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Coupling of Betatron and Phase Oscillations in a Synchrotron

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ABSTRACT

The coupled equations for radial position, momentum, and phase oscillations in a synchrotron are treated by use of difference equations. The effect of a radial variation in the accelerating radio-frequency voltage is included in the treatment; any resultant damping of one type of oscillation is shown necessarily to be accompanied by equal anti-damping of oscillations of the other type. A simple treatment of the adiabatic variation of parameters for systems of linear differential or difference equations is given.

Coupling of Betatron and Phase Oscillations in a Synchrotron

The equations of motion in a synchrotron have been treated by many authors.[1] The accelerating radiofrequency voltage, which is applied across one or more narrow gaps, is Fourier-analyzed and only the synchronous component considered. In addition, the separation of the motion into betatron and synchrotron oscillations appears rather arbitrary. We felt it worthwhile to treat the problem in a more direct fashion. We consider, for simplicity, a machine in which the accelerating rf voltage is applied across a narrow gap at some azimuth, say $\Theta = 0$. The applied voltage varies sinusoidally in time, but can also vary radially. The magnetic field B_0 at the fixed "equilibrium" radius R_0, and the radiofrequency ω_s, will be tracked to fit an ideal particle which is continuously accelerated so that its momentum p_s (speed v_s and total energy $\mathcal{E}_s$) just match to give a circular orbit of radius R_0 at all times. Since the particles in the actual machine are given impulses at $\Theta = 0$, none of the actual particles will follow the ideal circular orbit.

Except at $\Theta = 0$, the equations of motion are just those of free oscillation:

$$\frac{d^2x}{d\theta^2} + (1-n)x = \frac{\Delta p}{p_s} \qquad (1)$$

$$\Delta p = p - p_s \qquad (1a)$$

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(1) R. Q. Twiss and N. H. Frank, RSI 20, 1 (1949).

where x is the ratio of the radial displacement from the equilibrium orbit to R_o, n is the field index, and p is the momentum of the particle. We have limited ourselves to the linear approximation. We also have for the speed:

$$v = r\dot{\theta} \tag{2}$$

$$\dot{\theta} = \frac{v}{r} = \frac{v_s + \Delta v}{R_0(1+x)} \simeq \frac{v_s}{R_0}\left(1 + \frac{\Delta v}{v_s} - x\right) \tag{2a}$$

The radiofrequency will be tracked to equal $\omega_s = \frac{v_s}{R_0}$ so that the ideal particle always reaches the gap at the same phase φ_s. From 2a:

$$\omega_s\, dt = d\theta\left(1 - \frac{\Delta v}{v_s} + x\right) = d\theta\left(1 - \frac{\mathcal{E}_0^2}{\mathcal{E}_s^2}\frac{\Delta p}{p_s} + x\right) \tag{3}$$

where $\mathcal{E}_0$ is the rest energy. Integrating Equation 3 over θ from 0 to 2π, the left side gives $\delta\varphi$, the increase in phase φ for one turn:

$$\delta\varphi = 2\pi + \int_0^{2\pi} d\theta\left(x - \frac{\mathcal{E}_0^2}{\mathcal{E}_s^2}\frac{\Delta p}{p_s}\right) \tag{4}$$

We denote x and $x' = \frac{dx}{d\theta}$ just after passing the gap on the ν'th turn by x_ν, x'_ν. By integrating Equation 1, we find their values after one turn around the machine, $x_{\nu+1}$, $x'_{\nu+1}$, which are linear functions of x_ν, x'_ν, and $\left(\frac{\Delta p}{p_s}\right)_\nu$. $\delta\varphi = \varphi_{\nu+1} - \varphi_\nu$ is given by Equation 4. Using for x the solution of Equation 1 for the ν'th turn, $\varphi_{\nu+1}$ is expressible linearly in terms of x_ν, x'_ν, $\left(\frac{\Delta p}{p_s}\right)_\nu$, φ_ν. Finally, we must obtain an equation for the momentum. The particle will receive an increment of energy after it has completed the ν'th turn. The amount it then receives will depend on the radius $x_{\nu+1}$ at which it crosses the gap, and the phase $\varphi_{\nu+1}$. Up to linear terms in the displacement, this energy increment is $e\left(V_0+V_0' x\right)_{\nu+1}\sin\varphi_{\nu+1}$ where V_0 and V_0' are the voltage and voltage gradient at R_0. Meanwhile, the ideal particle receives the energy increment $e\,V_0\cdot\sin\varphi_s$. Thus the change in the energy error $\Delta\mathcal{E}=\mathcal{E}-\mathcal{E}_s$ is:

$$(\Delta\mathcal{E})_{\nu+1}-(\Delta\mathcal{E})_{\nu}=e\left(V_0+V_0' x_{\nu+1}\right)\sin\varphi_{\nu+1}-e\,V_0\sin\varphi_s \quad (5)$$

