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STATISTICAL VALIDATION OF STOCHASTIC MODELS

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Norman F. Hunter[1], Patrick Barney[2], Carlos Ferregut[3], Luis Perez[3], Thomas L. Paez[2]

[1] Engineering Science and Analysis Division
Los Alamos National Laboratory
Los Alamos, New Mexico

[2] Experimental Structural Dynamics Department
Sandia National Laboratories
Albuquerque, New Mexico

[3] Department of Civil Engineering
University of Texas at El Paso
El Paso, Texas

ABSTRACT

It is common practice in structural dynamics to develop mathematical models for system behavior, and we are now capable of developing stochastic models, i.e., models whose parameters are random variables. Such models have random characteristics that are meant to simulate the randomness in characteristics of experimentally observed systems. This paper suggests a formal statistical procedure for the validation of mathematical models of stochastic systems when data taken during operation of the stochastic system are available. The statistical characteristics of the experimental system are obtained using the bootstrap, a technique for the statistical analysis of non-Gaussian data. We propose a procedure to determine whether or not a mathematical model is an acceptable model of a stochastic system with regard to user-specified measures of system behavior. A numerical example is presented to demonstrate the application of the technique.

NOMENCLATURE

B	number of bootstrap replicates
CDF	cumulative distribution function
$\boldsymbol{F}$	a CDF
$\hat{\boldsymbol{F}}$	an empirical CDF
$\boldsymbol{K}$	stiffness matrix
K	kernel function
KDE	kernel density estimator
$\boldsymbol{M}$	mass matrix
PDF	probability density function
$\boldsymbol{X}$	a data sample
$\boldsymbol{X}^{*b}, b=1,...,B$	a bootstrap sample
$(\boldsymbol{X},\boldsymbol{Y})=(x_1,...,x_n,y_1,...,y_n)$	input/response pairs
f	a PDF
$\hat{f}$	approximation to a PDF
h	window width in KDE
m	number of modes
n	number of data points
s(.)	function defining a statistic
$x_j, j=1,...,n$	a data sample
θ	a ststistic of interest
$\hat{\theta}^*(b), b=1,...,B$	bootstrap replicates of statistic of interest
$\omega_k, k=1,...,m$	modal frequencies

1. INTRODUCTION

Development of a stochastic system model is commonly guided by a balance between two requirements: (1) the need to represent reality, reflected by the measured data, and (2) the pragmatic need for a relatively simple mathematical model. Therefore, the validation of a model would depend on the degree of uncertainty associated with the measured data reflecting system behavior and the number of basic variables, parameters and complexity of their interrelationships that have been included in the model. It is obvious from this that it is not particularly helpful to try to validate a model by calculating the differences of the results from the measured data. However, any alternative validation scheme should have a level of sophistication which does not alter the pragmatic level of complexity that characterizes the model. Further, it would be convenient if the model validation scheme makes full use of the information provided by the measured data.

The authors have proposed a statistical validation methodology for deterministic models of dynamic systems (Paez, et.al., 1996, Barney, et.al., 1997). The methodology is based on the bootstrap. The bootstrap was developed by Efron (1979), and is clearly described by Efron and Tibshirani (1993). It is a technique for the statistical analysis of non-Gaussian statistics of measured data. The statistical validation methodology uses the bootstrap with data measured experimentally to estimate confidence intervals for measures of the system behavior. Then, these same measures are evaluated from the mathematical model and located relative to the confidence intervals. If the measures of system behavior predicted by the deterministic model fall within the confidence intervals the model is accepted at the significance level represented by the interval, otherwise the model is rejected.

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This paper extends the proposed methodology to the validation of stochastic models of system performance and applies the extended methodology to problems in structural vibrations. The proposed procedure includes the following steps. First, identify one or more measures of system performance as the basis for validation of the mathematical model. Next, using the bootstrap and the experimental data from the physical system derive marginal or joint probability density functions (PDF) or cumulative distribution functions (CDF) of the statistics of interest. Using the stochastic model built, obtain the probabilistic functions of the same measures of system performance. Once the experimental and theoretical PDFs or CDFs are found, perform a goodness of fit test to accept or reject the hypothesis that the stochastic mathematical model is a satisfactory representation of the physical system behavior. In addition to visual inspection a Chi-square or a Kolmogorov-Smirnov test can be run to execute the hypothesis test.

