roughly as damped exponentials (if B is a pure imaginary), or as damped exponentials multiplied by an oscillating function if B_ν is complex. Since explicit expressions for B_ν are obtained most readily when the number of groups is infinite, a closer discussion of the non-asymptotic solutions will be given after the $n = \infty$ equations have been derived.

The characteristic equation for the case $n = \infty$, that is the Fermi picture, is easily derived from (7.54) by passing to the limit in which the number of groups becomes infinite but the sum of the non-thermal slowing down areas remain finite and equal to the age, i.e.,

$$\lim_{n \to \infty} \sum_{i=1}^{n-1} L_i^2 = \tau.$$

Upon taking logarithms of both sides of (7.54) we have

$$\ell nk = \sum_{i=1}^{n-1} \ell n(1 + L_i^2 B_\nu^2) + \ell n(1 + L_n^2 B_\nu^2) \quad .$$

In the limit, each L_i^2 becomes vanishingly small; thus

$$\ell n(1 + L_i^2 B_\nu^2) \longrightarrow L_i^2 B_\nu^2$$

and therefore, in the limit

$$\ell nk = \tau B_\nu^2 + \ell n(1 + L_n^2 B_\nu^2);$$

or

$$\frac{k\, e^{-\tau B_\nu^2}}{(1 + L_n^2 B_\nu^2)} = 1 \qquad (7.58)$$

which is the Fermi characteristic equation.

Since ℓnk is a multiple valued function, it is clear from (7.58) that there are infinitely many B_ν^2. These roots can be written down explicitly in case $L_n^2 = 0$; i.e., the thermal diffusion length is zero. In that case

$$B_\nu^2 = \frac{|\ell nk| + 2\pi\nu i}{\tau} \qquad (7.59)$$

where $|\ell nk|$ denotes the real logarithm of k. From (7.59) it follows that

$$B_\nu = \pm (a_\nu + b_\nu i)$$

where

$$a_\nu = \sqrt{\frac{\sqrt{|\ell nk|^2 + 4\pi^2\nu^2} + |\ell nk|}{2\tau}}$$

$$b_\nu = \sqrt{\frac{\sqrt{|\ell nk|^2 + 4\pi^2\nu^2} - |\ell nk|}{2\tau}} \quad .$$

For large ν, the roots approach

$$B_\nu \longrightarrow \pm\sqrt{\frac{\pi\nu}{\tau}}\,(1 + i).$$

The characteristic functions, $Z_\nu(B_\nu \underline{r})$ will have various forms depending on the geometry. Thus in a plane geometry,

$$Z_\nu(B_\nu x) = e^{\pm\, ia_\nu x}\; e^{\pm\, b_\nu x}$$

which is a damped wave with a relaxation length which approaches

$$\sqrt{\frac{\tau}{\pi\nu}}$$

as ν increases. The damping becomes stronger as ν increases -- that is, the neutron distribution is more and more adequately represented by fewer and fewer Z_ν's as the distance from the interface increases.

Methods for Solving the Multi-group Equations

The multi-group equations constitute a set of n differential equations (in n-group theory) with the boundary conditions $\Phi_i = 0$ on outermost boundary, and Φ_i and $D_i x$ (normal derivative of Φ_i) continuous across interfaces between regions. Considerable literature devoted to procedures for solving the multi-group equations has grown up. All of the many different schemes which have been used can be classified under three major headings:

1) Solution in closed form

2) Solution by iteration

3) Solution by expansion in characteristic functions.

Various combinations of the major methods -- for example, iteration of characteristic function expansions -- have also been found useful.

Each group equation is, in general, a partial differential equation in x, y, and z. If the chain reacting system has sufficient symmetry -- for example,

a finite cylinder surrounded by cylindrical reflectors of the same height as the reactor -- then each group equation can be reduced to an ordinary differential equation in, say, r, and the boundary conditions need be satisfied on surfaces r = constant. The methods which will be described in this section apply only to such one-coordinate reactors for which the multi-group system is a system of ordinary differential equations.

Solution in Closed Form

This method, an example of which has been worked out in detail on pages 15,16 consists simply in writing down the neutron distribution in each region as a linear combination, with undetermined coefficients, A_i, of the fundamental solutions of the group equations, and then determining the A_i by requiring that the boundary conditions be satisfied. As was shown in (7.24) the fundamental solutions $Z_\nu(r)$ satisfy

$$\Delta Z_\nu(\underline{r}) + B_\nu^2 Z_\nu(\underline{r}) = 0$$

where B_ν^2 are the roots (either real or complex) of an equation like (7.54), provided, of course, D_i, σ_i, k_i and p_i are constant throughout each region. We assume this to be the case, and confine the discussion to a chain reacting system consisting of m distinct regions, in each of which all properties are constant.

Each group equation, being of second order, will have two solutions, one of which is regular (or, for a slab, symmetric), the other irregular (or antisymmetric). We list these solutions for the three geometries in the following tables:

Table I

Functions $Z_\nu(r)$ in Various Geometries

	Regular (or symmetric)			Irregular (or anti-symmetric)		
Slab	$\cos B_\nu r$	or	$\cosh B_\nu r$	$\sin B_\nu r$	or	$\sinh B_\nu r$
Cylinder	$J_o(B_\nu r)$	or	$I_o(B_\nu r)$	$Y_o(B_\nu r)$	or	$K_o(B_\nu r)$
Sphere	$\frac{\sin B_\nu r}{r}$	or	$\frac{\sinh B_\nu r}{r}$	$\frac{\cos B_\nu r}{r}$	or	$\frac{\cosh B_\nu r}{r}$

Since there are n equations in each region, the neutron density in each group will in general be a 2n-fold linear combination of functions listed in Table I. The group equations are a coupled set; all of the neutron groups can therefore be expressed in terms of one of the groups, and in each region only 2n of the constants A_n will be arbitrary.

For simplicity, we assume the system is symmetric about the central region. Then, in the central region, the coefficients of the irregular solution must vanish either because of symmetry (in the slab case) or because the flux must remain finite (in sphere and cylinder cases). This reduces the number of A_i in the central region from 2n to n. Since there are 2n unknown A_i's in each of the other regions, the total number of A_i's in a system containing m separate regions will be $2n(m-1) + n$ or $(2m-1)n$.

At each interface between two regions there are 2n equations which must be satisfied by the A_i's in order to ensure continuity of flux and of current; at the outermost boundary there are n relations which ensure that the flux of each group vanishes there. In all then, there are $2n(m-1) + n = n(2m-1)$ linear homogeneous relations between the A_i's. These can be solved for $(2m-1)n-1$ A_i in terms of, say, A_1, provided the determinant of the coefficients vanishes. This

determinantal equation, of which (7.42) is a simple example, involves transcendental functions of the critical size, R_c, and of the critical multiplication constant, k. The value of k for which the determinant vanishes is of course the critical multiplication constant for the given sized system.

The actual solution of the determinantal critical equation presents considerable analytical difficulties as soon as the number of groups exceeds two. In most cases it is easier to solve the critical equation for k. having assumed a value of the critical size, than to solve for the size, given k.

The formal work of setting up the critical equation can be much systematized, especially if there are many regions, by the use of matrix notation.* We denote the j^{th} group neutron flux in the k^{th} region, $\Phi_j^{(k)}(r)$, as

$$\Phi_j^{(k)}(\underline{r}) = A_1^{(k)} Y_{1j}^{(k)}(r) + A_2^{(k)} Y_{2j}^{(k)}(r) + \ldots + A_{2n}^{(k)} Y_{2n,j}^{(k)}(r) \qquad j = 1 \ldots n \tag{7.60}$$

where each function $Y_{ij}(r)$ is proportional to a fundamental solution $Z_\gamma(r)$, the constant of proportionality being determined so that the functions $\Phi_j^{(k)}(r)$ satisfy the group equations. For example, in a two-group theory, according to (7.25),

$$Y_{11} = Z_1(B_1 r) \qquad Y_{21} = Z_2(B_2 r)$$

$$Y_{12} = \frac{D_1 B_1^2 + N\sigma_1}{\frac{k}{p} N\sigma_2} Z_1(B_1 r) \qquad Y_{22} = \frac{D_1 B_2^2 + N\sigma_1}{\frac{k}{p} N\sigma_2} Z_2(B_2 r) \tag{7.61}$$

In an n-group theory, there are n Φ_j's; that is, j runs from 1 to n.

It is convenient to introduce the "flow" functions

$$\Phi_{j+n}^{(k)}(r) = D_j^{(k)} \Phi_j'(r) \tag{7.62}$$

---

* A. S. Householder and H. L. Garabedian, MonP-202; MonP-246.

Then, in matrix notation, the neutron fluxes and the flows can be abbreviated

$$\underline{\Phi}^{(k)}(r) = \underline{\underline{Y}}^{(k)}_{(r)} \underline{A}^{(k)} \tag{7.63}$$

where the vector $\underline{\Phi}^{(k)}$ denotes $(\Phi_1^{(k)}, \Phi_2^{(k)}, \ldots \Phi_{2n}^{(k)})$, the vector $\underline{A}^{(k)}$ denotes $(A_1^{(k)}, A_2^{(k)}, \ldots A_{2n}^{(k)})$, and $\underline{\underline{Y}}^{(k)}_{(r)}$ is the matrix of function $Y_{ij}^{(k)}(r)$.

Suppose there are m regions, the coordinates of the interface between the k^{th} and k + 1st regions being denoted by r_k. At each interface, since fluxes and flows are continuous,

$$\underline{\underline{Y}}^{(k)}(r_k)\ \underline{A}^{(k)} = \underline{\underline{Y}}^{(k+1)}(r_k)\ \underline{A}^{(k+1)}$$

i.e.,

$$\underline{A}^{(k+1)} = \left[\underline{\underline{Y}}^{(k+1)}(r_k)\right]^{-1} \underline{\underline{Y}}^{(k)}(r_k)\ \underline{A}^{(k)} \tag{7.64}$$

where $\left[\underline{\underline{Y}}^{(k+1)}(r_k)\right]^{-1}$ is the matrix reciprocal to $\underline{\underline{Y}}^{(k+1)}(r_k)$. By successively applying the boundary conditions to the 1st, 2nd, etc. boundaries, we obtain finally

$$\underline{A}^{(m)} = \underline{\underline{X}}\ \underline{A}^{(1)}$$

where

$$\underline{\underline{X}} = \underline{\underline{Y}}^{(m)}(r_{m-1})^{-1} \underline{\underline{Y}}^{(m-1)}(r_{m-1})\ \underline{\underline{Y}}^{(m-1)}(r_{m-2})^{-1} \ldots \underline{\underline{Y}}^{(2)}(r_2)\ \underline{\underline{Y}}^{(2)}(r_1)^{-1}\ \underline{\underline{Y}}^{(1)}(r_1). \tag{7.65}$$

Thus the fluxes and flows in the 1st region are

$$\underline{\Phi}^{(1)}(r) = \underline{\underline{Y}}^{(1)}(r)\ \underline{A}^{(1)} \tag{7.66}$$

and the fluxes and flows in the m^{th} region are

$$\underline{\Phi}^{(m)}(r) = \underline{\underline{Y}}^{(m)}(r)\ \underline{\underline{X}}\ \underline{A}^{(1)} = \underline{\underline{H}}\ \underline{A}^{(1)} \tag{7.67}$$

where

$$\underline{\underline{H}} \equiv \underline{\underline{Y}}^{(m)}(\underline{r})\ \underline{\underline{X}} \tag{7.68}$$

is a 2n x 2n matrix.

