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Theory of Confinement in the Stellarator

by

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Preface

This analysis of confinement in the stellarator has been prepared primarily for the Sherwood Handbook. Since the theory presented is new in many respects, this material is also being given separate distribution as a Project Matterhorn report.

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THEORY OF CONFINEMENT IN THE STELLARATOR

1. Basic Principles

The fundamental element in any thermonuclear reactor is the magnetic field configuration used to confine the fully ionized reacting gas, or plasma, at the required temperature of 10^8 degrees K or more. The Princeton program at Project Matterhorn has been concentrated on magnetic fields produced by external coils, with the magnetic lines of force everywhere parallel to the walls of a closed, endless tube inside which the gas is confined. A device based on this type of magnetic configuration has been called a "stellarator". In the present report the theory of confinement in a stellarator will be discussed.

In this first section the principles used in analyzing confinement are treated and applied to geometrically simple systems. Discussion of the simple torus is required to indicate why a more complicated configuration appears to be required for effective confinement.

The objective of any confinement theory is to show theoretically that the number of particles which strike the wall of the tube is negligibly small. Such a proof cannot be based on the macroscopic equations only; the distribution of particle velocities is usually not known, in view of the long mean free path, and particles moving in a particular direction at a particular velocity may conceivably reach the wall even though the mean macroscopic velocity of all particles in each volume element is very small. On the other hand, analysis of single-particle orbits only is also insufficient; the particles must be assumed to move in given electric and magnetic fields, $\underset{\sim}{E}$ and $\underset{\sim}{B}$, and these quantities are determined by the cooperative effect of many particles, through their dependence on the macroscopic current, j and its divergence. Thus both the microscopic picture, based on particle orbits, and the macroscopic picture, based on the field equations and on the electric current, must be used to demonstrate confinement.

To handle this problem exactly a detailed solution of the Boltzmann equation would be required. In a complicated system this would be a very difficult task. Instead we shall employ a less general method. First we use the macroscopic equations to demonstrate that an equilibrium solution, satisfying certain restrictions, is possible. Second we discuss the motion of free particles in the electric and magnetic fields determined from the macroscopic equations. Since the macroscopic equations, in the form used, are not valid in the most general situations, certain conceivable equilibrium states could not be analyzed by this method. However, any equilibrium state which can be analyzed by this method and for which confinement can be established theoretically should be a valid equilibrium. The stability of such an equilibrium is, of course, a different matter; stability problems are discussed in a separate chapter in the Handbook, and will not be treated here.

1.1 Macroscopic equations and underlying assumptions

We turn, then to the macroscopic equations which will be employed. These have been derived elsewhere (Spitzer, 1956). The assumptions made here in deriving these equations are as follows:

(a) Over a distance of one Larmor radius the relative change of all quantities is small.

(b) The quantities m_e/m_i and $Zm_e p_i/m_i p_e$ may be neglected compared to unity; subscripts e and i refer to electrons and positive ions, respectively.

(c) All macroscopic quantities are independent of time at each position.

(d) The transverse and longitudinal pressures, p_t and p_ℓ, are equal.

(e) The electric resistivity, η, is negligibly small, and the mean free path is much greater than the Larmor radius.

(f) The mean macroscopic velocity, $\underset{\sim}{v}$, vanishes.

Assumption (a), which is basic in any analysis using the macroscopic equations, has a number of important consequences. First, since the sheath thickness, h, is much less than the Larmor radius of a positive ion for any well developed plasma, assumption (a) requires approximate electrical neutrality, with n_e nearly equal to Zn_i. Second, when the mean free path is much longer than the Larmor radius, as implied by assumption (e), assumption (a) leads to the result that the stress tenor is diagonal, (Watson, 1956; Chew, Goldberger and Low, 1956), provided that the principal axis is parallel to the magnetic field; the three components are thus p_ℓ parallel to the field and p_t in the two directions perpendicular to the field. Third, it follows from this assumption that the components transverse to the magnetic field of both the mean current, $\underset{\sim}{j}$, and the mean velocity, $\underset{\sim}{v}$, are small compared to the root mean square velocity. In any device much larger than the Larmor radius, assumption (a) seems legitimate, except in a boundary layer or sheath near the wall.

Assumption (b) is trivial. This assumption, like many of the subsequent ones, is not vital, and could be relaxed without substantial modification of the macroscopic equations.

Assumption (c) is certainly valid in any steady state. If diffusion to the walls is present, produced by finite η, a source of hot particles within the plasma must be assumed to maintain a steady state. The most serious effect of this assumption is to exclude hydromagnetic instabilities and electrostatic oscillations; while the former are treated elsewhere, the possible effect of the latter on confinement is not understood.

