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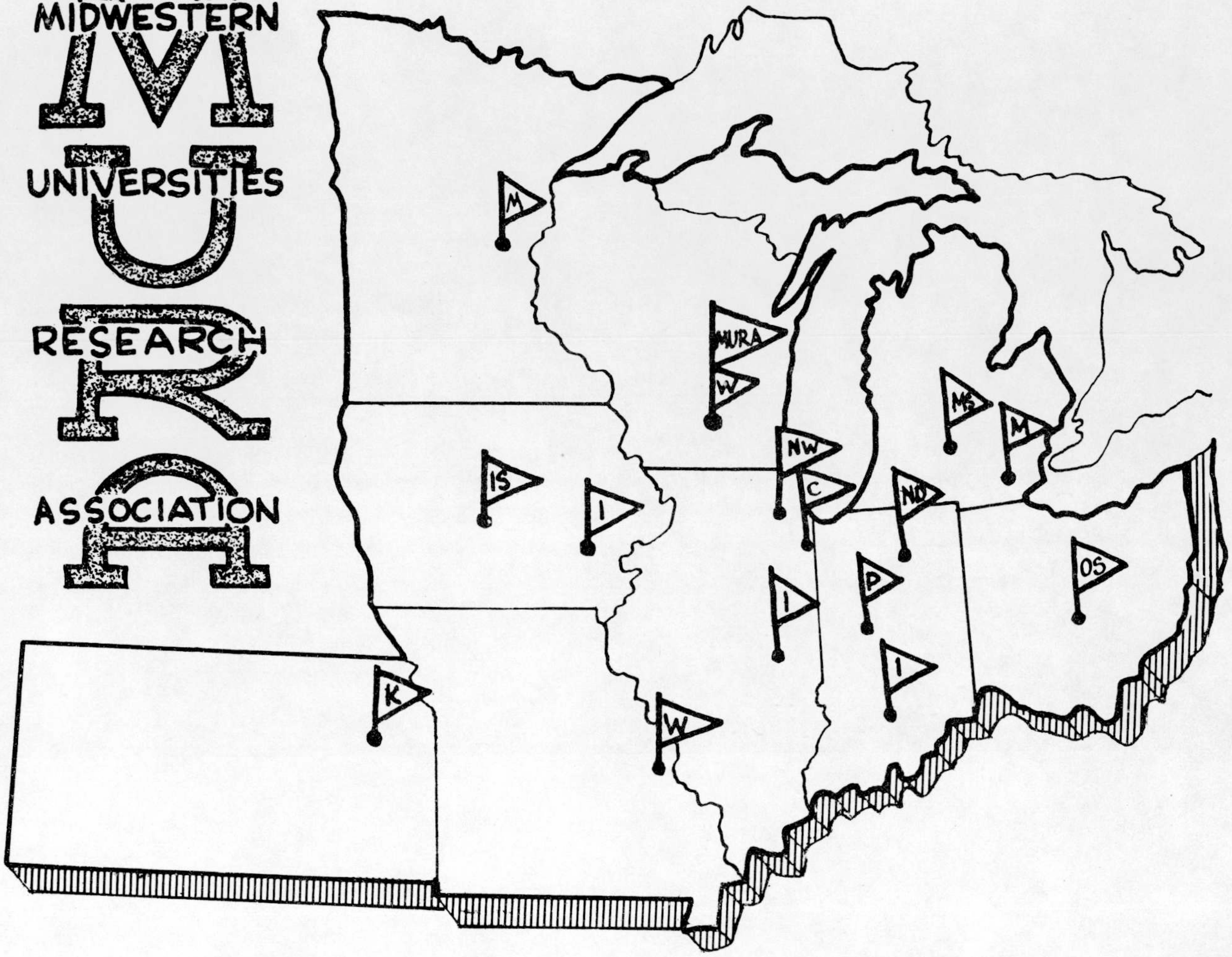

SCALING RADIAL SECTOR FFAG ACCELERATORS WITHOUT MEDIAN PLANE SYMMETRY

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ABSTRACT

The effect of radial median plane fields on particle orbits of the two-way FFAG accelerator is considered. The equilibrium orbit and linear motion about it are treated. It is found that the circumference factor can be lowered, but that the vertical focusing is seriously weakened.

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** On leave from the State University of Iowa.

I. INTRODUCTION

This report describes a preliminary investigation of particle orbits in FFAG fields which do not possess median plane symmetry. The impetus for this investigation was the realization that many of the difficult problems of the Ohkawa two-way accelerator[1,2] are related to its large circumference factor. The two-way accelerator derives its guide field from the difference in vertical field between positive and negative magnets along a "scalloped" equilibrium orbit. If radial fields on the median plane are added, they give rise to additional vertical fields off the median plane and to additional equilibrium orbit scalloping off that plane. The average vertical field on the equilibrium orbit can be increased in this way without increasing the magnitude of the field, so that the circumference factor is lowered. Such fields have, of course complications over more symmetric fields, perhaps the most notable being the existence of twice as many essential resonances.

The success of such an idea rests ultimately on the question of orbit stability. In the present report we treat only the linear problem in scaling radial sector accelerators. The one-way spiral sector accelerator is not in need of much help in the matter of circumference factor. In a two-way accelerator spiral sectors are not the striking refinement they are in the one-way case, for by the nature of the two-way accelerator the negative magnets cannot be eliminated. Spiral sectors do aid vertical focusing in the two-way accelerator, but at the expense of greater complication in magnet construction.

It should be noted that a special case of a one-way spiral sector accelerator has been treated by Ohkawa[3]. The treatment of fields in the present investigation, in particular the splitting of the median plane field into vertical and radial parts, derives from his work.

The conclusion drawn from this work is somewhat negative. It appears that the gain in the circumference factor of the two-way accelerator is achieved at the expense of the already weak vertical focusing. It is planned to extend this work to the non-scaling case in the hope that some amelioration of this difficulty can be found.