In the following a brief description of the bootstrap is presented. Next, it is shown how the bootstrap can be used to generate a sample of statistics of system behavior measures from limited modal test data. Next, stochastic mathematical models are discussed. Then, the framework for statistical validation of random models is developed. Finally, the methodology is applied to the validation of a stochastic finite element model of an aluminum beam.

2. THE BOOTSTRAP

The bootstrap is a data-based technique for estimating the accuracy of parameters derived from probability distributions. The bootstrap was developed by Efron (1979), and is readily applicable to estimating the accuracy of the mean estimate, the variance estimate, and the estimates of other probability distribution moments, as well as more complex statistics of random variables and random processes. The bootstrap is well suited to the estimation of bias, standard error, and confidence intervals of parameters derived from measured data. In the process of estimating confidence intervals we approximate the sampling distribution of the statistic of interest - this will be discussed in more detail later in this section and in the following section.

A bootstrap analysis is based on a sequence of data values, $x_j, j=1,\dots,n$. We assume that these values are produced by a source with an unknown probability distribution. Our only knowledge of the source is the measured sequence of data values. Each observed data point is assigned a probability of occurrence of 1/n, where n is the number of measured data points. A bootstrap sample is created by selecting at random, with replacement, n elements from the measured data set. The creation of a bootstrap sample is illustrated in Figure 1. This procedure is readily implemented using a uniform random number generator which selects, with equal probability, integer values in the range 1 to n. Sampling is done with replacement, so each bootstrap sample may have several occurrences of some data values and other data values may be absent.

$\hat{F} = \boldsymbol{X} = (x_1, x_2, \dots, x_{16})$ *(Samples have equal probability)*

$\downarrow$ *Creation of bootstrap sample is accomplished through random selection among elements of* $\boldsymbol{X}$. *For example, let* $\boldsymbol{X} = (x_1, \dots, x_{16})$. *A potential bootstrap sample is shown below. The sample contains* 16 *elements.*

$\boldsymbol{X}^* = (x_2, x_7, x_4, x_{11}, \dots, x_4)$

Figure 1. Obtaining a bootstrap sample.

In a typical bootstrap analysis, numerous bootstrap samples, $\boldsymbol{X}^{*b}, b=1,\dots,B$, are created. The statistic of interest is computed for each bootstrap sample; the resulting quantities are known as bootstrap replicates of the statistic of interest. The bootstrap replicates are denoted $\hat{\theta}^*(b), b=1,\dots,B$, and are defined

$$\hat{\theta}^*(b) = s\left(\boldsymbol{X}^{*b}\right) \qquad b=1,\dots,B \tag{1}$$

where s(.) denotes the formula applied to the data to compute the statistic.

The hope in bootstrap analysis is that the bootstrap replicates are governed by a probability law that is approximately the same as the theoretical sampling distribution of the statistic of interest. The sampling distribution of the statistic of interest is the probability law that governs realizations of the statistic - the realizations that could be generated if unlimited data from the underlying source were available.

One class of applications of the bootstrap seeks to estimate standard error, confidence intervals, and bias of the statistic of interest. The standard error is estimated using the bootstrap replicates in the usual formula for the standard deviation. Confidence intervals are estimated by sorting the bootstrap replicates and defining intervals associated with percentiles in the sorted list. However, this is not the class to be investigated here, so we will not elaborate on this further.

Bootstrap sampling provides an optimal estimate of the probability density function which characterizes the data source given that our knowledge of the source is limited to the measured data. Computation of a statistic from the bootstrap samples simulates computation of the same statistic on samples drawn from the real world distribution. Properties of the 'real world' distribution are estimated in the 'bootstrap world' as illustrated in Figure 2.