If (d) is not made, then in a converging or diverging magnetic field, the divergence of the stress tensor will include a term proportional to $p_t - p_\ell$. In such a case the axes in which the stress tensor is diagonal change with position, producing off-diagonal components in Cartesian coordinates. Thus the assumption that p_t and p_ℓ are equal simplifies the equations. In a steady state one would expect that even infrequent collisions would make p_t and p_ℓ equal, and hence this assumption appears a natural one.

Assumption (e) is approximately valid when η is small. The effect of a small finite η, together with the associated diffusion velocity and rate of injection, may be evaluated by a perturbation analysis of the solution for $\eta = o$ (Kruskal, 1955). If confinement can be demonstrated for zero η, it would appear that introduction of a small finite η will not impair the confinement. Thus assumption (e) simplifies the treatment substantially without any essential loss.

Assumption (f) replaces the more usual one (which partly results from assumption (a)) that quadratic terms in $\underset{\sim}{v}$ and $\underset{\sim}{j}$ are negligible. This more stringent condition is not so arbitrary as might first appear. In fact, the near-vanishing of $\underset{\sim}{v}$ is a simple consequence of the equations of motion (Spitzer, 1956, Section 3.3), provided that there is no additional effect tending to produce a separation of charge. The argument is that in most heating methods there are no appreciable forces tending to produce any momentum in any particular direction, and hence the macroscopic velocity must be vanishingly small. Thermonuclear reactions, which accelerate positive charges to large energies, favouring their escape from the gas, do produce some separation of electric charge, and hence $\mathbf{v}$ does not vanish completely in a thermonuclear reactor. Even in this case, however, $\mathbf{v}$ is relatively small, and we shall ignore it here. A fuller analysis of the effects associated with such macroscopic velocities would be desirable. The detailed mechanisms responsible for the vanishing of $\mathbf{v}$, and for the associated electric field, are reviewed in Section 4.2.

On the basis of these assumptions, the equation of motion and the generalized Ohm's Law become, in Gaussian units

$$\mathbf{j} \times \mathbf{B} = c \nabla p \quad , \tag{1}$$

$$-\nabla U = \mathbf{E} = \frac{1}{e n_e} \nabla p_i \quad , \tag{2}$$

while Maxwell's equations yield

$$\nabla \times \mathbf{B} = \frac{4\pi \mathbf{j}}{c} \quad , \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \quad . \tag{4}$$

Poisson's Law is not needed, as it merely gives the charge density, which is not otherwise of importance. It may be noted that equation (1) indicates that p is constant along a line of force, as a result of assumptions (c) and (f). Thus the problem of transport along the magnetic field, which is not easily handled within the macroscopic theory (Brueckner and Watson, 1956; Chew, Goldberger and Low, 1956), does not arise. We require that equations (1) through (4) be satisfied in the equilibrium state.

1.2 Motion of free particles

The basic principles affecting the motion of individual free particles are readily written down. If $\mathbf{w}_\perp$ is the component of particle velocity perpendicular to $\mathbf{B}$, we may write for the magnetic moment,

$$\mu = \tfrac{1}{2} m w_{\perp}^2 = \text{constant} . \tag{5}$$

This result has been proved (Kruskal, 1957) to all orders of ak, where a is the Larmor radius, and 1/k is the local e-folding distance of B. Thus the deviations from (5) must be very small indeed. We may also assume that the motion of a guiding center is independent of the phase of the gyrating electron, again to all orders of ak. This result was first indicated in an idealized case (Kruskal, 1951) and has now been established generally (Kruskal, 1957). Thus to a high approximation the motion of a guiding center is independent of phase and possesses two simple integrals, the magnetic moment μ, and the total energy W, where

$$W = \tfrac{1}{2} m w^2 + Z e U , \tag{6}$$

where U is the potential energy, in e. s. u.; Z is -1 for an electron.

The velocities of the guiding centers are governed by the usual equations for the drift velocity (Alfvén, 1950; Spitzer, 1956). With crossed electric and magnetic fields.

$$w_D = c \frac{E \times B}{B^2} . \tag{7}$$

We shall refer to this drift motion as an "electric drift". In an inhomogeneous magnetic field,

$$\frac{m dw_{\parallel}}{dt} = - \mu \frac{B \cdot \nabla B}{B} , \tag{8}$$

where $w_{\parallel}$ is the longitudinal velocity, parallel to B. For the transverse drift velocity, $w_{D\perp}$, we have the two familiar drifts. Firstly, the drift due to $\nabla_{\perp} B$, the gradient of the scalar field in the plane perpendicular to B, is given by

$$\underset{\sim}{w}_{D\perp} = \frac{w_\perp^2}{\omega_c B} \underset{\sim}{\nabla}_\perp B \; ; \tag{9}$$

