4

ORNL-2542
Controlled
Thermonuclear Processes

STABILITY EFFECTS OF SURFACE CURRENTS ON A PINCHED PLASMA CONTAINING A UNIFORM VOLUME CURRENT

O. G. Harrold, Jr.

OAK RIDGE NATIONAL LABORATORY
operated by
UNION CARBIDE CORPORATION
for the
U.S. ATOMIC ENERGY COMMISSION

Printed in USA. Price 0.75 cents. Available from the

Office of Technical Services
U. S. Department of Commerce
Washington 25, D. C.

ORNL-2542
Controlled
Thermonuclear Processes

Contract No. W-7405-eng-26

Applied Nuclear Physics Division

STABILITY EFFECTS OF SURFACE CURRENTS ON A PINCHED PLASMA CONTAINING A UNIFORM VOLUME CURRENT

O. G. Harrold, Jr.*

Date Issued

JUN 10 1958

OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee
operated by
UNION CARBIDE CORPORATION
for the
U.S. Atomic Energy Commission

*Consultant from the University of Tennessee.

Abstract

Rosenbluth (LA-2030) has investigated the stability of the pinch for an equilibrium case which has sheet currents at the surface of the pinch and a uniform pressure inside the plasma. In addition, there was an axial magnetic field in the plasma interior, a different axial field in the outer region, and a conducting wall. R. J. Tayler (AERE-T/R-1984) has also investigated the pinch stability by the method of normal modes for an equilibrium case in which the plasma contains a uniform volume current in the axial direction and a pressure which varies parabolically in the radial direction. However, although axial magnetic fields were considered in this case they were taken to be equal in the interior and exterior and no external conducting wall was used.

In this calculation Tayler's analysis is extended to include a finite axial field in the plasma interior with no external axial field. In addition, there is now an external conducting wall. The resulting dispersion relation is discussed.

Table of Contents

1. Introduction

The stability of a pinched conducting fluid under the combined influence of a longitudinal magnetic field and a conducting exterior wall or shell has been studied by M. Rosenbluth.[1] A set of conditions allowing complete stability were found. These conditions included an assumption of constant pressure in the plasma and a limitation on the pinch radius. A jump or discontinuity of the longitudinal field is permitted at the plasma surface. The stability of a pinched conducting fluid has also been studied by R. J. Tayler[2] in a somewhat similar situation. Tayler restricted himself in the above papers to the case of a continuous longitudinal field at the plasma surface, but attains greater generality in another direction in that internal currents are permitted in the plasma steady state.

In the present paper we consider an idealized steady state pinched plasma that contains internal currents but allows surface currents that produce a discontinuity in the longitudinal field at the plasma surface. It turns out that by an analysis essentially similar to Tayler's we are able to extend somewhat the conditions of stability found by Rosenbluth.[3] The plasma is at rest initially. Using normal modes the behavior of a so-called "kink"-type perturbation is investigated. There is some reason to believe that perhaps this type of perturbation is most capricious from the standpoint of stability. It turns out that for a pressure distribution that is nearly constant and for a not too severe pinch all wave lengths are

---

1. Marshall Rosenbluth, Stability of the Pinch, LA-2030 (June 19, 1956).
2. R. J. Tayler, Hydromagnetic Instabilities of a Cylindrical Gas Discharge, AERE-1859 (Jan. 1956).
3. Since the work described in this paper was completed, a paper by R. J. Tayler in the Proc. of the Physical Society, Section B, Vol. 70, Part II, Nov. 1, 1957, "The Influence of an Axial Magnetic Field on the Stability of a Constricted Gas Discharge," has appeared. The specific case considered below is apparently not treated by Tayler.

stable. In the limit of constant pressure our dispersion relation passes over into that given by Rosenbluth[1] for the magnetohydrodynamic picture.

In the last section an estimate is made of the ratio of volume current density to surface current density parallel to the axis at which value instability sets in. The limitation on volume current density may alternatively be regarded as a limitation on the pinch radius.