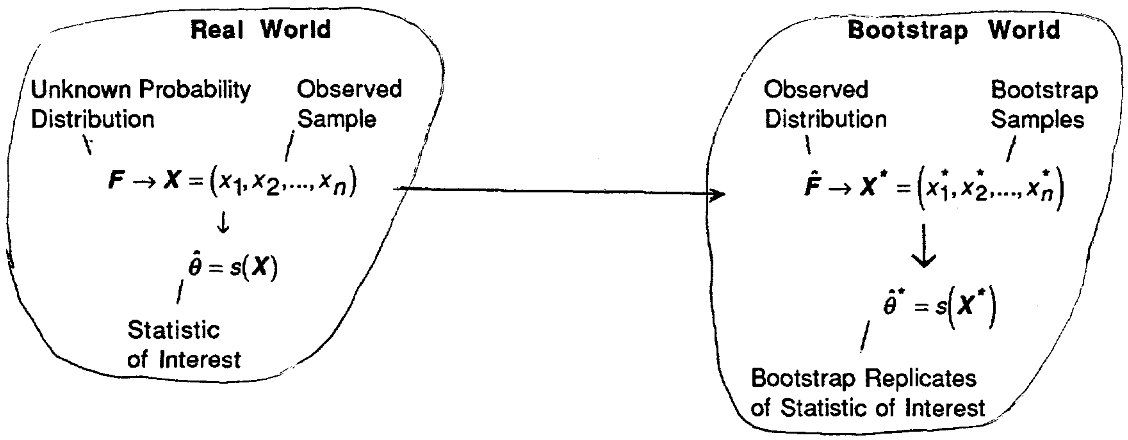

Figure 2. The bootstrap approximation to the real world. The observed distribution is our best estimate of the true distribution. The observed sample is X, and the statistic of interest $\hat{\theta} = s(X)$ can be computed based on this. In the bootstrap world the observed data are used to generate as many bootstrap samples X^* as we wish. Each bootstrap sample is used in the formula $\hat{\theta}^* = s(X^*)$ to compute a bootstrap replicate of the statistic of interest. The bootstrap replicates are used to analyze the standard error, confidence intervals and bias of the statistical estimator.

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3. SAMPLING DISTRIBUTION OF MEASURE OF MECHANICAL SYSTEM BEHAVIOR

The previous section briefly introduced the bootstrap and noted that it is a technique for the accuracy analysis of statistics of random data. Among other things, it can be used to estimate standard error and the confidence intervals of statistical estimators. However, in the process of establishing these estimates the bootstrap generates information that can be used to approximate the sampling distribution of the statistic of interest. Because our goal is to assess the accuracy of a stochastic mathematical model of a physical system, it is this distribution in which we are interested, and therefore, we now focus on its approximation.

Recall that to use the bootstrap, it is necessary to build an ensemble of bootstrap replicates of the statistic of interest. We will describe later in this section how to use vibration data to create bootstrap replicates of quantities that influence the dynamic behavior of structural systems. Now though, we assume that we have bootstrap replicates of a statistic of interest and describe the tools that are used to approximate the probability density function (PDF) of the source of the replicates.

There are several approaches to the approximation of the PDF of a random variable based on realizations measured from the random source, and some of these are described in Silverman, 1986. In this investigation we use the kernel density estimator (KDE). As in the previous section, denote the bootstrap replicates $\hat{\theta}^*(b), b = 1, \ldots, B$. The KDE based on these data is defined

$$\hat{f}(\alpha) = \frac{1}{Bh} \sum_{b=1}^{B} K\left(\frac{\alpha - \hat{\theta}^*(b)}{h}\right) \qquad -\infty < \alpha < \infty \qquad (2)$$

where $K(.)$ is the kernel function, and basically, must satisfy the requirements to be a valid PDF (see Wirsching, Paez, and Ortiz, 1995, for these requirements), and h is the window width and must be nonnegative. In the applications to follow, we define the kernel function to have the form of the standard normal PDF. Equation (2) is the estimator of the PDF of the bootstrap replicates.