$w_\perp$ is the transverse velocity with which the particle gyrates around the line of force, and ~~where~~ ω_c is the cyclotron frequency.

$$\omega_c = \frac{ZeB}{mc} \; . \tag{10}$$

Secondly, we have the drift produced by motion along a curved drive of force.

$$w_{D\perp} = \frac{w_{\parallel}^2}{\omega_c R} \; , \tag{11}$$

where R is the curvature of the line of force. The direction of these drifts is perpendicular both to $\underset{\sim}{B}$ and to either $\underset{\sim}{\nabla}_\perp B$ or to $\underset{\sim}{R}$; particles of opposite sign drift in opposite directions. We shall refer to these two drift motions as "gradient drift" and "curvature drift" respectively. This completes our survey of the basic principles.

1.3 <u>Equilibrium in infinite cylinder</u>

We pass on now to an application of these basic principles to two simple geometries, the infinite cylinder, and the torus. In the infinite cylinder, with the z axis parallel to the cylinder, only B_z is assumed present, equation (4) is satisfied if B_z is independent of z, and equations (1) and (3) yield

$$p + \frac{B_z^2}{8\pi} = \text{constant} \; . \tag{12}$$

If p_m denotes the maximum value of p in the cylinder, we define

$$\beta = \frac{8\pi p_m}{B_{zw}^2} \tag{13}$$

where B_{zw} is the value of B_z at the walls, where p is assumed to vanish.

Evidently β cannot exceed unity. We see that p is an arbitrary function of r and θ over the cross-section, subject only to the requirement, by assumption (a), that $p/\nabla p$ greatly exceed the Larmor radius, a. The variation of p over the cross-section will presumably depend on the mechanisms of diffusion and injection, and need not be considered in evaluating the confinement.

If we take the curl of equation (2), we see that the electron density n_e must be constant along an isobar. If this condition is not fulfilled, as for example, if T varies along a surface of constant n_e, the macroscopic velocity $\underset{\sim}{v}$ cannot vanish, and it is not obvious that a steady state is possible. If we assume, then, that n_e is constant along an isobar, and that as a result, T and n_e are both constant along isobaric surfaces, then equation (2) requires that the electric potential U is also constant on each isobaric surface. From equation (12) it follows that B_z is also constant on each isobaric surface.

We now apply the microscopic picture to this problem. Since $\underset{\sim}{B} \cdot \underset{\sim}{\nabla} B$ vanishes and the lines of force are straight, $w_{\parallel}$ is constant in time and also no curvature drift is present. Since U and $\underset{\sim}{B}$ are both constant along each isobaric surface, the electric drift and gradient drift are both in directions perpendicular to z and to ∇p, and are thus parallel to the local isobaric surface. Evidently these motions do not affect the distribution of guiding centers and do not impair the confinement.

It may be remarked that if T is everywhere constant, the electric potential is such that the density distribution of positive ions follows the Boltzmann distribution,

$$n_i \propto \exp(-ZeU/kT) \qquad (14)$$

Equation (14) appears naturally from the condition that the positive ions have no mean motion. Evidently equation (14) cannot be satisfied at the boundary of the plasma, where n_i is assumed to vanish, without infinite potentials. In fact as n_i approaches zero the basic assumption (a) must fail, since $p/\nabla p$ must become less than the Larmor radius. Thus the macroscopic equations cannot be used all the way to the wall, and a more detailed analysis is required for the outermost plasma layer.

One may raise the question whether in a higher approximation the individual particles may move across the isobaric surfaces. In the case of rotational symmetry about the cylindrical axis it is easily demonstrated (Spitzer, 1951) that the orbit of each charged particle is rigorously confined to the region between two flux tubes.

We conclude that confinement in the infinite cylinder has been amply demonstrated. The one region of uncertainty is the structure of the outer plasma boundary, where the macroscopic equations no longer apply.

1.4 Problem of the simple torus

Next we consider a toroidal system in which the lines of force are all circles; we denote by R the radius of curvature of a line of force. The plasma is assumed confined within a tube whose cross-section has the radius r_1; we shall call r_1 the "minor radius" of the torus. The value of R for the line of force centered in the middle of the tube cross-section will be denoted by R_1, and called the "major radius" of the torus. We introduce in addition to R the coordinates φ and z, where z is measured along the axis of rotational symmetry and φ is the angle of rotation around this axis. We assume that all quantities are independent of φ.