II. FIELDS AND POTENTIALS

Things begin in a conventional manner. The field components in cylindrical coordinates (r, θ, z) are expandable in powers of z/r. Thus

$$\left\{\begin{aligned} B_r &= -B_0\left(\frac{r}{r_0}\right)^k \sum_{m=0}^{\infty} R_m(\theta)\left(\frac{z}{r}\right)^m \\ B_\theta &= -B_0\left(\frac{r}{r_0}\right)^k \sum_{m=0}^{\infty} \Theta_m(\theta)\left(\frac{z}{r}\right)^m \\ B_z &= -B_0\left(\frac{r}{r_0}\right)^k \sum_{m=0}^{\infty} Z_m(\theta)\left(\frac{z}{r}\right)^m . \end{aligned}\right. \tag{2.1}$$

The coefficient is negative ($B_0 > 0$) because a downward field is required to bend positively charged particles around a circle in the increasing θ direction.

The relevant Maxwell's equations $\nabla \cdot \underline{B} = 0$ and $\nabla \times \underline{B} = 0$ give relations between the coefficients of (2.1). Here (and throughout) total derivatives with respect to θ are denoted by primes.

$$\left\{\begin{aligned} Z_m' &= (m+1)\,\Theta_{m+1} \\ (m+1)R_{m+1} &= (k-m)\,Z_m \\ (k+1-m)\,\Theta_m &= R_m' \\ (k+1-m)\,R_m + \Theta_m' &+ (m+1)\,Z_{m+1} = 0 . \end{aligned}\right. \tag{2.2}$$

Two symmetries are evident from (2. 2). R_m and θ_m of even (odd) m are related only to R_n and θ_n of even (odd) n and to Z_p of odd (even) p. Further, R_m even (odd) in θ is equivalent to Z_{m-1} even (odd) and to θ_m odd (even) in θ.

Because of the first symmetry, the field can be split into two parts which satisfy respectively on the median plane ($z \equiv 0$) the conditions

$$B_r = B_\theta = 0 \qquad \text{(I)}$$

and

$$B_z = 0 \qquad \text{(II)}.$$

Fields of type I are determined everywhere (barring convergence difficulties) by specification of B_z on the median plane, while fields of type II are determined everywhere by specification of B_r on the median plane, except for azimuthal fields which are independent of θ.* Such fields are not governed by the Maxwellian restrictions (2. 2). They are not of much interest in the present work, which is directed toward two-way accelerators.

Since only fields periodic in θ are of interest, the coefficients of (2. 1) can be expanded in Fourier series.

$$\begin{aligned} R_m &= \sum_{n=-\infty}^{\infty} \xi_{m,n} e_n \\ \Theta_m &= \sum_{n=-\infty}^{\infty} \eta_{m,n} e_n \\ Z_m &= \sum_{n=-\infty}^{\infty} \zeta_{m,n} e_n, \end{aligned} \qquad (2.3)$$

where

$$e_n = e^{in\theta} \qquad (2.4)$$

*I am indebted to Dr. L. C. Teng for calling this exception to my attention.

For fields of type I a little manipulation gives the recursion relations

$$\left\{\begin{aligned} \zeta_{m+2,n} &= \frac{1}{(m+1)(m+2)}\left[n^2-(k-m)^2\right]\zeta_{m,n} \\ \xi_{m+1,n} &= \frac{k-m}{m+1}\,\zeta_{m,n} \\ \eta_{m+1,n} &= \frac{in}{m+1}\,\zeta_{m,n}, \end{aligned}\right. \tag{2. 5}$$

which show by repeated application that all the fields of type I can be expressed in terms of the $\zeta_{o,n}$.

For fields of type II,

$$\left\{\begin{aligned} \xi_{m+2,n} &= \frac{1}{(m+1)(m+2)}\,\frac{k-1-m}{k+1-m}\left[n^2-(k+1-m)^2\right]\xi_{m,n} \\ \eta_{m,n} &= \frac{in}{k+1-m}\,\xi_{m,n} \\ \zeta_{m+1,n} &= \frac{1}{(m+1)(k+1-m)}\left[n^2-(k+1-m)^2\right]\xi_{m,n}, \end{aligned}\right. \tag{2. 6}$$

so that all fields of type II (except for those of the form $\eta_{m,o}$) can be expressed in terms of the $\xi_{o,n}$.

$\underline{B}$ can be derived from a vector potential $\underline{A}$ which is of direct interest because it appears in the Lagrangian. The vector potential components can be expanded in a manner quite analogous to the field expansion above. That is, we define

$$\begin{cases} A_r = -r_0 B_0 \left(\frac{r}{r_0}\right)^{k+1} \sum\limits_{m=0}^{\infty} \sum\limits_{n=-\infty}^{\infty} \varphi_{m,n}\, e_n \left(\frac{z}{r}\right)^m \\ A_\theta = -r_0 B_0 \left(\frac{r}{r_0}\right)^{k+1} \sum\limits_{m=0}^{\infty} \sum\limits_{n=-\infty}^{\infty} \psi_{m,n}\, e_n \left(\frac{z}{r}\right)^m \\ A_z = -r_0 B_0 \left(\frac{r}{r_0}\right)^{k+1} \sum\limits_{m=0}^{\infty} \sum\limits_{n=-\infty}^{\infty} \chi_{m,n}\, e_n \left(\frac{z}{r}\right)^m . \end{cases} \tag{2.7}$$

The potential components are related to the field components (from $\underline{B} = \nabla \times \underline{A}$) by

$$\begin{cases} \xi_{m,n} = in\,\chi_{m,n} - (m+1)\,\psi_{m+1,n} \\ \eta_{m,n} = (m+1)\,\varphi_{m+1,n} - (k+1-m)\,\chi_{m,n} \\ \zeta_{m,n} = (k+2-m)\,\psi_{m,n} - in\,\varphi_{m,n} \;, \end{cases} \tag{2.8}$$

while from $\nabla \cdot \underline{A} = 0$,

$$(k+2-m)\,\varphi_{m,n} + in\,\psi_{m,n} + (m+1)\,\chi_{m+1,n} = 0. \tag{2.9}$$

From (2.8) and (2.9) recursion relations can be derived:

$$\begin{cases} (m+1)(m+2)\,\psi_{m+2,n} + \left[(k+2-m)^2 - n^2\right]\psi_{m,n} = (k+2-m)\,\zeta_{m,n} - (m+1)\,\xi_{m+1,n} \\ (m+1)(m+2)\,\varphi_{m+2,n} + \left[(k+2-m)^2 - n^2\right]\varphi_{m,n} = (m+1)\,\eta_{m+1,n} - \frac{in(k-m)}{k+2-m}\,\zeta_{m,n} \\ (m+1)(m+2)\,\chi_{m+2,n} + \left[(k+1-m)^2 - n^2\right]\chi_{m,n} = in\,\xi_{m,n} - (k+1-m)\,\eta_{m,n} . \end{cases} \tag{2.10}$$

The r.h.s. of the third of Eqs. (2.10) is identically zero, from the second of Eqs. (2.2).

