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TWO-GROUP DIFFUSION THEORY FOR A RING OF CYLINDRICAL RODS

by

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ABSTRACT

The conditions for criticality and the resulting flux distribution have been obtained in the two-group diffusion theory approximation for a ring of N equally spaced, identical cylindrical rods embedded symmetrically in a radially bare cylinder. The system is uniform axially and of either finite or infinite height. Either or both of the two media of the system may be multiplying. The method used is a generalization of the Nordheim-Scalettar method for the solution of the control rod problem of similar geometry. In satisfying each of the various boundary conditions use is made of the Bessel function addition theorems to center all terms in the general solution at the appropriate line of symmetry. The results are obtained in terms of a Fourier expansion of the angular dependence of the flux about each rod, which in application must be cut off after some early term in the infinite series. The order of the critical determinant is equal to twice the number of angular terms retained.

I. INTRODUCTION

The problem of calculating the effect of a ring of N equally spaced, identical rods embedded in a reactor core most commonly arises in connection with a ring of control rods in a thermal reactor. The problem may be solved in one- or two-group diffusion theory by the use of the Nordheim-Scalettar method.

In this method the flux is determined in the region exterior to the rods, subject to the usual boundary conditions, i.e., vanishing of the flux at the outer extrapolated boundary, and, in addition, subject to certain internal boundary conditions imposed by the presence of the thermally black rods. The essential features of the Nordheim-Scalettar method are the inclusion in the general solution of singular terms centered at the rods, and the use of the Bessel function addition theorems to center all terms at an appropriate point to satisfy each of the various boundary conditions.

In design work on Argonaut, a low-power research reactor with an annular loading reflected internally and externally by graphite, a fuel arrangement was considered which consisted of sectors of fuel and sectors of graphite interspersed regularly in the annulus. Except for the shape of the fuel boxes, the geometry was similar to that of the Nordheim-Scalettar problem. The possibility of solving the problem with similar mathematical techniques was considered and led to the present investigation.

The present calculation differs from the Nordheim-Scalettar calculation in that we consider a ring of rods in which diffusion theory is applicable both inside and outside the rods. Since the method may be useful in a wider variety of reactor problems (e.g., interaction of several cylindrical reactors embedded in a common reflector, ring of grey control rods), we permit either or both of the regions to be multiplying, whereas in the Nordheim-Scalettar method the outside region is of necessity multiplying. We retain the full angular dependence throughout the entire formalism, so that the formal solution is exact for the two-group problem formulated. Approximation is involved only in that in practice no more than the first few terms in the Fourier expansion of the flux about each rod would be retained.

II. GEOMETRICAL NOTATION (Figure 1)

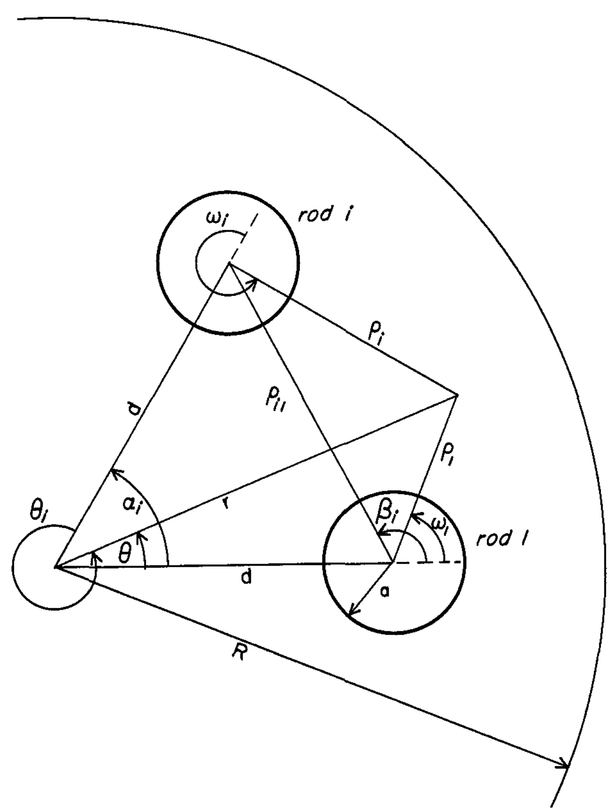

Fig. 1

S

N number of cylindrical rods in the ring

R extrapolated bare radius of the system

H extrapolated bare height of the system

a radius of a cylindrical rod

d radius of the ring on which the centers of the rods are located

α_i angle between the line from the origin to the center of rod 1, and the line from the origin to the center of rod i $= \frac{(i-1)\,2\pi}{N}$

β_i angle at which rod 1 sees the center of rod i $= \frac{\pi}{2} + \frac{(i-1)\pi}{N}$

ρ_{i1} distance from center of rod i to center of rod 1 $= 2d \sin\left[\frac{(i-1)\pi}{N}\right]$

r distance from the origin to a field point

θ angle of a field point at the origin, measured relative to the line from the origin to the center of rod 1

θ_i angle of a field point at the origin, measured relative to the line from the origin to the center of rod i

ρ_i distance from the center of rod i to the field point

ω_i angle of the field point measured at the center of rod i relative to the line from the origin to the center of rod i .

III. TWO-GROUP NOTATION

Writing the steady-state, two-group diffusion equations for the slow and fast fluxes, ϕ_S and ϕ_F, in a general form, we have for each homogeneous region:

$$
\begin{aligned}
D_S \nabla^2 \phi_S + D_S H_{SS} \phi_S + D_S H_{SF} \phi_F &= 0 \quad ; \\
D_F \nabla^2 \phi_F + D_F H_{FS} \phi_S + D_F H_{FF} \phi_F &= 0 \quad ,
\end{aligned}
\tag{1}
$$

where D_S and D_F are the slow and fast diffusion constants, and the H's define the transfer coefficients.

