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THE NEUTRON VELOCITY SPECTRUM IN A HEAVY MODERATOR

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THE NEUTRON VELOCITY SPECTRUM IN A HEAVY MODERATOR

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ABSTRACT

The differential equation developed by Wilkins to represent the velocity spectrum of neutrons in a heavy moderator is investigated for the case of 1/v absorption. An exact solution to terms of second order in the absorption parameter allows an accurate determination of the asymptotic neutron density. For large absorption parameters a numerical integration can yield higher accuracy. The analytic solution is applied to the calculation of the total migration area of neutrons from a mono-energetic source.

I. GENERAL DISCUSSION

It has been shown by Wilkins[1] and by Hurwitz[2, 3] that the neutron spectrum in a heavy moderator in the presence of an absorption cross section with a pure inverse velocity dependence can be reduced to the solution of the following differential equation:

$$xN''(x) + (2x^2 - 1)N'(x) + (4x - \Delta)N(x) = 0. \qquad \text{...(1)}$$

In this expression x is the velocity variable normalized to unity at the velocity corresponding to the energy kT, T is the moderator temperature, $N(x)$ is the number of neutrons per unit of x, and Δ is the absorption parameter. In this case $\Delta = 2ma$ where m = moderator mass in units of the neutron mass, and a is the ratio of absorption to scattering cross sections at the energy kT. The scattering cross section σ_s is assumed constant over the entire velocity interval, and the absorption cross section is $\sigma_a(x) = \sigma_o/x = a\sigma_s/x$.

Equation (1) reduces, in the case $\Delta = 0$, to the equation:

$$xN''(x) + (2x^2 - 1)N'(x) + 4xN(x) = 0 \quad . \qquad \text{...(2)}$$

The complete solution of this equation is:

$$N(x) = a_1 x^2 e^{-x^2} + a_2 [x^2 e^{-x^2} Ei(x^2) - 1] \quad . \qquad \text{...(3)}$$

The second term of the solution is negative at $x = 0$ and positive for large x. Hence a_2 must be set equal to zero for a physically significant solution. However, the singular component of the solution should reappear when absorption is introduced into the physical problem. This singular component, which behaves like $1/(x^2 - 2)$ for large values of x, represents the slowing down distribution of neutrons from a source at high energies and therefore can not be part of a steady solution unless there is absorption to remove those neutrons which become thermalized. Otherwise, a steady source of neutrons at high energy would produce a build-up of thermal neutrons to infinite amplitude. This second solution could also appear if there were an absorption of neutrons below some velocity $x = x_1$ and no absorption above that velocity. Then, in the region of no absorption, Eq. (3) would represent the neutron distribution; the parameters a_1 and a_2 would be determined by the physical conditions of the particular problem.

II. SERIES SOLUTION

In the presence of absorption, $\Delta \neq 0$, Eq. (1) canbe integrated once to give (with the boundary condition that $N(x)$ approaches zero as x goes to zero)

$$xN'(x) + 2(x^2 - 1)N(x) = \Delta \int_o^x N(t)dt \,. \qquad \ldots(4)$$

The number of neutrons which slow down per second past the velocity x must be equal to the total absorption of neutrons at velocities less than x. This is simply stating the law of the conservation of neutrons. Denoting the slowing down density at velocity x by $q(x)$, we have

$$q(x) = \int_o^x a\sigma_s N(t)dt = \frac{\Delta\sigma_s}{2m} \int_o^x N(t)dt$$

$$\ldots(5)$$

$$= \frac{\sigma_s}{m}\left[(x^2 - 1)N(x) + \frac{1}{2}\, xN'(x)\right] \,.$$

Since this equation expresses a relation between $q(x)$ and $N(x)$ and its first derivative in which Δ does not appear, we can conclude that the expression is valid for the case of zero absorption, even though it was derived from a consideration of absorption rates. In this way we can conclude that the coefficient a_2 in Eq. (3) is given by $a_2 = mq/\sigma_s$, where q is the slowing down density (q is a constant since there is no absorption). This relation should also exist in the high energy portion of a spectrum with $\Delta \neq 0$, since the ratio of absorption to scattering is assumed to fall off as $1/x$, and hence will be negligible for sufficiently large values of x. The same conclusion may also be reached by the observation that, for large values of x we may neglect $\Delta << 4x$ and hence reduce Eq. (1) to Eq. (2). Thus, in the limit of large x the solution of Eq. (1) must also approach the solution of Eq. (2).

Equation (5) may be formally integrated a second time to give an integral equation for $N(x)$:

$$N(x) = x^2 e^{-x^2} \left[\frac{4}{\sqrt{\pi}} + \Delta \int_o^x \frac{1}{u^3} e^{u^2} \int_o^u N(t) dt\, du \right] . \qquad \ldots(6)$$

The constant of integration of Eq. (6) has been chosen to be such that the Maxwell component of the neutron flux is normalized. It has been pointed out by Nelkin[3] that this expression can serve as a basis for an iterative solution for $N(x)$. In particular we can write

$$N(x) = x^2 e^{-x^2} [\mu_0(x) + \Delta\mu_1(x) + \Delta^2\mu_2(x) + \ldots] \qquad \ldots(7)$$

and we then obtain, by inserting this into Eq. (6) and equating coefficients in Δ;

$$\mu_0 = 4/\sqrt{\pi}$$

$$\mu_{r+1}(x) = \int_o^x \frac{e^{u^2}}{u^3} \int_o^u \mu_r(t) t^2 e^{-t^2} dt\, du \quad . \qquad \ldots(8)$$

In particular we find

$$\mu_1(x) = \int_o^x e^{u^2}\left[\frac{H(u) - uH'(u)}{u^3}\right] du\ , \qquad \text{...(8.1)}$$

where

$$H(u) = \frac{2}{\sqrt{\pi}} \int_o^u e^{-t^2}\, dt = erf(u) \qquad \text{...(8.2)}$$

The solution for $\Delta \neq 0$ may also be developed directly in a power series in x. It is most convenient to write $N(x) = x^2 e^{-x^2} M(x)$, which converts the differential equation, Eq. (1), into

$$xM''(x) + (3 - 2x^2)M'(x) - \Delta M(x) = 0\ . \qquad \text{...(9)}$$

The solution of this equation (see Appendix A) is:

$$M(x) = \sum a_n x^n = \sum \Delta^r \mu_r(x)\ ,$$

where

$$a_n = \frac{1}{n(n+2)}\left[2(n-2)a_{n-2} + \Delta a_{n-1}\right]\ . \qquad \text{...(10)}$$

Choosing the normalization that $a_0 = 4/\sqrt{\pi}$ we find, Ref. 1,

$$a_1 = \frac{4}{\sqrt{\pi}} \cdot \frac{\Delta}{3}$$

$$a_2 = \frac{4}{\sqrt{\pi}} \cdot \frac{\Delta^2}{24}$$

$$a_3 = \frac{4}{\sqrt{\pi}} \cdot \frac{\Delta}{45}\left(2 + \frac{\Delta^2}{8}\right)$$

$$a_4 = \frac{4}{\sqrt{\pi}} \frac{19\Delta^2}{2160}\left(1 + \frac{\Delta^2}{76}\right)$$

$$a_5 = \frac{4}{\sqrt{\pi}} \frac{2\Delta}{105}\left(1 + \frac{7}{36}\Delta^2 + \frac{\Delta^4}{5760}\right), \; etc. \qquad \ldots(11)$$

It is apparent from the structure of these coefficients that they break down into two series, each of which contains only even, or only odd, powers of Δ. Thus if we attempt to extract from this power series expansion the successive approximations of Eq. (7), we will find that $\mu_1(x)$ contains the leading term $\frac{4x}{3\sqrt{\pi}}$ and only odd powers of x. The second approximation function, $\mu_2(x)$, will have the leading term $x^2/6\sqrt{\pi}$, and will contain only even powers of x. Similarly, $\mu_3(x)$ will begin with $x^3/90\sqrt{\pi}$, and $\mu_4(x)$ will begin with $x^4/2160\sqrt{\pi}$. It is, in fact, fairly easy to extract completely the first two approximation functions from the power series solution.