We choose a gauge such that

$$\begin{cases} \varphi_{0,n} = 0 \\ \psi_{0,n} = \dfrac{\zeta_{0,n}}{k+2} \\ \chi_{0,n} = 0 \; . \end{cases} \tag{2.11a}$$

Then from (2.10) or from (2.8) and (2.9),

$$\begin{cases} \varphi_{1,n} = \eta_{0,n} \\ \psi_{1,n} = -\xi_{0,n} \\ \chi_{1,n} = -in\,\zeta_{0,n}/k+2 \; . \end{cases} \tag{2.11b}$$

$$\begin{cases} \varphi_{2n} = \frac{1}{2}\left[\dfrac{-ink}{k+2}\zeta_{0,n} + \eta_{1,n}\right] = \dfrac{in\,\zeta_{0,n}}{k+2} \\ \psi_{2,n} = \frac{1}{2}\left[\dfrac{n^2}{k+2}\zeta_{0,n} - \xi_{1,n}\right] = \dfrac{n^2 - k(k+2)}{2(k+2)}\zeta_{0,n} \\ \chi_{2,n} = 0 \; . \end{cases} \tag{2.11c}$$

$$\begin{cases} \varphi_{3,n} = \frac{1}{3}\eta_{2,n} = \dfrac{in}{6(k+1)}\left[n^2 - (k+1)^2\right]\xi_{0,n} \\ \psi_{3,n} = -\frac{1}{3}\xi_{2,n} = -\frac{1}{6}\dfrac{k-1}{k+1}\left[n^2 - (k+1)^2\right]\xi_{0,n} \\ \chi_{3,n} = \dfrac{n^2 - k^2}{6(k+2)}\,in\,\zeta_{0,n} \; . \end{cases} \tag{2.11d}$$

III. EQUATIONS OF MOTION

The equations of motion of a charged particle of momentum p in a static magnetic field are derivable from the Lagrangian

$$L = r_0 p \left\{ \sqrt{(1+x)^2 + x'^2 + y'^2} + \frac{\alpha}{r_0 B_0} \left[x' A_r + (1+x) A_\theta + y' A_z \right] \right\}, \tag{3.1}$$

where the relative deviations x and y from a circle of radius r_o in the median plane are defined by

$$\begin{cases} r = r_0 (1+x) \\ z = r_0 y \end{cases} \tag{3.2}$$

and

$$\alpha = \frac{e r_0 B_0}{cp} \tag{3.3}$$

is the parameter first introduced by Ohkawa[1]. α is a measure of the circle about which the motion is expanded. It can have either sign; negative values of α correspond to $eB_0/p < 0$. A change of sign of α describes a particle of opposite sign in the same field, a particle of the same charge in a field of opposite sign or a particle of the same charge in the same field circulating in the opposite sense of rotation.*

Different circles of expansion r_o and r_1 corresponding to α_0 and α_1 of the same sign are related for a scaling field by

$$\frac{\alpha_1}{\alpha_0} = \left(\frac{r_1}{r_0}\right)^{k+1}, \tag{3.4}$$

since for a given field $B_0 \sim r_0^{\,k}$. Thus a motion of a given sense of circulation can be expanded about any radius.

*It was first pointed out by Parzen that changing the sign of α is equivalent to choosing the opposite sign of the radical in (3.1), which is equivalent to motion in the opposite direction.

The magnitude of α depends on the units in which the magnetic field is given, since the field always occurs in the equations of motion multiplied by α. If a new scale is chosen for the magnetic field such that $\underline{B} \to a\underline{B}$, then $\alpha \to \alpha/a$.

α is also related to the circumference factor

$$C = \mathrm{Max}\left(\frac{r_e(\theta)}{\rho(\theta)}\right), \tag{3.5}$$

where $r_e\ (\theta)$ is the radius of the equilibrium orbit of a particle of given momentum and $\rho\,(\theta)$ is the radius of curvature of that equilibrium orbit, both at azimuth θ. Now

$$\rho = \frac{cp}{e\,|\underline{B}(r_e, \theta, z_e)|}.$$

Suppose that the maximum of (3.5) occurs at $\theta = \theta_o$. Choose $\alpha = \alpha_e$ such that $r = r_e\ (\theta_o)$. Then

$$C = \alpha_e \frac{|\underline{B}(r_e(\theta_o), \theta_o, z_e(\theta_o))|}{B_o}. \tag{3.6}$$

This is a generalization of the usual formula to include fields of type II. Note that r_e is the radius in cylindrical coordinates. The height of the orbit above the median plane enters only in the calculation of the field.

The equations of motion following from (3.1) are

$$\left\{\begin{aligned} \left(\frac{x'}{\sqrt{(1+x)^2 + x'^2 + y'^2}}\right)' &= \frac{1+x}{\sqrt{(1+x)^2 + x'^2 + y'^2}} + \frac{\alpha}{B_o}\left[(1+x)B_z - y'B_\theta\right] \\ \left(\frac{y'}{\sqrt{(1+x)^2 + x'^2 + y'^2}}\right)' &= \frac{\alpha}{B_o}\left[x'B_\theta - (1+x)B_r\right]. \end{aligned}\right. \tag{3.7}$$

In a two-way accelerator Eqs. (3.7) must be invariant under the combined transformation $\alpha \rightarrow -\alpha$, $\theta \rightarrow -\theta$. This can be accomplished by constructing B_z so that it is odd about some point θ_1. Then, from Maxwell's equations, B_r is also odd and B_θ is even about θ_1.