In the present geometry all terms in the solution have a factor $\cos\left(\frac{\pi}{H}z\right)$ (where z is measured from the center plane) which describes the axial dependence of the flux. Omitting this factor, the solution is of the form:

$$\phi_S = L + M \quad ;$$

$$\phi_F = S_1 L + S_2 M \quad ; \qquad (2)$$

where L and M are solutions of

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) L + \left(b_1^2 - \frac{\pi^2}{H^2}\right) L = 0 \quad ;$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) M + \left(b_2^2 - \frac{\pi^2}{H^2}\right) M = 0 \quad .$$

The bucklings are given by:

$$b_1^2 = \tfrac{1}{2}(H_{SS} + H_{FF}) + \tfrac{1}{2}\sqrt{(H_{SS} - H_{FF})^2 + 4 H_{SF} H_{FS}} \quad ;$$

$$b_2^2 = \tfrac{1}{2}(H_{SS} + H_{FF}) - \tfrac{1}{2}\sqrt{(H_{SS} - H_{FF})^2 + 4 H_{SF} H_{FS}} \quad , \qquad (3)$$

where b_1^2 is the asymptotic buckling. The group ratios are given by:

$$S_1 = \frac{b_1^2 - H_{SS}}{H_{SF}} \quad ;$$

$$S_2 = \frac{b_2^2 - H_{SS}}{H_{SF}} \quad . \qquad (4)$$

In equations of interest in reactors not both H_{SS} and H_{FF} are positive, from which it follows that b_2^2 is always negative.

For a medium with multiplication constant $k_\infty > 1$, b_1^2 is positive; otherwise it is negative. For L the solution is composed of terms of the form:

$$Y_n\left(\sqrt{b_1^2 - \frac{\pi^2}{H^2}}\, r\right) e^{in\theta} \quad ;$$

$$J_n\left(\sqrt{b_1^2 - \frac{\pi^2}{H^2}}\, r\right) e^{in\theta} \quad , \qquad (5a)$$

if the asymptotic buckling in the radial plane, $b_1^2 - \frac{\pi^2}{H^2}$ is positive, and

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$$K_n\left(\sqrt{\left|b_1^2 - \frac{\pi^2}{H^2}\right|}\, r\right) \quad e^{in\theta} \quad ;$$

$$I_n\left(\sqrt{\left|b_1^2 - \frac{\pi^2}{H^2}\right|}\, r\right) \quad e^{in\theta} \quad , \tag{5b}$$

if $b_1^2 - \frac{\pi^2}{H^2}$ is negative.

For M, the terms are of the form in Eq. (5b) with b_2^2 replacing b_1^2.

The two possible signs for $b_1^2 - \frac{\pi^2}{H^2}$ in each of the core and reflector give rise to four possible cases. Of these only three are really of interest, since obviously the case where $b_1^2 - \frac{\pi^2}{H^2} < 0$ in both regions cannot correspond to a critical system.

In the subsequent development all the various possibilities are considered simultaneously, the appropriate choice for each of the cases always being obvious.

IV. GENERAL SOLUTION FOR RING OF RODS

We refer to the medium contained in the cylindrical rods as the core (designated by subscript C) and the medium exterior to the rods as the reflector (designated by subscript R).

We will also use the following notation:

$$\ell^2 = \left| b_{1C}^2 - \frac{\pi^2}{H^2} \right|$$

$$m^2 = - b_{2C}^2 + \frac{\pi^2}{H^2} \tag{6}$$

$$\lambda^2 = \left| b_{1R}^2 - \frac{\pi^2}{H^2} \right|$$

$$\mu^2 = - b_{2R}^2 + \frac{\pi^2}{H^2} \quad , \tag{7}$$

where ℓ^2, m^2, λ^2, and μ^2 are all positive; ℓ, m, λ, μ are to be taken as the positive roots.

We limit the general solution to terms that satisfy the symmetry of the problem.

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For the solution in the reflector, the singular terms are centered, as in the Nordheim-Scalettar method, at the rod centers. The regular terms are centered at the origin. It is not necessary to center any regular solutions at the rod centers since such terms can be incorporated into the terms centered at the origin.

We give the core solution for a typical rod and omit the subscript on ρ and ω. The core solution is, of course, limited to the regular terms measured relative to the rod centers. We thus obtain:

$$L_R = \sum_{j=0}^{\infty} A_j \sum_{i=1}^{N} \begin{Bmatrix} K_j(\lambda\rho_i) \\ Y_j(\lambda\rho_i) \end{Bmatrix} \cos j\omega_i + \sum_{n=0}^{\infty} B_n \begin{Bmatrix} I_{Nn}(\lambda r) \\ J_{Nn}(\lambda r) \end{Bmatrix} \cos Nn\theta \quad (8)$$

$$M_R = \sum_{j=0}^{\infty} C_j \sum_{i=1}^{N} K_j(\mu\rho_i) \cos j\omega_i + \sum_{n=0}^{\infty} D_n I_{Nn}(\mu r) \cos Nn\theta \quad (9)$$

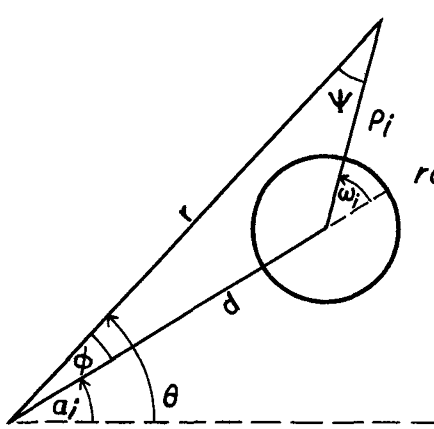
$$L_C = \sum_{k=0}^{\infty} E_k \begin{Bmatrix} I_k(\ell\rho) \\ J_k(\ell\rho) \end{Bmatrix} \cos k\omega \quad (10)$$

$$M_C = \sum_{k=0}^{\infty} F_k I_k(m\rho) \cos k\omega \quad , \quad (11)$$

where the A's, ..., F's are arbitrary constants to be determined by the boundary conditions, i.e., the fast and slow fluxes vanishing at the outer extrapolated boundary, and the fast and slow fluxes and currents continuous across the rod interfaces.