The first two approximation functions may most easily be found by defining

$$b_n = a_{2n+1} \; ; \qquad \ldots(12.1)$$

$$c_n = a_{2n} \; . \qquad \ldots(12.2)$$

These substitutions convert Eq. (10) into the coupled set of equations:

$$b_n = \frac{1}{(2n+1)(2n+3)}\left[2(2n-1)b_{n-1} + \Delta c_n\right] \; ; \qquad \ldots(10.1)$$

$$c_n = \frac{1}{2n(2n+2)}\left[2(2n-2)c_{n-1} + \Delta b_{n-1}\right] \; . \qquad \ldots(10.2)$$

Since c_n is of order Δ^2, we can neglect the term in Δc_n, which will be of order Δ^3, and hence obtain

$$b_n = \frac{2(2n-1)b_{n-1}}{(2n+1)(2n+3)} = \frac{\Delta}{(2n+1)\Gamma(n+\frac{5}{2})} \qquad \text{...(13)}$$

The second recursion formula, (10.2), is more complicated since Δb_{n-1} and c_{n-1} are both of second order in Δ. However, the b_n are now known so that we have an inhomogeneous linear first-order difference equation to solve. This difference equation may be written:

$$n(n+1)c_n - (n-1)c_{n-1} = \frac{\Delta^2}{4(2n-1)\Gamma(n+\frac{3}{2})} \qquad \text{...(14)}$$

The homogeneous equation has the solution $\frac{1}{n(n+1)!}$; hence we let

$$c_n = \frac{e_n}{n(n+1)!} \qquad e_o = 0 \quad , \qquad \text{...(15)}$$

and the difference equation then becomes

$$e_n = e_{n-1} + \frac{\Delta^2}{4} \frac{n!}{(2n-1)\Gamma(n+\frac{3}{2})} \qquad \text{...(15.1)}$$

By summing these equations from $n = 1$ to $n = n$, we obtain

$$c_n = \frac{\Delta^2}{4n(n+1)!} \sum_{k=o}^{n-1} \frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})} . \qquad \text{...(16)}$$

The neutron density, accurate to second order in the absorption parameter, is therefore given by

$$N(x) = x^2 e^{-x^2} \left[\frac{4}{\sqrt{\pi}} + \Delta\mu_1(x) + \Delta^2\mu_2(x) + 0(\Delta^3) \right]$$

$$\mu_1(x) = \sum_{n=o}^{\infty} \frac{x^{2n+1}}{(2n + 1)\Gamma(n + \frac{5}{2})} \qquad \text{...(17)}$$

$$= \frac{4x}{3\sqrt{\pi}} \left[1 + \frac{2x^2}{15} + \frac{4x^4}{175} + \frac{8x^6}{2205} + \frac{16x^8}{31185} + \frac{32x^{10}}{495495} + \cdots \right] \qquad \text{...(17.1)}$$

$$\mu_2(x) = \frac{1}{4} \sum_{n=1}^{\infty} \frac{x^{2n}}{n(n + 1)!} \left[\sum_{k=o}^{n-1} \frac{(k + 1)!}{(2k + 1)\Gamma(k + \frac{5}{2})} \right]$$

$$= \frac{x^2}{6\sqrt{\pi}} \left[1 + \frac{19x^2}{90} + \frac{157x^4}{3780} + \frac{1263x^6}{176400} + \cdots \right] \qquad \text{...(17.2)}$$

The power series for $\mu_1(x)$ given in Eq. (17.1) may also be developed directly from Eq. (8.1). The details of this evaluation are given in Appendix B.

III. THE ASYMPTOTIC NEUTRON DENSITY

We are interested in the neutron density in the limit of large x corresponding to the normalized Maxwell distribution at small x. It is known from the elementary theory of slowing down in a stationary moderator that the neutron distribution is of the form C/x^2; we now wish to find C as a function of Δ to connect with the normalized Maxwell distribution, $\frac{4}{\sqrt{\pi}} x^2 e^{-x^2}$.

From Eq. (5) we see that the slowing down density at large velocities, $q(\infty)$, is a constant since the integral of the neutron density over all velocities is finite. If the energy of the neutron is much larger than the energy of the moderator atoms, $(x^2 >> 1)$, the kinetic energy of the moderator will be unimportant in determining the neutron distribution. Hence for large velocities, $N(x)$ must approach the form of the elementary solution. Therefore for large x we have:

$$N(x) \approx \frac{C}{x^2} ; \qquad \frac{1}{2} x N'(x) \approx - N(x)$$

therefore

$$C = \lim_{x \to \infty} [x^2 N(x)] = \frac{\Delta}{2} \int_0^\infty N(t) dt \quad , \qquad \ldots(18)$$

since

$$\frac{\Delta}{2} \int_0^\infty N(t) dt = \lim_{x \to \infty} \left[(x^2 - 1) N(x) + \frac{1}{2} x N'(x) \right]$$

$$= \lim_{x \to \infty} [(x^2 - 2) N(x)] = \lim_{x \to \infty} [x^2 N(x)] \quad .$$

It is also possible to argue from Eq. (3): for large x one can neglect $\Delta << 4x$, hence Eq. (3) is asymptotically the solution of Eq. (1). Thus, without recourse to the elementary theory of slowing down in a stationary moderator (except perhaps as a check) we can establish that $(x^2 - 2) N(x) = mq(\infty)/\sigma_s$ in the limit of large x. From Eq. (17) we obtain

$$C = \frac{\Delta}{2} \left[1 + \int_0^\infty x^2 e^{-x^2} [\Delta \mu_1(x) + \Delta^2 \mu_2(x) + \ldots] dx \right] \quad . \qquad \ldots(19)$$

The integrations may be easily carried out, and we find

$$C = \frac{\Delta}{2} \left[1 + \Delta \sum_{n=o} \frac{1}{(2n + 1) \Gamma (n + \frac{5}{2})} \int_o^\infty x^{2n+3} e^{-x^2} dx + \right.$$

$$\frac{\Delta^2}{4} \sum_{n=1} \frac{I_n}{n(n+1)!} \int_o^\infty x^{2x+2} e^{-x^2} dx + \dots \Bigg]$$

$$= \frac{\Delta}{2} \left[1 + \frac{\Delta}{2} \sum_{n=o}^{\infty} \frac{(n+1)!}{(2n+1)\Gamma(n+\frac{5}{2})} + \frac{\Delta^2}{8} \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{3}{2}) I_n}{n(n+1)!} + \dots \right] \qquad \dots(20)$$

where

$$I_n = \sum_{k=o}^{n-1} \frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})}$$

Appendix C presents the evaluation of these sums. The result, correct to second order, is:

$$C = \frac{\Delta}{2} [1 + 0.798873\Delta + 0.286606\Delta^2] \ . \qquad \dots(21)$$

This expression is a lower bound to C since it neglects the higher order terms, $\int_o^\infty x^2 e^{-x^2} \mu_n(x)\, dx$, all of which are positive.