IV. THE EQUILIBRIUM ORBIT

The first task of an analytic development is the seeking of a periodic solution of (3.7), the equilibrium orbit. In order to proceed it is necessary to expand the equations of motion in powers of x, y, x' and y'. It should be noted that expansion of the equations of motion can produce equations which differ from those derived from an expanded Lagrangian, because it is easy to transform the equations in such a manner as to lose their Lagrangian properties. The solutions will, of course, also differ.

We expand the Lagrangian (3.1)

$$L \cong 1 + x + \frac{1}{2}(x'^2 + y'^2) - \frac{1}{2}x(x'^2 + y'^2) + \frac{\alpha}{r_0 B_0}\left[x' A_r + (1+x) A_\theta + y' A_z \right], \tag{4.1}$$

where the square bracket is symbolic of the expanded vector potentials from (3.11). The approximate equations of motion derived from (4.1) are

$$\begin{cases} x'' = 1 + \frac{1}{2}x'^2 + x x'' - \frac{1}{2}y'^2 \\ \qquad - \alpha\left[(1+x)^{k+1} \sum_m Z_m \left(\frac{y}{1+x}\right)^m - y'(1+x)^k \sum_m \Theta_m \left(\frac{y}{1+x}\right)^m \right] \\ y'' = x'y' + x y'' - \alpha\left[x'(1+x)^k \sum_m \Theta_m \left(\frac{y}{1+x}\right)^m - (1+x)^{k+1} \sum_m R_m \left(\frac{y}{1+x}\right)^m \right], \end{cases} \tag{4.2}$$

where the square brackets are again symbolic of an expansion too laborious to write out here.

Since we seek a periodic solution, it is natural to resort to Fourier series. Thus we write

$$\left\{\begin{aligned} x &= {\sum_{n=-\infty}^{\infty}}' x_n e_n \\ y &= {\sum_{n=-\infty}^{\infty}}' y_n e_n . \end{aligned}\right. \tag{4.3}$$

Substitution of (4.3) into (4.2) and use of harmonic balance or use of (4.1) as the integrand of a variational principal with (4.3) as a trial solution both give an infinite set of equations for the x_n and y_n (given here correct through second order in x, y, x' and y')

$$\left\{\begin{aligned} -n^2 x_n = {} & \delta_{n0} - \alpha \zeta_{0,n} - \alpha(k+1) {\sum_m}' \zeta_{0,m} x_{n-m} + i\alpha {\sum_m}' m y_m \eta_{0,n-m} - \alpha {\sum_m}' \zeta_{1,m} y_{n-m} \\ & - \frac{1}{2} {\sum_m}' m(n+m) x_m x_{n-m} + \frac{1}{2} {\sum_m}' m(n-m) y_m y_{n-m} \\ & - \frac{1}{2}\alpha(k+1) k {\sum_{m,p}}' x_m x_p \zeta_{0,n-m-p} - \alpha k {\sum_{m,p}}' x_m y_p \zeta_{1,n-m-p} \\ & - \alpha {\sum_{m,p}}' y_m y_p \zeta_{2,n-m-p} + i\alpha k {\sum_{m,p}}' m y_m \eta_{0,p} x_{n-m-p} + i\alpha {\sum_{m,p}}' m y_m \eta_p y_{n-m-p} \\ & + \cdots \\ -n^2 y_n = {} & \alpha \xi_{0,n} + \alpha {\sum_m}' \xi_{1,m} y_{n-m} + \alpha(k+1) {\sum_m}' \xi_{0,m} x_{n-m} - i\alpha {\sum_m}' m x_m \eta_{0,n-m} \\ & - {\sum}' n m y_m x_{n-m} + \alpha {\sum_{m,p}}' y_m y_p \xi_{2,n-m-p} - ik\alpha {\sum_{m,p}}' m x_m x_p \eta_{0,n-m-p} \\ & + \frac{1}{2}\alpha(k+1) k {\sum_{m,p}}' x_m x_p \xi_{0,n-m-p} + \alpha k {\sum_{m,p}}' x_m y_p \xi_{1,n-m-p} \\ & - i {\sum_{m,p}}' m x_m \eta_{1,p} y_{n-m-p} + \cdots \end{aligned}\right. \tag{4.4}$$

In any reasonable accelerator the x_n and y_n are small compared to unity. A system of successive approximations based on this smallness is not difficult to construct. We will substitute the i^{th} approximation in the r.h.s. of (4.4) to determine the $(i+1)^{st}$.

Such solutions can be calculated using any value of α. It is interesting to choose $x_o = 0$ and calculate α from (4.4). In this way all the possible values of can be found. This method, which was used by Ohkawa[1], has also the advantage that all the contributions of x_o to (4.4) vanish, which speeds the convergence of the system of successive approximations. With $x_o = 0$, the difference between the $(i+1)^{st}$ and i^{th} approximations relative to the i^{th} is of order $\alpha k \tau / n^2$, where τ is the size of the largest field harmonic. This quantity has a value about 1/5 for any FFAG accelerator.

The sums in Eqs. (4.4) all run from $-\infty$ to ∞. α is calculated from the first of (4.4) with n = 0. The second of (4.4) is used to calculate y_o. In an accelerator where $y_o = 0$ (e.g., a two-way accelerator), the second of (4.4) is just a statement of the fact that the r.h.s. of the second of Eqs. (4.2) has zero mean on the equilibrium orbit.

We take as our initial solution $x_n^{(0)} = y_n^{(0)} = 0$. Then, from (4.4),

$$\begin{cases} x_n^{(1)} = \dfrac{\alpha \, \zeta_{0,n}}{n^2} \\ y_n^{(1)} = -\dfrac{\alpha \, \xi_{0,n}}{n^2} \end{cases} \qquad (4.5)$$

(4.5) may have an error of about 30%. A relatively simple solution with an error not much larger than 10% can be found by choosing the largest terms of $x_n^{(2)}$ and $y_n^{(2)}$. Such a solution is

$$
\left\{
\begin{aligned}
x_n &= \frac{\alpha}{n^2}\left\{ \zeta_{0,n} + \alpha(k+1)\sum_{m\neq n} \frac{\zeta_{0,m}\,\zeta_{0,n-m}}{(n-m)^2} - \right. \\
&\qquad \left. - \frac{\alpha}{k+1}\sum_{m\neq 0,n} \frac{\left[1-\frac{k+1}{m^2}+m(n-m)\right]\xi_{0,m}\,\xi_{0,n-m}}{(n-m)^2} \right\} \\
y_n &= -\frac{\alpha}{n^2}\left\{ \xi_{0,n} - \alpha\sum_{m\neq n} \frac{k\,\zeta_{0,m}\,\xi_{0,n-m} - (k+1)\,\xi_{0,m}\,\zeta_{0,n-m}}{(n-m)^2} \right. \\
&\qquad \left. + \alpha \sum_{m\neq 0,n} \frac{n-m}{m(k+1)}\,\zeta_{0,m}\,\xi_{0,n-m} \right\} .
\end{aligned}
\right.
\tag{4.6}
$$

The non-linear terms of (4.4) make no contribution in this approximation, because the x_n and y_n are so small.