The author is happy to acknowledge the helpful suggestions of Dr. Albert Simon who pointed out the existence and desirability of studying the problem treated below.

2. Geometry and Basic Physical Hypotheses

In addition to the assumption of a plasma describable by means of the magnetohydrodynamic equations, we explicitly assume:

1. Collisions are sufficiently frequent to justify the use of a scalar pressure.
2. The plasma is incompressible.
3. The plasma has infinite conductivity.
4. Displacement currents may be neglected.
5. The speed of light is infinite.

The geometry is that of the infinite cylinder. The plasma fills the region $0 \leq r \leq r_o$ initially. There is a vacuum region $r_o < r < r_1$ bounded by a conducting wall at $r = r_1$.

3. Plasma Equations and Boundary Conditions

Denote by $\underline{v}$ the velocity of the fluid, by p the scalar pressure and ρ the fluid density. Let ϵ denote the charge density, $\underline{E}$, $\underline{B}$ electromagnetic field quantities and $\underline{j}$ the current density. The magnetohydrodynamic equations in Gaussian units under the above limitations will take the form:

$$\rho \frac{d\underline{v}}{dt} = \frac{\underline{j} \times \underline{B}}{c} - \nabla p + \epsilon \underline{E} \tag{1}$$

$$\nabla \cdot \underline{v} = 0 \tag{2}$$

$$\underline{E} + \frac{(\underline{v} \times \underline{B})}{c} = 0 \tag{3}$$

$$\nabla \times \underline{B} = \frac{4\pi \underline{j}}{c} \tag{4}$$

$$\nabla \cdot \underline{B} = 0 \tag{5}$$

$$\nabla \times \underline{E} = \left(-\frac{1}{c}\right) \frac{\partial B}{\partial t} \tag{6}$$

$$\nabla \cdot \underline{E} = 4\pi \epsilon . \tag{7}$$

At an interface between plasma and vacuum Eqs. (8) and (11) below must be fulfilled. The values of a quantity in plasma and vacuum will be denoted by X^p, X^o, etc.[4] Let $\underline{n}$ denote the inward surface normal and let u be the component of plasma velocity along the inward normal. The boundary conditions have been derived by Kruskal and Schwarzschild and others.

$$\underline{n} \times (\underline{B}^p - \underline{B}^o) = \frac{4\pi j^*}{c} - (\underline{E}^p - \underline{E}^o)\left(\frac{u}{c}\right) \tag{8}$$

$$\underline{n} \cdot (\underline{B}^p - \underline{B}^o) = 0 \tag{9}$$

$$\underline{n} \times (\underline{E}^p - \underline{E}^o) = (\underline{B}^p - \underline{B}^o)\left(\frac{u}{c}\right) \tag{10}$$

$$j^* \times (\underline{B}^p + \underline{B}^o) = 2c\underline{n}(p^p - p^o) \tag{11}$$

At the wall, $r = r_1$, it is required that

$$\underline{n} \cdot \underline{B} = 0, \quad \underline{n} \times \underline{E} = 0 \tag{12}$$

where, again, $\underline{n}$ is a unit normal, this time at the wall.

4. The Steady State

In the plasma we assume $\underline{j} = (0, 0, j_z)$ and $\underline{B} = \left[0, B_\theta(r/r_o), B_z\right]$ where j_z, B_z, and B_θ are constants. From Eq. (1)

$$p = p_o\left(1 - \frac{r^2}{r_o^2}\right) + p_1 \tag{13}$$

4. A surface quantity is denoted by an asterisk, X^*.

where

(14) $$p_o = \frac{B_\theta^2}{4\pi}\ , \quad j_z = \frac{cB_\theta}{2\pi r_o}\ .$$

In the vacuum

(15) $$\underline{H} = \left[0,\ H_\theta\left(\frac{r_o}{r}\right),\ 0\right]$$

and

(16) $$p_1 = \frac{H_\theta^2 - B_\theta^2 - B_z^2}{8\pi}\ , \quad p_{max} = \frac{H_\theta^2 + B_\theta^2 - B_z^2}{8\pi}$$

The surface current is determined from Eq. (8),

(17) $$j_\theta^* = \frac{cB_z}{4\pi}\ , \quad j_z^* = \frac{c}{4\pi}\left(H_\theta - B_\theta\right)$$

A check shows that these quantities satisfy the steady state equations.