We can also obtain an upper bound for C. The successive approximation functions, $f_n(x) = \mu_n(x) x^2 e^{-x^2}$, all behave like $1/x^2$ as $x \to \infty$. It follows, in fact directly from Eq. (18), that for large values of x,

$$f_{n+1}(x) = \frac{1}{2x^2} \int_o^\infty f_n(u)\, du \ .$$

For small values of x we can easily show, using Eq. (8) and the observation that $\mu_n(x) \sim x^n$, that

$$\mu_n(x) \approx \frac{x}{n(n+2)} \mu_{n-1}(x) \ . \qquad \dots(22)$$

Hence, we will have an upper bound approximation for large values of x if we replace all the higher functions $\mu_n(x)$ by functions having the form of $\mu_2(x)$. Thus, we write

$$N(x) \approx x^2 e^{-x^2}\left[\frac{4}{\sqrt{\pi}} + \Delta\mu_1(x) + (1 + \phi)\Delta^2\mu_2(x)\right] . \qquad ...(23)$$

For large values of x we may therefore deduce:

$$C = \lim_{x\to\infty} x^2 N(x) = \frac{\Delta}{2}[1 + 0.798873(1 + \phi)\Delta] , \qquad ...(24.1)$$

where ϕ is a constant to be determined. Another expression for C is obtained from the integral of the neutron density:

$$C = \frac{\Delta}{2}\int_0^\infty N(x)dx = \frac{\Delta}{2}[1 + 0.798873\Delta + 0.286606(1 + \theta\phi)\Delta^2]; \; 0 < \theta < 1. \qquad ...(24.2)$$

The condition on θ, that it be bounded between 0 and 1, arises from the following argument: If we consider Eq. (23) to define a function $\phi(x)$ such that the equation is an exact expression for $N(x)$, we know that $\phi(x) = x/15$ for small values of x; for large values of x, $\phi(x) \to \phi(\infty) = \phi$. The expression for C given by Eq. (24.1) involves only the value of $\phi(\infty)$ while the expression for C given by Eq. (24.2) involves $\phi(x)$ in the integrand of an integral. Hence by the mean value theorem we introduce the fraction, θ. When we equate these two expressions we obtain

$$C = \frac{\Delta}{2}\left[1 + 0.798873\Delta + \frac{0.286606\Delta^2}{1 - 0.358744\theta\Delta}\right] \leq \frac{\Delta}{2}\left[\frac{1 + 0.440129\,\Delta}{1 - 0.358744\,\Delta}\right] .$$

A curve of C vs Δ is shown in Fig. 1. Actually we do not plot C directly but plot instead the quantity $\gamma(\Delta)$ from which C can be computed by the relation

$$C = \frac{\Delta}{2} \exp\left[\, \Delta\gamma(\Delta)\right] ,$$

where, from Eq. (24.2), we may deduce that for small values of Δ,

$$\gamma(\Delta) = 0.798873 - 0.032493\Delta + \ldots \quad .$$

The results of a mechanical integration of Eq. (4) using the Nordsieck Differential Analyzer is shown in Fig. 1. In obtaining this curve the neutron densities (Fig. 2) were calculated from the power series of Eq. (10) in the region of low x, and the solution was then extended mechanically beyond that point. For larger values of Δ this was not a sufficiently accurate procedure, and it was necessary to use the asymptotic solution given by Wilkins for the range $5 \leq x \leq \infty$. This was then matched at $x = 5$ to the result of the mechanical integration in order to determine the value of C from the limiting value of $x^2 N(x)$.

The accuracy of the numerical integration is good for large values of Δ; for smaller Δ's the numerical results exhibit errors of the order of a few per cent in the value of $\gamma(\Delta)$. Such an error, however, implies a quite accurate value of C. The solid line indicates a "best guess" at the true dependence based upon the known analytical limits for small Δ and the quite accurately determined numerical solution for large Δ.

IV. SLOWING DOWN DENSITY

The slowing down density, $q(x)$, is given in Eq. (5) as a linear combination of $N(x)$ and its first derivative. The simple Fermi expression for the slowing down gives $q(x)$ in terms of the neutron density itself. It is therefore instructive, in order to evaluate the differences between the present formulation and the simpler model, to see to what extent Eq. (5) may be replaced by an expression which involves only $N(x)$. Such a replacement will obviously be possible only in the