Expanding the sin function (we assume small oscillations, $\varphi_\nu-\varphi_s\ll 1$) and dropping products of small quantities, we have:

$$(\Delta\mathcal{E})_{\nu+1}-(\Delta\mathcal{E})_{\nu}=\left(e\,V_0\cos\varphi_s\right)\left(\varphi_{\nu+1}-\varphi_s\right)+\left(e\,V_0'\sin\varphi_s\right)x_{\nu+1} \quad (5a)$$

Measuring φ from φ_s, and using $\Delta\mathcal{E}=\frac{c^2 p_s^2}{\mathcal{E}_s}\frac{\Delta p}{p_s}$ we can write:

$$\left(\frac{c^2 p_s^2}{\mathcal{E}_s}\frac{\Delta p}{p_s}\right)_{\nu+1}-\left(\frac{c^2 p_s^2}{\mathcal{E}_s}\frac{\Delta p}{p_s}\right)_{\nu}=\left(e\,V_0\cos\varphi_s\right)\varphi_{\nu+1}+\left(e\,V_0'\sin\varphi_s\right)x_{\nu+1} \quad (6)$$

Our discussion shows that we obtain four coupled linear difference equations for the four quantities x_ν, x'_ν, φ_ν and

$$p_\nu=\left(\frac{\Delta p}{p_s}\right)_\nu \quad (7)$$

We now do the detailed problem for the ordinary synchrotron, where n is independent of azimuth. The solutions of Equation 1

$$\frac{d^2x}{d\theta^2}+(1-n)x=\Delta p/p_s=p \quad (8)$$

have the betatron frequency $\sqrt{1-n}$. The general solution of Equation 8 for initial conditions x_ν, x'_ν is

$$x=\frac{p_\nu}{1-n}+\left(x_\nu-\frac{p_\nu}{1-n}\right)\cos\sqrt{1-n}\,\theta+\frac{x'_\nu}{\sqrt{1-n}}\sin\sqrt{1-n}\,\theta \quad (9)$$

Setting $\theta = 2\pi$, $\psi = 2\pi\sqrt{1-n}$ we have

$$x_{\nu+1} = \frac{1-\cos\psi}{1-n} p_\nu + \cos\psi\, x_\nu + \frac{\sin\psi}{\sqrt{1-n}} x'_\nu \tag{10}$$

$$x'_{\nu+1} = \frac{\sin\psi}{\sqrt{1-n}} p_\nu - \sqrt{1-n}\,\sin\psi\, x_\nu + \cos\psi\, x'_\nu . \tag{11}$$

Substituting Equation 9 in Equation 4 and integrating

$$\varphi_{\nu+1} - \varphi_\nu = \left[2\pi\left(\frac{1}{1-n} - \frac{\varepsilon_0^2}{\varepsilon_s^2}\right) - \frac{\sin\psi}{(1-n)^{3/2}}\right]_\nu p_\nu + \frac{\sin\psi}{(1-n)^{1/2}} x_\nu + \frac{(1-\cos\psi)}{1-n} x'_\nu \tag{12}$$

Equations 6, 10, 11, 12 are the coupled difference equations describing the system. They can be simplified by the following scale changes. Replace

$$\sqrt{1-n}\; x_\nu \longrightarrow x_\nu$$

$$\frac{p_\nu}{\sqrt{1-n}} \longrightarrow p_\nu$$

$$\varphi_\nu(1-n) \longrightarrow \varphi_\nu$$

Our set of equations becomes

$$x_{\nu+1} = \cos\psi\; x_\nu + \sin\psi\; x'_\nu + (1-\cos\psi)\, p_\nu \tag{13}$$

$$x'_{\nu+1} = -\sin\psi\; x_\nu + \cos\psi\; x'_\nu + \sin\psi\; p_\nu \tag{14}$$

$$\left(\frac{c^2 p_s^2}{\varepsilon_s} p\right)_{\nu+1} - \left(\frac{c^2 p_s^2}{\varepsilon_s} p\right)_\nu = \frac{eV_0' \sin\varphi_s}{1-n}\, x_{\nu+1} + \frac{eV_0 \cos\varphi_s}{(1-n)^{3/2}}\, \varphi_{\nu+1} \tag{15}$$

$$\varphi_{\nu+1} - \varphi_\nu = \sin\psi\; x_\nu + (1-\cos\psi)\, x'_\nu + C_\nu\, p_\nu \tag{16}$$

where

$$C_\nu = 2\pi\,\gamma_\nu\,(1-n)^{3/2} - \sin\psi \tag{17}$$

$$\gamma_\nu = \left[\frac{1}{1-n} - \frac{\varepsilon_0^2}{\varepsilon_s^2}\right]_\nu \tag{17a}$$

