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Covariant Theory of Magnetic Monopole*

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Abstract

In the local quantum field theory of magnetic monopole proposed by Hagen, not all of the equations of motion are covariant under the Lorentz transformation. In this note we show that with the introduction of a shadow electromagnetic field quantized with "wrong sign", the locality in the sense of local commutation relations and local equations of motion is still retained, while the equations of motion become covariant under the Lorentz transformation.

The lack of symmetry in the usual Maxwell's equation for the field tensor $F^{\mu\nu}$ and its dual $\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}F_{\kappa\lambda}$ has led to the interesting postulate of the existence of magnetic monopoles first proposed by Dirac. Dirac's idea of magnetic monopole has further been generalized to the case where a particle is both electrically and magnetically charged. In all of the studies so far it seems to indicate that the locality is incompatible with the Lorentz group, both in classical theories and in quantum field theories. The first quantum field theory for magnetic monopole presented by Dirac was nonlocal and involved the introduction of nonphysical dynamical variables associated with strings. More recent formulations, either the electric charge and the magnetic charge treated as separate entities or a particle carrying both the electric and the magnetic charge, have avoided string variables. However the incompatibility of the locality and the Lorentz group is still retained. In the Lorentz covariant theories, they are either based upon a Hamiltonian density which is a nonlocal function of the field variables, and an independently posited nonlocal commutation relation, which together yield nonlocal field equations,[2)] or started from a local Lagrangian density which by applying the canonical quantization procedure yields nonlocal commutation relations between the potentials.[3)] A local theory in which the electric charge and the magnetic charge are separately attached to different particles has also be constructed.[4)] In this theory, starting from a local Lagrangian, one obtains local commutation relations, and local field equations. However, the field equations are not covariant under the Lorentz transformation. The incompatibility of the locality and the Lorentz group has also been observed in the classical theory of magnetic monopole. Röhlich[5] has shown that in the relativistic local theory, no action integral exists from which both the particle equations and the field equations can be derived. On the contrary, Yan[6] has constructed a relativistic classical theory with a nonlocal action.

In conventional classical theory of electron and quantum field theory, the requirement of locality always induces some unpleasant difficulties. Recent development in quantum field theory shows that a consistent local Lagrangian field theory can be constructed by making use of the indefinite metric and the idea of shadow states.[7] In connection with the theory of electromagnetism, it is very interesting to observe that in the dipole approximation electrodynamics, the runaway modes can be avoided if a shadow electromagnetic field quantized with "wrong sign" is introduced.[8] In the classical theory of electrons, it has also been shown that the introduction of a shadow electromagnetic field makes the electron stable in the point particle limit.[9] All of these studies indicate that within the framework of quantum field theory with indefinite metric one might be able to formulate a local, Lorentz covariant theory of magnetic monopole. In this note we will closely follow the model suggested by Hagen[4] and show that the terms which destroy the covariance of the field equations disappear if we introduce a shadow electromagnetic field. All of the commutation relations and the field equations are still local.

In Hagen's model, one starts with the Lagrangian

$$\begin{aligned} L_H = & \frac{1}{2} i\psi\beta\gamma^\mu\partial_\mu\psi - \frac{1}{2} m\psi\beta\psi + \frac{1}{2} i\psi'\beta\gamma^\mu\partial_\mu\psi' \\ & + \frac{1}{4} F^{\mu\nu}F_{\mu\nu} - \frac{1}{2} F^{\mu\nu}(\partial_\nu A_\mu - \partial_\mu A_\nu) \\ & + ej^\mu A_\mu + gF^{ok}\varepsilon^{k\ell m}\partial_\ell \nabla^{-2}\tilde{j}^m \\ & - \frac{1}{2} gF^{\ell m}\varepsilon^{k\ell m}\partial_k \nabla^{-2}\tilde{j}^o \quad . \end{aligned} \tag{1}$$