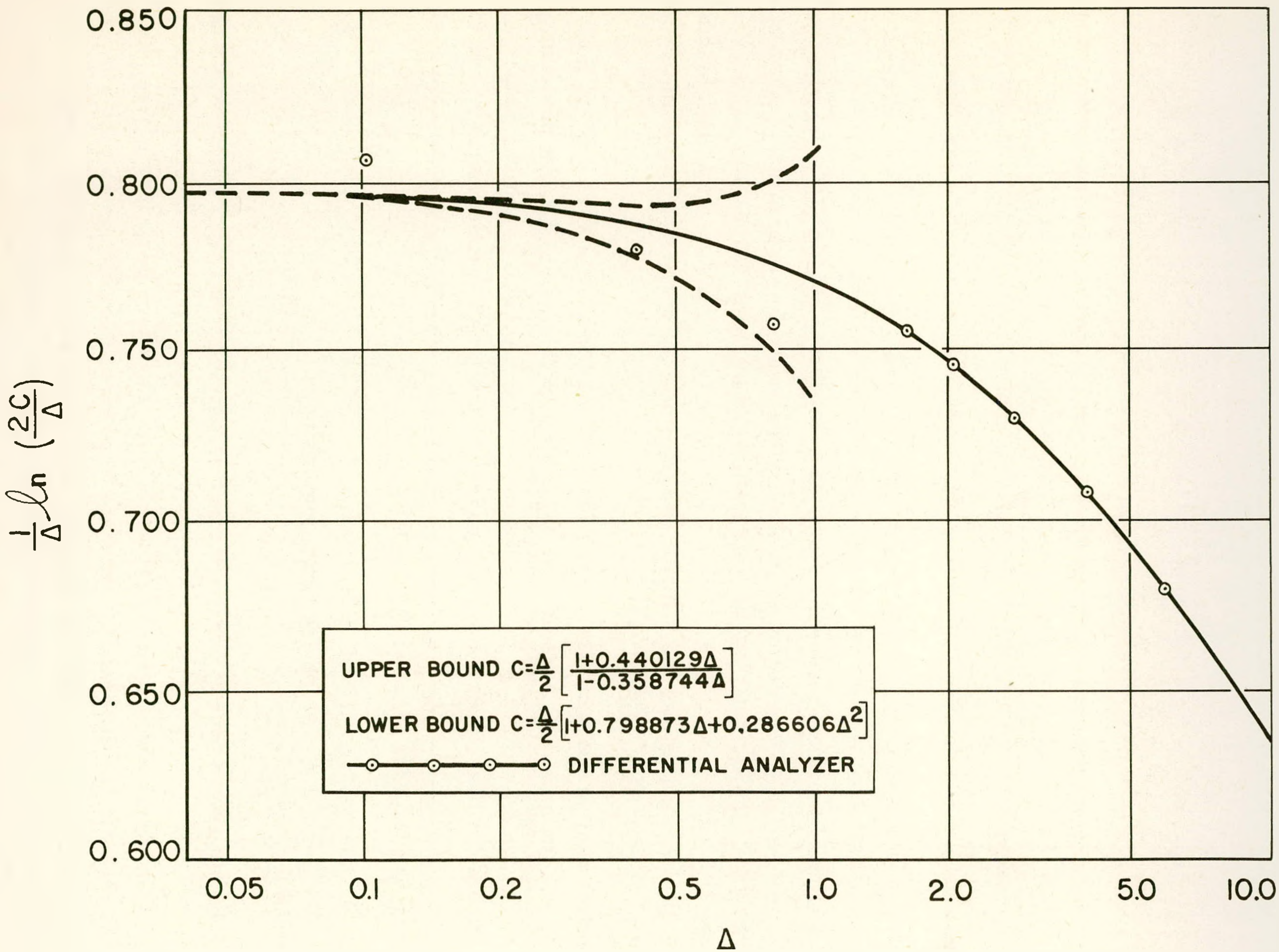

Fig. 1. Asymptotic Neutron Density, $N(x) = \frac{C}{x^2}$

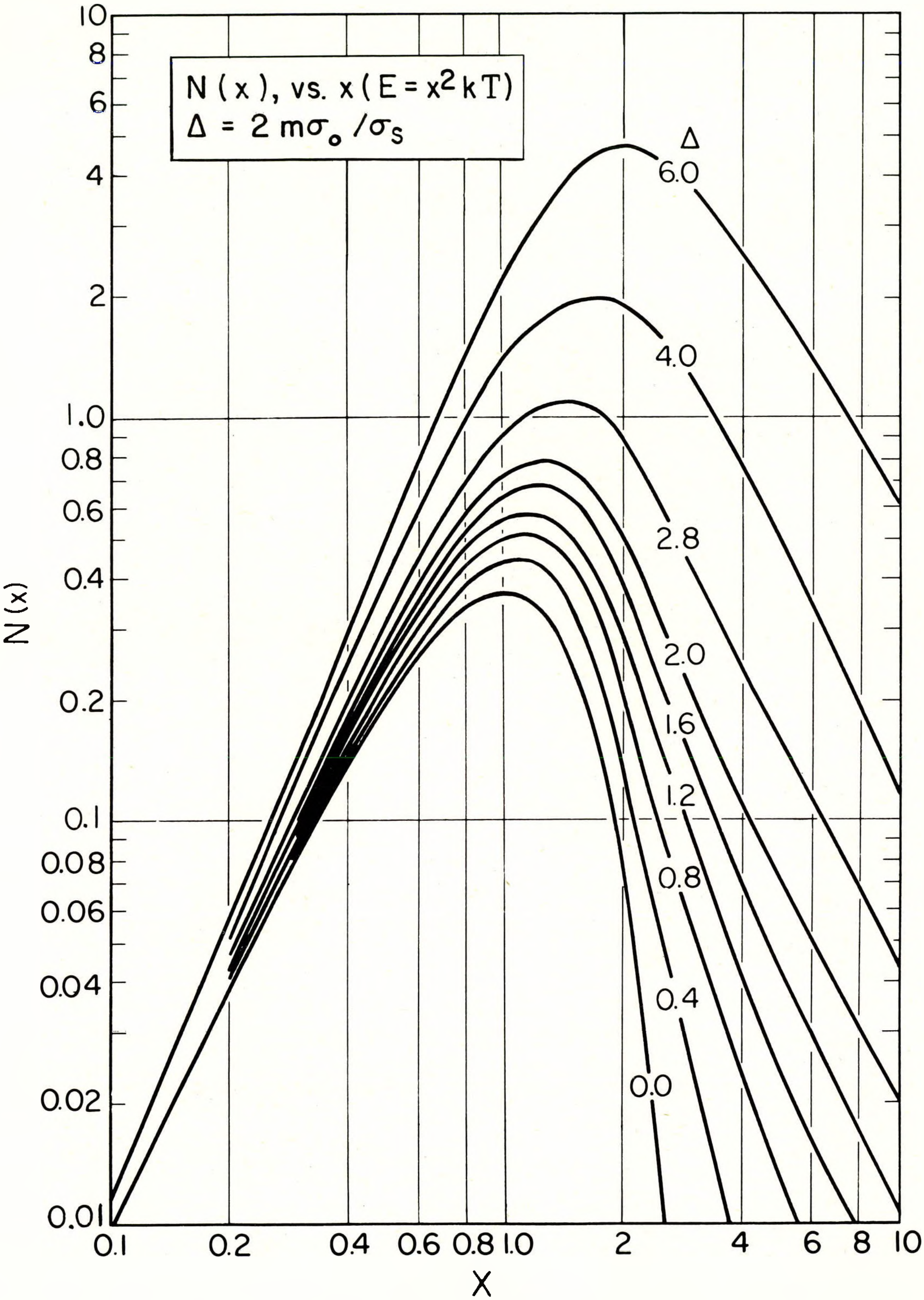

Fig. 2. Neutron Velocity Spectrum

asymptotic region. Here we have

$$N(x) = \frac{C}{x^2}\left[1 - \frac{\Delta}{2x} + \frac{2 + \frac{\Delta^2}{8}}{x^2} - \ldots\right] , \qquad \ldots(26)$$

and therefore,

$$N'(x) = -\frac{2C}{x^3}\left[1 - \frac{3\Delta}{4x} + \frac{4 + \frac{\Delta^2}{4}}{x^2} - \ldots\right] . \qquad \ldots(26.1)$$

These expressions are obtained directly from the differential equation, Eq. (2), when $N(x)$ is expressed as a power series in inverse powers of x in order to obtain an expansion which is valid near the point at infinity. The expansion has been given by Wilkins.[1,2]

If we want to replace $N'(x)$ by an expression involving only $N(x)$, our best possible choice is therefore

$$-\tfrac{1}{2}xN'(x) \approx N(x) \quad .$$

This substitution converts Eq. (5) into the approximate form:

$$q(x) = \frac{\sigma_s}{m}(x^2 - 2)N(x) \quad .$$

Since $q'(x) = \frac{\Delta\sigma_s}{2m}N(x)$, we are thus led to the following approximate differential equation for $N(x)$:

$$2(x^2 - 2)N'(x) + (4x - \Delta)N(x) = 0 \quad . \qquad \ldots(27.1)$$

This solution has been previously given.[4] Nelkin,[3] however, has pointed out that a more accurate representation is possible. Instead of approximating in Eq. (5), his procedure is equivalent to going back to the second order equation, Eq. (1). Then, in order to reduce it to a first order equation, one asks for the best approximation to $N''(x)$ in terms of a linear combination of $N(x)$ and $N'(x)$. From Eq. (26.1) one obtains

$$N''(x) = \frac{6C}{x^4}\left[1 - \frac{\Delta}{x} + \frac{5(16 + \Delta^2)}{12x^2} - \dots\right], \qquad \dots(26.2)$$

and hence deduces the approximation:

$$N''(x) \approx -\frac{6}{x^2}[N(x) + xN'(x)] = \frac{6C}{x^4}\left[1 - \frac{\Delta}{x} + \frac{3(16 + \Delta^2)}{8x^2} - \dots\right]$$