As before, we may eliminate $\underset{\sim}{j}$ from equations (1) and (3). The components B_r and B_z may be set equal to zero. If we take the curl of equation (1), we obtain, after some algebra

$$\frac{\partial B_\varphi^2}{R\,\partial z} = 0 \qquad (15)$$

We conclude that a solution of the equations is possible only if R is infinite, in which case we return to the infinite cylinder, or if B_φ is independent of z, in which case we can easily show that p must also be independent of z and confinement within a circular cross-section is not possible.

This failure to find an equilibrium corresponds to the presence of current divergences if an equilibrium is assumed. Consider the cross-section of the torus shown in Figure 1. The line OO' is the axis of symmetry.

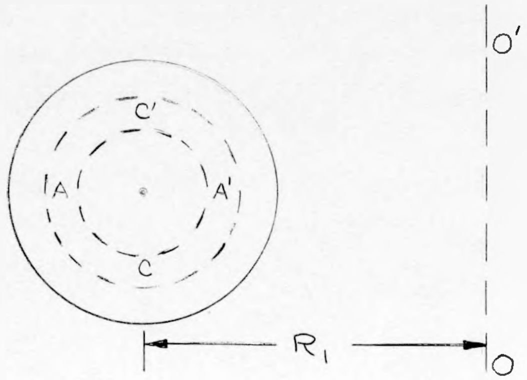

Figure 1
Current Divergences in the Torus

The two closed dashed curves represent the intersection of two isobaric surfaces with the cross-section of the tube. From equation (1) it follows that the electric current, j, is parallel to the isobaric surfaces. If an equilibrium is assumed, BR is the same at all points between these two surfaces, where B denotes the φ component of the magnetic field, and R is the distance from OO'. However, from equation (1) it follows that the total current between the two surfaces, which is simply $2\pi R j \Delta r$ equals $2\pi R \Delta p/B$ and therefore varies as R^2. Since the total current between the two surfaces at A is greater than at A', electric charge must accumulate at C, with a corresponding deficiency of charge

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at C'. Hence no equilibrium is possible.

This same result may be obtained, of course, from the microscopic picture. The curvature drift and the gradient drift have the same direction for positive particles, but are in the opposite direction for negative particles. A separation of charges results. As we shall see in the next section, the assumption of an original radial electrostatic field will change the microscopic picture somewhat, but will not alter the basic result found from the macroscopic equations, that an equilibrium solution of equations (1) through (4) is impossible if the lines of force are assumed circles about the axis of symmetry.

Let us discuss briefly what happens if an ionized gas is placed within such a toroidal system. The accumulation of charges will produce an electric field transverse to $\underset{\sim}{B}$. In a completely ionized gas this field produces a partial polarization of the plasma but no steady current. Thus the accumulating charge cannot be entirely neutralized, and the resultant electric drift is toward the outer wall of the torus. If this solid surface is non-conducting, the plasma will presumably be swept into the wall. If the wall is a perfect conductor, and hence at a uniform potential, the electric drift must be parallel to the wall, and on this basis we might expect the confinement to be unimpaired. However, we have seen that no equilibrium confinement is possible on the basis of the simple assumptions made above. Observations indicate that a plasma is in fact confined for appreciable periods in such a conducting toroidal tube. The nature of the quasi-equilibrium existing is obscure, but presumably the physical conditions are too complicated to be represented by the simplifying assumptions we have made. Effects in the plasma sheath near the wall and effects produced by plasma oscillations may play a dominant role. Equilibrium solutions can be obtained if macroscopic velocities are assumed, and the basic equations modified accordingly. However, the velocity fields required are rather special and it seems unlikely that such velocities would arise naturally.

To obtain equilibrium confinement under conditions which can be understood theoretically and which can readily be produced in practice, a more complicated magnetic configuration is required. Such a configuration is discussed in the next section.

2. Confining Field in the Stellarator

The confining field used in a stellarator is characterized by the existence of a so-called "rotational transform". This section describes what a rotational transform is, discusses the properties of magnetic fields possessing this characteristic, and analyzes different methods for producing a rotational transform. Confinement of a plasma in such magnetic fields is treated in the following section.

2.1 Rotational transform

In the torus considered above the magnetic lines of force were circles, centered at the axis of symmetry. Such a system is degenerate in that each line of force is closed after one revolution around the axis, OO' in Figure 1. If this degeneracy is removed, so that the lines of force are not closed, equilibrium confinement may, under certain circumstances, be possible.

The simplest way to remove this degeneracy is to add a current along the lines of force. Such a current produces components of $\underset{\sim}{B}$ encircling the current. These components, added to the confining field B_φ produced by external coils, produce lines of force which are helices bent into toroidal form as illustrated in Figure 2. Since the resultant lines of force are toroidally helical, the assumed currents along the lines of force must have the same geometry.

