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Mark V FFAG. Equations of Motion for Illiac Computation

A form proposed for the magnetic field in the median plane of the Mark V FFAG accelerator is (Laslett, MURA notes, 14 March 1955)

$$H_y = -(1+x)^k \left\{ 1 + f \sin \Phi \right\}, \quad \Phi = \varkappa \ln(1+x) - N\theta \tag{1}$$

(We adopt the notation $r = r_0(1 + x)$, $z = r_0 y$, and employ units in which $pc/e = r_0 = H_0 = 1$). Also, it has been suggested by Kerst that the addition of harmonic terms to the basic expression (1), e.g.

$$H_y = -(1+x)^k \left\{ 1 + f\left(\sin \Phi - 1/9 \sin 3\Phi\right) \right\} \tag{2}$$

may have the effect of increasing the largest amplitude of stable motion. The quantity $- 1/9 \sin 3\Phi$ in equation (2) represents the third harmonic in the Fourier development of a "saw-tooth" wave form.

It is proposed that a program for computation of orbits be based upon the following series developments of the field vectors: The magnetic field which has, in the median plane, the form

$$H_y = -(1+x)^k \left\{ \mu_0 \cos m\Phi + \nu_0 \sin m\Phi \right\}, \quad (y=0) \tag{3}$$

is given by

$$\vec{H} = -\nabla \psi = \nabla \times \vec{A} \quad , \quad \nabla \cdot \vec{H} = 0 \quad , \quad \nabla \times \vec{H} = 0$$

where

$$\psi = (1+x)^{k+1} \sum_{i=0}^{\infty} (\mu_i \cos m\Phi + \nu_i \sin m\Phi) \frac{(y/1+x)^{2i+1}}{(2i+1)!} \tag{4}$$

$$\left.\begin{aligned} A_\theta &= -(1+x)^{k+1} \sum_{i=0}^{\infty} (\alpha_i \cos m\Phi + \beta_i \sin m\Phi) \frac{(y/1+x)^{2i}}{(2i)!} \\ A_x &= -(1+x)^{k+1} \sum_{i=0}^{\infty} (\gamma_i \sin m\Phi + \delta_i \cos m\Phi) \frac{(y/1+x)^{2i}}{(2i)!} \end{aligned}\right\} \tag{5}$$

$$A_y = 0$$

The components of the field have the expansions

$$\left.\begin{aligned} H_x &= (1+x)^{k} \sum_{i=1}^{\infty} (\alpha_i \cos m\Phi + \beta_i \sin m\Phi) \frac{(y/1+x)^{2i-1}}{(2i-1)!} \\ H_\theta &= (1+x)^{k} m N \sum_{i=1}^{\infty} (\nu_{i-1} \cos m\Phi - \mu_{i-1} \sin m\Phi) \frac{(y/1+x)^{2i-1}}{(2i-1)!} \\ H_y &= -(1+x)^{k} \sum_{i=0}^{\infty} (\mu_i \cos m\Phi + \nu_i \sin m\Phi) \frac{(y/1+x)^{2i}}{(2i)!} \end{aligned}\right\} \tag{6}$$

The coefficients are given by the recurrence formulas

$$
\left.
\begin{aligned}
\mu_{i+1} &= \left\{ m^2(\kappa^2+N^2) - (k-2i)^2 \right\} \mu_i - 2m\kappa(k-2i)\nu_i \\
\nu_{i+1} &= 2m\kappa(k-2i)\mu_i + \left\{ m^2(\kappa^2+N^2) - (k-2i)^2 \right\} \nu_i \\
\alpha_{i+1} &= -(k-2i)\mu_i - m\kappa\nu_i , \qquad && \alpha_0 = 0 \\
\beta_{i+1} &= m\kappa\mu_i - (k-2i)\nu_i && \beta_0 = 0 \\
\gamma_{i+1} &= mN\mu_i && \gamma_0 = \frac{\mu_0}{mN} \\
\delta_{i+1} &= -mN\nu_i && \delta_0 = -\frac{\nu_0}{mN}
\end{aligned}
\right\} \quad \begin{aligned} &(7) \\ &(m \neq 0) \end{aligned}
$$

The azimuth-independent field (m = 0) is described in a similar way by

$$
\left.
\begin{aligned}
H_z &= -\tau_0(1+x)^k && (y=0) \\
\psi &= (1+x)^{k+1} \sum_{i=0}^{\infty} \tau_i \frac{(y/1+x)^{2i+1}}{(2i+1)!} \qquad && A_\theta = -(1+x)^{k+1} \sum_{i=0}^{\infty} \sigma_i \frac{(y/1+x)^{2i}}{(2i)!} \\
H_x &= (1+x)^k \sum_{i=1}^{\infty} \sigma_i \frac{(y/1+x)^{2i-1}}{(2i-1)!} , && H_y = -(1+x)^k \sum_{i=0}^{\infty} \tau_i \frac{(y/1+x)^{2i}}{(2i)!}
\end{aligned}
\right)
$$

$$\left.\begin{array}{l} A_x = A_y = H_\theta = 0 \\ \tau_{i+1} = -(k-2i)^2 \tau_i \quad , \quad \sigma_{i+1} = -(k-2i)\tau_i \quad , \quad \sigma_0 = \dfrac{\tau_0}{k+2} \end{array}\right\} \quad (8)$$

The series are convergent and represent the field provided $|y/(1+x)| < 1$. In truncating the series for numerical computation it is necessary, in order that the approximate $\vec{H}$ so obtained be <u>strictly</u> derivable from a vector potential, that the highest order terms retained be those explicitly written in eqs. (6), in which i has the <u>same</u> value in each equation.

The practical convergence of the series (6) has been examined in the case that the field in the median plane is

$$H_y = -(1+x)^{2.2}\left\{1 + .25 \sin(34.4 \ln(1+x) - 5\theta)\right\}$$

For $y/(1+x) = .1$, it is estimated that five-figure accuracy in ψ is obtained if terms up to $i = 8$ are retained. This set of parameters represents a rather unfavorable case from the point of view of convergence, and also the value .1 for amplitude/radius is rather larger than is likely to occur in practice. On this basis it seems reasonable to hope that accurate calculation with Illiac will be feasable. (J.N. Snyder, private communication.)

The equations of motion in Lagrangian form, viz

$$\frac{dx}{d\theta} = x' \qquad \frac{dy}{d\theta} = y'$$

$$\frac{d}{d\theta}\left(\frac{x'}{R}\right) = \frac{(1+x)}{R} + (1+x) H_y - y' H_\theta \qquad R = \sqrt{(1+x)^2 + y'^2 + x'^2}$$

$$\frac{d}{d\theta}\left(\frac{y'}{R}\right) = x' H_\theta - (1+x) H_x$$

are much simpler algebraically than the corresponding canonical equations, and are recommended for digital computation. The variables are not conjugate, but there seems to be no practical advantage in having them so, particularly in the case of a two-dimensional system. It may be mentioned also that study of modifications of the field in the azimuthal direction, (e.g. straight sections), is facilitated because the "velocity" components (x',y') are immediately available, without complicated algebraic transformations.

The expansion (4) for ψ may be used to construct a program for computation of equipotential surfaces by numerically solving the algebraic equation in y which results from truncating the series. The problem of practical convergence is obviously most critical in this case, but preliminary estimates have been encouraging.

In constructing the above formulation of the problem, the writer has been guided by the reports of Laslett, Vogt-Nilsen, and Akeley.

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