The relative error in this expression is therefore,

$$\frac{16 + \Delta^2}{24x^2} \quad .$$

When this expression for $N''(x)$ is inserted into Eq. (1) we obtain Nelkin's first order differential equation:

$$(2x^2 - 7)N' + \left(4x - \Delta - \frac{6}{x}\right)N = 0. \qquad \dots(27.2)$$

The adequacy of these two approximations may be judged by comparing the exact asymptotic expansion with the approximate expansions. The exact solution is:

$$N(x) = \frac{C}{x^2}\left[1 - \frac{\Delta}{2x} + \frac{\Delta^2 + 16}{8x^2} - \frac{\Delta(\Delta^2 + 76)}{48x^3} + \frac{\Delta^4 + 220\Delta^2 + 2304}{384x^4} - \dots\right]. \qquad \dots(28)$$

The approximation of Eq. (27.1) yields:

$$N(x) = \frac{C}{x^2}\left[1 - \frac{\Delta}{2x} + \frac{\Delta^2 + 16}{8x^2} - \frac{\Delta(\Delta^2 + 64)}{48x^2} + \frac{\Delta^4 + 160\Delta^2 + 1536}{384x^4} - \dots\right], \quad \dots(28.1)$$

while Eq. (27.2) yields:

$$N(x) = \frac{C}{x^2}\left[1 - \frac{\Delta}{2x} + \frac{\Delta^2 + 16}{8x^2} - \frac{\Delta(\Delta^2 + 76)}{48x^3} + \frac{\Delta^4 + 208\Delta^2 + 2112}{384x^4} - \dots\right]. \quad \dots(28.2)$$

Thus, Eq. (27.2) yields higher accuracy than Eq. (27.1). As Nelkin has noted, the price of this increase in accuracy is the loss of a simple relationship between the flux and the slowing down density. Whereas Eq. (27.1) corresponds to the relation

$$q(x) \approx \frac{\sigma_s}{m}(x^2 - 2)\, N(x) \quad , \quad \dots(5.1)$$

the best we can achieve from Eq. (27.2) is the expression

$$q(x) \approx \frac{\sigma_s}{m}\left[(x^2 - \frac{7}{2})\, N(x) + 3\int_x^{\infty} N(t)\,\frac{dt}{t}\right]. \quad \dots(5.2)$$

In comparison, the exact expression, Eq. (5), is probably simpler to use.

V. NEUTRON AGE

In addition to calculating the neutron spectrum, a major emphasis of the present study has been the determination of the effect of absorption on the migration area of neutrons. This quantity may be readily calculated from a consideration of the scattering collisions which the neutron makes during the slowing down process.

We have postulated a model in which the neutron scattering cross section of the moderating material is a constant independent of energy, and the neutron absorption cross section has an inverse velocity dependence on energy. The number of scattering collisions which occur per unit time in the medium in the velocity interval dx is $\sigma_s x N(x) dx$; the number of absorptions which occur in this interval is $a\sigma_s N(x)\,dx$. Thus, the total number of scattering collisions which occur in the medium below the velocity x_o is

$$R_s = \int_o^{x_o} \sigma_s x N(x) dx \quad ,$$

while the total number of neutrons absorbed in the moderating medium at a velocity less than x_o is

$$R_a = \int_o^{x_o} a\sigma_s N(x) dx \; .$$

The ratio of these two rates is the average number of scattering collisions which a neutron makes before it is absorbed. The mean square distance which the neutron travels from velocity x_o to absorption is then given by:

$$\overline{r^2} = \frac{2R_s}{\sigma_s^2 R_a} = \frac{2\int_o^{x_o} xN(x)\,dx}{a\sigma_s^2 \int_o^{x_o} N(x)\,dx} \qquad \ldots(29)$$

The migration area, $M^2(x)$, is one sixth of this quantity.

Equation (29) is not an exact expression for neutron age; it ignores quantities of order a. However, since our approximation parameter is $\Delta = 2ma$ and we are treating the large mass limit, $m >> 1$, it will be quite consistent with other approximations which we have made to neglect those terms which are smaller by the factor $1/m$ than those which are of immediate concern here.

The integration of the flux is carried out in Appendix D; the result, correct to second order in Δ, is:

$$\int_0^{x_o} xN(x)dx = \frac{2}{\sqrt{\pi}} + C\, ln\, x_o + 0.4908775\Delta + 0.230889\Delta^2 + \ldots \qquad \ldots(30)$$

The migration area therefore becomes

$$M^2(x_o) = \frac{1}{3\sigma_s \bar{\sigma}_a} + \frac{m}{3\sigma_s^2} [ln\, x_o - 0.821108 + 0.56345\Delta + \ldots] \qquad \ldots(31)$$

where $\bar{\sigma}_a = \frac{\sqrt{\pi}}{2} \alpha\sigma_s$ is the mean absorption cross section averaged over a Maxwell distribution.

The migration area of the neutrons from a source at high energy to absorption is therefore expressible as the sum of two terms: one is the usual thermal diffusion area of neutrons in a pure Maxwell distribution; the other is a Fermi Age term which depends logarithmically on the source energy. The usual development of the Fermi Age leads to an expression for the slowing down area which is given in terms of an integral over the logarithmic energy interval from the source energy to thermal energy. The exact value of this lower limit is not specified — nor is such a specification at all possible without a development similar to the one given here. Equation (24) allows us to define an effective value for the energy of the lower limit of the Fermi integral although, of course, there is no sharp distinction between the slowing down region and the thermal region. We may, however, specify an energy E_{eff} which may be used at the lower limit of the Fermi Age integral. We define the effective age as the total migration area minus the thermal diffusion area in a pure Maxwell spectrum. Thus,

$$\tau_{eff} = M^2(x_o) - L^2 = M^2(x_o) - \frac{1}{3\sigma_s \bar{\sigma}_a} ; \qquad \ldots(32)$$

and the effective lower limit of the Fermi age integral is then given by:

$$\tau_{eff} = \int_{E_{eff}}^{x_o^2 kT} \frac{1}{3\xi\sigma_s^2} \frac{dE}{E}, \quad \xi = \frac{2}{m}, \qquad \text{...(33)}$$

whence, from Eq. (31) we deduce:

$$E_{eff} = kT[exp(0.821108 - 0.56345\Delta)]^2 = 5.167kTe^{-1.127\Delta} \qquad \text{...(34)}$$

The separation of the migration area into a thermal diffusion area plus a slowing down age is an arbitrary procedure and the exact manner in which it should be done is not well defined. Thus, the only physically significant entity is the total migration area, and the separation made here is at best arbitrary if not completely artificial.