V. ADDITION THEOREMS

A. To center singular solutions from all the rods to origin.



r > d

Fig. 2

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In terms of Fig. 2, we have the following addition theorem: (see Appendix B).

$$\left\{\begin{matrix} K_j(\lambda\rho_i) \\ Y_j(\lambda\rho_i) \end{matrix}\right\} e^{ij\psi} = \sum_{k=-\infty}^{\infty} \left\{\begin{matrix} K_{j+k}(\lambda r)\, I_k(\lambda d) \\ Y_{j+k}(\lambda r)\, J_k(\lambda d) \end{matrix}\right\} e^{ik\phi} \quad .$$

We use an expansion which is valid outside the ring, i.e., $r>d$, since it will be used to satisfy the condition of the flux vanishing at the outer radius of system. Using:

$$\psi = \omega_i + \alpha_i - \theta$$

and

$$\phi = \theta - \alpha_i \quad ,$$

we obtain:

$$\left\{\begin{matrix} K_j(\lambda\rho_i) \\ Y_j(\lambda\rho_i) \end{matrix}\right\} e^{ij\omega_i} = \sum_{k=-\infty}^{\infty} \left\{\begin{matrix} K_{j+k}(\lambda r)\, I_k(\lambda d) \\ Y_{j+k}(\lambda r)\, J_k(\lambda d) \end{matrix}\right\} e^{i(j+k)(\theta-\alpha_i)} \quad .$$

Taking the real part

$$\left\{\begin{matrix} K_j(\lambda\rho_i) \\ Y_j(\lambda\rho_i) \end{matrix}\right\} \cos j\omega_i = \sum_{k=-\infty}^{\infty} \left\{\begin{matrix} K_{j+k}(\lambda r)\, I_k(\lambda d) \\ Y_{j+k}(\lambda r)\, J_k(\lambda d) \end{matrix}\right\} \cos (j+k)(\theta-\alpha_i)$$

$$= \sum_{n=-\infty}^{\infty} \left\{\begin{matrix} K_n(\lambda r)\, I_{n-j}(\lambda d) \\ Y_n(\lambda r)\, J_{n-j}(\lambda d) \end{matrix}\right\} \cos n(\theta-\alpha_i)$$

Summing over all rods and using:

$$\sum_{i=1}^{N} \cos n\,\alpha_i = \begin{cases} N, & \text{if } n \text{ is a multiple of } N \text{ (including } n=0) \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sum_{i=1}^{N} \sin n\,\alpha_i = 0$$

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we have:

$$\sum_{i=1}^{N}\begin{Bmatrix}K_j(\lambda\rho_i)\\ Y_j(\lambda\rho_i)\end{Bmatrix}\cos j\omega_i = N\sum_{n=-\infty}^{\infty}\begin{Bmatrix}K_{Nn}(\lambda r)\ I_{Nn-j}(\lambda d)\\ Y_{Nn}(\lambda r)\ J_{Nn-j}(\lambda d)\end{Bmatrix}\cos Nn\,\theta$$

With the notation:

$$\epsilon_n = 1, \quad n = 0$$
$$= 2, \quad n > 0 \tag{12}$$

this yields:

$$\sum_{i=1}^{N}\begin{Bmatrix}K_j(\lambda\rho_i)\\ Y_j(\lambda\rho_i)\end{Bmatrix}\cos j\omega_i = N\sum_{n=0}^{\infty}\frac{\epsilon_n}{2}\begin{Bmatrix}K_{Nn}(\lambda r)\,[I_{Nn+j}(\lambda d)+I_{Nn-j}(\lambda d)]\\ Y_{Nn}(\lambda r)\,[(-1)^j J_{Nn+j}(\lambda d)+J_{Nn-j}(\lambda d)]\end{Bmatrix}\cos Nn\,\theta \tag{13}$$

B. To center singular solutions from all the rods to rod 1.

To satisfy the boundary conditions at the surface of a rod we shall center all terms at the center of one of the rods. We select rod 1 and omit the subscripts on ρ_1 and ω_1 .

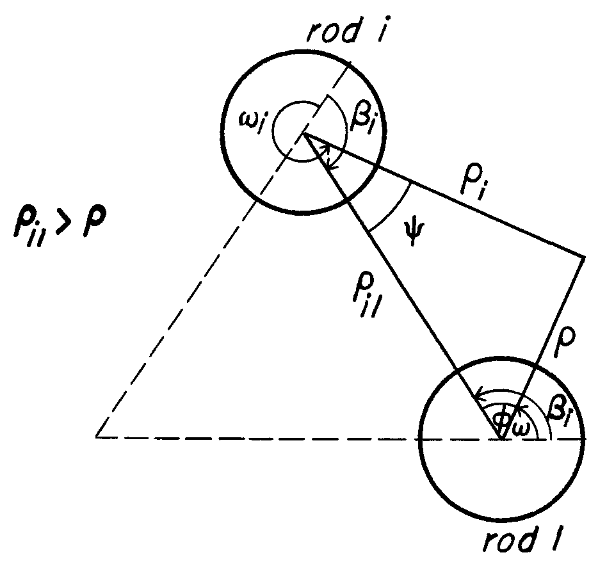

Fig. 3

In terms of Fig. 3, we have the following addition theorem (which is valid at the surface of rod 1, where it will be used to satisfy the interface conditions):

$$\begin{Bmatrix}K_j(\lambda\rho_i)\\ Y_j(\lambda\rho_i)\end{Bmatrix}e^{ij\psi} = \sum_{k=-\infty}^{\infty}\begin{Bmatrix}K_{j+k}(\lambda\rho_{i1})I_k(\lambda\rho)\\ Y_{j+k}(\lambda\rho_{i1})J_k(\lambda\rho)\end{Bmatrix}e^{ik\phi} .$$

Using:

$$\psi = \beta_i + \omega_i$$

and

$$\phi = \beta_i - \omega \quad ,$$

we obtain:

$$\begin{Bmatrix} K_j(\lambda\rho_i) \\ Y_j(\lambda\rho_i) \end{Bmatrix} e^{ij\omega_i} = \sum_{k=-\infty}^{\infty} \begin{Bmatrix} K_{j+k}(\lambda\rho_i)\, I_k(\lambda\rho) \\ Y_{j+k}(\lambda\rho_i)\, J_k(\lambda\rho) \end{Bmatrix} e^{-i[(j-k)\beta_i + k\omega]} \quad .$$