First we neglect the slow variation of the coefficients in Equations 15-17, drop the subscript on C_ν and replace 15 by

$$p_{\nu+1} = A x_{\nu+1} + p_\nu + B \varphi_{\nu+1} \tag{15a}$$

where

$$A = \frac{e V_0' \sin\varphi_s \, \mathcal{E}_s}{c^2 \beta_s^2 (1-n)}, \quad B = \frac{e V_0 \cos\varphi_s \, \mathcal{E}_s}{c^2 \beta_s^2 (1-n)^{3/2}} \tag{18}$$

Both A and B are of order energy gain per turn/total energy $\ll 1$. We find the eigenfrequencies and eigenvectors of the system 13, 14, 15a, 16. Let

$$x_\nu = \xi \lambda^\nu, \; x_\nu' = \xi' \lambda^\nu, \; p_\nu = \pi \lambda^\nu, \; \varphi_\nu = f \lambda^\nu \tag{19}$$

where ξ, ξ', π, f are the components of the eigenvector belonging to the eigenvalue λ, and substitute in the equations. The secular matrix is

$$\begin{pmatrix} \cos\psi - \lambda & \sin\psi & 1-\cos\psi & 0 \\ -\sin\psi & \cos\psi - \lambda & \sin\psi & 0 \\ A\lambda & 0 & 1-\lambda & B\lambda \\ \sin\psi & 1-\cos\psi & C & 1-\lambda \end{pmatrix} \tag{20}$$

Since A and B are small, we can find the roots of the secular equation by successive approximation. If we set A = B = 0, we find $\lambda = 1, 1, e^{\pm i\psi}$.

For the roots in the neighborhood of the double root 1, we set $\lambda = 1 + \epsilon$ in the secular equation. Keeping only the lowest power of ϵ, A, B, we find

$$\epsilon^2 + \epsilon^3 - \epsilon(A + 2BC') - BC' = 0 \tag{21}$$

where

$$C' = C + \sin\psi = 2\pi\gamma(1-n)^{3/2} \tag{22}$$

The lowest order equation would be

$$\epsilon^2 = BC' \quad , \quad \epsilon = \pm\sqrt{BC'} \tag{23}$$

Thus the oscillations will be stable if $BC' < 0$, i.e., $\gamma \cos \varphi_s < 0$.

Substituting 23 back into 21, we find the next approximation:

$$\epsilon^2 - \epsilon(A + BC') - BC' = 0$$

$$\epsilon = \frac{A + BC'}{2} \pm \sqrt{\frac{(A+BC')^2}{4} + BC'} \simeq \pm\sqrt{BC'} + \frac{A + BC'}{2} \tag{24}$$

or

$$\lambda \simeq 1 \pm \sqrt{BC'} + \frac{BC'}{2} + \frac{A}{2} \simeq \exp\left[\pm\sqrt{BC'} + \frac{A}{2}\right] \tag{25}$$

From 18 and 22,

$$|BC'| = \left| \frac{e V_0 \cos\varphi_s \, \mathcal{E}_s}{c^2 p_s^2} \cdot 2\pi\gamma \right| = \Omega^2 \tag{26}$$

where Ω is the synchrotron frequency. Thus the eigensolutions near $\lambda = 1$ correspond to synchrotron oscillations in the limit of zero coupling. From 25 it would appear that by choosing $A < 0$, these oscillations could be damped exponentially. However, we shall now show that the oscillations for the other pair of eigenfrequencies will be antidamped by the same factor. For the roots near $e^{\pm i\psi}$ we let $\lambda = e^{\pm i\psi}(1 + \epsilon)$. Again expanding the secular equation to lowest terms, we find $\epsilon = -A/2 \pm iB/2$ so that the roots are

$$\lambda \simeq \exp\left[\pm i\left(\psi + \frac{B}{2}\right) - \frac{A}{2}\right] \tag{27}$$

From Equations 25 and 27 we conclude that a radial variation of the accelerating voltage (which leads to the term $\pm A/2$) will cause damping of one pair of oscillations and an equal antidamping of the other pair. This effect was first discussed by Garren, et al.[2] From 27 we see that the second pair of roots correspond in the limit of zero coupling to the betatron oscillations.