5. The Perturbed State

Using a linear analysis, each perturbed variable is written

$$q = q_{equilibruim} + \hat{q}\ e^{\omega t}$$

where $\hat{q}$ depends only on space co-ordinates. The problem is to study the nature of ω. Limiting ourselves to a consideration of "kink"-type perturbations, we take

$$q = q_{equilibrium} + \tilde{q}(r)e^{i(\theta+kz)+\omega t}$$

with similar equations for vector quantities.

6. Perturbed Plasma Quantities

From $\nabla \cdot \underline{v} = 0$, $\underline{v} = \underline{v}_{eq} + \hat{v}$, $\nabla \cdot \hat{v} = 0$. From $\nabla \times \underline{E} = (-1/c)\, \partial \underline{B}/\partial t$ and $\underline{E} + \left[(\underline{v} \times \underline{B})/c\right] = 0$.

(18) $$\nabla \times (\hat{v} \times \underline{B}) = \omega \hat{B}.$$

From the equation of motion and $\nabla \times \underline{B} = r\pi \underline{j}/c$

(19) $$\rho\omega\hat{v} = \frac{1}{4\pi}\left[(\nabla \times \underline{B}) \times \underline{B}\right]_1 - (\nabla p)_1$$

where the subscript 1 means to pick out the first order terms. Rewriting the equation for $\omega\hat{B}$,

$$\omega\hat{B} = \nabla \times (\hat{v} \times \underline{B})$$

$$= (\underline{B} \cdot \nabla)\hat{v} - (\hat{v} \cdot \nabla)\underline{B}$$

Discarding exponential factors, we find

$$\omega\tilde{B} = i\tilde{v}\left(\frac{B_\theta}{r_o} + kB_z\right)$$

or

(20) $$- i\omega r_o \tilde{B} = B_\theta(1 + \alpha k)\tilde{v}$$

where $\alpha = r_o B_z/B_\theta$. In order to eliminate $\tilde{v}$ from Eq. (20), we curl Eq. (19) to find

(21) $$\frac{- i\rho_o r_o \omega^2}{B_\theta(1 + \alpha k)} \nabla \times \hat{B} = \frac{1}{4\pi} \nabla \times \left\{(\nabla \times \underline{B}) \times \underline{B}\right\}_1$$

The expression $\frac{1}{4\pi}\left\{(\nabla \times \underline{B}) \times \underline{B}\right\}_1$ will give, after discarding exponential factors, the following components

$$\hat{r}: \quad \frac{B_\theta}{r_o}\left[-2\tilde{B}_\theta + ik\alpha\tilde{B}_r - D(\alpha\tilde{B}_z) - D(r\tilde{B}_\theta) + i\tilde{B}_r\right]$$

$$\hat{\theta}: \quad \frac{B_\theta}{r_o}\left[2\tilde{B}_r - \frac{i\alpha}{r}\tilde{B}_z + i\alpha k\tilde{B}_\theta\right]$$

$$\hat{k}: \quad \frac{B_\theta}{r_o}\left[i\tilde{B}_z - ikr\tilde{B}_\theta\right]$$

where $D = d/dr$. The components of $\nabla \times \hat{B}$ in Eq. (21) give

$$\frac{i\tilde{B}_z}{r} - ik\tilde{B}_\theta$$

$$ik\tilde{B}_r - D\tilde{B}_z$$

$$\frac{1}{r}D(r\tilde{B}_\theta) - \frac{i}{r}\tilde{B}_r \,.$$

If, following Tayler, we set

$$\beta = \frac{2}{1 + \alpha k + \dfrac{4\pi\rho_o r_o^2\omega^2}{B_\theta^2(1+\alpha k)}} \tag{22}$$

the components of Eq. (21) above give

$$k\tilde{B}_\theta - \frac{1}{r}\tilde{B}_z - ik\beta\tilde{B}_r = 0 \tag{23}$$

$$ik\beta\tilde{B}_\theta + iD\tilde{B}_z + k\tilde{B}_r = 0 \tag{24}$$

$$iD\left(\frac{\tilde{B}_\theta}{r}\right) + D\left[\frac{D(r\tilde{B}_r)}{r}\right] - k^2\tilde{B}_r - ik^2\beta = 0 \tag{25}$$