Taking the real part and summing over all other rods to center all terms at rod 1:

$$\sum_{i=2}^{N} \begin{Bmatrix} K_j(\lambda\rho_i) \\ Y_j(\lambda\rho_i) \end{Bmatrix} \cos j\omega_i = \sum_{k=-\infty}^{\infty} \sum_{i=2}^{N} \begin{Bmatrix} K_{j+k}(\lambda\rho_{i1})\, I_k(\lambda\rho) \\ Y_{j+k}(\lambda\rho_{i1})\, J_k(\lambda\rho) \end{Bmatrix} \cos(j-k)\beta_i \cos k\omega$$

$$= \sum_{k=0}^{\infty} \frac{\epsilon_k}{2} \sum_{i=2}^{N} \left[\begin{Bmatrix} K_{j+k}(\lambda\rho_{i1}) \\ Y_{j+k}(\lambda\rho_{i1}) \end{Bmatrix} \cos(j-k)\beta_i + \begin{Bmatrix} K_{j-k}(\lambda\rho_{i1}) \\ (-1)^k\, Y_{j-k}(\lambda\rho_{i1}) \end{Bmatrix} \cos(j+k)\beta_i \right] \begin{Bmatrix} I_k(\lambda\rho) \\ J_k(\lambda\rho) \end{Bmatrix} \cos k\omega \quad . \tag{14}$$

C. To center regular solutions from origin to rod 1.

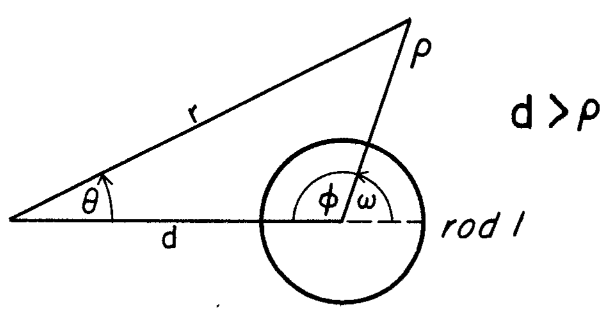

Fig. 4

In terms of Fig. 4, we have the following addition theorem (which is valid at the surface of rod 1, where it will be used to satisfy the interface conditions):

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$$\begin{Bmatrix} I_n(\lambda r) \\ J_n(\lambda r) \end{Bmatrix} \cos n\theta = \sum_{k=-\infty}^{\infty} \begin{Bmatrix} (-1)^k I_{n+k}(\lambda d)\, I_k(\lambda\rho) \\ J_{n+k}(\lambda d)\, J_k(\lambda\rho) \end{Bmatrix} \cos k\phi \quad .$$

From $\phi = \pi - \omega$ or $\cos k\phi = (-1)^k \cos k\omega$, we have:

$$\begin{Bmatrix} I_n(\lambda r) \\ J_n(\lambda r) \end{Bmatrix} \cos n\theta = \sum_{k=-\infty}^{\infty} \begin{Bmatrix} I_{n+k}(\lambda d)\, I_k(\lambda\rho) \\ (-1)^k J_{n+k}(\lambda d)\, J_k(\lambda\rho) \end{Bmatrix} \cos k\omega \quad .$$

Such terms will exist only if n be a multiple of N:

$$\begin{Bmatrix} I_{Nn}(\lambda r) \\ J_{Nn}(\lambda r) \end{Bmatrix} \cos Nn\,\theta = \sum_{k=-\infty}^{\infty} \begin{Bmatrix} I_{Nn+k}(\lambda d)\, I_k(\lambda\rho) \\ (-1)^k J_{Nn+k}(\lambda d)\, J_k(\lambda\rho) \end{Bmatrix} \cos k\omega$$

$$= \sum_{k=0}^{\infty} \frac{\epsilon_k}{2} \begin{Bmatrix} [I_{Nn+k}(\lambda d) + I_{Nn-k}(\lambda d)]\, I_k(\lambda\rho) \\ [(-1)^k J_{Nn+k}(\lambda d) + J_{Nn-k}(\lambda d)]\, J_k(\lambda\rho) \end{Bmatrix} \cos k\omega \quad . \tag{15}$$

VI. REFLECTOR SOLUTION WITH ALL TERMS CENTERED AT ROD 1

Centering all terms in Eq. (8) at the origin by the use of Eq. (13), we obtain, for $r > d$,

$$L_R(r,\theta) = \sum_{n=0}^{\infty} \left[B_n \begin{Bmatrix} I_{Nn}(\lambda r) \\ J_{Nn}(\lambda r) \end{Bmatrix} + \frac{N\epsilon_n}{2} \begin{Bmatrix} K_{Nn}(\lambda r) \\ Y_{Nn}(\lambda r) \end{Bmatrix} \sum_{j=0}^{\infty} A_j \begin{Bmatrix} I_{Nn+j}(\lambda d) + I_{Nn-j}(\lambda d) \\ (-1)^j J_{Nn+j}(\lambda d) + J_{Nn-j}(\lambda d) \end{Bmatrix} \right] \cos Nn\theta \,. \tag{16}$$

For the fast and slow fluxes to vanish at $r = R$, both L_R and M_R must vanish term by term in the Fourier expansion. Therefore,

$$B_n = -\frac{N\epsilon_n \begin{Bmatrix} K_{Nn}(\lambda R) \\ Y_{Nn}(\lambda R) \end{Bmatrix}}{2 \begin{Bmatrix} I_{Nn}(\lambda R) \\ J_{Nn}(\lambda R) \end{Bmatrix}} \sum_{j=0}^{\infty} A_j \begin{Bmatrix} I_{Nn+j}(\lambda d) + I_{Nn-j}(\lambda d) \\ (-1)^j J_{Nn+j}(\lambda d) + J_{Nn-j}(\lambda d) \end{Bmatrix} \quad . \tag{17}$$