Here ψ is the electron field and ψ' the magnetic monopole field. $F_{\mu\nu}$ is the electromagnetic field tensor and A_μ the corresponding potential. The electric current $j^\mu(x)$ is defined as

$$j^\mu(x) = \frac{1}{2} \psi\beta\gamma^\mu q\psi \quad , \tag{2}$$

where q is the imaginary antisymmetric charge matrix which acts in the two-dimensional internal space of the Hermitian field ψ.[10] The magnetic current $\tilde{j}^\mu(x)$ is defined as

$$\tilde{j}^{\mu}(x) = \frac{1}{2}\,\psi'\beta\gamma_5\gamma^{\mu}q'\psi' \tag{3}$$

where q' is the symmetrical matrix

$$q' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The magnetic current is designed in such a way that the invariance of the theory with respect to both parity and charge conjugation is preserved. Furthermore, the bare mass of the ψ' is taken to be zero in order to have $\partial_{\mu}\tilde{j}^{\mu} = 0$. Hagen has shown that not all of the equations of motion obtained from the Lagrangian (1) are covariant under the Lorentz transformation when $eg \neq 0$.

As in the classical theory of electrons and in the dipole approximation electrodynamics, let us now introduce a shadow electromagnetic potential B_{μ} with mass M, and modify the Lagrangian as follows

$$\begin{aligned} L = {} & \frac{1}{2}\, i\psi\beta\gamma^{\mu}\partial_{\mu}\psi - \frac{1}{2}\, m\psi\beta\psi + \frac{1}{2}\, i\psi'\beta\gamma^{\mu}\partial_{\mu}\psi' \\ & + \frac{1}{4}\, F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\, F^{\mu\nu}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}) \\ & - \frac{1}{4}\, G^{\mu\nu}G_{\mu\nu} + \frac{1}{2}\, M^2 B_{\mu}B^{\mu} + \frac{1}{2}\, G^{\mu\nu}(\partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}) \\ & + ej^{\mu}(A_{\mu} + B_{\mu}) \end{aligned}$$

$$+ g(F^{ok} + G^{ok})\varepsilon^{k\ell m}\partial_\ell \nabla^{-2}\tilde{j}^m$$

$$- \frac{1}{2} g(F^{\ell m} + G^{\ell m})\varepsilon^{k\ell m}\partial_k \nabla^{-2}\tilde{j}^o \quad . \tag{4}$$

Here $G^{\mu\nu}$ is the shadow electromagnetic field tensor. Note that the free Lagrangian for the shadow electromagnetic field has an opposite sign of the free Lagrangian of the physical electromagnetic field. This is due to the fact that the shadow field is to be quantized with "wrong sign".

The equations of motion implied by the Lagrangian (4) can easily be obtained

$$F^{ok} = -\partial_o A_k - \partial_k A^o + g\varepsilon^{k\ell m}\partial_\ell \nabla^{-2}\tilde{j}^m \tag{5.a}$$

$$F^{\ell m} = \partial_\ell A_m - \partial_m A_\ell + g\varepsilon^{k\ell m}\partial_k \nabla^{-2}\tilde{j}^o \tag{5.b}$$

$$G^{ok} = -\partial_o B_k - \partial_k B^o - g\varepsilon^{k\ell m}\partial_\ell \nabla^{-2}\tilde{j}_5^m \tag{5.c}$$

$$G^{\ell m} = \partial_\ell B_m - \partial_m B \quad - g\varepsilon^{k\ell m}\partial_k \nabla^{-2}\tilde{j}^o \tag{5.d}$$

$$[\gamma^\mu(\frac{1}{I}\partial_\mu - eq(A_\mu + B_\mu)) + m]\psi = 0 \tag{5.e}$$

$$[\gamma^\mu \frac{1}{i}\partial_\mu + g\gamma_5\gamma_k q'\varepsilon^{k\ell m}\partial_\ell \nabla^{-2}(F^{om} + G^{om})]\psi' = 0 \tag{5.f}$$