The divergence equation gives

$$D(r\tilde{B}_r) + i\tilde{B}_\theta + ikr\tilde{B}_z = 0. \tag{26}$$

The equations are not independent, of course, but we choose to use the most convenient three for solution.

Solving Eqs. (23) and (24) for B_θ and B_z in terms of B_r and substituting into Eq. (26),

$$r^2D^2\tilde{B}_z + rD\tilde{B}_z - \left[1 + (1 - \beta^2)k^2r^2\right]\tilde{B}_z = 0$$

Since B_z is bounded near $r = 0$ we take

$$B_z = B_{zo} i I_1 \left(kr\sqrt{1 - \beta^2}\right)$$

or, using a superscript to distinguish from vacuum quantities

$$\tilde{B}^p_r = \tilde{B}_{zo}\phi = \tilde{B}_{zo}\left\{\frac{I_1'(\sqrt{1 - \beta^2}\,kr)}{\sqrt{1 - \beta^2}} + \frac{\beta I_1(\sqrt{1 - \beta^2}\,kr)}{kr(1 - \beta^2)}\right\}$$

$$\tilde{B}^p_\theta = \tilde{B}_{zo}\psi = i\tilde{B}_{zo}\left\{\frac{I_1(\ )}{kr(1 - \beta^2)} + \frac{\beta}{\sqrt{1 - \beta^2}} I_1'(\ )\right\} \tag{27}$$

$$B^p_z = i\tilde{B}_{zo}I_1(\ ).$$

Use $\underline{E} = -(\underline{v} \times \underline{B})/c$, to find

$$\tilde{E}^p_r = \frac{i\omega r_o}{c(1 + \alpha k)}\left[\frac{\alpha}{r}\tilde{B}^p_\theta - \frac{r}{r_o}\tilde{B}^p_z\right]$$

$$\tilde{E}^p_\theta = \frac{-i\omega r_o}{c(1 + \alpha k)}\left[\frac{\alpha}{r}\tilde{B}^p_r\right] \tag{28}$$

$$\tilde{E}^p_z = \frac{i\omega r_o}{c(1 + \alpha k)}\left[\frac{r}{r_o}\tilde{B}^p_r\right]$$

If Eq. (19) is used (before curling) together with

(29) $$\tilde{v} = \frac{-i\omega r_o}{B_\theta(1 + \alpha k)} \tilde{B}$$

we get

(30) $$\tilde{p} = \frac{B_\theta}{4\pi r_o} \left[\left(\frac{2}{\beta k} - \alpha \right) \tilde{B}_z^p - r\tilde{B}_\theta^p \right] .$$

This completes the set of perturbed plasma quantities that we need.

7. Perturbed Field Quantities

Solving Maxwell's equations for the vacuum region, we find, setting $\sigma = 0$ and $\eta^2 = k^2 + \omega^2/c^2 \approx k^2$

(31) $$\begin{aligned} \tilde{E}_z &= E_{zo}^a K_1(kr) + E_{zo}^b I_1(kr) \\ \tilde{B}_z &= B_{zo}^a K_1(kr) + B_{zo}^b I_1(kr) \end{aligned}$$