We now center all of the terms in Eq. (8) at the center of rod 1 by the use of Eqs. (14), (15), and (17). We obtain for $\rho < \rho_{i1}$, d:

$$L_R(\rho,\omega) = \sum_{k=0}^{\infty} \sum_{j=o}^{\infty} A_j \left[\delta_{jk} \begin{Bmatrix} K_k(\lambda\rho) \\ Y_k(\lambda\rho) \end{Bmatrix} + T_{kj} \begin{Bmatrix} I_k(\lambda\rho) \\ J_k(\lambda\rho) \end{Bmatrix} \right] \cos k\omega \,, \tag{18}$$

where δ_{jk} is the Kronecker delta function, and

$$T_{kj} = \frac{\epsilon_k}{2} \sum_{i=2}^{N} \left[\begin{Bmatrix} K_{j+k}(\lambda\rho_{i1}) \\ Y_{j+k}(\lambda\rho_{i1}) \end{Bmatrix} \cos(j-k)\beta_i + \begin{Bmatrix} K_{j-k}(\lambda\rho_{i1}) \\ (-1)^k Y_{j-k}(\lambda\rho_{i1}) \end{Bmatrix} \cos(j+k)\beta_i \right]$$

$$- \frac{N\epsilon_k}{4} \sum_{n=0}^{\infty} \frac{\epsilon_n \begin{Bmatrix} K_{Nn}(\lambda R)\,[I_{Nn+k}(\lambda d)+I_{Nn-k}(\lambda d)]\,[I_{Nn+j}(\lambda d)+I_{Nn-j}(\lambda d)] \\ Y_{Nn}(\lambda R)\,[(-1)^k J_{Nn+k}(\lambda d)+J_{Nn-k}(\lambda d)]\,[(-1)^j J_{Nn+j}(\lambda d)+J_{Nn-j}(\lambda d)] \end{Bmatrix}}{\begin{Bmatrix} I_{Nn}(\lambda R) \\ J_{Nn}(\lambda R) \end{Bmatrix}} \tag{19}$$

In a similar manner we obtain:

$$M_R(r,\theta) = \sum_{n=0}^{\infty} \left\{ D_n I_{Nn}(\mu r) + \frac{N\epsilon_n}{2} K_{Nn}(\mu r) \sum_{j=0}^{\infty} C_j \,[I_{Nn+j}(\mu d) + I_{Nn-j}(\mu d)] \right\} \cos Nn\theta \quad ; \tag{20}$$

$$D_n = \frac{-N\epsilon_n K_{Nn}(\mu R)}{2 I_{Nn}(\mu R)} \sum_{j=0}^{\infty} C_j \,[I_{Nn+j}(\mu d) + I_{Nn-j}(\mu d)] \quad ; \tag{21}$$

$$M_R(\rho,\omega) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} C_j \,[\delta_{jk} K_k(\mu\rho) + U_{kj} I_k(\mu\rho)] \cos k\omega \; ; \tag{22}$$

$$U_{kj} = \frac{\epsilon_k}{2} \sum_{i=2}^{N} \left[K_{j+k}(\mu\rho_{i1}) \cos(j-k)\beta_i + K_{j-k}(\mu\rho_{i1}) \cos(j+k)\beta_i \right]$$

$$- \frac{N\epsilon_k}{4} \sum_{n=0}^{\infty} \frac{\epsilon_n K_{Nn}(\mu R)\,[I_{Nn+k}(\mu d)+I_{Nn-k}(\mu d)]\,[I_{Nn+j}(\mu d)+I_{Nn-j}(\mu d)]}{I_{Nn}(\mu R)} \;, \tag{23}$$

VII. BOUNDARY CONDITIONS AT SURFACE OF ROD - CRITICALITY CONDITION

Equations (2), (10), (11), (18), and (22) define the fluxes in the core and reflector at the interface in terms of a Fourier expansion of the angular dependence about each rod. For the fluxes and currents to be continuous across the interface for all angles, they must do so term by term in the Fourier expansion. The following four equations result for the cos $k\omega$ term:

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Slow flux

$$\sum_{j=0}^{\infty} A_j \left[\delta_{jk} \begin{Bmatrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{Bmatrix} + T_{kj} \begin{Bmatrix} I_k(\lambda a) \\ J_k(\lambda a) \end{Bmatrix} \right] + \sum_{j=0}^{\infty} C_j [\delta_{jk} K_k(\mu a) + U_{kj} I_k(\mu a)]$$

$$- E_k \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} - F_k I_k(ma) = 0 \tag{24a}$$

Fast flux

$$S_{1R} \sum_{j=0}^{\infty} A_j \left[\delta_{jk} \begin{Bmatrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{Bmatrix} + T_{kj} \begin{Bmatrix} I_k(\lambda a) \\ J_k(\lambda a) \end{Bmatrix} \right]$$

$$+ S_{2R} \sum_{j=0}^{\infty} C_j [\delta_{jk} K_k(\mu a) + U_{kj} I_k(\mu a)]$$

$$- S_{1C} E_k \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} - S_{2C} F_k I_k(ma) = 0 \tag{24b}$$

Slow current

$$D_{SR} \lambda \sum_{j=0}^{\infty} A_j \left[-\delta_{jk} \begin{Bmatrix} K_{k+1}(\lambda a) + K_{k-1}(\lambda a) \\ Y_{k+1}(\lambda a) - Y_{k-1}(\lambda a) \end{Bmatrix} + T_{kj} \begin{Bmatrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{Bmatrix} \right]$$

$$+ D_{SR} \mu \sum_{j=0}^{\infty} C_j [-\delta_{jk} (K_{k+1}(\mu a) + K_{k-1}(\mu a)) + U_{kj} (I_{k-1}(\mu a) + I_{k+1}(\mu a))]$$

$$- D_{SC} \ell E_k \begin{Bmatrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{Bmatrix}$$

$$- D_{SC}\, m\, F_k (I_{k-1}(ma) + I_{k+1}(ma)) = 0 \tag{24c}$$

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Fast current

$$S_{1R}D_{FR}\lambda\sum_{j=0}^{\infty}A_j\left[-\delta_{jk}\begin{Bmatrix}K_{k+1}(\lambda a)+K_{k-1}(\lambda a)\\ Y_{k+1}(\lambda a)-Y_{k-1}(\lambda a)\end{Bmatrix}+T_{kj}\begin{Bmatrix}I_{k-1}(\lambda a)+I_{k+1}(\lambda a)\\ J_{k-1}(\lambda a)-J_{k+1}(\lambda a)\end{Bmatrix}\right]$$