These equations can be recast into the more symmetrical form

$$\partial_\mu F^{\nu\mu} = ej^\nu \tag{6.a}$$

$$\partial_\mu \tilde{F}^{\nu\mu} = g\tilde{j}^\nu \tag{6.b}$$

$$\partial_\mu G^{\nu\mu} + M^2 B^\nu = -ej^\nu \tag{6.c}$$

$$\partial_\mu \tilde{G}^{\nu\mu} = -g\tilde{j}^\nu \tag{6.d}$$

$$[\gamma^\mu (\frac{1}{i}\,\partial_\mu - eq(A_\mu + B_\mu)) + m]\psi = 0 \tag{6.e}$$

$$\gamma^\mu (\frac{1}{i}\,\partial_\mu - gq'\gamma_5(\tilde{A}_\mu + \tilde{B}_\mu))\psi' = 0 \quad . \tag{6.f}$$

where

$$\tilde{A}_k = -\varepsilon^{k\ell m}\partial_\ell \nabla^{-2} F^{om} \tag{7.a}$$

$$\tilde{A}^o = g\nabla^{-2}\tilde{j}^o \tag{7.b}$$

$$\tilde{B}_k = -\varepsilon^{k\ell m}\partial_\ell \nabla^{-2} G^{om} \tag{7.c}$$

$$\tilde{B}^o = -g\nabla^{-2}\tilde{j}^o \quad . \tag{7.d}$$

The equal time commutation relations among the canonical variables $\psi(x)$, $\psi'(x)$, $F_T^{ok}(x)$, $A_k(x)$, $G^{ok}(x)$, and $B_k(x)$ can also easily be found.

$$\{\psi(x),\psi(x')\}\delta(x^o - x^{o\prime}) = \delta(\vec{x} - \vec{x}') \tag{8.a}$$

$$\{\psi'(x),\psi'(x')\}\delta(x^o - x^{o\prime}) = \delta(\vec{x} - \vec{x}') \tag{8.b}$$

$$[F^{ok}(x),A^{\ell}(x')]\delta(x^o - x^{o\prime}) = i\delta^T_{k\ell}(\vec{x} - \vec{x}') \tag{8.c}$$

$$[G^{ok}(x),B^{\ell}(x')]\delta(x^o - x^{o\prime}) = -i\delta_{k\ell}(x - x') \quad . \tag{8.d}$$

Note that in (8.d) we have used the "wrong sign" for the quantization of the shadow field. From (8) we can derive the following equal time commutation relations which will be useful in the later computation.

$$[\tilde{F}^{ok}_T(x),\tilde{A}^{\ell}(x')]\delta(x^o - x^{o\prime}) = -i\delta^T_{k\ell}(\vec{x} - \vec{x}') \tag{9.a}$$

$$[F^{ok}_T(x),\tilde{F}^{ok}_T(x')]\delta(x^o - x^{o\prime}) = -i\epsilon_{k\ell m}\partial_m\delta(\vec{x} - \vec{x}') \tag{9.b}$$

$$[A^k(x),\tilde{A}^{\ell}(x')]\delta(x^o - x^{o\prime}) = -i\epsilon_{k\ell m}\partial_m\nabla^{-2}\delta(\vec{x} - \vec{x}') \tag{9.c}$$

$$[\tilde{G}^{ok}_T(x),\tilde{B}^{\ell}(x')]\delta(x^o - x^{o\prime}) = i\delta^T_{k\ell}(\vec{x} - \vec{x}') \tag{9.d}$$

$$[G^{ok}(x),\tilde{G}^{o\ell}_T(x')]\delta(x^o - x^{o\prime}) = i\epsilon_{k\ell m}\partial_m\delta(\vec{x} - \vec{x}') \tag{9.e}$$

$$[B^k(x),\tilde{B}^{\ell}(x')]\delta(x^o - x^{o\prime}) = i\epsilon_{k\ell m}\partial_m\nabla^{-2}\delta(x - x') \tag{9.f}$$

To show that the equations of motion (6) are covariant under the Lorentz transformation, let us first construct the energy-momentum tensor operator $T^{\mu\nu}$ such that the generators

$$P^{\mu} = \int d^3x \; T^{o\mu}(x) \tag{10.a}$$

$$J^{\mu\nu} = \int d^3x \,[x^{\mu}T^{o\nu}(x) - x^{\nu}T^{o\mu}(x)] \tag{10.b}$$

satisfy the structure relations

$$[P^{\mu}, P^{\nu}] = 0 \tag{11.a}$$

$$-i[P_{\lambda}, J_{\mu\nu}] = g_{\lambda\nu}P_{\mu} - g_{\lambda\mu}P_{\nu} \tag{11.b}$$