The boundary conditions at the outermost boundary, namely, that the fluxes vanish, involve only the first n components of $\underline{\Phi}^{(m)}(r)$. If $\underline{\underline{H}}^*$ denotes the matrix formed from $\underline{\underline{H}}$ by suppressing the last n rows, then the first n components of $\underline{\Phi}^{(m)}(r)$, which we denote by $\underline{\Phi}^{*(m)}(r)$, are

$$\underline{\Phi}^{*(m)}(r) = \underline{\underline{H}}^*\ \underline{A}^{(1)}. \tag{7.69}$$

The boundary condition at the outermost boundary is therefore

$$\underline{\underline{H}}^*\ \underline{A}^{(1)} = 0\ . \tag{7.70}$$

Now, because of symmetry, or because of the condition of regularity at the origin, all $A_i^{(1)}$'s which multiply antisymmetric or irregular solutions vanish; i.e., the vector $\underline{A}^{(1)}$ really contains only n, instead of 2n, non-vanishing components. If $\underline{A}^{*(1)}$ denotes the column vector obtained from $\underline{A}^{(1)}$ by suppressing the vanishing n components, and if $\underline{\underline{H}}^{**}$ denotes the $\underline{\underline{H}}^*$ matrix with the corresponding n columns suppressed, then the boundary condition (7.70) is

$$\underline{\underline{H}}^{**}\ \underline{A}^{*(1)} = 0, \tag{7.71}$$

which is an n x n set of homogeneous linear equations. The compatibility condition,

$$\det \underline{\underline{H}}^{**} = 0 \tag{7.72}$$

is a transcendental equation in the critical size and the multiplication constants, and is evidently the critical equation.

The matrix notation is useful, first because it shows that the critical determinant $\left|\underline{\underline{H}}^{**}\right|$ is n x n regardless of the number of regions; and second, because it gives a systematic way of computing the critical equation. The

actual computation of $\underline{\underline{H}}^{**}$ involves, according to (7.65) and (7.68), the computation of the matrices $\underline{\underline{Y}}^{m}(r)$ and their inverses $\underline{\underline{Y}}^{m}(r)^{-1}$. Systematic computation forms for calculating these matrices have been developed by B. I. Spinrad.

An example of a problem solved by the matrix method follows:

We consider a slab reactor which is symmetric about the central plane and has a core and two reflectors on each side. The constants are given in Table II.

Table II

	Fast Group		Thermal Group	
	D_1	L_1^2	D_2	L_2^2
Region 1 (0-12 cm)	1.26	66	.28	3.6
Region 2 (12-40 cm)	.63	94	.54	430
Region 3 (40-60 cm)	1.1	100	.3	200

In the second and third regions the matrix $\underline{\underline{Y}}(r)$ has the form

$$\begin{vmatrix} \cosh(B_1r) & \sinh(B_1r) & 0 & 0 \\ S\cosh(B_1r) & S\sinh(B_1r) & \cosh(B_2r) & \sinh(B_2r) \\ B_1D_1\sinh(B_1r) & B_1D_1\cosh(B_1r) & 0 & 0 \\ SB_1D_2\sinh(B_1r) & SB_1D_2\cosh(B_1r) & B_2D_2\sinh(B_2r) & B_2D_2\cosh(B_2r) \end{vmatrix}$$

and its matrix reciprocal $\left|\underline{\underline{Y}}(r)\right|^{-1}$ is

$$\begin{vmatrix} \cosh(B_1r) & 0 & -\frac{1}{B_1D_1}\sinh(B_1r) & 0 \\ -\sinh(B_1r) & 0 & +\frac{1}{B_1D_1}\cosh(B_1r) & 0 \\ -S\cosh(B_2r) & \cosh(B_2r) & +\frac{S}{B_2D_1}\sinh(B_2r) & -\frac{1}{B_2D_2}\sinh(B_2r) \\ S\sinh(B_2r) & -\sinh(B_2r) & -\frac{S}{B_2D_1}\cosh(B_2r) & +\frac{1}{B_2D_2}\cosh(B_2r) \end{vmatrix}$$

where $B_1 = \frac{1}{L_1}$; $B_2 = \frac{1}{L_2}$; $S = \frac{D_1 L_2^2}{D_2(L_1^2 - L_2^2)}$. It is understood, of course, that in each region the constants appropriate to that region are used to compute B_1, B_2 and S. In this case, from (7.65),

$$\underline{\underline{X}}^* = \left[Y^3(40)\right]^{-1} Y^2(40) \left[Y^2(12)\right]^{-1} Y^{1*}(12)$$

where $\underline{\underline{Y}}^{1*}(r)$ is given by

$$\begin{vmatrix} \cos(B_1 r) & \cosh(B_2 r) & 0 & 0 \\ S_1 \cos(B_1 r) & S_2 \cosh(B_2 r) & 0 & 0 \\ -B_1 D_1 \sin(B_1 r) & +B_2 D_1 \sinh(B_2 r) & 0 & 0 \\ -S_1 B_1 D_2 \sin(B_1 r) & +S_2 B_2 D_2 \sinh(B_2 r) & 0 & 0 \end{vmatrix}$$

and $S_1 = \frac{D_1 L_2^2}{L_1^2 D_1} \frac{1}{1 + L_2^2 B_1^2}$, $S_2 = \frac{D_1 L_2^2}{L_1^2 D_2} \frac{1}{1 - L_2^2 B_2^2}$ and B_1 and B_2 satisfy the condition that $k = (1 + L_1^2 B_1^2)(1 + L_2^2 B_1^2) = (1 - L_1^2 B_2^2)(1 - L_2^2 B_2^2)$. From the X*, H** is computed by multiplying X* by $Y^{(3)}(60)$. By iteration, the determinant H** is made to vanish for $B_1 = .0403547$, $B_2 = .5427315$, corresponding to a value for k of 1.114. The coefficients A_i are determined from (7.65) in terms of an arbitrary scale factor A with the following result:

$$A^{(1)} = \begin{vmatrix} A \\ -.00013451\ A \\ 0 \\ 0 \end{vmatrix} \quad A^{(2)} = \begin{vmatrix} 2.8973\ A \\ -2.8977\ A \\ 2.9544\ A \\ -2.9591\ A \end{vmatrix} \quad A^{(3)} = \begin{vmatrix} 1.8906\ A \\ -1.8906\ A \\ +10.9195\ A \\ -10.924\ A \end{vmatrix}$$

Hence the flux distributions in the three regions are:

$$\Phi_1^1 = \cos(.04035r) - .0001345 \cosh(.54273r)$$

$$\Phi_2^1 = .24382 \cos(.04035r) + .0005461 \cosh(.54273r)$$

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$$\phi_1^2 = 2.8973 \cosh(.10314r) - 2.8977 \sinh(.10314r)$$

$$\phi_2^2 = -4.3259 \cosh(.10314r) + 4.3265 \sinh(.10314r) + 2.9544 \cosh(.048224r) - 2.9591 \sinh(.048224r)$$

$$\phi_1^3 = 1.8908 \cosh(.1r) - 1.8906 \sinh(.1r)$$

$$\phi_2^3 = -13.8642 \cosh(.1r) + 13.8644 \sinh(.1r) + 10.9195 \cosh(.07071r) - 10.924 \sinh(.07071r)$$

Solution by Iteration

When the number of groups exceeds three, or when the properties of each region vary continuously, the direct analytical method becomes unfeasible. Because the group equations form a coupled set, it is always possible to solve them by assuming a distribution for one neutron group and, using this as a source for the next less energetic group, computing the distribution in each successive group until the distribution in the original group is recomputed. The process is repeated until the iterated distribution does not change with further iterations.

Physically, the iteration method amounts to following the history of an initial neutron distribution as it converges onto the equilibrium distribution after successive generations of multiplication and diffusion in a critical system. Each iteration cycle corresponds to a single generation.

Iterations of the group equations may be carried out either analytically or numerically. The analytic solution of the equations is feasible provided each region is uniform, while, if the values of D, σ, etc. vary from point to point, numerical integration of the differential equations is necessary.

Ordinarily the critical multiplication constant, in addition to the neutron distributions, is unknown. It is therefore necessary to develop formulas for computing the critical value of k in terms of the iterated neutron distributions. In order to do this we introduce the Green's vector, $G_{il}(\underline{r},\underline{r}')$, for the group equations. Each function $G_{il}(\underline{r},\underline{r}')$ is defined as the flux of i^{th} group neutrons at $\underline{r}$ due to a unit source of group 1, i.e., the most energetic, neutrons at $\underline{r}'$. We confine the discussion to the case of slow fission alone with all fission neutrons being produced at the energy of the 1st group, and we neglect resonance capture. Then the source of fastest neutrons is $kN\sigma_n\Phi_n$, and the group equations are equivalent to the set of integral equations

$$\Phi_i(\underline{r}) = k\int N\sigma_n\Phi_n(\underline{r}')\, G_{il}(\underline{r},\underline{r}')d\underline{r}' \qquad (7.73)$$

the integral being taken over the reactor alone. For the slow neutron group,

$$\Phi_n(\underline{r}) = k\int N\sigma_n\Phi_n(\underline{r}')\, G_{nl}(\underline{r},\underline{r}')d\underline{r}' \qquad (7.74)$$

i.e., Φ_n satisfies a homogeneous integral equation with eigenvalue k.

The formulation of the group equations as a vector-integral equation immediately suggests a scheme for computing k by iteration. We define the ℓ^{th} iterate of Φ_n by the equation

$$\Phi_n^{(\ell)}(\underline{r}) = \int N\sigma_n\Phi_n^{(\ell-1)}(\underline{r}')\, G_{nl}(\underline{r},\underline{r}')d\underline{r}'. \qquad (7.75)$$

In the limit $\ell \to \infty$,

$$\lim_{\ell \to \infty} \frac{\phi_n^{(\ell-1)}}{\phi_n^{(\ell)}} = k$$

since the iterated functions converge onto the characteristic functions $\phi_n(\underline{r})$ which satisfy (7.74), here k is the critical, k_∞ required.