VI. NEUTRON TEMPERATURE

Several attempts have been made to describe the low energy portion of the neutron spectrum in terms of a Maxwellian distribution with a fictitious temperature.[6, 7, 8] Such a description is qualitatively attractive, but a close examination of Fig. 3 shows that it is deceptively so. Figure 3 shows the fraction of neutrons absorbed at energies less than x^2kT for different values of the absorption parameter Δ. If it were possible to define an effective neutron temperature, this would say that the curves for $\Delta \neq 0$ were obtainable from the curve for $\Delta = 0$ (pure Maxwellian - no absorption) by a simple change of scale. Since the abscissa in the figure is logarithmic, a change of scale would be equivalent to a rigid displacement required to bring that curve into correspondence with one of the other curves. This displacement is then directly a measure of the effective temperature. It is, however, apparent that no such simple displacement is possible. Thus, although it may be possible to define effective temperature by means of some appropriate analytical recipe, such a definition is ambiguous to the extent that different recipes will give quite different effective temperatures.

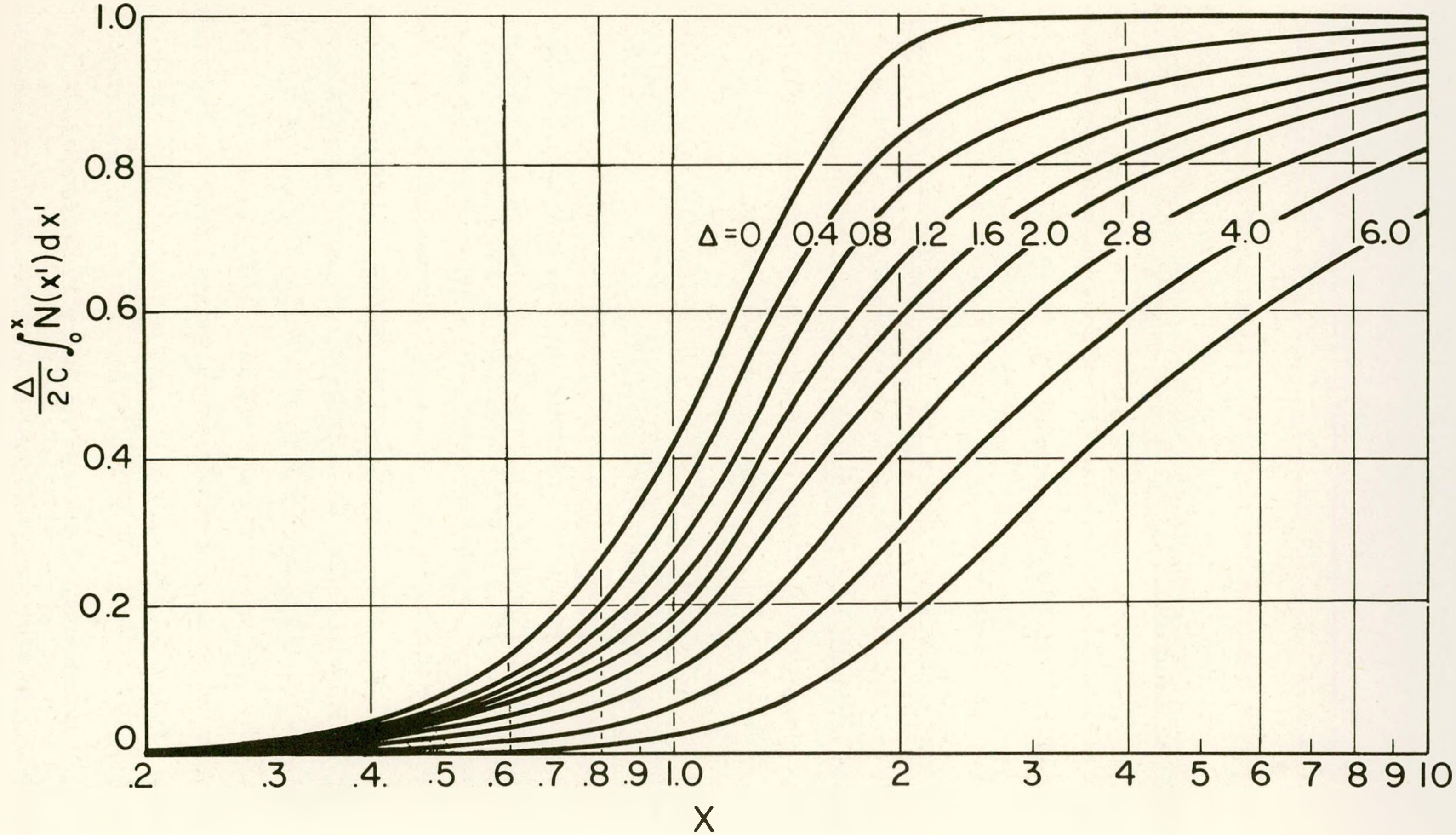

Fig. 3. Neutron Slowing Down Density

APPENDIX A

POWER SERIES SOLUTION FOR NEUTRON SPECTRUM

Equation (9) is:

$$xM''(x) + (3 - 2x)M'(x) - \Delta M(x) = 0.$$

If we assume that $M(x)$ can be expanded in a power series about the point $x = 0$, at which there is a simple pole, we write:

$$M(x) = \sum_{n=o} a_n x^{n+s}, \qquad \text{...}(A.1)$$

where s is as yet undetermined. Inserting this expression into the differential equation, we obtain

$$\sum_{n=o} [(n + s)(n + s - 1)x^{n-1} + (3 - 2x^2)(n + s)x^{n-1} - \Delta x^n] x^s a_n = 0. \qquad \text{...}(A.2)$$

We now collect like powers of x by rearranging the sum.

$$\sum x^n [(n + s + 1)(n + s)a_{n+1} + 3(n + s + 1)a_{n+1} - 2(n + s - 1)a_{n-1} - \Delta a_n] = 0,$$

so that

$$a_{n+1} = \frac{1}{(n + s + 1)(n + s + 3)} [2(n + s - 1)a_{n-1} + \Delta a_n]. \qquad \text{...}(A.3)$$

Since $a_{-2} = a_{-1} = 0$ while $a \neq 0$, we must have

$$s(s + 2) = 0.$$

In this way we find $s = 0$. (The second solution, corresponding to $s = -2$, is singular at the origin and hence will be excluded. It can not be obtained directly in a power series form and will not be further discussed here.)

The coefficients of the power series are then related by the recursion formula of Eq. (1).

APPENDIX B

FIRST ORDER CORRECTION TO MAXWELL DISTRIBUTION

Equation (8.1) is:

$$\mu_1(x) = \int_o^x e^{u^2}\left[\frac{H(u) - uH'(u)}{u^3}\right] du$$

where

$$H'(u) = \frac{2}{\sqrt{\pi}} e^{-u^2} = \frac{2}{\sqrt{\pi}} \sum \frac{(-)^n u^{2n}}{n!} \qquad ;$$

hence

$$H(u) - uH'(u) = \frac{2}{\sqrt{\pi}} \sum_{n=0} \frac{(-)^n}{n!} \left\{ \frac{u^{2n+1}}{2n+1} - u^{2n+1} \right\}$$

$$= \frac{4}{\sqrt{\pi}} \sum_{n=1} \frac{(-)^{n+1} u^{2n+1}}{(2n+1)(n-1)!} \quad ,$$

or

$$\frac{1}{u^3}[H(u) - uH'(u)] = \frac{4}{\sqrt{\pi}} \sum_{n=o} \frac{(-)^n u^{2n}}{(2n+3)n!}$$