$$-i[J_{\kappa\lambda}, J_{\mu\nu}] = g_{\mu\lambda}J_{\nu\kappa} - g_{\nu\lambda}J_{\mu\kappa} - g_{\mu\kappa}J_{\mu\lambda} + g_{\mu\kappa}J_{\mu\lambda} \tag{11.c}$$

of the inhomogeneous Lorentz group. This can easily be done by adding proper terms due to the shadow field to the energy-momentum tensor operator constructed by Hagen. We have therefore

$$T^{ok} = F_T^{o\ell}(\partial_k A_\ell - \partial_\ell A_k) + \frac{1}{2}\,\psi\,\frac{1}{i}\,\delta_k\psi + \frac{1}{2}\,\psi'\,\frac{1}{i}\,\delta_k\psi'$$
$$+ \frac{1}{2}\,\partial_\ell(\psi\,\frac{1}{2}\,\sigma_{k\ell}\psi) + \frac{1}{2}\,\partial_\ell(\psi'\,\frac{1}{2}\,\sigma_{k\ell}\psi')$$
$$- G_{k\ell}G^{o\ell} + B_k\partial_\ell G^{o\ell} \quad . \tag{12}$$

and

$$T^{oo}(x) = \frac{1}{2}\,\psi\beta\gamma^k(\frac{1}{i}\,\partial_k - eq(A_k + B_k))\psi + \frac{1}{2}\,m\psi\beta\psi$$

$$+ \frac{1}{2} \psi' \beta \gamma^k (\frac{1}{i} \partial_k - gq' \gamma_5 (\tilde{A}_k + \tilde{B}_k)) \psi'$$

$$+ \frac{1}{2} [(F^{ok})^2 + (\tilde{F}^{ok})^2]$$

$$- \frac{1}{2} [(G^{ok})^2 + (\tilde{G}^{ok})^2] - \frac{1}{2} M^2 B_k B^k - \frac{1}{2M^2} (j^o + \partial_k G^{ok})^2 \tag{13}$$

With $T^{ok}(x)$ given in (12), one can show by straightforward calculation that P_k and $J_{k\ell}$ indeed generate the group of spatial translations and rotations upon all the basic field operators of the theory. Furthermore, it can also be shown that with $T^{oo}(x)$ given in (13), P^o is indeed the time development operator of all operators in the theory, i.e.,

$$\int d^3x' [T^{oo}(x'), \chi(x)] = \frac{1}{i} \partial_o \chi(x)$$

for any operator $\chi(x)$. Finally one can show that J^{ok} transforms $F^{\mu\nu}$ and $G^{\mu\nu}$ as second rank tensors, and j^μ and $\tilde{j}^\mu$ as four-vectors. Thus the covariance of Eqs. (6.a) - (6.d) is established.

It remains to see whether Eqs. (6.e) and (6.f) are also covariant under pure Lorentz transformation. In the case $g = 0$, the vector potentials $A_\mu(x)$ and $B_\mu(x)$, and the electron field $\psi(x)$ satisfy the commutation relations

$$-i[J^{ok}, A^\ell] = (x^o \partial_k - x_k \partial^o) A_\ell - \delta_{k\ell} A^o + \partial_\ell a_k \tag{14.a}$$

$$-i[J^{ok}, B^\ell] = (x^o\partial_k - x_k\partial^o)B_\ell - \delta_{k\ell}B^o \tag{14.b}$$

$$-i[J^{ok}, \psi] = (x^o\partial_k - x_k\partial^o)\psi - \frac{1}{2}\beta x^k\psi + \frac{1}{2}eq\{a_k, \psi\} \tag{14.c}$$

where

$$a_k = [\partial_m\nabla^{-2}, x^k]F^{om}$$

One can explicitly show that the equation of motion

$$[\gamma^\mu(\frac{1}{i}\partial_\mu - eq(A_\mu + B_\mu)) + m]\psi = 0$$

is covariant with respect to the Lorentz transformation described by (14). Similarly, in the case $e = 0$, the vector potentials $\tilde{A}_\mu(x)$ and $\tilde{B}_\mu(x)$, and the magnetic monopole field $\psi'(x)$ satisfy the commutation relations