In order to compute the first iterate ($\ell = 1$), as defined by (7.75), from the group differential equations it is first necessary to set k = 1. Then a distribution $\phi_n^{(o)}(\underline{r})$ is assumed in the first equation, and the remaining equations are successively integrated until a new distribution $\phi_n^{(1)}(\underline{r})$ is found. If the assumed $\phi_n^{(o)}(\underline{r})$ had the shape of the characteristic function, then the value of k would immediately be

$$k = \frac{\phi_n^{(o)}(\underline{r})}{\phi_n^{(1)}(\underline{r})} .$$

It is unlikely that the first guess to $\phi_n^{(o)}$ is correct, and k will therefore usually vary from point to point. An average value of k can be defined as

$$k^{(\ell)} = \frac{\int \phi_n^{(\ell-1)}(\underline{r})d\underline{r}}{\int \phi_n^{(\ell)}(\underline{r})d\underline{r}} \qquad (7.76)$$

the integrations being over the pile. Evidently $k^{(\ell)}$ as defined by (7.76) converges to the true value of k as ℓ approaches ∞.

Numerical Iteration Method

The determination of the iterates $\phi_n^{(\ell)}$ by numerical integration of the group differential equations can be done whenever p, D, k, and $N\sigma$ are known functions of position in one coordinate cases. The details of the numerical integration will be given for the plane case; the extension to other one-coordinate

geometries follows easily. The group equations are written

$$\frac{d}{dx}\left(D_1 \frac{d\Phi_1}{dx}\right) - N\sigma_1\Phi_1 + kN\sigma_n\Phi_n = 0 \qquad (7.77)$$

$$\frac{d}{dx}\left(D_1 \frac{d\Phi_1}{dx}\right) - N\sigma_i\Phi_i + N\sigma_{i-1}\Phi_{i-1} = 0 . \qquad i = 2 \rightarrow n$$

So as to write each of these second order equations as two first order equations we introduce the flow I_i:

$$I_i = D_i \frac{d\Phi_i}{dx} ; \qquad (7.78)$$

then equations (7.77) become

$$\frac{dI_1}{dx} = N\sigma_1\Phi_1 - kN\sigma_n\Phi_n \qquad (7.79)$$

$$\frac{dI_i}{dx} = N\sigma_i\Phi_i - N\sigma_{i-1}\Phi_{i-1} .$$

For the numerical method equations (7.79) are replaced by difference equations. Thus if a small lattice spacing h is used,

$$\frac{d\Phi_i}{dx} \doteq \frac{\Phi_i(x + \frac{1}{2}h) - \Phi_i(x - \frac{1}{2}h)}{h} \qquad (7.80)$$

$$\frac{dI_i}{dx} \doteq \frac{I_i(x + \frac{1}{2}h) - I_i(x - \frac{1}{2}h)}{h}$$

and equations (7.78) and (7.79) can be written

$$I_1(x + h) = I_1(x) + hN\sigma_1\Phi_1(x + \tfrac{1}{2}h) - hkN\sigma_n\Phi_n(x + \tfrac{1}{2}h) \qquad (7.81)$$

$$I_i(x + h) = I_i(x) + hN\sigma_i\Phi_i(x + \tfrac{1}{2}h) - hN\sigma_{i-1}\Phi_{i-1}(x + \tfrac{1}{2}h) \qquad i = 2 \rightarrow n$$

$$\Phi_i(x + \tfrac{1}{2}h) = \Phi_i(x - \tfrac{1}{2}h) + \frac{h}{D_i} I_i(x).$$

Equations (7.81) are inhomogeneous in the sense that i-1st group fluxes appear as "sources" for i^{th} group fluxes. There will be a corresponding Φ_i^p and Φ_i^H. There is consequently a homogeneous solution I_i^H, found by putting $\Phi_{i-1} = 0$ in the i^{th} group equation, and a particular solution, I_i^p, found by keeping the Φ_{i-1} in the i^{th} group equation. The complete solution is I_i^p plus an arbitrary constant α times I_i^H. Similarly there will be a corresponding Φ_i^p and Φ_i^H. Thus

$$I_i^H(x + h) = I_i^H(x) + hN\sigma_i\Phi_i^H(x + \tfrac{1}{2}h)$$

$$I_1^p(x + h) = I_1^p(x) + hN\sigma_1\Phi_1^p(x + \tfrac{1}{2}h) - hN\sigma_n\Phi_n(x + \tfrac{1}{2}h) \qquad (7.82)$$

$$I_i^p = (x + h) = I_i^p(x) + hN\sigma_i\Phi_i^p(x + \tfrac{1}{2}h) - hN\sigma_{i-1}\Phi_{i-1}(x + \tfrac{1}{2}h).$$

We consider a reactor divided into regions in each of which all parameters are constant. To start the iteration, k is set equal to unity in the core and zero in the reflectors and the trial value of Φ_n chosen in the core is unity. From this by equation (7.82) I_1^H, I_1^p, Φ_1^H, Φ_1^p are determined for each lattice point. In crossing boundaries quadratic interpolation is used to obtain the proper value for Φ_1^H, Φ_1^p. At the outermost boundary α can then be chosen to make Φ_1 vanish there. From this the first iterate of Φ_1, $\Phi_1^{(')}$ can be evaluated and used in the same manner to determine $\Phi_2^{(')}$, ...$\Phi_n^{(')}$. The first approximation for k will then be, from equation (7.76),

$$k^1 \doteq \frac{\int \Phi_n^o(x)dx}{\int \Phi_n^1(x)dx}.$$

This first integration is over the multiplying region of the pile. This process can be repeated indefinitely to obtain better approximations for k, the accuracy of which is limited only by the coarseness of the lattice spacing.

To illustrate the method, the example treated in the previous section by the matrix method is solved by numerical iteration below. Table III gives the constants needed for the calculation.

Table III

	core	1st reflector	2nd reflector
h	1.2 cm	2.8 cm	4.0 cm
$hN\sigma_1$	.0229091	.0187660	.044
$hN\sigma_2$	.0933333	---------	-------
$\frac{h}{D_1}$	.052381	4.44444	3.63636

In order to start the iteration, values for $I_1^H(0)$, $I_1^{(p)}(0)$, $\Phi_1^H(\frac{h}{2})$ and $\Phi_1^p(\frac{h}{2})$ are needed in addition to the assumed $\Phi_2 = 1$. Since the pile is symmetric, $I_1^H(0)$ and $I_1^p(0)$ are equated to zero. $\Phi_1^H(\frac{h}{2})$ and $\Phi_1^p(\frac{h}{2})$ are arbitrarily set equal to unity. By means of equation (7.82) at intervals of h/2 either I_1^H and I_1^p or Φ_1^H and Φ_1^p are calculated in the core (Table IV) until $\Phi_1^H(12.6)$ and $\Phi_1^p(12.6)$ are obtained. By quadratic interpolation $\Phi_1^H(12)$ and $\Phi_1^p(12)$ are obtained. In the first reflector (Table V) $\Phi_1^{p,H}(12 + \frac{h}{2})$ must be so chosen that $\Phi_1^{p,H}(12)$ obtained by quadratic interpolation from $\Phi_1^{p,H}(12 - \frac{h}{2})$ and $\Phi_1^{p,H}(12 + \frac{3h}{2})$ (calculated from first reflector constants) is the same as the previously calculated $\Phi_1(12)$. In like manner the second reflector values of $\Phi_1^{p,H}$ and $I_1^{p,H}$ are obtained (Table VI). Then, since the flux vanishes at x = 60,

$$\alpha = \frac{-\phi_1^p(60)}{\phi_1^H(60)} = \frac{1149.322043}{466.547601} = 2.463448$$

and $\phi_1^{(1)}(x) = \phi_1^p(x) + 2.463448\ \phi_1^H(x)$. Thus $\phi_1^{(1)}$ is obtained and the procedure can be repeated to find $\phi^{(2)}$, $\phi^{(3)}$, etc. Figure (III) shows $\phi_1^{(1)}$, $\phi_1^{(2)}$, $\phi_1^{(3)}$ and $\phi_2^{(1)}$ and $\phi_2^{(2)}$. The resulting approximations for k, computed from (7.76), are respectively 1.1224, 1.1169, 1.1132 and 1.1149, which is in good agreement with the value 1.1140 obtained by the matrix method.

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Table IV

(Core Calculations)

x	Φ_1^H	I_1^H	Φ_2(assumed)	Φ_1^p	I_1^p	Φ_1'
0		0 (assumed)	1		0 (assumed)	
.6	1 (assumed)		1	1 (assumed)		3.463448
1.2		.0229091	1		-.0704242	
1.8	1.021818		1	.932929		3.450125
2.4		.0463180	1		-.142385	
3.0	1.065930		1	.797324		3.423188
3.6		.0707375	1		-.217452	
4.2	1.133299		1	.590227		3.382051
4.8		.1012751	1		-.297264	
5.4	1.229751		1	.307118		3.336546
6.0		.129448	1		-.383562	
6.6	1.353035		1	-.0581787		3.274953
7.2		.160445	1		-.478228	
7.8	1.505840		1	-.513634		3.195925
8.4		.194942	1		-.583328	
9.0	1.691499		1	-1.069185		3.097736
9.6		.233693	1		-.701155	
10.2	1.914064		1	-1.736951		2.978247
10.8		.277542	1		-.834280	
11.4	2.178390		1	-2.531503		2.834848
12.0	2.328376	.327447	1	-2.982825	-.985608	2.753009
	2.490244			-3.470177		

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Table V

(First Reflector Calculations)

x	Φ_1^H	I_1^H	Φ_1^p	I_1^p	Φ_1'
12.0	2.328376	.327447	-2.982825	-.985608	2.753009
13.4	3.088232		-5.227563		2.380137
14.8		.385401		-1.083708	
16.2	4.801124		-10.044038		1.783283
17.6		.475499		-1.272194	
19.0	6.914451		-15.698228		1.335165
20.4		.605256		-1.566787	
21.8	9.604475		-22.661719		.998409
23.2		.785494		-1.992057	
24.6	13.095556		-31.515297		.744929
26.0		1.031245		-2.583473	
27.4	17.678863		-42.997388		.553578
28.8		1.363007		-3.390362	
30.2	23.736666		-68.065648		.408403
31.6		1.808449		-4.480022	
33.0	31.774209		-77.976837		.297287
34.4		2.404724		-5.943335	
35.8	42.461861		-104.391633		.210970
37.2		3.201463		-7.902348	
38.6	56.691016		-139.513145		.142247
40.0	65.578730	4.265428	-161.437403	-10.520452	.112413
	75.648535		-186.270663		

Table VI

(Second Reflector Calculations)

x	Φ_1^H	I_1^H	Φ_1^p	I_1^p	Φ_1'
40	65.578730	4.265428	-161.437403	-10.520452	.112413
42	74.830658		-184.250484		.090979
44		7.557977		-18.627473	
46	102.314183		-251.986682		.059027
48		12.059801		-29.714887	
50	146.167961		-360.040708		.036519
52		18.491191		-45.556678	
54	213.408588		-525.701190		.019851
56		27.881169		-68.687530	
58	314.794556		-775.473777		.006363
60	384.375194	41.732129	-946.888450	-102.808376	0
	466.547601		-1149.322043		

The Variation Method

If the pile and the reflector have identical slowing down and transport properties, then the pile equation can be rigorously cast as an integral equation with symmetric kernel. All the usual methods, in particular, the variation method, are then available for computing the critical mass and the neutron distribution. The appropriate integral equation can be written down for any arbitrary slowing down function, and the equation is rigorous if the reflector is infinite in extent. For a finite reflector, an almost rigorous integral equation has been established provided the pile and reflector have sufficient geometric symmetry.

