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A Note on the Amplitude of Betatron Oscillations

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H. R. Crane has proposed that certain resonance orbits in a strong focussing synchrotron with non-linearities might be stable and that particle paths might lie close to and oscillate about these orbits. The most stable orbits might then be those where the betatron wavelength is an integral number of magnet periods. In the case of either this machine or a near-perfect linear strong focussing machine, the amplitudes of betatron oscillation would depend largely on injection conditions. Courant and Snyder (EDC/HSS-1) (referred to hereafter as C and S) consider the amplitude of oscillations resulting from an error in injection angle of particles injected at the equilibrium radius. The justification for this treatment is that the diameter of beam spots from Van de Graaf's and linear accelerators can be made small relative to the aperture of projected strong focussing machines, while the angular divergence of the beam is still of the order of .001 radians.

Let us reconsider the results of C and S on this point. In section 3 they show that

$$W = U/\beta$$

is a constant of the motion, where

$$U = y^2 + (\alpha y + \beta y')^2 \tag{1}$$

α and β are functions only of position within each sector and the values of n and N for the machine. At the center of a focussing or defocussing sector, β is maximum or minimum and α is zero.

At y max, $y' = \alpha = 0$, $U = y^2$, $W = y^2/\beta$ max.

At injection, $y = 0$, y' is the injection error, and $U = \beta^2 y'^2$, $W = \beta$ inj y'^2.

Since W is constant, y^2/β max $= \beta$ inj y'^2, or

$$y\ \text{max} = \sqrt{\beta\ \text{max}\,\beta\ \text{inj}}\ y' = \sqrt{\beta\ \text{max}\,\beta\ \text{inj}}\ R\,\delta \tag{2}$$

C and S assume β inj $= \beta$ max, corresponding to injection at the center of a focussing sector. If one injects at the center of a defocussing sector, β inj $= \beta$ min, where β min is determined from a matrix similar to that for β max.

$$k\beta^2\text{max} = \frac{\cosh\varphi\sin\varphi + \sinh\varphi}{\cosh\varphi\sin\varphi - \sinh\varphi}$$

$$k\beta^2\text{min} = \frac{\cos\varphi\sinh\varphi + \sin\varphi}{\cos\varphi\sinh\varphi - \sin\varphi}$$

In the above, $k = |n|$, $n_1 = -n_2 = n$, $\varphi = \pi\sqrt{k/N}$. C and S plot $k\beta^2_{max}$ versus φ. Figure 1 reproduces their graph including also $k\beta^2_{min}$. The gain in y max for a given injection error in y' using $\sqrt{\beta_{max}\beta_{min}}$ rather than $\sqrt{\beta^2_{max}}$ in (2) is a factor of 2.5 in the center of the nectie. Typical values of y max for both cases are given in Table 1 for a range of parameters.

Although these results hold rigidly only for a perfect, linear machine, there are probably equivalent values for β, etc. for small amounts of non-linearities and for the case of straight sections in the center of focussing and defocussing sectors.

Of course injecting at a position of minimum β for one coordinate means injecting at a position of maximum β for the other, hence the potential gain of a factor of 2.5 mentioned above would only be realized in one coordinate. This also means that coupling terms are important in the non-linear case and must be considered unless strong "lock-in" occurs.