In the general case where the bootstrap replicates are vector quantities we can define a multivariate KDE to approximate the multivariate PDF, and the validation to be described in the sequel can be carried out in a multi-dimensional space. For present purposes, however, we will

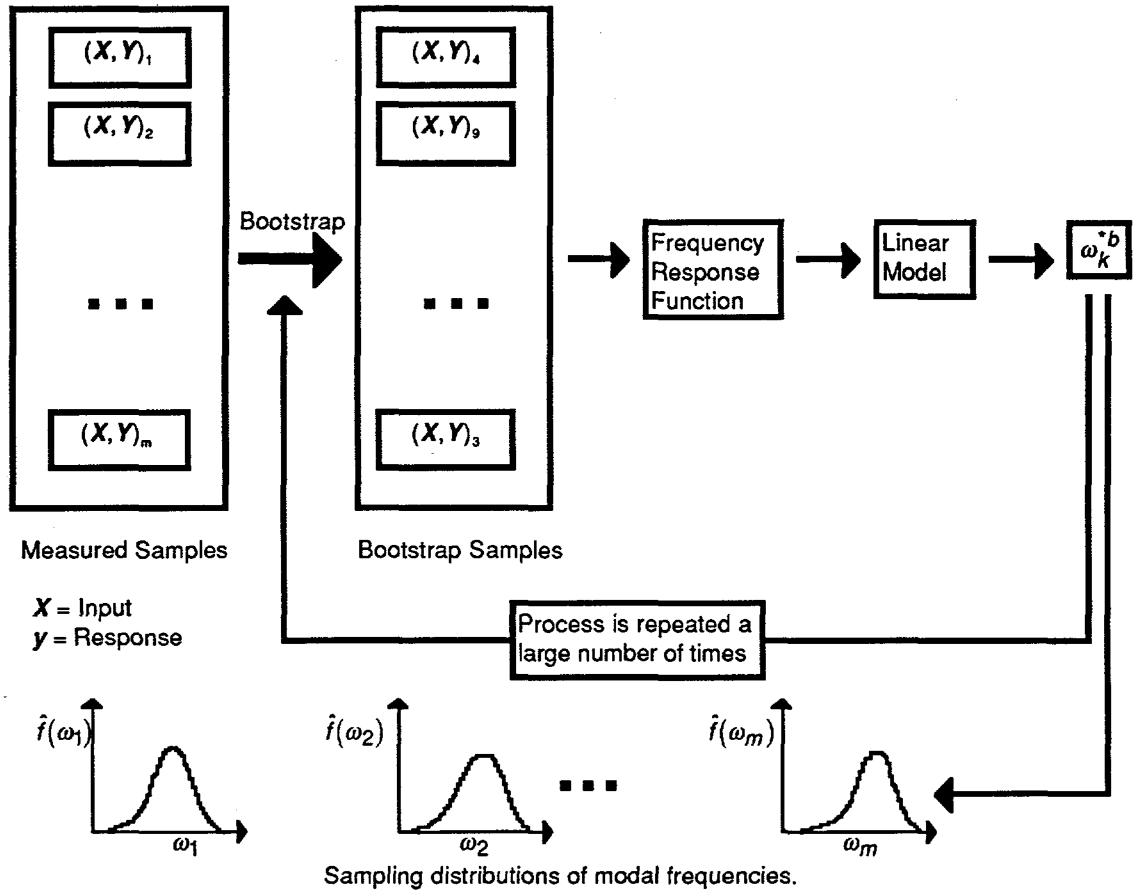

Figure 3. Schematic showing how the bootstrap is used to create bootstrap replicates of system modal frequencies.

be satisfied to consider individual variables and their univariate distributions.