To derive the integral equation for a uniform pile surrounded by an infinitely large reflector having the same slowing down properties as the pile, we begin with the usual pile integro-differential equation. In this case the infinite slowing down kernel is obviously correct since the slowing down medium extends to infinity without change. Thus we have

$$D\Delta\phi_s(\underline{r}) - N_p\sigma_p\phi_s(\underline{r}) + k\int_p N_p\sigma_p\phi_s(\underline{r}')\,P_\infty(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' = 0 \qquad \text{in pile}$$

$$D\Delta\phi_s(\underline{r}) - N_r\sigma_r\phi_s(\underline{r}) + k\int_p N_p\sigma_p\phi_s(\underline{r}')\,P_\infty(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' = 0 \qquad \text{in reflector} \tag{7.83}$$

the integral being taken over the pile, since fissions occur only there. Since the reflector extends to infinity, the asymptotic solution of (7.83), which vanishes periodically, is not acceptable by itself as the neutron distribution, although it will be a good approximation to the solution far from the pile-reflector interface.

The two equations (7.83) can be written as a single equation as follows:

$$D\Delta\phi_s(\underline{r}) - N_r\sigma_r\phi_s(\underline{r}) + N_d\sigma_d(\underline{r})\phi_s(\underline{r}) + k\int_p N_p\sigma_p\phi_s(\underline{r}')\,P_\infty(E_s,|\underline{r}-\underline{r}'|)d\underline{r}' = 0 \tag{7.84}$$

where

$$N_d\sigma_d(\underline{r}) = N_r\sigma_r - N_p\sigma_p \qquad \text{in pile}$$

$$N_d\sigma_d(\underline{r}) = 0 \qquad \text{in reflector.}$$

The last two terms in (7.84) can be viewed formally as space varying source terms in an absorbing system whose reciprocal diffusion length is

$$\kappa_r = \left(\frac{N_r\sigma_r}{D}\right)^{\frac{1}{2}}.$$

The integral term represents the source arising from neutrons which slow down. The term with $N_d\sigma_d$, represents a sink which takes account of the fact that the pile is a heavier absorber of thermals than is the reflector.

The point Green's function for the diffusion equation is (Table I , Chapter I)

$$G(|\underline{r}-\underline{r}'|) = \frac{e^{-\mathcal{K}_r|\underline{r}-\underline{r}'|}}{4\pi D|\underline{r}-\underline{r}'|} ;$$

hence the solution of (7.84) can be written

$$\bar{\phi}_s(\underline{r}) = (N_r\sigma_r - N_p\sigma_p)\int_p \bar{\phi}_s(\underline{r}')G(|\underline{r}-\underline{r}'|)d\underline{r}' + kN_p\sigma_p\int_\infty\int_p \bar{\phi}(\underline{r}')P_\infty(E_s,|\underline{r}''-\underline{r}'|)G(|\underline{r}-\underline{r}''|)d\underline{r}'d\underline{r}''. \tag{7.85}$$

The order of integration in the second integral can be interchanged, and the resulting infinite integral is, in principle, known. We denote this function by $H(\underline{r},\underline{r}')$:

$$H(\underline{r},\underline{r}') = \int_\infty P_\infty(E_s,|\underline{r}''-\underline{r}'|)G(|\underline{r}-\underline{r}''|)d\underline{r}''. \tag{7.86}$$

Since $H(\underline{r},\underline{r}')$ is the convolution of two displacement type kernels, it is symmetric in $\underline{r}$ and $\underline{r}'$.

Upon introduction of $H(\underline{r},\underline{r}')$ into (7.85) we can write

$$\frac{\bar{\phi}_s(\underline{r})}{N_p\sigma_p} = \frac{N_r\sigma_r - N_p\sigma_p}{N_p\sigma_p}\int_p \bar{\phi}_s(\underline{r}')\,G(|\underline{r}-\underline{r}'|)d\underline{r}' + k\int_p \bar{\phi}(\underline{r}')\,H(\underline{r},\underline{r}')d\underline{r}' \tag{7.87}$$

i.e., $\bar{\phi}_s(\underline{r})$ satisfies an integral equation with symmetric kernel. It should be noted that (7.87) defines $\bar{\phi}_s(\underline{r})$ both inside and outside the pile, even though the integrals extend only over the pile.

To compute the critical multiplication constant k, we multiply (7.87) by $\bar{\phi}_s(\underline{r})$ and integrate over the pile. Thus we obtain

$$k = \frac{\dfrac{1}{N_p\sigma_p}\int_P \Phi_s^2(\underline{r})d\underline{r} - \dfrac{N_r\sigma_r - N_p\sigma_p}{N_p\sigma_p}\iint_P \Phi_s(\underline{r}')\, G(|\underline{r}-\underline{r}'|)\, \Phi_s(\underline{r})d\underline{r}d\underline{r}'}{\iint_P \Phi_s(\underline{r}')\, H(\underline{r},\underline{r}')\, \Phi_s(\underline{r})d\underline{r}d\underline{r}'} . \qquad (7.88)$$

It is easy to show that k, considered as a functional of $\Phi_s(\underline{r})$ according to (7.88), is stationary with respect to small deviations of $\Phi_s(\underline{r})$ from the function which satisfies the integral equation (7.87). To demonstrate this we form the differential quotient

$$\delta k = \lim_{\varepsilon \to 0} \frac{k\left[\Phi_s(\underline{r}) + \varepsilon\eta(\underline{r})\right] - k\left[\Phi_s(\underline{r})\right]}{\varepsilon} \qquad (7.89)$$

where $\eta(\underline{r})$ is an arbitrary quadratically integrable function over P. A straightforward substitution of $\Phi_s + \varepsilon\eta$ into (7.89) in which use is made of the symmetry of the kernels $G(|\underline{r}-\underline{r}'|)$ and $H(\underline{r},\underline{r}')$ yields

$$\delta k = \frac{\int_P \eta(\underline{r})d\underline{r}\left[\int_P kH(\underline{r},\underline{r}')\, \Phi_s(\underline{r}')d\underline{r}' - \dfrac{\Phi_s(\underline{r})}{N_p\sigma_p} + \int_P \dfrac{N_r\sigma_r - N_p\sigma_p}{N_p\sigma_p}\, G(|\underline{r}-\underline{r}'|)\Phi_s(\underline{r}')d\underline{r}'\right]}{-\frac{1}{2}\iint_P \Phi_s(\underline{r})\, H(\underline{r},\underline{r}')\, \Phi_s(\underline{r}')d\underline{r}d\underline{r}'} . \qquad (7.90)$$

If $\Phi_s(\underline{r}')$ satisfies the pile equation (7.87), then, from (7.90), $\delta k = 0$ for any arbitrary variation function $\eta(\underline{r})$; i.e., the value of k as computed from (7.88) is stationary when the actual neutron distribution is used in (7.88). Equation (7.88) therefore constitutes a variation principle from which fairly good approximations to k can be computed by substituting convenient, but possibly incorrect distributions $\Phi_s(\underline{r})$. Better approximations to both the characteristic value and to the distribution can be obtained by assuming for $\Phi_s(\underline{r})$ a linear combination of simple quadratically integrable functions and determining the

coefficients of this combination so as to make (7.88) stationary with respect to variations of the coefficients. Such a procedure is merely the Ritz variation method.

It is, of course, always possible to iterate an assumed distribution to get a better approximation -- i.e., to substitute the first approximation into the integrals on the right side of (7.87), obtain a second approximation, and then repeat. At the n^{th} stage in the approximation the value of k appropriate for the (n + 1st) approximation is to be computed from the variation principle with the n^{th} iterate used for $\Phi_s(\underline{r})$.

The simplest function to use as a variation function is $\Phi_s(\underline{r}) = 1$. To this approximation the critical multiplication constant k is related to the pile dimensions by the equation

$$k N_p \sigma_p = \frac{V_p - (N_r \sigma_r - N_p \sigma_p) \iint_P G(|\underline{r}-\underline{r}'|) d\underline{r} d\underline{r}'}{\iint_P H(\underline{r},\underline{r}') d\underline{r} d\underline{r}'} \tag{7.91}$$

where V_p is the volume of the pile.

To illustrate the use of this simple critical equation we compute the critical size of a reflected slab for a one-group model both by use of (7.91) and by the exact equations. In the one-group model all neutrons are supposed to have the same energy -- that is, no slowing down occurs and therefore

$$P_\infty(E_s, |\underline{r}-\underline{r}'|) = \delta(|\underline{r}-\underline{r}'|). \tag{7.92}$$

Since the geometry is a slab, the plane kernel is appropriate. From the usual relation between point and plane kernel we have

$$G_{p\ell}(|x-x'|) = \frac{\mu_r}{2 N_r \sigma_r} e^{-\mu_r |x-x'|} \tag{7.93}$$

and

$$H_{p\ell}(x,x') = \int_{-\infty}^{\infty} P_{\infty\, p\ell}(E_s, |x+x''|)G_{p\ell}(|x''-x'|)dx'' = G_{p\ell}(|x+x'|). \qquad (7.94)$$

We denote the thickness of the slab by $2R_c$. Then

$$\iint_P \Phi_s(\underline{r})G(|\underline{r}-\underline{r}'|)\Phi_s(\underline{r}')d\underline{r}d\underline{r}' = \int_{-R_c}^{R_c}\int_{-R_c}^{R_c} G_{p\ell}(|x-x'|)dxdx'$$

$$= \frac{2}{\mathcal{H}_r N_r \sigma_r}\left[\mathcal{H}_r R_c - e^{-\mathcal{H}_r R_c}\sinh \mathcal{H}_r R_c\right] \qquad (7.95)$$

$$= \iint_P \Phi_s(\underline{r})H(\underline{r},\underline{r}')\Phi_s(\underline{r}')d\underline{r}d\underline{r}' .$$

Substituting these expressions into the critical condition (7.91) we obtain the following equation for R_c:

$$kN_p\sigma_p - N_p\sigma_p + N_r\sigma_r = \frac{R_c N_r \sigma_r}{R_c - \frac{e^{-\mathcal{H}_r R_c}}{\mathcal{H}_r}\sinh \mathcal{H}_r R_c} . \qquad (7.96)$$

If the slab thickness is small compared to the reflector diffusion length, $\mathcal{H}_r R_c \ll 1$; then (7.96) reduces to the following simple result:

$$R_c = \frac{\mathcal{H}_r}{B^2 + \mathcal{H}_r^2} \qquad (7.97)$$

where $B^2 = \frac{k-1}{L_p^2} = (k-1)\mathcal{H}_p^2$.