Figure 2
Toroidally helical line of force

A cross–section of this toroidal tube is shown in Figure 3. Let a particular line of magnetic force intersect this plane at the point 1 in the figure. This line of force, if followed around the tube for one "revolution" around the axis OO', will then intersect this same cross–section at a different point, designated as point 2 in Figure 3.

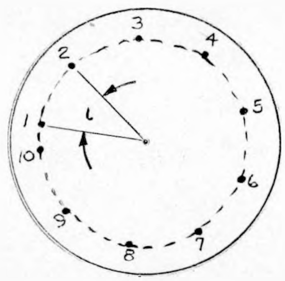

Figure 3
Successive intersections of a line of force with a cross–section

Because of the helical nature of the field, point 2 will, in general be rotated about the central region of the tube. Any point in the cross–sectional plane will similarly be transformed into another point after one revolution about the axis OO', (except possibly for some points near the outer boundary which may not return at all). Such a transformation of a plane into itself has been called an "H-transform". A number of important results about such transforms have been established by Kruskal (1952). These results follow from the condition that the density of magnetic lines of force (i.e., the value of B_ϕ) is a single-valued function of position, and from the assumption that the transform is primarily rotational, in the sense that at least the outer parts of the plane all rotate in the same direction in a single transformation. As we shall see in subsequent sub-sections, there are a variety of ways of achieving a rotational transform in a confining field which is topologically a torus. Hence the following analysis applies to all such systems, no matter how twisted and non-uniform they may be.

The first result is that at least one point in the plane must be transformed into itself. In most systems of practical interest the H-transform involves only small deformations of a plane, in addition to a general rotation, and there will be only one point that transforms into itself, and only one line of force that is closed on itself after one revolution around the axis of symmetry. This line is called the "magnetic axis", and should not be confused with the axis of symmetry of the torus. A motion which encircles the magnetic axis will be called "rotation", while motion parallel to the magnetic axis will be called "revolution". In the case of a toroidal system, with axial symmetry, the motion of revolution encircles the axis of symmetry; in a more general system, however, an axis of symmetry need not be present.

The second result is that any other point, when followed through successive transformations, will not move far from a single closed curve. This is illustrated in Figure 3, where the points generated by successive

H–transforms of point 1 all lie close to a single closed curve. Thus a single line of force, after many revolutions around the tube, generates a surface, which will be called a "magnetic surface".

This result is so important that we shall now describe in more detail what has actually been proved. Let us introduce coordinates r, θ in the cross–section plane depicted in Figure 3; r is essentially the minor radius, except that it is now measured from the magnetic axis rather than from the geometrical center of the tube cross–section. The value of $\Delta\theta$ between point 1 and its transform at point 2 is denoted by ι and is called the "transform angle". Let $\theta = 0$ at point 1. Let us assume that after n transforms of point 1, we return to a point n, whose value of θ is exactly zero. The distance Δr from point 1 to point n is called the "deviation from closure" of point n. Evidently, Δr measures how far the line of force has strayed from a single closed curve. It has been established that as n increases Δr decreases more rapidly than any power of $1/n$. Hence one may surmise that Δr varies as $\exp(-Kn)$, where K is some dimensionless constant.

The physical reason for this result can best be understood in the special case that the normal component of the magnetic field is constant over the cross–section plane. The analysis of more general systems may be reduced mathematically to a consideration of this special situation. In this case the density of points in the plane must remain constant in successive transformations. Let us now draw a closed curve in the cross–section plane connecting point 1 and its successive transformed points as smoothly as possible. Since the H–transform now preserves areas, the total area enclosed within this curve must remain constant in successive transformations. Hence all points on the curve cannot move inwards with successive transformations. If some move in, others must move out.

In the special case that the θ coordinate of every point returns to its original value after n transforms, it is possible for some points on the curve, together with all their transformed points, to move steadily in, while the points between move steadily out. Thus the closed curve develops wrinkles in successive transformations. As we have already seen, this rate of wrinkling decreases very rapidly with increasing n. In the more general case that the θ coordinate of a point never returns exactly to its initial value, (to within a multiple of 2π) one would expect a further averaging out of these radial motions to occur. We conclude that to a very high approximation the successive transforms of a single point do generate a closed curve, and that in a magnetic topography characterized by a rotational transform a line of force, followed for many revolutions around the tube, generates a magnetic surface.