The other components are

$$\tilde{B}_r = \frac{-i\omega}{rck^2} \tilde{E}_z - i\tilde{B}_z'$$

$$\tilde{B}_\theta = \frac{1}{rk} \tilde{B}_z + \frac{\omega}{ck} \tilde{E}_z'$$

$$\tilde{E}_r = \frac{i\omega}{rck^2} \tilde{B}_z - i\tilde{E}_z'$$

$$\tilde{E}_\theta = \frac{1}{rk} \tilde{E}_z - \frac{\omega}{ck} \tilde{B}_z'$$

To simplify notations we impose first the boundary conditions

$$B_r = E_\theta = E_z = 0$$

at $r = r_1$, hence Eq. (31) becomes

$$\tilde{E}_z^o = \frac{E_{zo}^b}{K_1(kr_1)} \left[I_1(kr)K_1(kr_1) - K_1(kr)I_1(kr_1) \right]$$

(31')

$$\tilde{B}_z^o = \frac{B_{zo}^b}{K_1'(kr_1)} \left[I_1(kr)K_1'(kr_1) - K_1(kr)I_1'(kr_1) \right]$$

Setting

$$Y(kr) = I_1(kr)K_1'(kr_1) - K_1(kr)I_1'(kr_1)$$

$$Z(kr) = I_1(kr)K_1(kr_1) - K_1(kr)I_1(kr_1)$$

we list the vacuum quantities as follows

$$\tilde{B}_r^o = \frac{-i\omega}{rck^2} \tilde{E}_z^o - i\tilde{B}_z^{o\,\prime}$$

$$\tilde{B}_\theta^o = \frac{1}{rk} \tilde{B}_z^o + \frac{\omega}{ck} \tilde{E}_z^{o\,\prime}$$

$$\tilde{B}_z^o = \frac{B_{zo}^b \; Y(kr)}{K_1'(kr_1)}$$

(32)

$$\tilde{E}_r^o = \frac{i\omega}{rck^2} \tilde{B}_z^o - i\tilde{E}_z^{o\,\prime}$$

$$\tilde{E}_\theta^o = \frac{1}{rk} \tilde{E}_z^o - \frac{\omega}{ck} B_z^{o\,\prime}$$

$$\tilde{E}_z^o = \frac{E_{zo}^b \; Z(kr)}{K_1(kr_1)}$$

It remains to impose the boundary conditions at $r = r_o$ on the perturbed plasma and vacuum quantities.

The surface of the plasma, in the perturbed state, is given by

$$r = r_o + \tilde{r}\, e^{i(\theta+kz)+\omega t}$$

The components of the surface normal are given initially by (-1, 0, 0) and the perturbations of the components are

$$\frac{iv_r}{\omega}\, e^{i(\theta+kz)+\omega t}\, (0, \frac{1}{r_o}, k)$$

8. Application of Boundary Conditions to Interface Between Plasma and Vacuum

Perturbation of $\underline{n} \cdot (\underline{B}^p - \underline{B}^o)$ and $\underline{n} \times (\underline{E}^p - \underline{E}^o) = \frac{u}{c} (\underline{B}^p - \underline{B}^o)$ gives

$$\underline{n} \cdot (\underline{B}^p - \underline{B}^o) + n \cdot (\underline{B}^p - \underline{B}^o) = 0$$

and

$$\tilde{E}_z^p - \tilde{E}_z^o = \frac{u}{c} (B_\theta - H_\theta) - (\tilde{E}_\theta^p - \tilde{E}_\theta^o) = \frac{u}{c} B_z .$$

Noting that $u = -\tilde{v}_r$ and substituting the appropriate quantities for $\tilde{B}^p - \tilde{B}^o$, $\tilde{E}^p - \tilde{E}^o$ we obtain 3 linear equations

$$\frac{\omega}{rck^2}\tilde{E}_z^o + \tilde{B}_z^{o\,\prime} - \frac{1}{\omega}\left(\frac{B_\theta - H_\theta}{r_o} + kB_z\right)\tilde{v}_r = i\tilde{B}_r^p$$