Clearly, if the process of successive approximations is continued, higher and higher powers of α enter the solution. Then the equation determining α is in our approximation a polynomial of infinite degree. There are therefore an infinite number of roots, i.e., values of α. To each real value of α there corresponds an equilibrium orbit, so we should not be surprised at the occurrence of multiple equilibrium orbits. However, most of these do not appear to be of practical interest, since $|\alpha| \gg 1$ and consequently the circumference factor is very large. Examination of the order of successive terms shows that only a few roots are of order unity. In addition, it is doubtful that the oscillations about many of these large α orbits are stable, which probably accounts for the fact that they have not been found.

The equation determining α is

$$1 - \alpha \zeta_{0,0} - \alpha^2 \left\{ (k + \tfrac{3}{2})(G_1^2 + G_2^2) \right\} + \ldots = 0, \tag{4.7}$$

where

$$\left\{ \begin{aligned} G_1^2 &= \sum_{m \neq 0} \frac{\zeta_{0,m}\, \zeta_{0,-m}}{m^2} \\ G_2^2 &= \sum_{m \neq 0} \frac{\xi_{0,m}\, \xi_{0,-m}}{m^2} . \end{aligned} \right. \tag{4.8}$$

The direction of the vertical field enters only in the calculation of α. If the sign of the vertical field is in the wrong direction for particles to circulate about the accelerator, the sign of every term of (4.7) changes and there are no real roots.

In the interesting special case of a two-way accelerator, the coefficients of all odd powers of α vanish. To very good accuracy,

$$\alpha = \pm \left[(k + \tfrac{3}{2})(G_1^2 + G_2^2) \right]^{-\frac{1}{2}} . \tag{4.9}$$

Since the two roots are equal and opposite, a particle has the same dynamical properties in either direction. It may be noted also that there are no real roots for $k < 0$, so that it is impossible to construct a two-way accelerator with negative momentum compaction.

V. LINEAR OSCILLATIONS ABOUT THE EQUILIBRIUM ORBIT

It is not difficult to introduce axes moving with the equilibrium orbit[4] as in the conventional case. However, for simplicity we expand in a less elegant manner following earlier work[5] on the spiral sector accelerator. We introduce new variables u, v by

$$\begin{cases} x = x^e(\theta) + u \\ y = y^e(\theta) + v \, , \end{cases} \tag{5.1}$$

where $x^e(\theta)$ and $y^e(\theta)$ are the solutions of the equilibrium orbit equations. Linear equations of motion follow from a Lagrangian containing only quadratic terms (terms independent of u, v, u' and v' give no contributions to the equations of motion and terms linear in these variables vanish because the equilibrium orbit is a solution of the equations of motion). Then

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2} a_1 u'^2 + a_2 u'v' + \tfrac{1}{2} a_3 v'^2 + a_4 uu' + a_5 uv' + a_6 u'v \\ & + a_7 vv' + \tfrac{1}{2} a_8 u^2 + a_9 uv + \tfrac{1}{2} a_{10} v^2 , \end{aligned} \tag{5.2}$$

where

$$\begin{cases} a_1 = \dfrac{\partial^2 L}{\partial x'^2} = \dfrac{(1+x)^2 + y'^2}{L_0^3} \\ a_2 = \dfrac{\partial^2 L}{\partial x' \partial y'} = -\dfrac{x'y'}{L_0^3} \\ a_3 = \dfrac{\partial^2 L}{\partial y'^2} = \dfrac{(1+x)^2 + x'^2}{L_0^3} \\ a_4 = \dfrac{\partial^2 L}{\partial x \partial x'} = -\dfrac{x'(1+x)}{L_0^3} + \dfrac{e}{cp}\dfrac{\partial A_r}{\partial x} \\ a_5 = \dfrac{\partial^2 L}{\partial x \partial y'} = -\dfrac{y'(1+x)}{L_0^3} + \dfrac{e}{cp}\dfrac{\partial A_3}{\partial x} \\ a_6 = \dfrac{\partial^2 L}{\partial y \partial x'} = \dfrac{e}{cp}\dfrac{\partial A_r}{\partial y} \\ a_7 = \dfrac{\partial^2 L}{\partial y \partial y'} = \dfrac{e}{cp}\dfrac{\partial A_3}{\partial y} \\ a_8 = \dfrac{\partial^2 L}{\partial x^2} = \dfrac{x'^2 + y'^2}{L_0^3} + \dfrac{e}{cp}\left[x' \dfrac{\partial^2 A_r}{\partial x^2} + \dfrac{\partial^2}{\partial x^2}\left((1+x)A_\theta\right) + y' \dfrac{\partial^2 A_3}{\partial x^2}\right] \\ a_9 = \dfrac{\partial^2 L}{\partial x \partial y} = \dfrac{e}{cp}\left[x' \dfrac{\partial^2 A_r}{\partial x \partial y} + \dfrac{\partial}{\partial x}\left((1+x)\dfrac{\partial A_\theta}{\partial y}\right) + y' \dfrac{\partial^2 A_3}{\partial y \partial x}\right] \\ a_{10} = \dfrac{\partial^2 L}{\partial y^2} = \dfrac{e}{cp}\left[x' \dfrac{\partial^2 A_r}{\partial y^2} + (1+x)\dfrac{\partial^2 A_\theta}{\partial y^2} + y' \dfrac{\partial^2 A_3}{\partial y^2}\right] \end{cases} \tag{5.3}$$

and

$$L_0 = \sqrt{(1+x)^2 + x'^2 + y'^2}. \tag{5.4}$$

All quantities appearing in the a_i are to be evaluated at the equilibrium orbit.