Since $\mathcal{H}_r R_c \ll 1$ implies $\mathcal{H}_r^2 \ll B^2 + \mathcal{H}_r^2$, i.e., $\mathcal{H}_r^2/B^2 \ll 1$, this result

can be further amplified to

$$R_c \approx \frac{\lambda_r}{B^2} \quad . \tag{7.98}$$

The assumption that the neutron distribution is flat evidently is most nearly correct when the critical thickness is small compared to 1/B because, in this case, the active portion of the reactor covers only a small part of the unreflected critical dimension. It is therefore to be expected that (7.98) is an accurate formula for a one-group reflected core which is small compared to the unreflected pile.

This is easily demonstrated by computing R_c exactly. In the core, (again dropping subscript s),

$$\phi_p(x) = \cos Bx \tag{7.99}$$

and in the reflector

$$\phi_r(x) = \alpha e^{-\lambda_r |x|} \quad . \tag{7.100}$$

Equating flux and current at the boundary R_c we obtain

$$B \tan BR_c = \lambda_r . \tag{7.101}$$

If the flux in the core is nearly constant, $BR_c \ll 1$, as can be seen from the fact that the core flux is a cosine. Replacing $\tan BR_c$ by BR_c, we find

$$R_c \approx \frac{\lambda_r}{B^2} \tag{7.102}$$

which is the same as the variational result.

The iterated neutron distribution which results from the use of a constant trial function is

$$\phi_s(x) = (N_r\sigma_r - N_p\sigma_p + kN_p\sigma_p) \int_{-R_c}^{+R_c} \frac{\lambda_r}{2N_r\sigma_r} e^{-\lambda_r |x-x'|} dx' \tag{7.103}$$

$$= \left[1 + (k-1)\frac{N_p\sigma_p}{N_r\sigma_r}\right]\left[1 - e^{-\lambda_r R_c}\cosh \lambda_r x\right] \qquad \text{inside core}$$

$$= \left[1 + (k-1)\frac{N_p\sigma_p}{N_r\sigma_r}\right]\left[\sinh \lambda_r R_c\right] e^{-\lambda_r |x|} \qquad \text{outside core}$$

The exact distribution and first iterate as given by (7.103) are compared in Figure 7-IV.

A better value for the critical size and neutron distribution can be obtained by setting

$$\phi_s(x) = 1 - \beta x^2 ,$$

substituting into the variation principle (7.98), and then choosing β so that $k(\beta)$ is stationary. The details of the computation are rather tedious and will not be reproduced here.

Extrapolation Distance by Two-Group Method*

In calculating the critical size of an unreflected pile it is customary to assume that the neutron density extrapolates to zero at a distance $0.71\,\lambda_{tr}$. In general λ_{tr} is a function of energy so that the effective boundary is different for each neutron energy. It is evidently useful to compute a suitable average extrapolation distance which, when used with the usual one-group equation, will give the correct critical mass for the actual unreflected pile. Such a computation is of greatest interest for a water moderated pile since the mean free path in water is very strongly energy dependent, and therefore the average extrapolation distance will differ significantly from either the fast or the slow lengths.

The general solution of this problem is essentially out of the question; however, by two-group theory it is possible to compute an approximate average extrapolation distance.

*cf. B. I. Spinrad, MonP-284.

We consider an infinite bare slab pile of actual height 2h, in which there is no resonance capture; we seek the equivalent extrapolated height 2H -- i.e., the extrapolated height of a pile which, by one-group theory, has the same critical mass as would be calculated by two-group theory for a pile of actual height 2h. It is assumed that $h \approx H \gg \lambda_{tr}$ and that k is close to unity.

In the equivalent one-group pile, the fast neutron flux is

$$\Phi(x) = \cos \frac{\pi x}{2H} \tag{7.104a}$$

while the slow neutron flux is, according to (7.34b)

$$\Phi_2(x) = A_1 \cos \frac{\pi x}{2H} \tag{7.104b}$$

where A_1 is approximately σ_1/σ_2 since $H \gg \frac{1}{N\sigma}$.

Actually the asymptotic distribution (7.104) does not hold to the pile edge because the extrapolation distance is different for fast and slow neutrons. The complete solution of (7.35) is

$$\left.\begin{aligned} \Phi_1(x) &= \cos \frac{\pi x}{2H} + b \cosh B_2' x \\ \Phi_2(x) &= A_1 \cos \frac{\pi x}{2H} + A_2\, b \cosh B_2' x \end{aligned}\right\} \tag{7.105}$$

where $A_2 \approx -\frac{L_1^2}{L_2^2}\frac{\sigma_1}{\sigma_2}$, $B_2' \approx \left[\frac{1}{L_1^2} + \frac{1}{L_2^2} + \frac{\pi^2}{(2H)^2}\right]^{\frac{1}{2}}$, and b is an arbitrary constant. The boundary conditions at x = h are

$$\left.\begin{aligned} \left.\frac{\Phi_1}{\Phi_1'}\right|_h &= -0.71\, \lambda_{tr\,1} \\ \left.\frac{\Phi_2}{\Phi_2'}\right|_h &= -0.71\, \lambda_{tr\,2} \end{aligned}\right\} \tag{7.106}$$

where $\lambda_{tr\ 1}$ and $\lambda_{tr\ 2}$ are respectively the fast and slow transport mean free paths. Upon substituting (7.105) into (7.106) there result two different linear equations for b. In order for these equations to be compatible, the following determinant must vanish:

$$\begin{vmatrix} \cos\frac{\pi h}{2H} - 0.71\,\lambda_{tr\ 1}\left(\frac{\pi}{2H}\sin\frac{\pi h}{2H}\right) & \cosh B_2' h + .71\,\lambda_{tr\ 1} B_2' \sinh B_2' h \\ A_1\left[\cos\frac{\pi h}{2H} - 0.71\,\lambda_{tr\ 2}\left(\frac{\pi}{2H}\sin\frac{\pi h}{2H}\right)\right] & A_2\left[\cosh B_2' h + .71\,\lambda_{tr\ 2} B_2' \sinh B_2' h\right] \end{vmatrix} = 0 \qquad (7.107)$$

We assume $B_2' h$ is sufficiently large so that $\sinh B_2' h \approx \cosh B_2' h \approx \frac{e^{B_2' h}}{2}$; and that $\frac{\pi}{2}\left(\frac{H-h}{h}\right)$ is small. Then replacing sines and cosines by the first term in their expansions, we find that the determinant reduces to

$$A_2(1 + 0.71\,B_2'\lambda_{tr\ 2})(H-h - 0.71\,\lambda_{tr\ 1}) = A_1(1 + 0.71\,B_2'\lambda_{tr\ 1})(H-h - 0.71\,\lambda_{tr\ 2})$$

which can be solved for H-h:

$$H-h \approx \frac{L_2^2(1 + 0.71\,B_2'\lambda_{tr\ 1})(0.71\,\lambda_{tr\ 2}) + L_1^2(1 + 0.71\,B_2'\lambda_{tr\ 2})(0.71\,\lambda_{tr\ 1})}{L_2^2(1 + 0.71\,B_2'\lambda_{tr\ 1}) + L_1^2(1 + 0.71\,B_2'\lambda_{tr'2})} \cdot \qquad (7.108)$$

If $L_1^2 \gg L_2^2$, as is the case in a water moderated reactor, then

$$H-h \approx 0.71\left[\lambda_{tr\ 1} + \frac{L_2^2}{L_1^2}\,\frac{\left(1 + \frac{0.71\,\lambda_{tr\ 1}}{L_2}\right)}{\left(1 + \frac{0.71\,\lambda_{tr\ 2}}{L_2}\right)}(\lambda_{tr\ 2} - \lambda_{tr\ 1})\right], \qquad (7.109)$$

provided $\left(\frac{2H}{\pi}\right)^2$ is also much greater than L_2^2. Since the pile is large compared to the transport mean free path, H-h can be identified with the extrapolation distance. According to (7.108), the extrapolation distance always lies between $0.71\,\lambda_{tr\ 1}$ and $0.71\,\lambda_{tr\ 2}$.

The results of this section can be summarized in the following remarks. If the fast and slow extrapolation distances are different, then the neutron distribution includes a non-asymptotic as well as an asymptotic part. The asymptotic distribution extrapolates to zero at a distance beyond the pile boundary which is intermediate between the fast and slow extrapolation distances, the exact distance being given by (7.108). Since the asymptotic distribution is the solution of the one-group pile equation, the critical size of a bare pile can therefore be determined by adding to its dimension the extrapolation distance (7.108), and treating the problem by one-group theory.

General Kernel in a System with Finite Reflector

Two general methods, aside from the group method, are available for determining the critical conditions in a reactor with finite reflector which has the same slowing down kernel as the reactor. The slowing down kernel can be completely general; the only requirement is that the extrapolation distance be independent of energy.

The first method[1], which is applicable to slabs or spheres but not to cylinders, is called the method of images and converges best when the reflector is large compared with a reflector migration length. The second method[2], which is applicable in principle to piles of arbitrary shape, is called the method of harmonics and converges best when the reflector is small compared to a reflector migration length. The relation between the two methods is rather analogous to the relation between the source-wise and the characteristic function representations of the solution of the heat conduction equation in a finite system.

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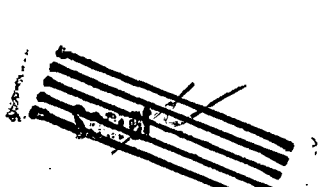
1 See H. L. Garabedian and A. M. Weinberg, MonP-434; H. L. Garabedian, MonP-435.