The components of the eigenvectors are easily found from Equation 20:

$$\xi : \begin{vmatrix} \sin\psi & 1-\cos\psi & 0 \\ \cos\psi-\lambda & \sin\psi & 0 \\ 1-\cos\psi & C & 1-\lambda \end{vmatrix} = \xi' : -\begin{vmatrix} \cos\psi-\lambda & 1-\cos\psi & 0 \\ -\sin\psi & \sin\psi & 0 \\ \sin\psi & C & 1-\lambda \end{vmatrix}$$

$$= \pi : \begin{vmatrix} \cos\psi-\lambda & \sin\psi & 0 \\ -\sin\psi & \cos\psi-\lambda & 0 \\ \sin\psi & 1-\cos\psi & 1-\lambda \end{vmatrix} = f : -\begin{vmatrix} \cos\psi-\lambda & \sin\psi & 1-\cos\psi \\ -\sin\psi & \cos\psi-\lambda & \sin\psi \\ \sin\psi & 1-\cos\psi & C \end{vmatrix} \tag{28}$$

Evaluating the determinants:

$$\frac{\xi}{(1-\cos\psi)(1-\lambda^2)} = \frac{\xi'}{-\sin\psi(1-\lambda)^2} = \frac{\pi}{(1-\lambda)(\lambda^2-2\lambda\cos\psi+1)} = \frac{f}{-\begin{bmatrix} C(\lambda^2-2\lambda\cos\psi+1) \\ +2\lambda\sin\psi(1-\cos\psi) \end{bmatrix}} \tag{29}$$

For the roots near unity we set $\lambda = 1 + \epsilon$. Keeping only the leading terms, we find

$$\xi = \frac{\epsilon}{C'} f \; ; \; \xi' = \frac{\epsilon^2 \sin\psi}{2C'(1-\cos\psi)} f \; ; \; \pi = \frac{\epsilon}{C'} f \tag{30}$$

For $\epsilon << 1$, we see that only the phase has a sizable oscillation amplitude; i.e., we have a synchrotron oscillation. Similarly for

$\lambda = e^{\pm i\psi}(1+\epsilon)$, we find: (31)

$$\xi = \pm i f \; ; \; \xi' = -f \; ; \; \pi = \left(\mp i - \frac{\sin\psi}{1-\cos\psi}\right)\epsilon f$$

(2) A. A. Garren, et al, UCRL-547, Dec., 1949.

so that the betatron oscillations are necessarily accompanied by comparable changes in φ.

We must now take account of the slow variation of the coefficients in our coupled equations. The WKB method can be extended easily to systems of linear differential or difference equations. The derivations are given in the Appendix. Applying this technique, we find

(a) for the roots near unity

$$x_\nu = \frac{\epsilon \varphi_\nu}{C'_\nu}, \quad x'_\nu = \frac{\epsilon^2 \sin\psi}{2C'_\nu(1-\cos\psi)}\varphi_\nu, \quad p_\nu = \frac{\epsilon}{C'_\nu}\varphi_\nu \tag{32}$$

$$\varphi_\nu = \sqrt{\Omega_{\nu+1}} \prod_{j=1}^{\nu}\left(1 \pm i\Omega_j + \frac{A_j - \Omega_j^2}{2}\right) \tag{33}$$

with

$$\epsilon = \pm i\Omega + \frac{A-\Omega^2}{2}$$

$$\Omega_j = -i\sqrt{B_j C'_{j-1}} \tag{34}$$

(b) for the roots near $e^{\pm i\psi}$

$$x_\nu = \pm i\varphi_\nu, \quad x'_\nu = -\varphi_\nu, \quad p_\nu = \left(\mp i - \frac{\sin\psi}{1-\cos\psi}\right)\epsilon\,\varphi_\nu \tag{35}$$

$$\varphi_\nu = e^{\pm i\nu\psi}\prod_{j=1}^{\nu}\left(1 - \frac{A_j}{2} \pm i\frac{B_j}{2}\right) \tag{36}$$

with

$$\epsilon = -\frac{A}{2} \pm iBC' \tag{37}$$

Appendix

WKB Method for systems of linear differential or difference equations.

1. Differential Equations.

Consider the system

$$\dot{y}_i = \sum_j a_{ij} y_j \qquad (A.1)$$

where the coefficients a_{ij} are slowly varying functions of t. Let

$$y_i = x_i e^{i\int Y dt} \qquad (A.2)$$

where x_i and Y are slowly varying. Differentiating,

$$\dot{y}_i = (\dot{x}_i + iYx_i) e^{i\int Y dt} \qquad (A.2a)$$

Substituting in A.1:

$$\dot{x}_i = \sum_j a_{ij} x_j - iYx_i = \sum_j (a_{ij} - iY\delta_{ij}) x_j \qquad (A.3)$$