The linear equations following from (5. 2) are

$$\begin{cases} a_1 u'' + a_2 v'' = -a_1' u' + (a_5 - a_6 - a_2')v' + (a_8 - a_4')u + (a_9 - a_6')v \\ a_2 u'' + a_3 v'' = (a_6 - a_5 - a_2')u' - a_3' v' + (a_9 - a_5')u + (a_{10} - a_7')v . \end{cases} \tag{5.5}$$

The coefficients appearing in (5. 5) can be evaluated in terms of the median plane fields by use of (2. 11). These coefficients are again essentially expansions in powers of $\alpha k r/n^2$. We expand only through first order in this quantity. This means that we need only the crudest approximation (4. 5) for the equilibrium orbits, since more accurate expressions are of higher order in $\alpha k r/n^2$.

We define Fourier coefficients of the a_i by

$$a_i = \sum_{n=-\infty}^{\infty} a_{i,n} e_n . \tag{5.6}$$

Then

$$\begin{cases} a_{1,n} = \delta_{n0} - \dfrac{\alpha \zeta_{0,n}}{n^2} + \dfrac{3}{2}\alpha^2 \displaystyle\sum_{m \neq 0,n} \dfrac{\zeta_{0,m}\zeta_{0,n-m}}{m(n-m)} + \dfrac{1}{2}\alpha^2 \displaystyle\sum_{m \neq 0,n} \dfrac{\xi_{0,m}\xi_{0,n-m}}{m(n-m)} \\ a_{2,n} = \alpha^2 \displaystyle\sum_{m \neq 0,n} \dfrac{\zeta_{0,m}\xi_{0,n-m}}{m(n-m)} \\ a_{3,n} = \delta_{n0} - \dfrac{\alpha \zeta_{0,n}}{n^2} + \dfrac{1}{2}\alpha^2 \displaystyle\sum_{m \neq 0,n} \dfrac{\zeta_{0,m}\zeta_{0,n-m}}{m(n-m)} + \dfrac{3}{2}\alpha^2 \displaystyle\sum_{m \neq 0,n} \dfrac{\xi_{0,m}\xi_{0,n-m}}{m(n-m)} \end{cases}$$

$$
\left\{
\begin{aligned}
a_{4,n} &= -\frac{i\alpha \zeta_{0,n}}{n} + \frac{i\alpha^2 k}{k+1} \sum_{m \neq n} \frac{m}{(n-m)^2}\, \xi_{0,m}\, \xi_{0,n-m} \\
a_{5,n} &= \frac{i\alpha \xi_{0,n}}{n} + \frac{i\alpha^2 k}{k+2} \sum_{m \neq n} \frac{m}{(n-m)^2}\, \zeta_{0,m}\, \xi_{0,n-m} \\
a_{6,n} &= -\frac{i\alpha n}{k+1}\, \xi_{0,n} - i\alpha^2 \frac{k^2+4k+2}{(k+1)(k+2)} \sum_{m \neq n} \frac{m\, \xi_{0,m}\, \zeta_{0,n-m}}{(n-m)^2} \\
a_{7,n} &= \frac{in\alpha\, \zeta_{0,n}}{k+2} + i\alpha^2 \frac{k}{k+2} \sum_{m \neq n} \frac{m}{(n-m)^2}\, \zeta_{0,m}\, \zeta_{0,n-m} \\
a_{8,n} &= -\frac{\alpha k^2}{k+2}\, \zeta_{0,n} - \alpha^2 \sum_{m \neq 0,n} \frac{\zeta_{0,m}\, \zeta_{0,n-m} + \xi_{0,m}\, \xi_{0,n-m}}{m(n-m)} \\
&\quad - \frac{\alpha^2 k^3}{k+2} \sum_{m \neq n} \frac{\zeta_{0,m}\, \zeta_{0,n-m}}{(n-m)^2} - \alpha^2 (k-1)^2 \sum_{m \neq n} \frac{\xi_{0,m}\, \xi_{0,n-m}}{(n-m)^2} \\
a_{9,n} &= \alpha(k+1)\, \xi_{0,n} + \alpha^2 (k+1) k \sum_{m \neq n} \frac{\xi_{0,m}\, \zeta_{0,n-m}}{(n-m)^2} \\
&\quad + \alpha^2 \sum_{m \neq 0} \frac{n-m}{m} \left[\xi_{0,n-m}\, \zeta_{0,m} + \frac{k+1}{k+2}\, \xi_{0,m}\, \zeta_{0,n-m} \right] \\
&\quad - \alpha^2 \frac{k}{k+2} \sum_{m \neq n} \frac{[m^2 - k(k+2)]}{(n-m)^2}\, \zeta_{0,m}\, \xi_{0,n-m} \\
a_{10,n} &= -\alpha \frac{n^2 - k(k+2)}{k+2}\, \zeta_{0,n} + \frac{2\alpha^2}{k+2} \sum_{m \neq 0} \frac{n-m}{m}\, \zeta_{0,m}\, \zeta_{0,n-m} \\
&\quad - \frac{k}{k+2}\, \alpha^2 \sum_{m \neq n} \frac{m^2 - k(k+2)}{(n-m)^2}\, \zeta_{0,m}\, \zeta_{0,n-m} \\
&\quad - \alpha^2 \frac{k-1}{k+1} \sum_{m \neq n} \frac{m^2 - (k+1)^2}{(n-m)^2}\, \xi_{0,m}\, \xi_{0,n-m}\, .
\end{aligned}
\right.
\qquad (5.7)
$$