We now show how bootstrap replicates of dynamic systems performance data can be developed in a very practical framework. The overall process is illustrated in Figure 3. To commence the development we assume that measured inputs and outputs from the system to be characterized are available. Denote the collection of inputs and output pairs as $(\boldsymbol{X},\boldsymbol{Y})=(x_1,...,x_n,y_1,...,y_n)$ where y_1 is the output function corresponding to the input function x_i. These pairs may be, for example, input and response segments of a stationary random process, shock inputs and their corresponding responses, or any other form of input and corresponding responses. We assume that one or more statistics of these data are the measures of system behavior or parameter estimates of interest. For example, if the system being modeled is a linear structural system and X represents the excitation function and Y the response function (e.g. acceleration) then a frequency response function (FRF) can be derived by combining all the x,y pairs the usual way (Wirsching, Paez, and Ortiz, 1995). Once the FRF is known a linear model can be used to extract the modal frequencies of the system $\omega_k, k=1,...,m$. We could also extract mode shapes and modal frequencies from a linear model. If we assume that the response is nonlinear then parameters of nonlinear input/output models can be established, or parameters of the response only can be established. In figure 3 and the following discussion we assume that the modal frequencies are the statistics of interest of the random input/output data.

Given that the natural frequencies can be estimated using the original set of measurements $(\boldsymbol{X},\boldsymbol{Y})$, they can also be estimated using a bootstrap sample of this original set. (Hunter and Paez, 1995, and Paez and Hunter, 1996) A bootstrap sample of the input and output set can be denoted $(\boldsymbol{X},\boldsymbol{Y})^{*b}, b=1,...,B$ where each of the x,y pairs is drawn randomly from the elements of the original set $(\boldsymbol{X},\boldsymbol{Y})$ as shown graphically in Figure 3. Using this bootstrap sample, a bootstrap replicate of the statistic of interest is computed. In the case of the structural system mentioned above, this would be a bootstrap replicate of natural frequencies $\omega_k^{*b}, b=1,...,B, k=1,...,m$, of the system. Any number B of bootstrap replicates can be generated using this approach. The collection of bootstrap replicates generated can be used in Eq. (2) to approximate the PDF of the sampling distribution of each ω_k. This will be done later in the numerical example.

4. STOCHASTIC MATHEMATICAL MODELS

Our objective in this study is to establish a technique for the validation of stochastic mathematical models, and we have shown in the previous sections how to develop an estimate for the probability distribution of one or more measure of the character of a physical system (modal frequencies, in the examples). It now remains to show how to establish the probability distributions of these same measures of character of a structure, but now based on the stochastic mathematical model.

A stochastic mathematical model of a physical system is normally (1) a set of differential equations with random variable or random process parameters, (2) the solution of a set of differential equations involving random variable or random process terms, or (3) an empirical expression characterizing physical system response and involving random variable or random process terms. The developments and applications of stochastic mathematical models are so broad that we cannot provide a general discussion here. We can, however, briefly show the form of a stochastic finite element model of a structural dynamic system, and explain how the stochastic model leads to random characteristics.

The equation of motion for an undamped, unforced, linear structural dynamic system is

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} = 0 \qquad (3)$$

where $\boldsymbol{x}$ is an Nx1 vector of displacement responses, $\boldsymbol{M}$ is the NxN system mass matrix, $\boldsymbol{K}$ is the NxN system stiffness matrix, N is the number of degrees of freedom in the model, dots denote differentiation with respect to time. See Clough and Penzien (1975) for further details and discussion of this equation of motion. If the system is time invariant, i.e., its characteristics do not change with time, then the mass and stiffness matrices are populated with constants. When the system is assumed to be stochastic, then some or all of the elements in the mass and stiffness matrices are random variables. (The random elements do not vary in time, but rather assume one constant random value for all time.)

The modal frequencies of the system described in Eq. (3) are related to the roots of the characteristic equation

$$\left| -\omega^2 \boldsymbol{M} + \boldsymbol{K} \right| = 0 \qquad (4)$$

The equation has N roots, and usually the lowest m are evaluated. The modal frequencies can be denoted $\omega_k, k = 1,\ldots,m$. When any of the elements in $\boldsymbol{M}$ of $\boldsymbol{K}$ is random, then all the modal frequencies are random.