$$+S_{2R}D_{FR}\mu\sum_{j=0}^{\infty}C_j\left[-\delta_{jk}(K_{k+1}(\mu a)+K_{k-1}(\mu a))+U_{kj}(I_{k-1}(\mu a)+I_{k+1}(\mu a))\right]$$

$$-S_{1C}D_{FC}\,\ell E_k\begin{Bmatrix}I_{k-1}(\ell a)+I_{k+1}(\ell a)\\ J_{k-1}(\ell a)-J_{k+1}(\ell a)\end{Bmatrix}$$

$$-S_{2C}D_{FC}\,m\,F_k(I_{k-1}(ma)+I_{k+1}(ma))=0 \qquad (24d)$$

We may define the ν'th order approximation as one in which all cos $k\omega$ terms are discarded for $k>\nu$. We define this independently of the number of terms that are taken to evaluate the infinite series that occur in T_{kj} and U_{kj} when the outer radius of the system if finite. It is assumed that in an approximation of any order sufficient terms are taken to obtain T_{kj} and U_{kj} accurately. In practice, the terms in T_{kj} and U_{kj} converge very rapidly and in most cases the contributions from all higher order terms are negligible compared to the term with n = 0.

We can, if we wish, consider Eqs. (24) as the final result. In the ν'th order approximation we would have a set of 4 $(\nu + 1)$ linear homogeneous equations to be solved for 4 $(\nu + 1)$ unknowns. The criticality condition would correspond to the vanishing of the determinant of the matrix of coefficients. To obtain the flux distribution the arbitrary constants can then be determined, subject to an arbitrary normalization factor.

VIII. REDUCTION OF ORDER OF CRITICAL DETERMINANT

The order of the critical determinant can be reduced by a factor of two through the following algebraic steps which eliminate the core constants.

In a straightforward manner we segregate the A terms from the C terms in Eqs. (24) and obtain:

$$\sum_{j=0}^{\infty}A_j\left[\delta_{jk}\frac{\begin{Bmatrix}K_k(\lambda a)\\ Y_k(\lambda a)\end{Bmatrix}}{\begin{Bmatrix}I_k(\lambda a)\\ J_k(\lambda a)\end{Bmatrix}}+T_{kj}\right]=E_k\left(\frac{S_{1C}-S_{2R}}{S_{1R}-S_{2R}}\right)\frac{\begin{Bmatrix}I_k(\ell a)\\ J_k(\ell a)\end{Bmatrix}}{\begin{Bmatrix}I_k(\lambda a)\\ J_k(\lambda a)\end{Bmatrix}}$$

$$+F_k\left(\frac{S_{2C}-S_{2R}}{S_{1R}-S_{2R}}\right)\frac{I_k(ma)}{\begin{Bmatrix}I_k(\lambda a)\\ J_k(\lambda a)\end{Bmatrix}} \qquad (25a)$$

$$\sum_{j=0}^{\infty} C_j \left[\delta_{jk} \frac{K_k(\mu a)}{I_k(\mu a)} + U_{kj} \right] = - E_k \left(\frac{S_{1C} - S_{1R}}{S_{1R} - S_{2R}} \right) \frac{\begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix}}{I_k(\mu a)}$$

$$- F_k \left(\frac{S_{2C} - S_{1R}}{S_{1R} - S_{2R}} \right) \frac{I_k(ma)}{I_k(\mu a)} \qquad (25b)$$

$$\sum_{j=0}^{\infty} A_j \left[- \delta_{jk} \frac{\begin{Bmatrix} K_{k+1}(\lambda a) + K_{k-1}(\lambda a) \\ Y_{k+1}(\lambda a) - Y_{k-1}(\lambda a) \end{Bmatrix}}{\begin{Bmatrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{Bmatrix}} + T_{kj} \right]$$

$$= E_k \frac{\ell}{\lambda} \frac{S_{1C} \frac{D_{FC}}{D_{FR}} - S_{2R} \frac{D_{SC}}{D_{SR}}}{S_{1R} - S_{2R}} \frac{\begin{Bmatrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{Bmatrix}}{\begin{Bmatrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{Bmatrix}}$$

$$+ F_k \frac{m}{\lambda} \frac{S_{2C} \frac{D_{FC}}{D_{FR}} - S_{2R} \frac{D_{SC}}{D_{SR}}}{S_{1R} - S_{2R}} \frac{I_{k-1}(ma) + I_{k+1}(ma)}{\begin{Bmatrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{Bmatrix}} \qquad (25c)$$

$$\sum_{j=0}^{\infty} C_j \left[- \delta_{jk} \frac{K_{k+1}(\mu a) + K_{k-1}(\mu a)}{I_{k-1}(\mu a) + I_{k+1}(\mu a)} + U_{kj} \right]$$

$$= - E_k \frac{\ell}{\mu} \frac{S_{1C} \frac{D_{FC}}{D_{FR}} - S_{1R} \frac{D_{SC}}{D_{SR}}}{S_{1R} - S_{2R}} \frac{\begin{Bmatrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{Bmatrix}}{I_{k-1}(\mu a) + I_{k+1}(\mu a)}$$

$$- F_k \frac{m}{\mu} \frac{S_{2C} \frac{D_{FC}}{D_{FR}} - S_{1R} \frac{D_{SC}}{D_{SR}}}{S_{1R} - S_{2R}} \frac{I_{k-1}(ma) + I_{k+1}(ma)}{I_{k-1}(\mu a) + I_{k+1}(\mu a)} \qquad (25d)$$