(33) $$-\tilde{E}_z^o + \frac{1}{c}(B_\theta - H_\theta)\tilde{v}_r = \frac{-i\omega r_o}{c(1+\alpha k)}\tilde{B}_r^p$$

$$\frac{1}{r_o k}\tilde{E}_z^o - \frac{\omega}{ck}\tilde{B}_z^{o\,\prime} + \frac{B_z}{c}\tilde{v}_r = \frac{-i\omega\alpha}{c(1+\alpha k)}\tilde{B}_r^p$$

whose solutions are given by

$$\tilde{E}_z^o = \left(\frac{i\tilde{B}_r^p}{B_\theta + r_o kB_z}\right)\left(\frac{r_o \omega}{c} H_\theta\right)$$

$$(34) \qquad \tilde{B}_z^{o\,\prime} = \left(\frac{i\tilde{B}_r^p}{B_\theta + r_o kB_z}\right)(H_\theta)$$

$$\tilde{v}_r = \left(\frac{i\tilde{B}_r^p}{B_\theta + r_o kB_z}\right)(-\omega r_o)$$

where we have used our hypotheses that $\frac{\omega^2}{c^2k^2} << 1.$

Before using the last boundary condition, namely the one which arises by elimination of j^*, we rewrite B_z^o in a more convenient form. By differentiation of $\tilde{B}_z^o = B_{zo}^b Y(kr)/K_1'(kr_1)$ and substituting $kr = kr_o$ and comparing with the result of solving the 3 linear equations we find

$$(35) \qquad \tilde{B}_z^o = \left[H_\theta \frac{i\tilde{B}_r^p}{B_\theta + r_o kB_z}\right] \frac{Y(kr)}{Y'(kr_1)}$$

where all quantities within the bracket have been evaluated at $r = r_o$.

Elimination of j^* between the first and fourth boundary condition gives

$$(36) \qquad [\tilde{\underline{B}}] \cdot \langle\underline{B}\rangle + [\underline{B}] \cdot \langle\tilde{\underline{B}}\rangle = -8\pi[\tilde{p}]$$

where $[\;]$ means the jump in the quantity in question in passing from vacuum into plasma, and $\langle\rangle$ means the arithmetic mean on the two sides of the surface. We list the explicit quantities needed and note that since the plasma surface has moved there is a first order contribution to $[\tilde{B}]$ and $\langle\tilde{B}\rangle$ above that due to local change of these quantities.

$$\left[\underline{B}\right] = \left[\tilde{B}^p_r - \tilde{B}^o_r,\ \ \tilde{B}^p_\theta - \tilde{B}^o_\theta - \frac{u}{r_o\omega}(B_\theta + H_\theta),\ \ \tilde{B}^p_z - \tilde{B}^o_z\right]$$

$$\langle \underline{B} \rangle = \left(0,\ \ \frac{B_\theta + H_\theta}{2},\ \ \frac{B_z}{2}\right)$$

$$\left[\underline{B}\right] = (0,\ \ B_\theta - B_\theta,\ \ B_z)$$

$$\langle \tilde{B} \rangle = \left[\frac{\tilde{B}^p - \tilde{B}^o_r}{2},\ \ \frac{\tilde{B}^p_\theta + \tilde{B}^o_\theta}{2} + \frac{u}{2r_o\omega}(B_\theta - H_\theta),\ \ \frac{\tilde{B}^p_z + \tilde{B}^o_z}{2}\right]$$

Substituting these expressions into Eq. (36) and using the value of $\left[\tilde{p}\right]$ from Eq. (30) we find

$$(37)\qquad \left(\frac{2B_\theta}{\beta}\right)\left(\frac{2\tilde{B}^p_z}{r_o k}\right) = B_z\tilde{B}^p_z + B_\theta\tilde{B}^p_\theta + H_\theta\tilde{B}^o_\theta + \frac{u}{r_o\omega}(2B_\theta H_\theta)$$