There are many potential approaches to the probabilistic characterization of the modal frequencies. Among these are (1) Monte Carlo, (2) first and second order reliability methods (FORM/SORM) (Madsen, Krenk, and Lind, 1986), (3) the advanced mean value (AMV) method, (Wu and Wirsching, 1987), and others. We choose to analyze the probabilistic character of the modal frequencies using the AMV method. The basic idea behind the AMV method is that it finds an approximate expression for the modal frequencies in terms of the random variables in $\boldsymbol{M}$ and $\boldsymbol{K}$, then iteratively approximates the probability distribution of each modal frequency using a transformation to standard normal space. It is used to establish the approximate CDF of each modal frequency.

$$\hat{F}_{\omega_k}(\alpha) = P(\omega_k \le \alpha) \qquad \alpha \le 0, k = 1,\ldots,m \qquad (5)$$

This function can be approximately differentiated to obtain the approximate PDF for comparison to the bootstrap-based PDF in Eq. (2), or the bootstrap-based PDF can be integrated to obtain the data-based CDF approximation, and that can be compared to Eq. (5). Such a comparison is carried out later in the numerical example.

5. VALIDATION OF STOCHASTIC MODELS

The validation of a stochastic model is carried out by comparing the probability distributions of some measures of its performance to the probability distributions of the corresponding measures obtained from experimental data from a physical system. The comparison can be performed using PDFs or CDFs, and these can be marginal or joint. When the individual characteristics of model and system are to be compared, then the marginal distributions are used. When the joint behavior of multiple characteristics is to be compared, then the joint distributions are used. An example of the former is the comparison of individual modal frequencies. An example of the latter is the comparison of mode shapes. Of course, the stochastic mathematical model is judged valid with respect to the characteristics tested when its probabilistic character matches the probabilistic character of the experimental system closely enough. Otherwise, it is judged not representative.

The comparison that is the basis of the validation can be done in several ways. The first possibility is a visual comparison of PDFs or CDFs. This can be used when marginal distributions are to be compared. More formally, distributions can be compared using the chi squared test, the Kolmogorov-Smirnov test, or probability paper (Ang and Tang, 1975). Further, the distributions can be compared using the bootstrap approach, an extension of the development presented here. Both visual and probability paper comparisons are used in the following numerical example.

6. NUMERICAL EXAMPLE

This example applies the stochastic model validation technique developed in the previous sections to the validation of a stochastic finite element model of a beam. The beam was configured as a cantilever and tested in the laboratory. Its parameters are listed in Table 1. The clamped end was attached to the fixture using a bolted steel sandwich device with neoprene pads between the fixture and the beam. The number of pads was varied randomly from one test to the next to simulate the randomness that might be encountered in a complex physical experiment.

Table 1. Parameters of beam tested in laboratory.

length	12 inches
depth	0.125 inch
width	1 inch
material	aluminum
modulus of elasticity (E)	10^7 psi
mass density	2.53×10^{-4} lb-sec^2/in^4

The beam was tested by applying an initial deflection, releasing it, and observing the decayed response acceleration. This was repeated 10 times, and the response was measured each time.

A stochastic finite element model of the beam was created. The model uses 48 Bernoulli beam elements with a translational and a rotational degree of freedom at each node. The cantilever condition is modeled using a torsional spring in the rotational degree of freedom at the clamped end. The torsional spring stiffness is taken to be a normally distributed random variable with a mean of 657.5 in-lb/rad and a variance of $(13.5)^2$ (in-lb/rad)^2. All the other finite element model parameters are taken as deterministic and based on the parameters in Table 1.

The data obtained in the experiment were used with the bootstrap approach to characterize the sampling distribution of the first modal frequency. The KDE estimate of the PDF is shown as the solid line in Figure 4. An AMV method analysis of the finite element model was performed to characterize the first modal frequency, and the results were used to approximate the PDF. The PDF approximation is shown as the dashed line in Figure 4. The curves match well.

The two distributions were also compared using a probability paper approach. Probability paper can be constructed as described in Ang and Tang (1975). The comparison is based on the formula

$$\beta = \hat{F}_{\omega_1}^{-1}\left(\hat{F}(\alpha)\right) \quad (6)$$

where $\hat{F}(.)$ is the CDF approximation corresponding to the PDF approximation in Eq. (2), and $\hat{F}_{\omega_1}(.)$ is the CDF approximation of Eq. (5). Clearly, if the two CDFs were identical, then the plot of β versus α would be a straight line. The nearness of the plot to a straight line reflects the accuracy of the approximation of $\hat{F}_{\omega_1}(.)$ to $\hat{F}(.)$. Figure 5 shows the result obtained when the approximate CDFs developed in this example were used in Eq. (6). The curve spans abscissa values corresponding to $\hat{F}(.)$ in the range [0.001,0.999]. Clearly, the approximation is a relatively good one over the entire range.