The expression for $\mu_1(x)$ can therefore be written in the form:

$$\mu_1(x) = \frac{4}{\sqrt{\pi}} \int_o^x \sum_{r=o} \frac{u^{2r}}{r!} \sum_{n=o} \frac{(-)^n u^{2n}}{n!(2n+3)}$$

$$= \frac{4}{\sqrt{\pi}} \int_o^x \sum_{n,k} \frac{(-)^n u^{2k} du}{n!(k-n)!(2n+3)} \quad .$$

We now make the substitution:

$$\frac{1}{2n+3} = \int_o^1 \xi^{2n+2} d\xi \, ,$$

and interchange the order of integration and summation. This strategem allows us to carry out the summation over the n-index easily and reduces the ξ- integration to a beta-function:

$$\mu_1(x) = \frac{4}{\sqrt{\pi}} \int_o^x \sum_{n,k} \frac{(-)^n}{n!(k-n)!} \int_o^1 \xi^{2n+2} d\xi u^{2k} du$$

$$= \frac{4}{\sqrt{\pi}} \int_o^x \sum_{k=o} \int_o^1 \xi^2 (1-\xi^2)^k d\xi \frac{u^{2k}}{k!} du$$

$$= \frac{2}{\sqrt{\pi}} \int_o^x \sum_{k=o} \frac{B(\frac{3}{2}\; k+1)}{k!} u^{2k} du$$

$$\mu_1(x) = \frac{2}{\sqrt{\pi}} \sum_{k=o} \frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{5}{2})} \int_o^x u^{2k} du$$

$$\mu_1(x) = \sum_{k=o} \frac{x^{2k+1}}{(2k+1)\Gamma(k+\frac{5}{2})} \quad .$$

APPENDIX C

EVALUATION OF $\frac{\Delta}{2}\int_o^{\infty} N(x)dx = C$

From Eq. (20) we have:

$$C = \frac{\Delta}{2}\left[1 + \frac{\Delta}{2}\sum_{n=o}^{\infty}\frac{(n+1)!}{(2n+1)\Gamma(n+\frac{5}{2})} + \frac{\Delta^2}{8}\sum_{n-1}^{\infty}\frac{\Gamma(n+\frac{3}{2})}{n(n+1)!}I_n + \ldots\right] ,$$

where $I_n = \sum_{k=o}^{n=1}\frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})}$.

1. The first order term corresponds to the evaluation of I_{∞}; we can write this in the form:

$$I_{\infty} = \sum_{n=o}^{\infty}\frac{(n+1)!}{(2n+1)\Gamma(n+\frac{5}{2})}$$

Now

$$\int_o^{\pi/2} sin^{2n+3}\theta d\theta = \frac{\sqrt{\pi}\,(n+1)!}{2\Gamma(n+\frac{5}{2})} ,$$

hence

$$I_{\infty} = \frac{2}{\sqrt{\pi}}\int_o^{\pi/2}\sum_{n=o}\frac{sin^{2n+3}\theta}{2n+1}d\theta$$

$$= \frac{2}{\sqrt{\pi}} \int_o^{\pi/2} sin^2\theta \; tanh^{-1}(sin\theta) \, d\theta \quad .$$

We now make the Gudermannian transformation:

$$tanh\,\phi = sin\,\theta$$

$$cosh\,\phi = sec\,\theta$$

$$sech\,\phi = cos\,\theta \quad ;$$

then $$I_\infty = \frac{2}{\sqrt{\pi}} \int_o^\infty \frac{tanh^2\,\phi}{cosh\,\phi} \phi \, d\,\phi \ ,$$

$$= \frac{2}{\sqrt{\pi}} \int_o^\infty \phi \; sinh\,\phi \; \frac{sinh\,\phi}{cosh^3\,\phi} \, d\phi \ ,$$

$$I_\infty = \frac{1}{\sqrt{\pi}} \left[1 + \int_o^\infty \phi \; sech\,\phi \, d\phi \right] .$$

The integration by parts has reduced the problem to the calculation of a fairly well behaved integral. Several methods exist for evaluating this integral; the most direct is to expand $sech\,\phi$ as a series in $e^{-\phi}$.

$$\int_o^\infty \phi \; sech\,\phi \, d\phi = 2 \int_o^\infty \phi e^{-\phi} \; [1 - e^{-2\phi} + e^{-4\phi} - \dots] \, d\phi$$

$$= 2\left[1 - \frac{1}{9} + \frac{1}{25} - \frac{1}{49} + \frac{1}{81} - \ldots\right] .$$

The series in the square brackets defines Catalan's constant. Its value[5] is

$$G = 0.91596\ 55941\ 18\ \ldots.$$

Hence we have

$$\frac{1}{2} I_\infty = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2} + G\right) = 0.79887\ 3038\ \ldots$$

2. The second order term can be best evaluated by interchanging the order of summation. The second order term is, therefore,

$$\frac{\Delta^2}{8} \sum_{n=1}^{\infty} \sum_{k=o}^{n=1} \frac{(k+1)!\Gamma(n+\frac{3}{2})}{(2k+1)\Gamma(k+\frac{5}{2})n(n+1)!}$$

$$= \frac{\Delta^2}{8} \sum_{k=o} \frac{(k+1)!\, S_k}{(2k+1)\Gamma(k+\frac{5}{2})}$$

where

$$S_k = \sum_{n=k+1} \frac{\Gamma(n+\frac{3}{2})}{n(n+1)!}$$

To evaluate S_k , we make use of the integral representation of the Beta-function.

It follows directly, then, that

$$\frac{\Gamma(n+\frac{3}{2})}{(n+1)!} = \frac{1}{\sqrt{\pi}} \int_o^1 u^{n+1/2}(1-u)^{-1/2}\, du ,$$

and therefore,

$$S_k = \frac{1}{\sqrt{\pi}} \int_o^1 \sqrt{\frac{u}{1-u}} \left\{ ln \frac{1}{1-u} - u - \frac{1}{2} u \ldots \frac{1}{k} u^k \right\} du$$

and

$$S_o = \frac{1}{\sqrt{\pi}} \int_o^1 \sqrt{\frac{u}{1-u}} \, ln \frac{1}{1-u} \, du = \sqrt{\pi} \left(ln 2 + \frac{1}{2} \right) .$$

This integral is tabulated by Grobner and Hofreiter;[5] then with

$$s_n = \frac{\Gamma(n + \frac{3}{2})}{(n+1)!} \frac{(2n+1)}{(2n+2)} s_{n-1} \quad ;$$

we can write

$$S_k = S_{k-1} - \frac{1}{k} s_k \quad .$$

We further improve the convergence of the summation by noting that

$$S_k \rightarrow \sum_{k+1} \frac{1}{n^{3/2}} \approx \frac{2}{(k + \frac{1}{2})^{1/2}} \quad ;$$

hence

$$\frac{(k+1)! \, S_k}{(2k+1)\Gamma(k + \frac{5}{2})} \rightarrow \frac{4}{(2k+1)^2} \quad .$$

The second order term can therefore be written in a more convenient form for computation:

$$\frac{\Delta^2}{2}\left[\sum_{k=o}\frac{1}{(2k+1)^2}+\sum_{k=o}\frac{1}{(2k+1)^2}\left\{\frac{(k+1)!(2k+1)S_k}{4\Gamma(k+\frac{5}{2})}-1\right\}\right]$$

The first sum is well known; its value is $\pi^2/8 = 1.23370\ 05501\ldots$.
The second sum is rapidly convergent and can be easily computed. Using ten terms and estimating a correction for the sum of the remaining terms gives us -0.660488. The second order term therefore is computed to be $0.286606\Delta^2$.