From Eq. (22)

$$\frac{2B_\theta}{\beta} = \frac{4\pi\rho_o r_o^2\omega^2}{B_\theta + r_o k B_z} + B_\theta + r_o k B_z$$

hence

$$\left(\frac{4\pi\rho_o r_o^2\omega^2}{B_\theta + r_o k B_z}\right)\left(\frac{2B^p_z}{r_o k}\right) = -\,B_z\tilde{B}^p_z + B_\theta\left(\tilde{B}^p_\theta - \frac{2\tilde{B}^p_z}{r_o k}\right) + \frac{2uB_\theta H_\theta}{r_o\omega} + H_\theta\tilde{B}^o_\theta$$

or

$$(38)\quad \frac{4\pi\rho_o r_o^2\omega^2}{(B_\theta + r_o k B_z)}\left(\frac{2}{r_o k B_z}\right) = -\,1 + \frac{B_\theta}{B_z}\left(\frac{\tilde{B}^p_\theta}{\tilde{B}^p_z} - \frac{2}{r_o k}\right) + \frac{B_\theta}{B_z}\left(\frac{2uH_\theta}{r_o\omega\tilde{B}^p_z}\right) + \frac{H_\theta}{B_z}\ \frac{\tilde{B}^o_\theta}{\tilde{B}^p_z}$$

From the formulas for $\tilde{B}^p$, $\tilde{B}^o$ [Eqs. (27-32)] this equation may be written

$$\frac{4\pi\rho_o r_o^2\omega^2}{\left(r_o k + \frac{B_\theta}{B_z}\right)}\left(\frac{2}{r_o k H_\theta^2}\right) = -\alpha_p^2$$

(39)
$$+\left[\frac{I_1'(\sqrt{1-\beta^2}\,kr_o)}{I_1(\quad)} + \frac{\beta}{kr_o(1-\beta^2)}\right]\frac{1}{r_o k(r_o k + \frac{B_\theta}{B_z})}\;\frac{Y(kr_o),(kr_1)}{Y'(kr_o)(kr_1)}$$

$$+\alpha_p^2\frac{B_\theta}{B_z}\left[\frac{2\beta^2-1}{kr_o(1-\beta^2)} + \frac{\beta}{\sqrt{1-\beta^2}}\;\frac{I_1'(\sqrt{1-\beta^2}\,kr_o)}{I_1(\;)} + \frac{2H_\theta}{B_\theta + r_o kB_z}\left\{\frac{I_1'(\quad)}{I_1(\quad)} + \frac{\beta}{kr_o(1-\beta^2)}\right\}\right]$$

where $\alpha_p = B_z/H_\theta$. This is the general dispersion relation with a plasma current density[5]

$$j_z = \frac{cB_\theta}{2\pi r_o} \quad .$$

To treat the constant pressure case one lets $B_\theta \longrightarrow 0$ ($p_o = B_\theta^2/4\pi$) to obtain the following simplified dispersion relation:

(40)
$$\frac{8\pi\rho_o r_o^2\omega^2}{(r_o k)^2 H_\theta^2} = -\alpha_p^2 + \left[\frac{I_1'(kr_o)}{kr_o I_1(\quad)}\right]\left[\frac{Y(kr_o)}{r_o kY'(kr_o)}\right]$$

If one sets $\omega^2 = 0$, the resulting limiting equation agrees with that of

---

5. In any finite machine $|k| \geqq \delta > 0$. Hence for B_θ sufficiently small $\left(r_o k + \frac{B_\theta}{B_z}\right)(r_o k)$ is positive and it is unnecessary to analyze separately the case $r_o k + \frac{B_\theta}{B_z} = 0$.

Rosenbluth[1] based on considerations of a virtual displacement of a hydrodynamic plasma.

9. Onset of Instability

Suppose the volume current density is increased. We seek the critical ratio of surface current density to volume current density for the kink-type perturbation to become unstable.

From $4\pi j^*/c = \underline{n} \times (\underline{B}^p - \underline{B}^o)$ we get

$$j^*_\theta = \frac{cB_z}{4\pi} = \frac{c\alpha_p H_\theta}{4\pi}$$

(41)

$$j^*_z = \frac{c}{4\pi}(H_\theta - B_\theta)$$

From $\int j_z dA = \int \underline{H} \cdot d\underline{s}$,

$$j_z = \frac{c}{2\pi r_o} B_\theta \ .$$

Hence, eliminating H_θ and B_θ in Eq. (41)

(42) $$j^*_z = \frac{j^*_\theta}{\alpha_p} - \frac{r_o j_z}{2}$$

An increase in j_z can evidently be brought about only if at least one of the other quantities in Eq. (42) is suitably altered. If we assume the surface currents unaltered, j^*_θ and j^*_z remain the same and α_p must decrease. Since B_z is fixed, and α_p decreases, H_θ must increase, as may also be seen from Eq. (41).