First we neglect the time derivatives $\dot{x}_i$:

$$\sum_j (a_{ij} - iY\delta_{ij}) x_j = 0 \qquad (A.4)$$

The Y and x_i determined from this first approximation will be denoted by $\overline{Y}$, $\overline{x_i}$, i.e.,

$$\sum_j (a_{ij} - i\overline{Y}\delta_{ij}) \overline{x}_j = 0$$

Thus $\overline{Y}$ is a root of the equation

$$|a_{ij} - i\overline{Y}\delta_{ij}| = |A - i\overline{Y} I| \equiv D(\overline{Y}) = 0 \qquad (A.5)$$

where $A = (a_{ij})$, I is the unit matrix and $|\;|$ means determinant = D. The $\overline{x_i}$ are proportional to the cofactors of any row, say the k'th:

$$\overline{x_j} = f M_{kj} \qquad (A.6)$$

where M_{kj} is the cofactor of the kj element in $D(\bar{Y})$. Differentiating A.6,

$$\dot{\bar{x}}_j/\bar{x}_j = \dot{f}/f + \dot{M}_{kj}/M_{kj} \qquad \text{(A.7)}$$

Now we use this first approximation for the $\bar{x}_j$ in A.3 and find

$$\sum_j \left(a_{ij} - iY\delta_{ij} - \frac{\dot{f}}{f}\delta_{ij} - \frac{\dot{M}_{kj}}{M_{kj}}\delta_{ij}\right)x_j = 0 \qquad \text{(A.8)}$$

Let $Y = \bar{Y} + \epsilon$. Then

$$\sum_j \left(a_{ij} - i\bar{Y}\delta_{ij} - \delta_{ij}\left\{i\epsilon + \dot{f}/f + \dot{M}_{kj}/M_{kj}\right\}\right)x_j = 0 \qquad \text{(A.8a)}$$

The secular equation for ϵ is:

$$\left| a_{ij} - i\bar{Y}\delta_{ij} - \delta_{ij}\left(i\epsilon + \dot{f}/f + \dot{M}_{kj}/M_{kj}\right)\right| = 0 \qquad \text{(A.9)}$$

ϵ, $\dot{f}/f$ and $\dot{M}_{kj}/M_{kj}$ are small quantities; expanding to lowest order we get:

$$\left| a_{ij} - i\bar{Y}\delta_{ij}\right| - \sum_j \left(i\epsilon + \dot{f}/f + \dot{M}_{kj}/M_{kj}\right) M_{jj} = 0 \qquad \text{(A.10)}$$

The first term is zero according to Equation A.5. Solving for ϵ:

$$\epsilon = \left[\dot{f}/f + \sum_j \frac{\dot{M}_{kj}}{M_{kj}} M_{jj} \Big/ \sum_j M_{jj}\right] \qquad \text{(A.11)}$$

$Y = \bar{Y} + \epsilon$. Substituting 11, and 6 in Equation A2:

$$y_i = f M_{ki} \exp i\left[\int dt\left(\bar{Y} + i\left\{\dot{f}/f + \sum_j \frac{\dot{M}_{kj}}{M_{kj}} M_{jj} \Big/ \sum_j M_{jj}\right\}\right)\right]$$

$$= M_{ki}\left[\exp i\int \bar{Y}\,dt\right] \exp - \int dt \left\{\sum_j \frac{\dot{M}_{kj}}{M_{kj}} M_{jj} \Big/ \sum_j M_{jj}\right\} \qquad \text{(A.12)}$$

Note that the factor f drops out. The result is independent of the index k, since a change of k is equivalent to multiplying by a common factor f.

As examples, we treat some simple cases. (We omit comment, but label equations with their numbers in the general derivation).

$$\dot{y}_1 = -a_{12} y_2 \,, \quad \dot{y}_2 = a_{21} y_1 \tag{1}$$

$$\dot{x}_1 = -i Y x_1 - a_{12} x_2$$
$$\dot{x}_2 = a_{21} x_1 - i Y x_2 \tag{3}$$

$$(\overline{Y})^2 = a_{12} a_{21} \,; \quad \overline{Y} = \pm \sqrt{a_{12} a_{21}} \tag{5}$$

Using the minors of the first row in 3,

$$\overline{x}_1 = a_{12} \,, \quad \overline{x}_2 = -i \overline{Y} = \mp i \sqrt{a_{12} a_{21}} \tag{6}$$

$$\dot{\overline{x}}_1 / \overline{x}_1 = \dot{a}_{12}/a_{12} \,; \quad \dot{\overline{x}}_2 / \overline{x}_2 = \tfrac{1}{2}\left(\dot{a}_{12}/a_{12} + \dot{a}_{21}/a_{21}\right) \tag{7}$$

$$-\left(\dot{a}_{12}/a_{12} + i Y\right) X_1 - a_{12} X_2 = 0 \tag{8}$$
$$a_{21} X_1 - \left(i Y + \tfrac{1}{2}\left\{\dot{a}_{12}/a_{12} + \dot{a}_{21}/a_{21}\right\}\right) X_2 = 0$$

$$Y = \overline{Y} + \epsilon \,; \quad -\left(\dot{a}_{12}/a_{12} + i\epsilon + i\overline{Y}\right) X_1 - a_{12} X_2 = 0 \tag{8a}$$