2.2 Methods for producing a rotational transform

We have seen that in a toroidal system a rotational transform is produced by a current in the φ direction; i.e., with a component parallel to the magnetic field. The transform angle ι may be computed simply if the current density j_φ is assumed uniform and if the field produced by this current is computed as though the major radius of the torus were infinite. Evidently if we follow a single line of force around the torus, the changes of φ, the position angle about at the axis of symmetry, and of θ, the position angle about the magnetic axis, are related by

$$\frac{r d\theta}{B_\theta} = \frac{R d\varphi}{B_\varphi} \quad , \qquad (16)$$

Introducing the usual formula for B_θ in terms of j_φ, we obtain

$$L = \int_{\varphi=0}^{\varphi=2\pi} d\theta = \frac{2\pi L j_\varphi}{B_\varphi} , \tag{17}$$

where L, the length around the torus is evidently given by

$$L = 2\pi R \quad . \tag{18}$$

Equation (17) is valid for any system in which (a) the minor radius is small compared to the major radius of curvature of the magnetic axis, and (b) the magnetic axis lies in a single plane.

To produce a rotational transform in this way requires currents parallel to the magnetic axis. For confinement of a gas in a system designed to produce power, such a current has the disadvantage that it must be transient. The electromotive force required to produce such a current may easily by produced by changing the flux which threads the magnetic axis, and in practice this flux can be increased only up to a certain limit. To permit confinement in a steady state it is desirable to produce a rotational transform in the absence of plasma currents.

The simplest way to produce a rotational transform in a vacuum field is to twist a torus out of a single plane. It is readily shown that virtually any such distortion will remove the degeneracy and can produce a rotational transform.

The simplest such system is the figure-eight, historically the first geometry proposed for a practical stellarator. The topography is indicated in Figure 4. The curving end sections are each tilted at an angle, α , to the parallel planes in which sections AA' and CC' are placed.

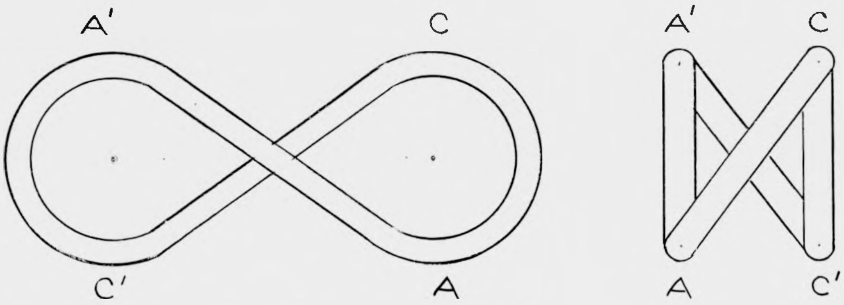

Figure 4
Geometry of the Figure-8 Stellarator

To show that a rotational transform is present, cross-section planes at A, C, C' and A' are indicated in Figure 5, as seen from the end of the device. The point O represents the magnetic axis, while the point 1 represents the successive intersections of a single line of force with each of the four planes. The solid lines represent the path followed by the magnetic axis. The transformation of plane A into plane C (and from C' into A') simply reflects one plane about an axis inclined at an angle α , while the transformation from C to C' (and from A' to A) is an identity.

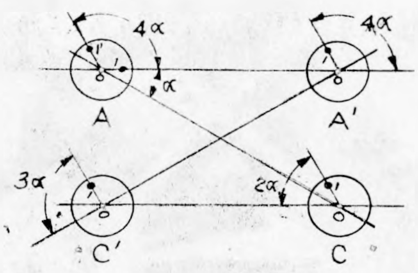

Figure 5
Cross-Section Planes Showing Rotational Transform

Evidently the line of force which passes through point 1 in plane A, and is then followed through one revolution around the tube, intersects plane A again in a point 1', rotated by a transform angle ι. Examination of the figure shows that for this geometry.

$$\iota = 4\alpha \,, \tag{19}$$

Modifications of this geometry, with a number of practical advantages, have been proposed by Stix (B–64 stellarator) and Coor (Etude stellarator). The existence of a rotational transform in a stellarator has been illustrated by use of an electron beam. If appreciable gas is present in the tube, the successive passages of the electron beam past a viewing point provide a good visual demonstration of the transform angle.