2 See G. Goertzel and H. L. Garabedian, ORNL-30.



The Method of Images

We consider an infinite slab pile with reflectors of equal thickness on each side. Since the pile reflector is finite, the slowing down kernel for the system of pile and reflector is symmetric but it is not of displacement character. This circumstance arises because the probability that a neutron born at x arrives at x' depends not only on the distance between x and x' but also on the distance of each of these points from the extrapolated boundary. However, at least in the case of a slab pile surrounded by a finite reflector it is possible to write the pile equation in such a way that the displacement character of the kernel is restored, and in fact, that the kernel in the finite system is expressed as an explicit superposition of infinite system kernels. The finite problem is thus reduced to an equivalent infinite one, and all the techniques -- in particular the variation method -- which were used for the infinite problem become available for this one.

The device by which the finite kernel is expressed as a sum of infinite system kernels is analogous to the one used in the method of images in potential theory. The neutron density in a finite slab pile with a finite reflector must satisfy the pile equation and vanish on the extrapolated boundary of the reflector. Such a solution can be viewed as arising from an infinite sequence of identical piles and reflectors set side by side. That periodic solution of the pile equation in such an infinite array which oscillates with wave length equal to the thickness of pile reflector, and in which the neutron density alternates in sign in each successive pile and reflector interval, is evidently an analytic continuation of the actual solution in the finite system. Thus the problem in the finite system can be replaced by the problem of seeking periodic solutions in a periodic structure which extends from $-\infty$ to $+\infty$. It is this extension of the finite system to include all space which makes it possible to express the kernel as a superposition of infinite system displacement kernels.

The alternating positive and negative neutron densities serve as sources and sinks which, from symmetry, must yield zero neutron density on the reflector boundary (Fig. IV). When the reflector thickness vanishes, the proper periodic

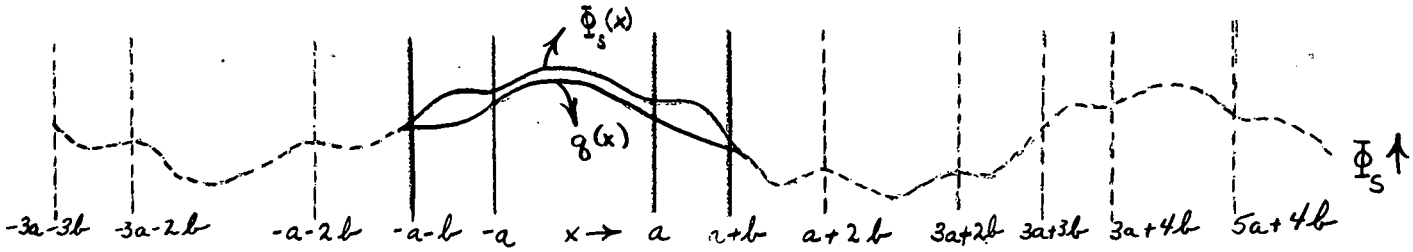


Figure 7-V

solution is cos Bx which is also the asymptotic solution. Then, as shown in Chapter III , the slowing down density is everywhere proportional to the thermal neutron density. In the case of the reflected pile, the correct periodic solution is not in general a simple cosine function and hence the proportionality between slowing down density and slow neutron density is destroyed -- that is, the asymptotic solution does not persist throughout the system. However, because of the high symmetry of the plane problem it is still possible to construct the image system which leads to a solution satisfying the boundary conditions.

The decomposition of the solution into a superposition of images converges well only if the successive image piles are far apart. This is the case if the reflector is thick compared to the reflector migration length, for in this event the neutrons from neighboring image piles are "insulated" from each other. When the reflector is thin, many images are required to represent the correct solution, and in this case it is to be expected that the method will become unwieldy.

Since, from symmetry considerations, the infinite sequence of piles and reflectors described in Fig. V is equivalent to a single pile with reflector, provided $\Phi_s(x)$ has the periodicity shown in Fig. V , it is a straightforward

matter to write down the thermal slowing down density $q(E_s,x)$. Suppose $\Phi_s(x)$ is the actual slow neutron density in the reactor and reflector, and its analytic continuation to infinity. The reactor is assumed to extend from -a to +a, the reflector being of thickness b. The function $\Phi_s(x)$ will have the general shape shown in Fig. V : it is periodic and goes through zero at the reflector edge. The integral

$$q(E_s,x) = \int_{-\infty}^{\infty} K(x')\, N_p\sigma_p\, \Phi_s(x')\, P_\infty(E_s,|x-x'|)dx' ,$$

where $K(x') = k$ inside the reactor or any of its images, and $K = 0$ in the reflector or any of its images, vanishes on the edge of the reflector by symmetry and is therefore the slowing down density for thermal neutrons.

The pile equation, accordingly, can be written

$$D\Delta\Phi_s(x) - N_r\sigma_r\Phi_x(x) + N_d\sigma_d(x)\,\Phi_s(x) + \int_{-\infty}^{\infty} K(x')N_p\sigma_p\Phi_s(x')\,P_\infty(E_s,|x-x'|)dx' = 0 \qquad (7.110)$$

where $N_r\sigma_r$ and $N_d\sigma_d$ are defined as in equation (7.84). We seek a periodic function $\Phi_s(x)$ which is alternately positive and negative in neighboring "image" piles, i.e., $\Phi_s(x)$ must satisfy

$$\Phi_s(x + i\delta) = (-1)^i\Phi_s(x) \qquad (7.111)$$

where $\delta = 2(a + b)$ is the total thickness of the reactor plus the two reflectors. Then clearly

$$\int_{-\infty}^{\infty} K(x')\, N_p\sigma_p\Phi_s(x')\, P_\infty(E_s,|x-x'|)dx' = kN_p\sigma_p\int_{-a}^{a}\Phi_s(x')\sum_{i=-\infty}^{\infty}(-1)^i P_\infty(E_s,|x+i\delta-x'|)dx' \quad ; \qquad (7.112)$$

i.e., the appropriate finite slowing down kernel, $P(E_s,x,x')$ is

$$P(E_s,x,x') = \sum_{i=-\infty}^{\infty} (-)^i P_\infty (E_s,|x + i\delta - x'|). \qquad (7.113)$$

Hence the pile equation becomes

$$D\Delta\Phi_s(x) - N_r\sigma_r\Phi_s(x) + N_d\sigma_d(x)\,\Phi_s(x) + kN_p\sigma_p\int_{-a}^{a}\Phi_s(x')\,P(E_s,x,x')dx' = 0; \qquad (7.114)$$

this equation resembles (7.84) and can be treated similarly. We first convert (7.114) to an equivalent integral equation by introducing the Green's function

$$G(|x-x'|) = \frac{\mu_r}{2N_r\sigma_r}\,e^{-\mu_r|x-x'|},$$

so that

$$\Phi_s(x) = \int_{-\infty}^{\infty} N_d\sigma_d(x')\,\Phi_s(x')\,G(|x-x'|)dx' + kN_p\sigma_p\int_{-\infty}^{\infty}\int_{-a}^{a}\Phi_s(x')\,P(E_s,x'',x')\,G(|x''-x'|)dx'dx''. \qquad (7.115)$$

Since $N_d\sigma_d = N_r\sigma_r - N_p\sigma_p$ in pile

$= 0$ in reflector,

and since $\Phi_s(x)$ has the periodicity implied by (7.111),

$$\int_{-\infty}^{\infty} N_d\sigma_d(x')\,\Phi_s(x')\,G(|x-x'|)dx' = (N_r\sigma_r - N_p\sigma_p)\int_{-a}^{a}\Phi_s(x')G(x,x')dx' \qquad (7.116)$$

where

$$G(x,x') = \sum_{i=-\infty}^{\infty}(-1)^i\,G(|x + i\delta - x'|).$$

Hence the integral equation is

$$\frac{\Phi_s(x)}{N_p\sigma_p} = \frac{N_r\sigma_r - N_p\sigma_p}{N_p\sigma_p}\int_{-a}^{a}\Phi_s(x')\,G(x,x')dx' + k\int_{-a}^{a}\Phi_s(x')\,H(x,x')dx' \qquad (7.117)$$

where

$$H(x,x') = \int_{-\infty}^{\infty} P(E_s,x'',x')\, G(|x''-x|)dx'' =$$

$$= \sum_{i=-\infty}^{\infty} (-1)^i \int_{-\infty}^{\infty} P_\infty (E_s, |x' + i\delta - x''|)\, G(|x'' - x'|)dx'' .$$

The critical value of k is again determined by

$$k = \frac{\frac{1}{N_p\sigma_p}\int_{-a}^{a} \Phi_s^2(x)dx - \frac{N_r\sigma_r - N_p\sigma_p}{N_p\sigma_p}\int_{-a}^{a}\int \Phi_s(x')\, G(x,x')\, \Phi_s(x)dxdx'}{\int_{-a}^{a}\int \Phi_s(x')\, H(x',x)\, \Phi_s(x)dx'dx} \qquad (7.118)$$

and k, as determined from (7.118), is stationary when $\Phi_s(x)$ satisfies the integral equation (7.117).

The Harmonics Method*

The image method does not converge very well when the reflector is so small (i.e., small compared to a migration area) that the "image piles" affect the neutron distribution sensibly. In this event there is available a scheme, called the method of harmonics, which consists essentially of expanding the neutron distribution in an orthogonal set of functions. The orthogonal functions are characteristic functions for a uniform pile with the same dimensions as the system pile and reflector. The method bears a close resemblance to the harmonics method of solving the multi-group equations. It is less general than the group method in that the slowing down properties must be independent of position; it is more general in that the slowing down kernel can be arbitrary.

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* G. Goertzel and H. L. Garabedian, ORNL-30; also, H. C. Schweinler, MonP-152.

To describe the method, we consider the pile equation

$$D\Delta\Phi_s(\underline{r}) - N\sigma_a\Phi_s(\underline{r}) + q(E_s,\underline{r}) = 0, \tag{7.119}$$

$$q(E,\underline{r}) = \int_{\text{pile}} kN\sigma_a\Phi_s(\underline{r}')\, P(E,\underline{r},\underline{r}')d\underline{r}' \tag{7.120}$$

where $P(E,\underline{r},\underline{r}')$ is the finite slowing down kernel. The absorption cross-section $N\sigma_a$ is considered to be a function of space: thus a uniform pile surrounded by a reflector is a special case of (7.119).

The function $kN\sigma_a\Phi_s$ is first represented as a series of orthonormal functions $Z_i(\underline{r})$. The $Z_i(\underline{r})$ are defined to be normalized solutions of the equation

$$\Delta Z_i + B_i^2 Z_i = 0$$

which vanish on the extrapolated outer boundary of the reflector and are regular inside the pile.