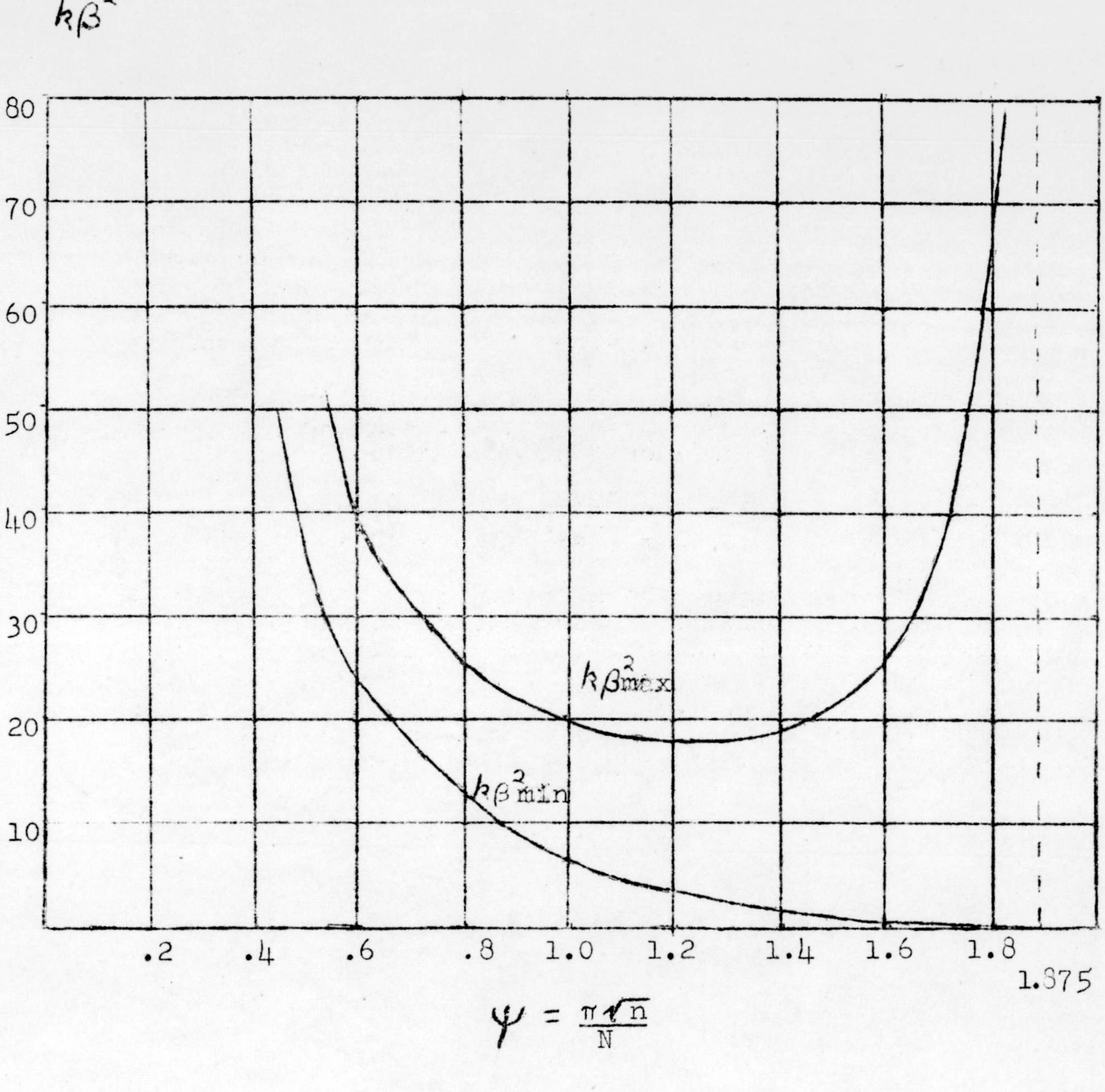
Quite apart from the above discussion, an observation on the form of non-linearities might be made. Powell shows (MAC JLP-1) the equations of motion in a non-linear field to be:

$$\ddot{r} + (1-n)r + er(3z^2-r^2) = 0$$
$$\ddot{z} + nz + ez(3r^2-z^2) = 0$$

Since e has the same sign in both expressions, a positive e can provide a net focussing force for both r and z, providing they have comparable average amplitudes. Previous work on the allowable amount of non-linearity, besides assuming no cross term, has assumed that e changes sign with n. It would appear superficially more desirable to keep e positive in sectors of both positive and negative n so as to provide a continuous net focussing force.

Nov. 18, 1953

FIGURE 1



$k\beta^2$
80
70
60
50
40
30
20
10
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
1.875
$k\beta^2_{max}$
$k\beta^2_{min}$
$\psi = \frac{\pi\sqrt{n}}{N}$

TABLE I

Resonance conditions and maximum amplitude of oscillations resulting from injection conditions.

σ	π	$2\pi/3$	$\pi/2$	$2\pi/5$	$\pi/3$	$2\pi/7$	$\pi/4$
$\lambda = \frac{2\pi}{\sigma}$	2	3	4	5	6	7	8
Ψ ($\cos\sigma = \cos\Psi \cosh\Psi$)	1.875	1.73	1.57 $\left(=\frac{\pi}{2}\right)$	1.42	1.31	1.22	1.15
$\sqrt{n}/N$ ($\Psi = \pi\sqrt{n}/N$)	.6	.55	.5	.45	.42	.39	.38
$n\beta^2 max$ (C&S)	∞	44	24	19	18	17	17
$n\beta^2 min$	0	.4	1.0	1.8	2.5	3.5	5
$n\beta max \beta min$	?	4.0	4.8	5.8	6.9	7.8	9.5
$\sqrt{n\beta max \beta min}$		2.0	2.2	2.4	2.6	2.8	3.1
$\sqrt{n\beta^2 max}$		6.8	5	4.5	4.2	4.0	4.0
n = 625							
$y' = R\delta = 10$cm							
ymax (β min)		.8	.88	.96	1.04	1.12	1.24
ymax (C&S)		2.8	2.0	1.8	1.7	1.6	1.6
N	42	45	50	55	60	64	65
$Q = N/\lambda$	21	15	12	11	10	9	8
n = 1000							
$y' = R\delta = 10$cm							
ymax (β min)		.6	.66	.72	.78	.84	.93
ymax (C&S)		2.3	1.7	1.5	1.4	1.3	1.3
N	53	58	63	70	75	81	83
$Q = N/\lambda$	27	19	16	14	12	11	10

$$ymax(\beta\, min) = \sqrt{\beta\, max\, \beta\, min}\; y'\, inj$$

$$ymax(C\&S) = \beta\, max\; y'\, inj$$