Another method of providing a rotational transform is by means of a transverse magnetic field, whose direction rotates with distance along the magnetic axis. We shall follow a point along a line of force in this situation and show that a transform angle appears. We treat here the infinite cylinder, and let z represent distance parallel to the cylinder axis; in each plane perpendicular to the magnetic axis we use coordinates r and θ as before. The coordinates of a point moving along a line of force are given by

$$dr = \frac{B_r}{B_z} dz \,, \quad r d\theta = \frac{B_\theta}{B_z} dz \,. \tag{20}$$

Suppose now that B_r and B_θ vary as the sine and cosine, respectively, of $\ell\theta - kz$. A variation of about this type may be produced by 2ℓ helically wound wires outside a toroidal tube, with opposite currents in adjacent wires, and with a pitch of $2\pi\ell/k$; alternatively, ℓ such wires all with currents in the same direction may be used. Evidently as z increases by $2\pi/k$, the field direction rotates through an angle 2π. Solution of Laplace's equation shows that for kr_1 small, where r_1 is the outer tube radius, B_r and B_θ for a vacuum field are given by

$$B_r = B_t \left(\frac{r}{r_1}\right)^{\ell-1} \sin(\ell\theta - kz) \quad , \tag{21}$$

$$B_\theta = B_t \left(\frac{r}{r}\right)^{\ell-1} \cos(\ell\theta - kz) \quad , \tag{22}$$

where B_t is a constant, equal to the maximum value of the transverse field at r equal to r_1. There is also a component of B_z associated with B_r and B_θ, but its magnitude is less by a factor kr.

Let us now follow a point whose initial coordinates are r_o and θ_o. Equation (20) may be integrated at once to zero order in $r - r_o$ and $\theta - \theta_o$. In this order the point simply moves in a circle. Solving next to higher order in $r - r_o$ and $\theta - \theta_o$, we must take into account that for ℓ equal to 2 or more, B_θ is larger in magnitude on the outside of the circle where B_θ is positive, than it is on the inside where it is negative. As a result the positive values of $d\theta/dz$ found from equation (20) more than offset the negative ones, and θ increases. According to Johnson and Oberman (1957), a detailed integration gives for ι the following results

$$\iota = \pi \left(\frac{B_t}{kr_1 \; B_z}\right)^2 \left(\frac{r}{r_1}\right)^{2\ell-4} \left\{ 2(\ell - 1) + k^2 r^2 + \ldots \right\} \quad . \tag{23}$$

Terms of order kr_1 have been neglected in this expression. The term in $(kr)^2$ is included to give results for ℓ equal to unity; in this case a rotational transform arises from the variation of B_z in a helical magnetic field. The configuration for which ℓ is unity, with a helical magnetic axis, and its use for confining a plasma has been extensively studied by H. Koenig (1956b). For small kr it would appear that an appreciable ι is more readily obtained with transverse fields of higher multiplicity.

The primary importance of these transverse field lies in their stabilizing action. The theory of hydromagnetic instabilities is discussed elsewhere in this Handbook, but the stabilizing effect of the transverse fields is so important that the topic will be treated briefly here.

Instabilities are most marked if the lines of force in the dense plasma region can exchange places with the lines of force outside the plasma. In this situation, when the plasma pressure is very small the magnetic field is clearly neutral against such interchanges, since the magnetic field at each point after the interchange is the same as before. Hence the de-stabilizing effect of even a slight plasma pressure can produce instabilities. Evidently the cylinder is neutral against all interchanges, if the lines of force are all straight and parallel to the cylinder axis. We have already seen that in this case the lines of force can interchange places without any change of energy. When a bulge (a region of weaker field) is present in the cylinder, the plasma is still neutral against such interchanges if the plasma pressure is negligible, but any finite pressure will produce an instability.

If the magnetic configuration is such that interchanges are not possible, the situation is different. In this case as the plasma pressure becomes small, and the magnetic field approaches the vacuum value (provided we assume that no currents flow along the lines of force), the system is clearly stable. If no interchanges are possible any perturbation will increase the magnetic energy, since the vacuum field is always a configuration of minimum energy. To counteract the stabilizing influence of the magnetic field an appreciable plasma pressure is necessary. If β is defined as in equation (13), we therefore conclude that there exists a critical value of β, which may be denoted by β_c. For β less that β_c, the confined plasma will be stable against hydromagnetic disturbances.

It is evident that if the transform angle ι varies with r, the distance from the magnetic axis, interchanges are impossible, except for the trivial rotation of a magnetic surface about the magnetic axis. Hence if a plasma of negligible pressure is placed in such a system, the plasma should be stable against any hydromagnetic disturbances which might move the plasma toward the wall. It is evident from equation (23) that ι does in fact vary with r. For small kr_1, $d\iota/dr$ in the body of the plasma is greatest for ℓ equal to 3. Calculations by Johnson and Oberman (1957) indicate that with attainable transverse fields of this multiplicity, a critical β of at least 0.1 can be achieved.

3. Plasma Equilibrium in the Stellarator

We now examine the equilibrium of an ionized gas in a magnetic field configuration characterized by a rotational transform. The existence of a solution for the macroscopic equations will be discussed first. A more detailed consideration of this solution will then show the existence of secondary currents within the gas, will examine the effects of such current, and will analyze the way in which transverse fields and "scallops" can reduce these effects. Discussion of the confinement for single free particles in a stellarator is postponed to the following section.