Expanding the determinant,

$$a_{21} X_1 - \left(i\overline{Y} + i\epsilon + \tfrac{1}{2}\left\{\dot{a}_{12}/a_{12} + \dot{a}_{21}/a_{21}\right\}\right) X_2 = 0$$

$$i\epsilon(2i\overline{Y}) + i\overline{Y}\left(\tfrac{3}{2}\, \dot{a}_{12}/a_{12} + \tfrac{1}{2}\, \dot{a}_{21}/a_{21}\right) = 0 \tag{10}$$

$$\epsilon = i\left[\tfrac{3}{4}\, \dot{a}_{12}/a_{12} + \tfrac{1}{4}\, \dot{a}_{21}/a_{21}\right] \tag{11}$$

$$y_1 = a_{12} \exp i \int dt \left[\pm \sqrt{a_{12} a_{21}} + i\left(\tfrac{3}{4}\, \dot{a}_{12}/a_{12} + \tfrac{1}{4}\, \dot{a}_{21}/a_{21}\right)\right] \tag{12}$$
$$= \left(a_{12}/a_{21}\right)^{1/4} \exp \pm i \int dt \sqrt{a_{12} a_{21}}$$
$$y_2 = \mp i \left(a_{21}/a_{12}\right)^{1/4} \exp \pm i \int dt \sqrt{a_{12} a_{21}}$$

Another example

$$\dot{y}_1 = a_{11} y_1 + a_{12} y_2 \quad ; \quad \dot{y}_2 = a_{21} y_1 + a_{22} y_2 \tag{1}$$

$$\dot{x}_1 = (a_{11} - i\Upsilon) x_1 + a_{12} x_2$$

$$\dot{x}_2 = a_{21} x_1 + (a_{22} - i\Upsilon) x_2 \tag{3}$$

$$(a_{11} - i\bar{\bar{\Upsilon}})(a_{22} - i\bar{\bar{\Upsilon}}) - a_{12} a_{21} = 0$$

$$\bar{\bar{\Upsilon}} = -i\bar{\bar{T}}/2 \pm \sqrt{d - T^2/4} \tag{5}$$

where $T = a_{11} + a_{22}$; $d = a_{11} a_{22} - a_{12} a_{21}$.

$$\bar{\bar{X}}_1 = -a_{12} = M_{21} \; ; \; \bar{\bar{X}}_2 = a_{11} - i\bar{\bar{\Upsilon}} = M_{22} \tag{6}$$

$$\dot{\bar{\bar{X}}}_1 / \bar{\bar{X}}_1 = \dot{M}_{21} / M_{21} \quad ; \quad \dot{\bar{\bar{X}}}_2 / \bar{\bar{X}}_2 = \dot{M}_{22} / M_{22} \tag{7}$$

Substitute in 3 to get

$$(a_{11} - i\bar{\bar{\Upsilon}} - i\epsilon - \dot{M}_{21}/M_{21}) x_1 + a_{12} x_2 = 0$$

$$a_{21} x_1 + (a_{22} - i\bar{\bar{\Upsilon}} - i\epsilon - \dot{M}_{22}/M_{22}) x_2 = 0 \tag{8a}$$

Expand.

$$\epsilon = i \, \frac{(a_{22} - i\bar{\bar{\Upsilon}}) \dot{M}_{21}/M_{21} + (a_{11} - i\bar{\bar{\Upsilon}}) \dot{M}_{22}/M_{22}}{a_{11} + a_{22} - 2i\bar{\bar{\Upsilon}}} \tag{11}$$

Substitute for $\bar{\bar{Y}}$ from 5:

$$\epsilon = \frac{i}{2}\left(\dot{M}_{21}/M_{21} + \dot{M}_{22}/M_{22}\right) \mp \frac{a_{11} - a_{22}}{4\sqrt{d - T^2/4}}\left(\dot{M}_{22}/M_{22} - \dot{M}_{21}/M_{21}\right)$$

$$y_1 = M_{21} \exp i \int (\bar{\bar{\Upsilon}} + \epsilon)\, dt = \left(\frac{M_{21}}{M_{22}}\right)^{1/2} \exp i \int dt \left[\bar{\bar{\Upsilon}} \mp \frac{a_{11} - a_{22}}{4\sqrt{d - T^2/4}}\left(\frac{\dot{M}_{22}}{M_{22}} - \frac{\dot{M}_{21}}{M_{21}}\right)\right]$$

$$y_2 = \left(\frac{M_{22}}{M_{21}}\right)^{1/2} \quad '' \quad '' \quad '' \tag{12}$$