We now subtract Eq. (25c) from Eq. (25a), and Eq. (25d) from Eq. (25b), and obtain

$$A_k a_k (S_{1R} - S_{2R}) = E_k e_k + F_k f_k \quad ;$$

$$C_k c_k (S_{2R} - S_{1R}) = E_k g_k + F_k h_k \quad , \qquad (26)$$

where

$$a_k = \left\{ \begin{matrix} \dfrac{K_k(\lambda a)}{I_k(\lambda a)} + \dfrac{K_{k+1}(\lambda a) + K_{k-1}(\lambda a)}{I_{k-1}(\lambda a) + I_{k+1}(\lambda a)} \\ \\ \dfrac{Y_k(\lambda a)}{J_k(\lambda a)} + \dfrac{Y_{k+1}(\lambda a) - Y_{k-1}(\lambda a)}{J_{k-1}(\lambda a) - J_{k+1}(\lambda a)} \end{matrix} \right\} \tag{27a}$$

$$c_k = \frac{K_k(\mu a)}{I_k(\mu a)} + \frac{K_{k+1}(\mu a) + K_{k-1}(\mu a)}{I_{k-1}(\mu a) + I_{k+1}(\mu a)} \tag{27b}$$

$$e_k = (S_{1C} - S_{2R}) \frac{\left\{ \begin{matrix} I_k(\ell a) \\ J_k(\ell a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_k(\lambda a) \\ J_k(\lambda a) \end{matrix} \right\}} - \frac{\ell}{\lambda}\left(S_{1C}\frac{D_{FC}}{D_{FR}} - S_{2R}\frac{D_{SC}}{D_{SR}}\right) \frac{\left\{ \begin{matrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{matrix} \right\}} \tag{27c}$$

$$f_k = (S_{2C} - S_{2R}) \frac{I_k(ma)}{\left\{ \begin{matrix} I_k(\lambda a) \\ J_k(\lambda a) \end{matrix} \right\}} - \frac{m}{\lambda}\left(S_{2C}\frac{D_{FC}}{D_{FR}} - S_{2R}\frac{D_{SC}}{D_{SR}}\right) \frac{I_{k-1}(ma) + I_{k+1}(ma)}{\left\{ \begin{matrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{matrix} \right\}} \tag{27d}$$

$$g_k = (S_{1C} - S_{1R}) \frac{\left\{ \begin{matrix} I_k(\ell a) \\ J_k(\ell a) \end{matrix} \right\}}{I_k(\mu a)} - \frac{\ell}{\mu}\left(S_{1C}\frac{D_{FC}}{D_{FR}} - S_{1R}\frac{D_{SC}}{D_{SR}}\right) \frac{\left\{ \begin{matrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{matrix} \right\}}{I_{k-1}(\mu a) + I_{k+1}(\mu a)} \tag{27e}$$

$$h_k = (S_{2C} - S_{1R}) \frac{I_k(ma)}{I_k(\mu a)} - \frac{m}{\mu}\left(S_{2C}\frac{D_{FC}}{D_{FR}} - S_{1R}\frac{D_{SC}}{D_{SR}}\right) \frac{I_{k-1}(ma) + I_{k+1}(ma)}{I_{k-1}(\mu a) + I_{k+1}(\mu a)} \tag{27f}$$

We can now solve Eq. (26) for E_k and F_k in terms of A_k and C_k:

$$E_k = (-p_k A_k + q_k C_k)(S_{1R} - S_{2R})$$

$$F_k = (-s_k A_k + t_k C_k)(S_{1R} - S_{2R}) \quad , \tag{28}$$

where

$$p_k = -a_k h_k/(e_k h_k - f_k g_k) \tag{29a}$$

$$q_k = c_k f_k/(e_k h_k - f_k g_k) \tag{29b}$$

$$s_k = a_k g_k/(e_k h_k - f_k g_k) \tag{29c}$$

$$t_k = -c_k e_k/(e_k h_k - f_k g_k) \quad . \tag{29d}$$

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We may now substitute back these expressions for E_k and F_k into Eqs. (25a) and (25b), yielding for our final result the following criticality condition:

$$\sum_{j=0}^{\infty} A_j \left[\delta_{jk} \left(\begin{Bmatrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{Bmatrix} + p_k(S_{1C} - S_{2R}) \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} + s_k(S_{2C} - S_{2R}) I_k(ma) \right) + T_{kj} \begin{Bmatrix} I_k(\lambda a) \\ J_k(\lambda a) \end{Bmatrix} \right]$$

$$-C_k \left[q_k(S_{1C} - S_{2R}) \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} + t_k(S_{2C} - S_{2R}) I_k(ma) \right] = 0 \tag{30a}$$

$$\sum_{j=0}^{\infty} C_j \left[\delta_{jk} \left(K_k(\mu a) + q_k(S_{1C} - S_{2R}) \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} + t_k(S_{2C} - S_{1R}) I_k(ma) \right) + U_{kj} \, I_k(\mu a) \right]$$

$$-A_k \left[p_k(S_{1C} - S_{1R}) \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} + s_k(S_{2C} - S_{1R}) I_k(ma) \right] = 0 \quad . \tag{30b}$$

For most practical applications, the second-order approximation should suffice. In problems where the rod size is small compared to the inter-rod distances and to the distance to the outer boundary, even the zero-order approximation may be adequate. When this is not the case, the cos ω term should largely account for the over-all radial dependence of the flux which results from the smoothed-out interaction with all the rods, and also the over-all leakage effects out of the system. The cos 2ω term should largely account for the angular dependence resulting from the interaction with nearest rods and which peaks at $\sim\omega = \pm \frac{\pi}{2}$ ($\sim$angle at which rod sees nearest neighbors if $N \gg 1$). There is little reason, however, to expect (in general) the cos 2ω contribution to be small compared to the cos ω contribution.

APPENDIX A

ONE-GROUP RESULTS

The one-group case is, of course, much simpler than the two-group case. Rather than re-derive this case completely, we can extract the results from the previous analysis. We associate L with the complete solution, ϕ, and ℓ and λ with the radial bucklings in the core and reflector, respectively.