If we specialize to the case of a large accelerator (k >> 1), many terms of (5. 5) may be neglected. For example

$$a_1 - 1 \sim a_3 - 1 \sim \frac{\alpha \tau}{n^2} \sim \frac{1}{n\sqrt{k}} ;$$

$$a_1' \sim a_2' \sim a_3' \sim \frac{1}{\sqrt{k}} ,$$

so that

$$a_1' u' \sim a_2' u' \sim a_2' v' \sim a_3' v' \sim \nu / \sqrt{k} ,$$

where ν is the betatron oscillation wavelength. Such terms are of order unity, while the leading terms are of order k. In the same way

$$\frac{(a_5 - a_6) v'}{(a_8 - a_4') u} \sim \frac{\nu n}{k^2} .$$

To good accuracy, the equations of motion are

$$\begin{cases} u'' = K u + L v \\ v'' = M u + N v , \end{cases} \tag{5. 8}$$

where

$$\begin{cases} K = a_8 - a_4' = \sum_{n=-\infty}^{\infty} K_n e_n \\ L = a_9 - a_6' = \sum_{n=-\infty}^{\infty} L_n e_n \\ M = a_9 - a_5' = \sum_{n=-\infty}^{\infty} M_n e_n \\ N = a_{10} - a_7' = \sum_{n=-\infty}^{\infty} N_n e_n . \end{cases} \tag{5. 9}$$

Floquet's theorem applies to (5. 8). That is, there is a solution of the form

$$
\begin{aligned}
u &= e^{i\nu\theta} \sum_{n=-\infty}^{\infty} u_n e_n \\
v &= e^{i\nu\theta} \sum_{n=-\infty}^{\infty} v_n e_n .
\end{aligned}
\tag{5.10}
$$

If we substitute (5.10) into (5.8) and equate coefficients of e_n, we find

$$
\begin{aligned}
-(\nu+n)^2 u_n &= \sum_{m=-\infty}^{\infty} \left(K_{n-m} u_m + L_{n-m} v_m \right) \\
-(\nu+n)^2 v_n &= \sum_{m=-\infty}^{\infty} \left(M_m u_{n-m} + N_m v_{n-m} \right).
\end{aligned}
\tag{5.11}
$$

Eqs. (5.11) are exact. If we now assume that u_o and v_0 are large compared to the other Fourier coefficients and substitute $u_n^{(0)} = \delta_{n0} u_0$, $v_n^{(0)} = \delta_{n0} v_0$ on the r.h.s. of (5.11), we find

$$
\begin{aligned}
u_n^{(1)} &= -\frac{1}{(\nu^{(0)}+n)^2} \left(K_n u_0 + L_n v_0 \right) \\
v_n^{(1)} &= -\frac{1}{(\nu^{(0)}+n)^2} \left(M_n u_0 + L_n v_0 \right),
\end{aligned}
\tag{5.12}
$$

and, from Eqs. (5.11) with n = 0,

$$
(\nu^{(0)})^2 = -\frac{1}{2} \left\{ K_0 + N_0 \pm \sqrt{(K_0 - N_0)^2 + 4 L_0 M_0} \right\}.
\tag{5.13}
$$

Substitution of (5.12) in the r.h.s. of (5.11) gives in the same way a second approximation for ν,

$$
\begin{aligned}
(\nu^{(1)})^2 = \frac{1}{2} \Bigg\{ & \sum_{m \neq 0} \frac{K_m K_{-m} + 2 L_m M_{-m} + N_m N_{-m}}{(\nu^{(0)}+m)^2} - K_0 - N_0 \\
& \pm \sqrt{\left(K_0 - N_0 + \sum_{m \neq 0} \frac{N_m N_{-m} - K_m K_{-m}}{(\nu^{(0)}+m)^2} \right)^2 + 4 \left(L_0 - \sum_{m \neq 0} \frac{K_m L_{-m} + L_{-m} N_m}{(\nu^{(0)}+m)^2} \right) \left(M_0 - \sum_{m \neq 0} \frac{K_m M_{-m} + M_{-m} N_m}{(\nu^{(0)}+m)^2} \right)} \Bigg\}.
\end{aligned}
\tag{5.14}
$$

(5.14) is equivalent to the usual smooth approximation[6] except that we have not neglected $\nu^{(0)}$ compared to m in the denominators. In this way it is akin to, though presumably not as accurate as Vogt-Nilsen's treatment.[7]

In our approximation, where $1/k$ is neglected compared to unity

$$K_m = -N_m \qquad (m \neq 0)$$

from

$$\frac{\partial B_3}{\partial x} = \frac{\partial B_r}{\partial y},$$

and

$$L_0 = M_0$$

from

$$\left\langle \frac{\partial B_3}{\partial y} + \frac{\partial B_r}{\partial x} \right\rangle_\theta = 0$$

since

$$\left\langle \frac{\partial B_\theta}{\partial \theta} \right\rangle = 0$$

Then (5.14) reduces to

$$(\nu^{(1)})^2 = \sum_{m \neq 0} \frac{K_m K_{-m} + L_m M_{-m}}{(\nu^{(0)} + m)^2} - \frac{1}{2}(K_0 + N_0) \pm \frac{1}{2}\sqrt{(K_0 - N_0)^2 + 4M_0^2}\,. \tag{5.15}$$

VI. THE TWO-WAY ACCELERATOR

For the case of a two-way accelerator with $k \gg 1$, we have

$$\alpha = \pm \frac{1}{\sqrt{k\,(G_1^2 + G_2^2)}} \tag{6.1}$$

defining the average radius of the equilibrium orbit and, from (5.7),

$$
\begin{cases}
K_0 = -\alpha^2 k^2 (G_1^2 + G_2^2) \\
K_n = -N_n = -\alpha k \zeta_{0,n} \\
N_0 = \alpha^2 k^2 (G_1^2 + G_2^2) - \alpha^2 (F_1^2 + F_2^2) \\
L_0 = M_0 = -\alpha^2 F_{12}^2 \\
L_n = \alpha \dfrac{k^2 - n^2}{k} \xi_{0,n} \\
M_n = \alpha k \xi_{0,n} ,
\end{cases}
\tag{6.2}
$$

where G_1 and G_2 are defined in (4.8) and

$$
\begin{cases}
F_1^2 = \sum\limits_{m \neq 0}{}' \zeta_{0,m} \zeta_{0,-m} \\
F_2^2 = \sum\limits_{m \neq 0}{}' \xi_{0,m} \xi_{0,-m} \\
F_{12}^2 = \sum\limits_{m \neq 0} \xi_{0,m} \zeta_{0,-m} .
\end{cases}
\tag{6.3}
$$