The function $\Phi_s(\underline{r})$ can be expanded in a series of $Z_i(\underline{r})$:

$$\Phi_s(\underline{r}) = \sum_i C_i Z_i(\underline{r}) \tag{7.121}$$

where

$$C_i = \int \Phi_s(\underline{r})\, Z_i(\underline{r})d\underline{r}\,, \tag{7.122}$$

the integration extending to the outer edge of the system. Now if we put

$$N\sigma_a\Phi_s(\underline{r}) = \sum_i E_i Z_i(\underline{r}) \tag{7.123a}$$

$$kN\sigma_a\Phi_s(\underline{r}) = \sum_i A_i Z_i(\underline{r}), \tag{7.123b}$$

then, upon comparing (7.122) and (7.123) we easily obtain

$$N\sigma_a\Phi_s(\underline{r}) = \sum_i \sum_j C_i \eta_{ij} Z_j(\underline{r}) \tag{7.124a}$$

$$kN\sigma_a\Phi_s(\underline{r}) = \sum_i \sum_j C_i \xi_{ij} Z_j(\underline{r}) \tag{7.124b}$$

where

$$\eta_{ij} = \int N\sigma_a Z_i(\underline{r})\, Z_j(\underline{r})d\underline{r} \qquad (7.124c)$$

$$\xi_{ij} = \int kN\sigma_a Z_i(\underline{r})\, Z_j(\underline{r})d\underline{r} \,. \qquad (7.124d)$$

Equation (7.124b) defines the analytic continuation of $kN\sigma_a\Phi_s(\underline{r})$ over all space. Hence the slowing down density can be written

$$q(E,\underline{r}) = \int_{\text{pile}} kN\sigma_a\Phi_s(\underline{r}')\, P(E,\underline{r},\underline{r}')d\underline{r}' = \int_{\substack{\text{all}\\ \text{space}}} kN\sigma_a\Phi_s(\underline{r}')\, P_\infty(E,|\underline{r}-\underline{r}'|)d\underline{r}' \qquad (7.125)$$

where $P_\infty(E,|\underline{r}-\underline{r}'|)$ is the slowing down kernel in the infinite system. By the fundamental theorem of pile theory,

$$\int_{\substack{\text{all}\\ \text{space}}} Z_i(r')\, P_\infty(E,|r-r'|)\, dr' = \bar{P}_\infty(E,B_i^2)\, Z_i(\underline{r}) \qquad (7.126)$$

where $\bar{P}_\infty$ denotes the three dimensional Fourier transform of P_∞,

$$\bar{P}_\infty(E,B_j^2) = 4\pi \int_0^\infty P_\infty(E,r)\, \frac{\sin B_j r}{B_j r}\, r^2 dr. \qquad (7.127)$$

Substituting (7.124b) into (7.125) and using (7.126), we obtain

$$q(E_s,\underline{r}) = \sum_i \sum_j C_i \xi_{ij}\, \bar{P}_\infty(E_s,B_j^2)\, Z_j(\underline{r}). \qquad (7.128)$$

Substituting (7.128), (7.124a) and (7.121) into the pile equation (7.119) we obtain

$$-D\sum_i B_i^2 C_i Z_i(\underline{r}) - \sum_i \sum_j C_i \eta_{ij} Z_j(\underline{r}) + \sum_i \sum_j C_i \xi_{ij} \bar{P}_\infty(E_s,B_j^2) Z_j(\underline{r}) = 0. \qquad (7.129)$$

Since the $Z_i(\underline{r})$ are orthonormal, equation (7.129) can hold only if the coefficients of each Z_i vanish. This leads to

$$-DB_i^2 C_i \delta_{ij} - \sum_i C_i \left[\eta_{ij} - \xi_{ij} \bar{P}_\infty (E_s, B_j^2)\right] = 0 \qquad (7.130)$$

which is an infinite set of homogeneous linear equations in C_i. Equation (7.130) has a non-trivial solution if and only if the determinant of coefficients vanishes; i.e.,

$$\left| DB_i^2 \delta_{ij} + \eta_{ij} - \xi_{ij} \bar{P}_\infty (E_s, B_j^2) \right| = 0. \qquad (7.131)$$

Since ξ_{ij} depends on k according to (7.124d), equation (7.131) may be viewed as the critical equation; for a pre-assigned $N\sigma_a$ and pile radius, for example, it determines the critical value of k.

The Modified Pile Equation

In the first section of this chapter it was shown that, in a large, uniform reactor the asymptotic slow neutron flux satisfies the one-group pile equation

$$\Delta \bar{\Phi}_s(\underline{r}) + \frac{k-1}{M^2} \bar{\Phi}_s(\underline{r}) = 0. \qquad (7.132)$$

We shall now show that an equation of similar form holds asymptotically in a large reactor in which the absorption cross-section is a function of position but the slow diffusion coefficient, i.e., the transport mean free path, is independent of position, and the multiplication constant is everywhere close to unity. A large reactor in which the concentration of fissionable material is variable is an example of such a system.

The asymptotic pile equation can be written

$$D\Delta \bar{\Phi}_s(\underline{r}) - N\sigma_a(\underline{r}) \bar{\Phi}_s(\underline{r}) + \int \frac{k}{p}(\underline{r}') N\sigma_a(\underline{r}') \bar{\Phi}_s(\underline{r}') P_\infty (E_s, |\underline{r}-\underline{r}'|) d\underline{r}' = 0. \qquad (7.133)$$

If $k(\underline{r})$ is everywhere close to unity, it is permissible to expand the quantity $kN\sigma_a\Phi_s(\underline{r}')$ around the point $\underline{r}$ in Taylor's series and keep only the first two terms, since, as will be seen presently, the curvature of $kN\sigma_a\Phi_s(\underline{r})$ at any point is less than $\frac{k-1}{M^2}$ and is therefore small when k is close to one. Expanding in Taylor's series we have, as in (7.1),

$$kN\sigma_a\Phi_s(\underline{r}') = kN\sigma_a\Phi_s(\underline{r}) + (\underline{r}'-\underline{r}) \cdot \nabla kN\sigma_a\Phi_s(\underline{r}) + \frac{1}{2}\left[(\underline{r}'-\underline{r}) \cdot \nabla\right]^2 kN\sigma_a\Phi_s(\underline{r}) + \ldots\ldots \quad (7.134)$$

Upon substituting into the pile equation we obtain

$$D\Delta\Phi_s(\underline{r}) - N\sigma_a(\underline{r})\,\Phi_s(\underline{r}) + k(\underline{r})N\sigma_a(\underline{r})\Phi_s(\underline{r}) + \Delta k(\underline{r})\,\frac{\overline{r^2}}{6}\,(E_s)N\sigma_a(\underline{r})\Phi_s(\underline{r}) = 0. \quad (7.135)$$

Since $D = L^2(\underline{r})N\sigma_a(\underline{r})$ is assumed to be independent of position, it can be introduced under the Δ operator. The resulting equation is

$$\Delta\left[L^2N\sigma_a + k\,\frac{\overline{r^2}}{6}\,(E_s)N\sigma_a\right]\Phi_s(\underline{r}) + (k-1)N\sigma_a\Phi_s(\underline{r}) = 0 \quad (7.136)$$

which, upon putting

$$\nu(\underline{r}) = M^2(\underline{r})N\sigma_a(\underline{r})\Phi_s(\underline{r}) \quad (7.137)$$

where $M^2 = L^2 + \frac{k\,\overline{r^2}}{6}\,(E_s)$, becomes

$$\Delta\nu(\underline{r}) + \left[\frac{k(\underline{r}) - 1}{M^2(\underline{r})}\right]\nu(\underline{r}) = 0. \quad (7.138)$$

The quantity $\nu(\underline{r})$ is seen to satisfy the one-group pile equation in a large reactor with a variable distribution of fissionable material. According to (7.138) the buckling of $\nu(\underline{r})$ is $(k-1)/M^2$; this justifies the expansion in Taylor's series when $k - 1 \ll 1$.

It must be emphasized that the simple one-group equation is satisfied by $\nu(\underline{r})$ only asymptotically -- i.e., far from a boundary, since otherwise $P_{\infty}(E_s, |\underline{r}-\underline{r}'|)$ must be replaced by the finite slowing down kernel. Also, the equation is satisfied only in regions where the curvature of the neutron density is small, since otherwise it is not correct to cut off the Taylor's expansion at two terms. In a heterogeneous lattice where the slow neutron density shows marked fluctuations, the details of these fluctuations cannot be computed from (7.138) since (7.138) was derived on the assumption that the quantity $kN\sigma_a\Phi_s(\underline{r})$ varies slowly over the pile.

The modified pile equation is useful in computing the critical mass and asymptotic neutron distribution in a system in which the disposition of fissionable material changes, but rather slowly, over the pile. Thus in a water moderated thermal neutron reactor, $\frac{\overline{r^2}}{6} \gg L^2$. If the fissionable and other absorbing material is disposed so that $k(\underline{r}) = k$, a constant, then equation (7.138) reduces to

$$\Delta\nu(\underline{r}) + \frac{k-1}{\overline{r^2}/6}\,\nu(\underline{r}) \approx 0; \qquad (7.139)$$

i.e., in a slab geometry

$$\nu(x) = A\cos Bx$$

where $B^2 = \frac{k-1}{\overline{r^2}/6}$ and A is a constant. Hence

$$\Phi_s(x) = \frac{\nu(x)}{M^2(x)N\sigma_a(x)} \approx \frac{A\cos BX}{\frac{\overline{r^2}}{6}N\sigma_a(x)} \qquad (7.140)$$

i.e., the slow neutron density is inversely proportional to the capture cross-section.