We have, equivalent to Eq. (24), the following two interface conditions at the rod surface:

$$\sum_{j=0}^{\infty} A_j \left[\delta_{jk} \left\{ \begin{matrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{matrix} \right\} + T_{kj} \left\{ \begin{matrix} I_k(\lambda a) \\ J_k(\lambda a) \end{matrix} \right\} \right] - E_k \left\{ \begin{matrix} I_k(\ell a) \\ J_k(\ell a) \end{matrix} \right\} = 0 \qquad \text{(A1a)}$$

$$D_R \lambda \sum_{j=0}^{\infty} A_j \left[-\delta_{jk} \left\{ \begin{matrix} K_{k+1}(\lambda a) + K_{k-1}(\lambda a) \\ Y_{k+1}(\lambda a) - Y_{k-1}(\lambda a) \end{matrix} \right\} + T_{kj} \left\{ \begin{matrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{matrix} \right\} \right]$$

$$- D_C \ell E_k \left\{ \begin{matrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{matrix} \right\} = 0 \qquad \text{(A1b)}$$

or

$$\sum_{j=0}^{\infty} A_j \left[\delta_{jk} \frac{\left\{ \begin{matrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_k(\lambda a) \\ J_k(\lambda a) \end{matrix} \right\}} + T_{kj} \right] = E_k \frac{\left\{ \begin{matrix} I_k(\ell a) \\ J_k(\ell a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_k(\lambda a) \\ J_k(\lambda a) \end{matrix} \right\}} \qquad \text{(A2a)}$$

$$\sum_{j=0}^{\infty} A_j \left[-\delta_{jk} \frac{\left\{ \begin{matrix} K_{k+1}(\lambda a) + K_{k-1}(\lambda a) \\ Y_{k+1}(\lambda a) - Y_{k-1}(\lambda a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{matrix} \right\}} + T_{kj} \right] = E_k \frac{D_C \ell}{D_R \lambda} \frac{\left\{ \begin{matrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{matrix} \right\}} \qquad \text{(A2b)}$$

Subtracting Eq. (A2b) from Eq. (A2a) and using notation analogous to the two-group case we obtain:

$$a_k = \frac{\left\{ \begin{matrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_k(\lambda a) \\ J_k(\lambda a) \end{matrix} \right\}} + \frac{\left\{ \begin{matrix} K_{k+1}(\lambda a) + K_{k-1}(\lambda a) \\ Y_{k+1}(\lambda a) - Y_{k-1}(\lambda a) \end{matrix} \right\}}{\left\{ \begin{matrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{matrix} \right\}} , \qquad \text{(A3a)}$$

$$e_k = \frac{\begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix}}{\begin{Bmatrix} I_k(\lambda a) \\ J_k(\lambda a) \end{Bmatrix}} - \frac{D_C \ell}{D_R \lambda} \frac{\begin{Bmatrix} I_{k-1}(\ell a) + I_{k+1}(\ell a) \\ J_{k-1}(\ell a) - J_{k+1}(\ell a) \end{Bmatrix}}{\begin{Bmatrix} I_{k-1}(\lambda a) + I_{k+1}(\lambda a) \\ J_{k-1}(\lambda a) - J_{k+1}(\lambda a) \end{Bmatrix}} \qquad \text{(A3b)}$$

and

$$E_k = \frac{a_k}{e_k} A_k = -p_k A_k \quad . \qquad \text{(A4)}$$

Substitution of the expression for E_k into Eq. (A1a) yields the one-group criticality condition:

$$\sum_{j=0}^{\infty} A_j \left[\delta_{jk} \left\{ \begin{Bmatrix} K_k(\lambda a) \\ Y_k(\lambda a) \end{Bmatrix} + p_k \begin{Bmatrix} I_k(\ell a) \\ J_k(\ell a) \end{Bmatrix} \right\} + T_{kj} \begin{Bmatrix} I_k(\lambda a) \\ J_k(\lambda a) \end{Bmatrix} \right] = 0 \quad . \qquad \text{(A5)}$$

The order of the critical determinant is just equal to the number of angular terms retained.

APPENDIX B

BESSEL FUNCTION RELATIONSHIPS AND ADDITION THEOREMS

$$J_{-n}(x) = (-1)^n J_n(x)$$

$$I_{-n}(x) = I_n(x)$$

$$Y_{-n}(x) = (-1)^n Y_n(x)$$

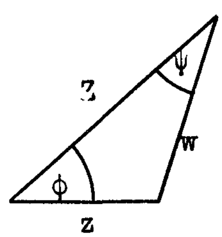
$$K_{-n}(x) = K_n(x)$$

$$\frac{d}{dx} J_n(\kappa x) = \frac{\kappa}{2} [J_{n-1}(\kappa x) - J_{n+1}(\kappa x)]$$

$$\frac{d}{dx} I_n(\kappa x) = \frac{\kappa}{2} [I_{n-1}(\kappa x) + I_{n+1}(\kappa x)]$$

$$\frac{d}{dx} Y_n(\kappa x) = -\frac{\kappa}{2} [Y_{n+1}(\kappa x) - Y_{n-1}(\kappa x)]$$

$$\frac{d}{dx} K_n(\kappa x) = -\frac{\kappa}{2} [K_{n+1}(\kappa x) + K_{n-1}(\kappa x)]$$



$Z > z$

$$J_n(w) \left\{ \begin{matrix} e^{in\psi} \\ \cos n\psi \end{matrix} \right\} = \sum_{k=-\infty}^{\infty} J_{n+k}(Z)\, J_k(z) \left\{ \begin{matrix} e^{ik\phi} \\ \cos k\phi \end{matrix} \right\}$$

$$I_n(w) \left\{ \begin{matrix} e^{in\psi} \\ \cos n\psi \end{matrix} \right\} = \sum_{k=-\infty}^{\infty} (-1)^k\, I_{n+k}(Z)\, I_k(z) \left\{ \begin{matrix} e^{ik\phi} \\ \cos k\phi \end{matrix} \right\}$$

$$Y_n(w) \left\{ \begin{matrix} e^{in\psi} \\ \cos n\psi \end{matrix} \right\} = \sum_{k=-\infty}^{\infty} Y_{n+k}(Z)\, J_k(z) \left\{ \begin{matrix} e^{ik\phi} \\ \cos k\phi \end{matrix} \right\}$$

$$K_n(w) \left\{ \begin{matrix} e^{in\psi} \\ \cos n\psi \end{matrix} \right\} = \sum_{k=-\infty}^{\infty} K_{n+k}(Z)\, I_k(z) \left\{ \begin{matrix} e^{ik\phi} \\ \cos k\phi \end{matrix} \right\}$$