APPENDIX D

INTEGRATION OF $\int_o^{x_o} xN(x)\,dx$.

From Eq. (17) we can write

$$\int_o^{x_o} xN(x)dx = \frac{4}{\sqrt{\pi}} \int_o^{x_o} x^3 e^{-x^2} dx + \Delta \int_o^{x_o} \sum_{n=o} \frac{x^{2n+4} e^{-x^2}\, dx}{(2n+1)\Gamma(n+\frac{5}{2})}$$

$$+ \Delta^2 \int_o^{x_o} \sum_{n=1}^{\infty} \sum_{k=o}^{n-1} \frac{(k+1)!\; x^{2n+3} e^{-x^2}}{(2k+1)\Gamma(k+\frac{5}{2})\, n(n+1)!}\, dx$$

$$= \frac{2}{\sqrt{\pi}} \left[1 - (1 + x_0^2)\, e^{-x_0^2} \right] + \Delta\, K_1 + \Delta^2 K_2 \ .$$

We shall treat K_1 and K_2 separately.

The integrand of K_1 approaches $\frac{1}{2x}$ for large values of x and K_1 is therefore logarithmically divergent. However, this divergence can be controlled by adding and subtracting from the integrand an expression which is on the one hand easily integrable and on the other hand represses the divergent part of K_1. Such an expression is

$$\frac{1 - e^{-u}}{2u}$$

Therefore we shall write, replacing x^2 by u,

$$K_1 = \frac{1}{2} \int_o^{x_0^2} \left[\sum_{n=0}^{\infty} \frac{u^{n+3/2}}{(2n+1)\Gamma(n+\frac{5}{2})} - \frac{e^u - 1}{2u} e^{-u} du \right]$$

$$+ \frac{1}{4} \int_o^1 \frac{1-e^{-u}}{u} du + \frac{1}{2} \ln x_o - \frac{1}{4} \int_1^{x_0^2} \frac{e^{-u}}{u} du \quad .$$

We now expand the function $e^u - 1$ in a power series. Also we recognize that

$$\int_0^1 \frac{1-e^{-u}}{u} du - \int_1^{\infty} \frac{e^{-u}}{u} du = \gamma = 0.5772157 \ldots$$

and write

$$K_1 = \frac{1}{2} \int_0^{x_0^2} \sum_{n=0}^{\infty} \left\{ \frac{u^{n+3/2}}{(2n+1)\Gamma(n+\frac{5}{2})} - \frac{u^n}{2(n+1)!} \right\} e^{-u} du + \frac{1}{4} \gamma + \frac{1}{2} \ln x_o \quad .$$

We are interested primarily in the limit, as x_o becomes very large, of the quantity $K_1 - \ln x_o$. This expression is finite and hence we may extend the integral to infinity and interchange the order of summation and integration:

$$K_1 = \frac{1}{2} \sum_{n=o}^{\infty} \left\{ \frac{1}{2n+1} - \frac{1}{2(n+1)} \right\} + \frac{1}{2} \ln x_o + \frac{1}{4} \gamma \quad .$$

It is easy to see that the summation which now appears in K_1 is simply the series for $1/2 \ln 2$; hence,

$$K_1 = \frac{1}{2}\left(ln2 + \frac{1}{2}\gamma + ln\, x_o \right)$$

$$= 0.4908775 + \frac{1}{2}\, ln\, x_o \quad .$$

We must follow the same procedure for the evaluation of K_2. The asymptotic behavior of $\mu_2(x)$ can be obtained from Eq. (8). We know from the calculation in Appendix C that

$$\int_o^u \mu_1(t)t^2 e^{-t^2}\, dt \to 0.798873 \text{ as } u \to \infty ,$$

whence Eq. (8) in the limit of large x, can be written:

$$\mu_2(x) = \frac{1}{2}(0.798873) \int_o^{x^2} \frac{e^{x^2 - v}}{(x^2 - v)^2}\, dv \quad ;$$

$$\mu_2(x) \approx \frac{1}{2}(0.798873) \frac{e^{x^2}}{x^4} \int_o^{x^2} \left(1 + \frac{2v}{x^2} + \ldots\right) e^{-v}\, dv$$

$$\approx \frac{1}{2}(0.798873) \frac{e^{x^2}}{x^4} \quad .$$

The integrand of K_2 therefore approaches $\frac{0.798873}{2x}$ for large values of x. We add and subtract as before and obtain:

$$K_2 = \frac{1}{4} \int_o^{\infty} e^{-u} \sum_{n=1}^{\infty} \left\{ \frac{1}{n} \sum_{k=o}^{n-1} \frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})} - \frac{1}{n+2} \sum_{o}^{\infty} \frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})} \right\} \frac{u^{n+1}}{(n+1)!}\, du$$

$$+\frac{1}{2}(0.798873)(\gamma + 2\, ln\, x_o) \quad ,$$

where, for symmetry reasons we have retained the summation

$\frac{1}{2}\sum_{k=o}^{\infty}\frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})} = 0.798873$ inside the integral. The integral is convergent and we can write:

$$K_2 = \frac{1}{4}\sum_{n=1}\left\{\frac{1}{n}\sum_{k=o}^{n-1}\frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})} - \frac{1}{n+2}\sum_{o}^{\infty}\frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})}\right\}$$

$$+\frac{0.798873}{2}(\gamma + 2\, ln\, x_o) \quad .$$

The summation which appears here can be simplified somewhat by writing:

$$K_2 = \frac{1}{4}\sum_{n=1}\left\{\frac{1}{n} - \frac{1}{n+2}\right\}\sum_{o}^{\infty}\frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})} - \frac{1}{4}\sum_{n=1}\frac{1}{n}\sum_{k=n}^{\infty}\frac{(k+1)!}{(2k+1)\Gamma(k+\frac{5}{2})}$$

$$+\frac{0.798873}{2}(\gamma + 2\, ln\, x_o) \quad .$$

But $$\sum_{n=1}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{3}{2}$$

so that

$$K_2 = \frac{0.798873}{2}\left[\gamma + 2\, ln\, x_o + \frac{3}{2} - \frac{1}{s_o}\sum_{n=1}\frac{s_n}{n}\right] \quad ,$$

in which

$$s_{n+1} = s_n - \frac{t_n}{2n + 1} \quad , \qquad s_o = \frac{3}{2}(G + \frac{1}{2}) = 2.12394\ 83913 \quad ,$$

$$t_n = \frac{2(n + 1)}{2n + 3} t_{n-1} \qquad t_o = 1$$

so that finally

$$K_2 = \frac{0.798873}{2}(\gamma + 0.000818 + 2\ ln\ x_o) \quad .$$

Therefore,

$$\int_o^{x_o} xN(x)dx = \frac{2}{\sqrt{\pi}} + C\ ln\ x_o + \frac{1}{2}\left(ln\ 2 + \frac{1}{2}\gamma\right)\ \Delta + 0.798873\left(\frac{1}{2}\gamma + .000409\right)\Delta^2 + \ldots$$

$$= \frac{2}{\sqrt{\pi}} + C\ ln\ x_o + 0.4908775\Delta + 0.230889\Delta^2 + \ldots$$

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