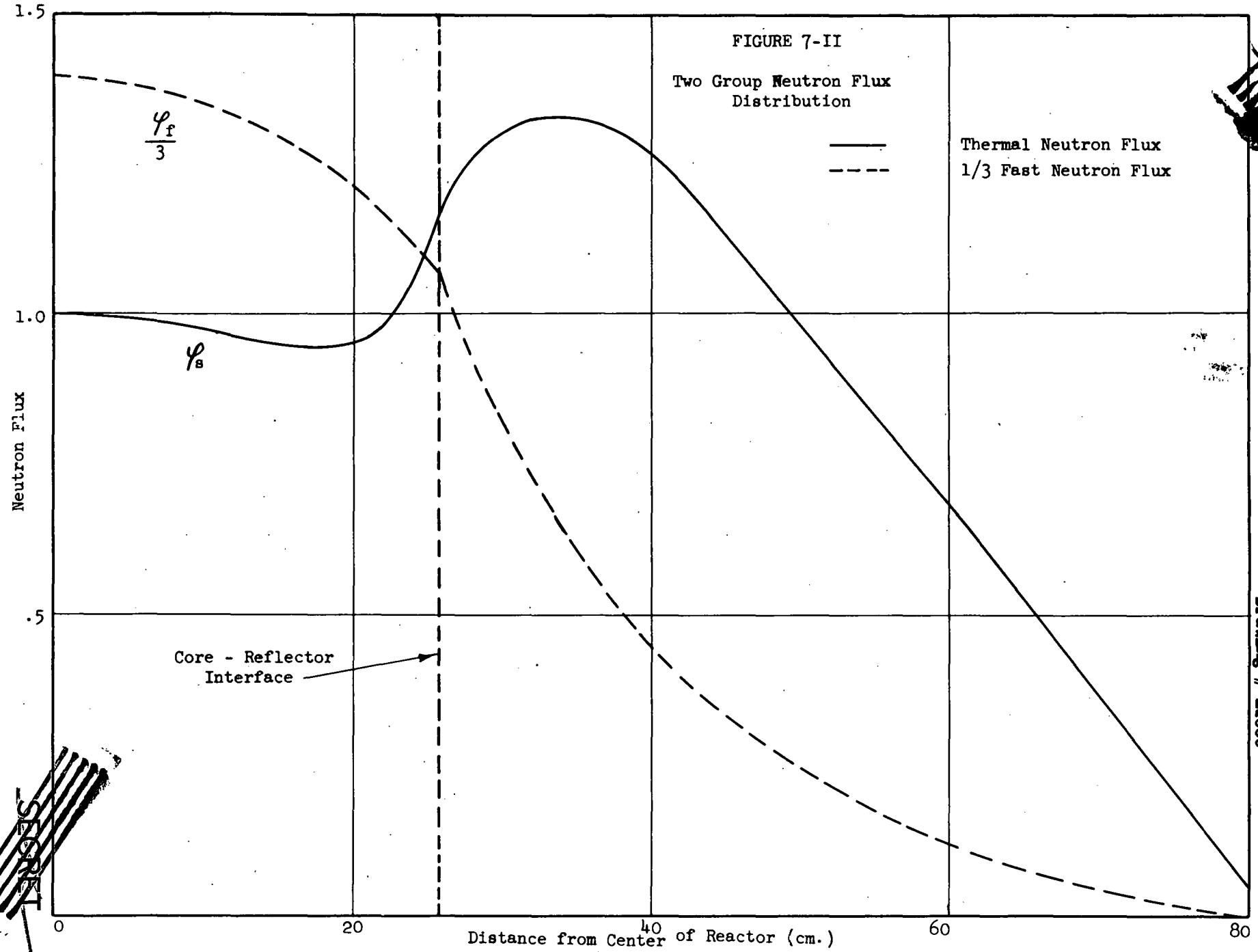

FIGURE 7-II

Two Group Neutron Flux Distribution

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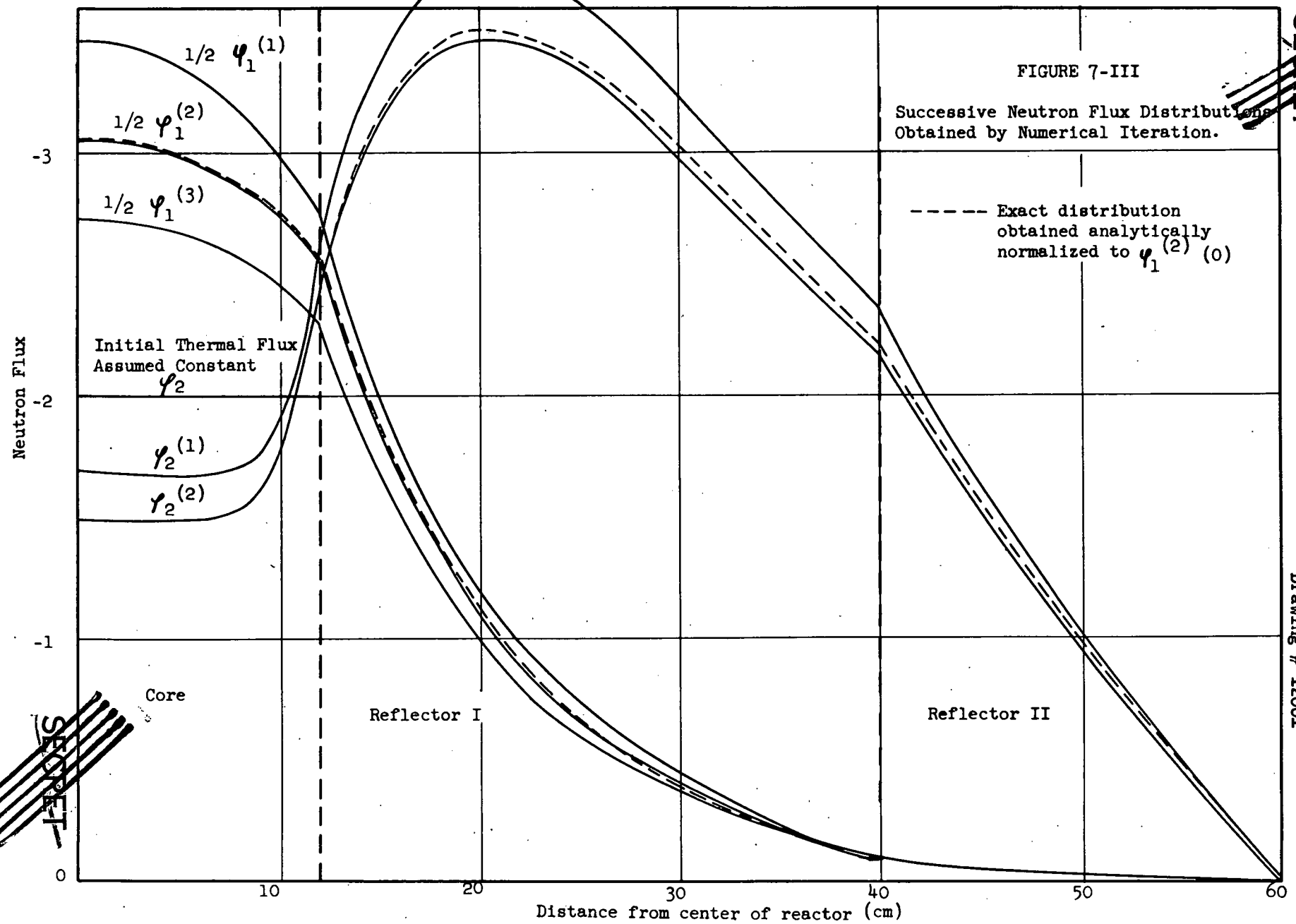

FIGURE 7-III

Successive Neutron Flux Distributions Obtained by Numerical Iteration.

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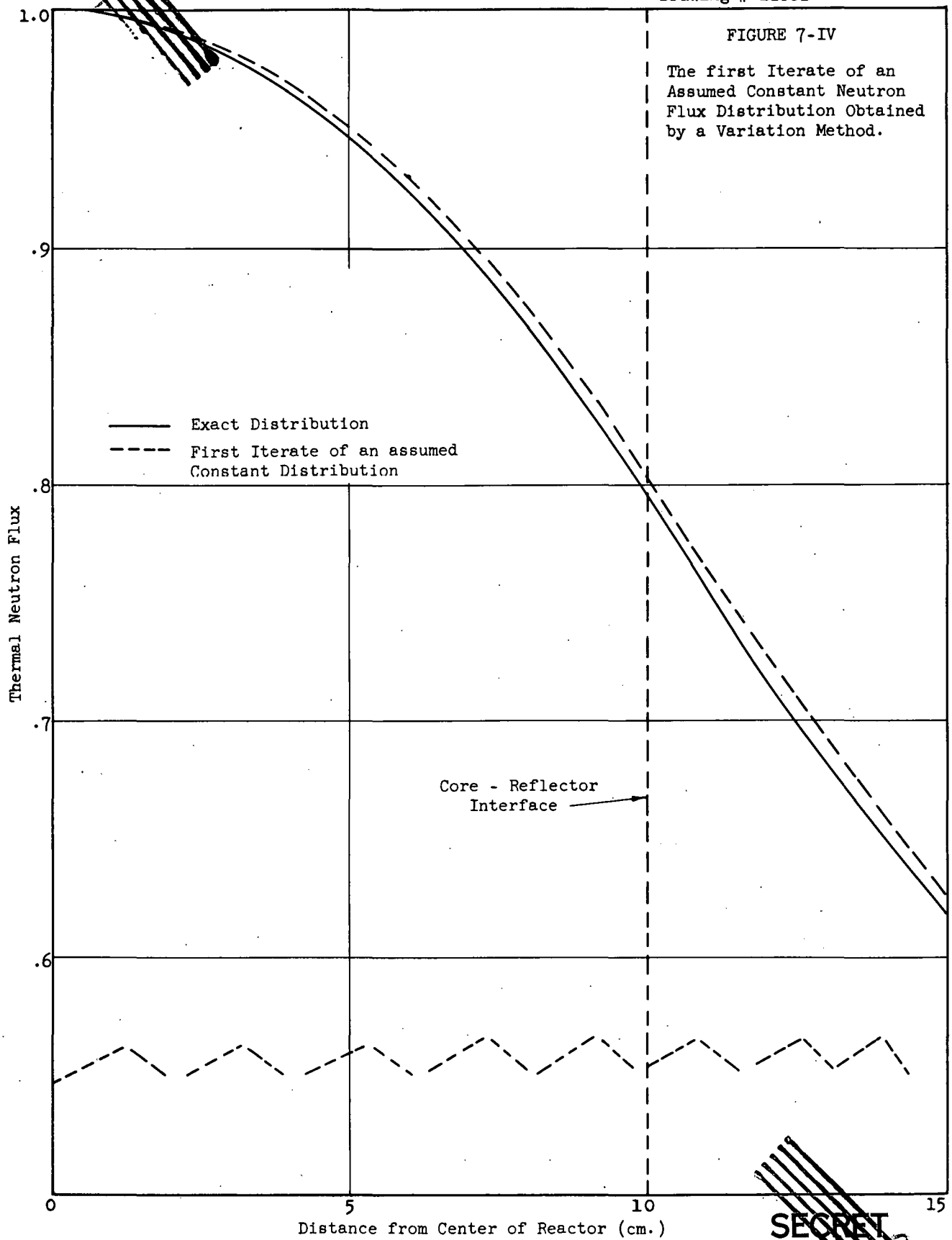
Drawing # 12002

FIGURE 7-IV

The first Iterate of an Assumed Constant Neutron Flux Distribution Obtained by a Variation Method.



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