If $d-\frac{T^2}{4} \geqq 0$, the extra term in the exponent contributes only a phase change. If $a_{11} = a_{22}$, the extra term drops out.

$$d-\frac{T^2}{4} = -a_{12}a_{21} - \frac{1}{4}(a_{11}-a_{22})^2$$

so if $a_{11} = a_{22}$, $d-\frac{T^2}{4} = -a_{12}a_{21}$

From 5 and 6 $$\frac{M_{12}}{M_{21}} = -\frac{\frac{a_{11}-a_{22}}{2} \mp i\sqrt{d-\frac{T^2}{4}}}{a_{12}}$$

If $a_{11} = a_{22}$,

$$\frac{M_{12}}{M_{21}} = \left(\frac{a_{21}}{a_{12}}\right)^{1/2}$$

II. Difference Equations.

Consider a system of difference equations

$$y^i_{\nu+1} - y^i_\nu = \sum_j a^{ij}_\nu y^j_\nu \tag{A1'}$$

where ν denotes the lattice point and the superscripts distinguish the variables. The a_ν^{ij} are slowly varying functions of ν. Let

$$y^i_\nu = x^i_\nu \prod_{n=0}^{\nu-1}(1+i\gamma_n) \tag{A2'}$$

Substitute in 1':

$$(1+i\gamma_\nu)x^i_{\nu+1} - x^i_\nu = \sum_j a^{ij}_\nu x^j_\nu \tag{A3'}$$

$$(1+i\gamma_\nu)(x^i_{\nu+1}-x^i_\nu) = \sum_j a^{ij}_\nu x^j_\nu - i\gamma_\nu x^i_\nu = \sum_j (a^{ij}_\nu - i\gamma_\nu \delta^{ij})x^j_\nu \tag{A3a'}$$

Neglect the first differences:

$$\sum_j (a^{ij}_\nu - i\gamma_\nu \delta^{ij})x^j_\nu = 0 \tag{A4'}$$

Call the root $\overline{Y}_\nu$ and the minors M_ν^{kj}:

$$|A_\nu - i\overline{Y}_\nu I| = 0 \qquad \text{(A5')}$$

$$x^j_\nu = M^{kj}_\nu \qquad \text{(A6')}$$

$$\frac{x^j_{\nu+1} - x^j_\nu}{x^j_\nu} = \frac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu} \qquad \text{(A7')}$$

Substitute $\overline{Y}$ and A7' on the left of A3a':

$$\sum_j \left[a^{ij}_\nu - \left\{iY_\nu + (1+i\overline{Y}_\nu)\left(\frac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu}\right)\right\}\delta^{ij}\right] x^j_\nu = 0 \qquad \text{(A8')}$$

$$Y_\nu = \overline{Y}_\nu + \epsilon_\nu$$

$$\left|a^{ij}_\nu - \left\{i\overline{Y}_\nu + i\epsilon_\nu + (1+i\overline{Y}_\nu)\left(\frac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu}\right)\right\}\delta^{ij}\right| = 0 \qquad \text{(A9')}$$

ϵ_ν and $\dfrac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu}$ are small quantities. Expanding to first order and using A5':

$$\sum_j \left[i\epsilon_\nu + (1+i\overline{Y}_\nu)\left(\frac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu}\right)\right] M^{jj}_\nu = 0 \qquad \text{(A10')}$$

$$\epsilon_\nu = i(1+i\overline{Y}_\nu)\left[\sum_j \frac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu} M^{jj}_\nu \Big/ \sum_j M^{jj}_\nu\right] \qquad \text{(A11')}$$

$$Y_\nu = \overline{Y}_\nu + \epsilon_\nu,$$

$$1 + iY_\nu = 1 + i\overline{Y}_\nu + \epsilon_\nu = (1+i\overline{Y}_\nu)\left(1 - \sum_j \frac{M^{kj}_{\nu+1} - M^{kj}_\nu}{M^{kj}_\nu} M^{jj}_\nu \Big/ \sum_j M^{jj}_\nu\right)$$

$$y^i_\nu = M^{ki}_\nu \left[\prod_n^{\nu-1} (1+i\overline{Y}_n)\right]\left[\prod_n^{\nu-1}\left(1 - \sum_j \frac{M^{kj}_{n+1} - M^{kj}_n}{M^{kj}_n} M^{jj}_n \Big/ \sum_j M^{jj}_n\right)\right] \qquad \text{(A12')}$$