From (5.15)

$$
\nu^2 = k + \frac{1}{2k} \frac{F_1^2 - F_2^2}{G_1^2 + G_2^2} \pm \frac{1}{2} \sqrt{\left[\frac{1}{k} \frac{F_1^2 + F_2^2}{G_1^2 + G_2^2} - 2k \right]^2 + 4 \left(\frac{F_{12}^2}{k(G_1^2 + G_2^2)} \right)^2} .
\tag{6.4}
$$

In the case where $\xi_{0,n} = 0$, this reduces to the well-known result

$$
\nu^2 = \begin{cases}
2k = \nu_x^2 \\
\dfrac{1}{k} \dfrac{F_1^2}{G_1^2} = \nu_y^2 .
\end{cases}
\tag{6.5}
$$

In the case where $\zeta_{0,n} = 0$ (purely radial field in the median plane)

$$\nu^2 = \begin{cases} 2k - \frac{1}{k}\frac{F_2^{\,2}}{G_2^{\,2}} \\ 0 \end{cases}, \tag{6.6}$$

so that there is no stability for one mode of oscillation and it is impossible to construct a two-way accelerator with purely radial median plane fields.

Let us consider for simplicity a field with a single harmonic and even symmetry. That is, we take

$$\begin{cases} \zeta_{0,n} = \frac{1}{2} f \left(\delta_{n,N} + \delta_{n,-N} \right) \\ \xi_{0,n} = \frac{1}{2} g \left(\delta_{n,N} + \delta_{n,-N} \right). \end{cases} \tag{6.7}$$

Then

$$G_1^{\,2} = \frac{1}{2}\frac{f^2}{N^2} \quad ; \quad F_1^{\,2} = \frac{1}{2} f^2 ;$$

$$G_2^{\,2} = \frac{1}{2}\frac{g^2}{N^2} \quad ; \quad F_2^{\,2} = \frac{1}{2} g^2 ;$$

$$F_{12}^{\,2} = \frac{1}{2} f g \quad ; \quad \alpha = \pm \sqrt{\frac{2N^2}{k(f^2+g^2)}}$$

Field maxima occur at the centers of positive magnets, i.e., at $\theta = 2\pi p/N$ (p integral). We calculate the maximum value of the field to first order in $\alpha k/N^2$.

$$\left|\frac{B_3}{B_0}\right|_{max.} = \sum_{n=-\infty}^{\infty}{}' \left\{ \zeta_{0,n} + \alpha k \sum_{m \neq n}{}' \frac{\zeta_{0,m}\,\zeta_{0,n-m}}{(n-m)^2} - \frac{\alpha}{k+1} \sum_{m \neq n}{}' \frac{m^2-(k+1)^2}{(n-m)^2}\, \zeta_{0,m}\,\zeta_{0,n-m} \right\}$$

$$= f + \alpha\left[\frac{k}{N^2}(f^2+g^2) - \frac{g^2}{k}\right]$$

$$\left|\frac{B_r}{B_0}\right|_{max.} = \sum_{n=-\infty}^{\infty}{}' \left\{ \xi_{0,n} + \alpha k \sum_{m \neq n}{}' \frac{\zeta_{0,m}\,\zeta_{0,n-m}}{(n-m)^2} - \alpha k \sum_{m \neq n} \frac{\zeta_{0,m}\,\xi_{0,n-m}}{(n-m)^2} \right\}$$

$$= g .$$

Then, using (3.6) and (6.1), we find

$$C = \sqrt{\frac{2N^2}{k}\left\{ \frac{\left[f + \alpha\left(\frac{k}{N^2}(f^2+g^2) - \frac{g^2}{k}\right)\right]^2 + g^2}{f^2+g^2} \right\}}$$

From (6.4)

$$\nu^2 = k + \frac{N^2}{2k}\left(\frac{f^2-g^2}{f^2+g^2}\right) \pm \frac{1}{2}\frac{N^2}{k}\sqrt{\left(1-\frac{2k^2}{N^2}\right)^2 + 4\left(\frac{fg}{f^2+g^2}\right)^2} .$$

In the following table we give C and the values of ν for k = 200 and $k/N^2 = 0.04$ as functions of g/f.

g/f	C	ν_1	ν_2
0	9.07	20	5.00
0.5	8.80	19.9	4.40
1	8.47	19.7	3.41
2	8.02	19.5	2.10
∞	7.28	13.7	0

It is apparent from the table that the expected gain in circumference factor is accompanied by a weakening of vertical focusing. The same discouragement can be achieved by inspection of (6.4).

The radial motion has a focusing average term arising from the equilibrium orbit scalloping. The vertical motion has this same average term with the opposite (defocusing) sign and Thomas focusing terms also arising from orbit scalloping (these last are the same as edge focusing). The alternating gradient terms add focusing in both dimensions, cancelling the effect of the defocusing vertical average term, but the Thomas focusing which arises from vertical orbit scalloping has a defocusing effect on the vertical motion, weakening the already small Thomas focusing. The term in (6.4) containing F_{12} is also a Thomas term and is always small compared to the k term also under the radical.

It appears that radial median plane fields offer little practical advantage for the two-way accelerator. The vertical scalloping can be used to reduce beam-beam interaction,[2] but only at the expense of vertical focusing.

REFERENCES

1. T. Ohkawa, R. S. I. 29, 108 (1958).
2. Proposal for a High Energy Research Facility, Midwestern Universities Research Association (1958)
3. T. Ohkawa, MURA-100 (TO-5, Internal, Unpublished) March 1, 1958.
4. See for example T. Sigurgeirsson, CERN Report 55-14, (CERN/T/TS-3, Unpublished) or the discussion on pp. 70 ff. of "The Alternating Gradient Synchrotron" (unpublished notes) by M. Hamermesh. A different system leading to non-orthogonal coordinates is given by E. D. Courant and H. S. Snyder, Annals of Physics 3, 1, (1958) (Appendix B).
5. F. T. Cole, MURA-95 (FTC-3, Unpublished) January 19, 1956.
6. Symon et al, P. R. 103, 1837 (1956) (Appendix A).
7. N. Vogt-Nilsen, MURA-118 (NVN-3, Unpublished) May, 1956.