From Eq. (42)

$$\alpha_p = \frac{j^*_\theta}{j^*_z + \frac{r_o j_z}{2}}$$

Rosenbluth shows[1] that if α_p^2 is greater than some minimum critical value between 1/2 and 1 (depending on the ratio of plasma radius to wall radius), then the plasma is stable against perturbations for all k and $m = 0, \pm 1, \ldots$ where $m = \pm 1$ corresponds to the kink-type instabilities now being considered. Further, a plasma with $\alpha_p' > \alpha_p$ as above is stable. If we denote the minimum value of α_p that corresponds to stability by α_{p_c} in the no volume current case an estimate may be made of the maximum allowable volume current density j_z before instabilities set in. That is, if

$$\frac{j_\theta^*}{j_z^*} > \alpha_{p_c} ,$$

then the critical current is given by

$$\frac{r_o j_z}{2} = \frac{j_\theta^*}{\alpha_{p_c}} - j_z^* . \tag{43}$$

It is not clear that the parameter α_p retains the same general significance as in the constant pressure case. However, on account of the simple way in which α_p enters in the dispersion relation [Eq. (39)] it seems likely that for m = 1 (kink perturbation) this is the case. This is particularly true when $r_o j_z/2$ is small compared to the surface currents. In that case B_θ/B_z is small and the last term in Eq. (39) may be neglected.

ORNL-2542
Controlled Thermonuclear Processes
TID-4500 (13th ed., Rev.)
February 15, 1958

INTERNAL DISTRIBUTION

1. C. E. Center
2. Biology Library
3. Health Physics Library
4-5. Central Research Library
6. Reactor Experimental Engineering Library
7-26. Laboratory Records Department
27. Laboratory Records, ORNL R.C.
28. A. M. Weinberg
29. L. B. Emlet (K-25)
30. J. P. Murray (Y-12)
31. J. A. Swartout
32. E. H. Taylor
33. E. D. Shipley
34. A. H. Snell
35. E. P. Blizard
36. M. L. Nelson
37. W. H. Jordan
38. G. E. Boyd
39. R. A. Charpie
40. S. C. Lind
41. F. L. Culler
42. A. Hollaender
43. J. H. Frye, Jr.
44. M. T. Kelley
45. J. L. Fowler
46. R. S. Livingston
47. K. Z. Morgan
48. T. A. Lincoln
49. A. S. Householder
50. C. P. Keim
51. C. S. Harrill
52. C. E. Winters
53. D. S. Billington
54. D. W. Cardwell
55. D. Phillips
56. A. J. Miller
57. M. J. Skinner
58. R. R. Dickison
59. J. A. Harvey
60. A. Simon
61. F. C. Maienschein
62. C. E. Clifford
63. A. D. Callihan
64. R. R. Coveyou
65. W. Zobel
66. F. L. Keller
67. C. D. Zerby
68. D. K. Trubey
69. S. K. Penny
70. R. W. Peelle
71. F. L. Friedman (consultant)
72. H. Goldstein (consultant)
73. H. Hurwitz, Jr. (consultant)
74. L. W. Nordheim (consultant)
75. R. R. Wilson (consultant)
76. ORNL - Y-12 Technical Library, Document Reference Section

EXTERNAL DISTRIBUTION

77. O. G. Harrold, Jr., University of Tennessee
78. Division of Research and Development, AEC, ORO
79-656. Given distribution as shown in TID-4500 (13th ed., Rev., Feb. 15, 1958) under Controlled Thermonuclear Processes category (75 copies - OTS)