$$-i[J^{ok}, \tilde{A}^\ell] = (x^o\partial_k - x_k\partial^o)\tilde{A}^\ell - \delta_{k\ell}\tilde{A}^o + \partial_\ell\tilde{a}_k \tag{15.a}$$

$$-i[J^{ok}, \tilde{B}^\ell] = (x^o\partial_k - x_k\partial^o)\tilde{B}^\ell - \delta_{k\ell}\tilde{B}^o + \partial_\ell\tilde{b}_k - M^2\epsilon^{\ell mn}[\partial_m\nabla^{-2}, x^k]B^n \tag{15.b}$$

$$-i[J^{ok}, \psi'] = (x^o\partial_k - x_k\partial^o)\ ' - \frac{1}{2}\beta x^k\psi' + \frac{ig\gamma_5 q'}{2}\ {}_k + \tilde{b}_k, \psi'\} \tag{15.c}$$

where

$$\tilde{a}_k = -[\partial_m\nabla^{-2}, x^k]\tilde{F}^{om}$$

$$\tilde{b}_k = -[\partial_m\nabla^{-2}, x^k]\tilde{G}^{om}$$

and the equation of motion

$$[\gamma^{\mu} \frac{1}{i} \partial_{\mu} - gq'\gamma_5(\tilde{A}_{\mu} + \tilde{B}_{\mu})]\psi'(x) = 0$$

is covariant with respect to the Lorentz transformation described by (15).

In the case of nonvanishing electric and magnetic couplings, ψ and ψ' satisfy the commutation relations (14.c) and (15.c) respectively. However, instead (14.a), (14.b) (15.a) and (15.b), A_{μ}, B_{μ}, $\tilde{A}_{\mu}$, and $\tilde{B}_{\mu}$ satisfy the commutation relations

$$-i[J^{ok},A^{\ell}] = (x^{o}\partial_{k} - x_{k}\partial^{o})A_{\ell} - \delta_{k\ell}A^{o} + \partial_{\ell}a_{k} + g\epsilon^{\ell mn}[\partial_{m}\nabla^{-2},x^{k}]\tilde{j}^{n} \quad (16.a)$$

$$-i[J^{ok},B^{\ell}] = (x^{o}\partial_{k} - x_{k}\partial^{o})B_{\ell} - \delta_{k\ell}B^{o} - g\epsilon^{\ell mn}[\partial_{m}\nabla^{-2},x^{k}]\tilde{j}^{n} \quad (16.b)$$

$$-i[J^{ok},\tilde{A}^{\ell}] = (x^{o}\partial_{k} - x_{k}\partial^{o})\tilde{A}^{\ell} - \delta_{k\ell}\tilde{A}^{o} + \partial_{\ell}\tilde{a}_{k} + e\epsilon^{\ell mn}[\partial_{m}\nabla^{-2},x^{k}]j^{n} \quad (16.c)$$

$$-i[J^{ok},\tilde{B}^{\ell}] = (x^{o}\partial_{k} - x_{k}\partial^{o})\tilde{B}^{\ell} - \delta_{k\ell}\tilde{B}^{o} + \partial_{\ell}\tilde{b}_{k} - \epsilon^{\ell mn}[\partial_{m}\nabla^{-2},x^{k}] (ej^{n} + M^{2}B^{n}) \quad (16.d)$$

In the original model of Hagen, where there is no shadow potentials, the last term proportional to g in (16.a) and the last term proportional to e in (16.c) destroy the covariance of the equations of motion (6.e) and (6.f), since they induce a direct interaction proportional to eg between the fields ψ and ψ'. However, with the introduction of the shadow potential, due to the linear sum of A_μ and B_μ, and $\tilde{A}_\mu$ and $\tilde{B}_\mu$ with equal weight in the equations of motion, the unpleasant terms proportional to eg induced by the Lorentz transformation (16) cancel out each other and the covariance of the equations of motion (6.e) and (6.f) are therefore preserved. It is important to emphasize that the cancellation of the noncovariant terms happens only when the electric current (magnetic current) is coupled to the linear sum of the physical potential (physical field tensor) and the shadow potential (shadow field tensor) with the same weight. This establishes the electromagnetic character of the auxiliary shadow potential and field tensor in terms of the coupling constants.

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(10) We use a Majorana representation of the Dirac algebra and the metric (1,1,1,-1).