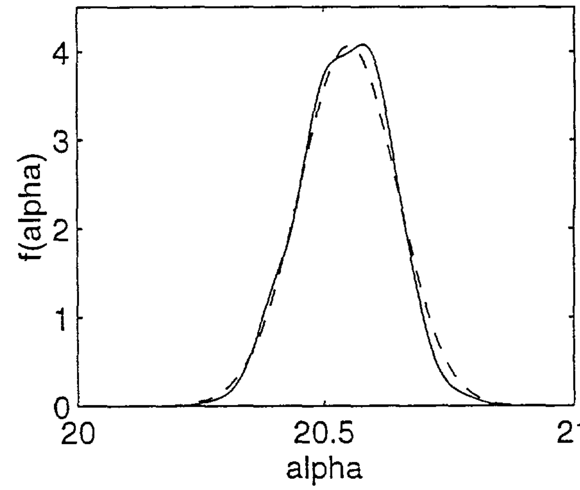

Figure 4. Comparison of the PDF approximation of the stochastic mathematical model to the bootstrap-based PDF approximation.

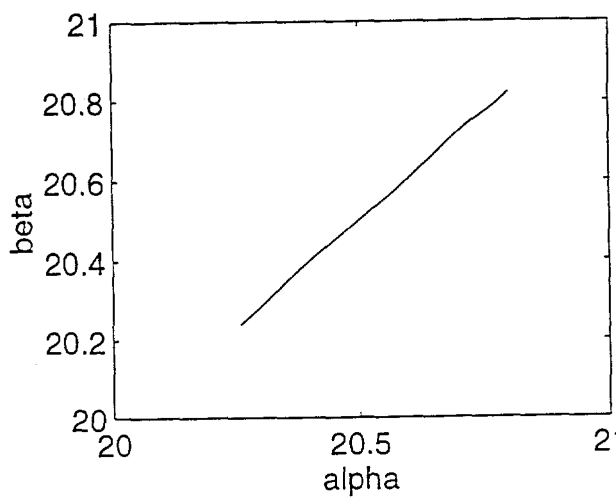

Figure 5. Probability paper type comparison of the approximate stochastic mathematical model CDF to the bootstrap-based CDF, based on Eq. (6).

Other comparisons of the two distributions could also be performed. It would be desirable to establish a confidence region for the CDF $\hat{F}(.)$ and observe whether or not the cdf $\hat{F}_{\omega_1}(.)$ lies within that region. In this way a hypothesis test on the stochastic mathematical model could be performed and we could reject or accept the stochastic model on that basis.

7. CONCLUSION

We have extended in this paper the approach to statistical model validation developed by the authors in previous investigations. The approach is based on the bootstrap method for statistical analysis and the probabilistic analysis of a stochastic mathematical model. We chose to execute the latter analysis here using the advanced mean value method. The validation approach accounts for randomness in real system characteristics and the randomness included in the mathematical model. The approach is systematic in that it is based on a well established statistical analysis procedure, and any of a number of well established probabilistic analysis procedures. The approach is computer intensive with regard to both the bootstrap analysis and the probabilistic system analysis of the stochastic model. However, its advantage is that it properly accounts for the non-Gaussian nature of the data and the mathematical model.

It must be emphasized that the analyst who uses the proposed procedure for statistical model validation must be judicious in his or her choice of the specific measures and the number of measures of model performance used to validate the model. The number of measures should be neither too great nor too small, and should reflect the importance of the application. The specific measures of performance used should reflect the analyst's expectations of the model. Some measures of performance will be easier to validate than others. However, when detailed model behavior is validated, model performance in the simulation of detailed behavior will be anticipated to be accurate.

8. ACKNOWLEDGMENT

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