Again, we take an example:

$$\begin{cases} y^1_{\nu+1} - y^1_\nu = -a^{12}_\nu y^2_\nu \\ y^2_{\nu+1} - y^2_\nu = a^{21}_\nu y^1_\nu \end{cases} \tag{1'}$$

$$\begin{cases} (1+iY_\nu)x^1_{\nu+1} - x^1_\nu = -a^{12}_\nu x^2_\nu \\ (1+iY_\nu)x^2_{\nu+1} - x^2_\nu = a^{21}_\nu x^1_\nu \end{cases} \tag{3'}$$

$$\begin{cases} (1+iY_\nu)(x^1_{\nu+1} - x^1_\nu) = -iY_\nu x^1_\nu - a^{12}_\nu x^2_\nu \\ (1+iY_\nu)(x^2_{\nu+1} - x^2_\nu) = a^{21}_\nu x^1_\nu - iY_\nu x^2_\nu \end{cases} \tag{3a'}$$

Neglecting first differences:

$$\overline{Y}_\nu = \pm\sqrt{a^{12}_\nu a^{21}_\nu} \tag{A5'}$$

$$\frac{x^1_\nu}{x^2_\nu} = \frac{i\,a^{12}_\nu}{\overline{Y}_\nu} = \pm i\sqrt{a^{12}_\nu / a^{21}_\nu}$$

Choose the factor f so that

$$x^1_\nu = \pm i\sqrt{a^{12}_\nu}\,,\quad x^2_\nu = \sqrt{a^{21}_\nu} \tag{6'}$$

$$\frac{x^1_{\nu+1} - x^1_\nu}{x^1_\nu} = \sqrt{a^{12}_{\nu+1}/a^{12}_\nu} - 1 = \sqrt{1 + \frac{a^{12}_{\nu+1} - a^{12}_\nu}{a^{12}_\nu}} - 1 \simeq \frac{1}{2}\,\frac{a^{12}_{\nu+1} - a^{12}_\nu}{a^{12}_\nu} \tag{7'}$$

Similarly $$\frac{x^2_{\nu+1} - x^2_\nu}{x^2_\nu} = \frac{1}{2}\,\frac{a^{21}_{\nu+1} - a^{21}_\nu}{a^{21}_\nu}.$$

Substitute from 5' and 7' on the left side of 3a':

$$-\left(iY_\nu + \{i\overline{Y}_\nu + 1\}\tfrac{1}{2}\left\{\frac{a^{12}_{\nu+1} - a^{12}_\nu}{a^{12}_\nu}\right\}\right)x^1_\nu - a^{12}_\nu x^2_\nu = 0$$

$$a^{21}_\nu x^1_\nu - \left(iY_\nu + \{i\overline{Y}_\nu + 1\}\tfrac{1}{2}\left\{\frac{a^{21}_{\nu+1} - a^{21}_\nu}{a^{21}_\nu}\right\}\right)x^2_\nu = 0 \tag{8'}$$

$$Y_\nu = \overline{Y}_\nu + \epsilon_\nu$$

Expanding the determinant,

$$\epsilon_\nu = \frac{i}{4}\left(1+i\overline{Y}_\nu\right)\left(\frac{a^{12}_{\nu+1}-a^{12}_\nu}{a^{12}_\nu}+\frac{a^{21}_{\nu+1}-a^{21}_\nu}{a^{21}_\nu}\right) \tag{11'}$$

$$\begin{aligned} 1+iY_\nu &= 1+i\overline{Y}_\nu+i\epsilon \\ &= \left(1+i\overline{Y}_\nu\right)\left(1-\frac{1}{4}\left\{\frac{a^{12}_{\nu+1}-a^{12}_\nu}{a^{12}_\nu}+\frac{a^{21}_{\nu+1}-a^{21}_\nu}{a^{21}_\nu}\right\}\right) \\ &\simeq \left(1+i\overline{Y}_\nu\right)\left(\frac{a^{12}_{\nu+1}}{a^{12}_\nu}\,\frac{a^{21}_{\nu+1}}{a^{21}_\nu}\right)^{-1/4} \end{aligned}$$

$$\prod_{n=0}^{\nu-1}\left(1+iY_n\right)=\left[\prod_n^{\nu-1}\left(1+i\overline{Y}_n\right)\right]\left(\frac{a^{12}_0\,a^{21}_0}{a^{12}_\nu\,a^{21}_\nu}\right)^{1/4}$$

Omitting constant factors,

$$\begin{aligned} y^1_\nu &= \pm i\left(a^{12}_\nu/a^{21}_\nu\right)^{\frac{1}{4}}\prod_n^{\nu-1}\left(1+i\overline{Y}_n\right) \\ y^2_\nu &= \left(a^{21}_\nu/a^{12}_\nu\right)^{\frac{1}{4}}\prod_n^{\nu-1}\left(1+i\overline{Y}_n\right) \end{aligned} \tag{12'}$$