3.1 Solution of the macroscopic equations

Equations (1) through (4), given in Section 1, define the problem; we assume that a rotational transform exists and that a single line of force generates a magnetic surface. To prove the existence of a solution under these conditions, with appropriate boundary conditions, is not trivial. Indeed, as we shall see below it is not yet clear whether any solution exists for values of the gas pressure, p, between certain limits. The uniqueness of a solution, if it exists, has been investigated by Kruskal (1955), who has also considered the effect of finite resistivity, together with the associated diffusion velocity and the rate of injection required in a steady state. Here we shall demonstrate the existence of a solution for sufficiently low p, following an earlier analysis by Spitzer (1952).

To solve equations (1) through (4) we proceed by a process of iteration. Let subscripts zero refer to quantities in the vacuum field; evidently p_o, U_o and j_o (in the vacuum) all vanish. We shall then define j_n, B_n and U_n by the following equations.

$$j_n \times B_{n-1} = c \nabla p_n \quad , \; n \geq 1 \; , \tag{24}$$

$$\nabla U_n = \frac{1}{en_e} \nabla p_{in} \quad , \; n \geq 1 \; , \tag{25}$$

$$\nabla \times B_n = 4\pi j_n / c \; , \quad n \geq o \; , \tag{26}$$

$$\nabla \cdot B_n = 0 \; , \quad n \geq o \; , \tag{27}$$

where p_n, and p_{in}, are not as yet defined. For n equal to zero equations (26) and (27) are the familiar ones for the vacuum field, B_o, which may be assumed to be determined by a given distribution of current, j_o, external to the plasma. If these equations can be solved by iteration for all values of n, and if the solutions approach a limit uniformly for increasing n, with p_n and p_{in} determined in some way, then this limit is a solution of equations (1) through (4). We shall show that such a limit exists for sufficiently small values of p_n.

Equation (25) may be integrated directly if T is a known function of

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p_1; for an isothermal gas equation (14) is again obtained. Since the electric potential U is not needed in the subsequent analysis, equation (25) will not be considered further.

We discuss the solution of the remaining equations in the case n equal to one. To obtain a solution one must choose p_1 as a function of position. From equation (24) it follows that $\mathbf{B}_0 \cdot \nabla p_1$ must vanish, and hence p_1 must be constant along each magnetic surface of the vacuum field. The variation of p_1 from one magnetic surface to the next is arbitrary; we shall assume any smooth distribution.

Once p_1 is assumed, then from equation (24) $\mathbf{j}_1$ may be determined uniquely, provided we assume that $\mathbf{j}_1 \cdot \mathbf{B}_0$ vanishes when averaged over the volume between two magnetic surfaces. To show this, we decompose $\mathbf{j}_1$ into two components, $\mathbf{j}_\perp$, the transverse component, perpendicular to $\mathbf{B}_0$, and $j_\parallel$, the longitudinal component, parallel to $\mathbf{B}_0$. If we take the cross product of $\mathbf{B}$ with equation (24), we obtain

$$j_\perp = \frac{\mathbf{B}_0 \times \nabla p_1}{B_0^2} , \tag{28}$$

In general the divergence of $\mathbf{j}_\perp$ will not vanish. However, the integral of $\nabla \cdot \mathbf{j}_\perp$ over the volume between two magnetic surfaces must necessarily vanish. This result follows from integrating $\nabla \cdot \mathbf{j}_\perp$ over such a volume and using Gauss's Theorem to express the integral in terms of $\mathbf{j}_\perp \cdot d\mathbf{S}$, where $d\mathbf{S}$ is a surface element, integrated over the two bounding magnetic surfaces. Since p_1 is constant over a magnetic surface, ∇p_1 is parallel to $d\mathbf{S}$; since from equation (28) $\mathbf{j}_\perp \cdot \nabla p$ vanishes, $\mathbf{j}_\perp \cdot d\mathbf{S}$ also vanishes. Hence there is no current perpendicular to the magnetic surface and $\nabla \cdot \mathbf{j}_\perp$ vanishes when integrated over the volume between two magnetic surfaces.

It follows that we can always find a $j_{\parallel}$ along the lines of force such that $\nabla \cdot j_{\perp}$ vanishes, and hence

$$\nabla \cdot j_{\perp} = - \nabla \cdot j_{\parallel} . \qquad (29)$$

Since the right-hand side of equation (29) is known from equation (28), $j_{\parallel}$ can, in principle, be determined simply by integrating indefinitely along a line of force. As shown by Kruskal (1955) such an integration will yield a single-valued function, $j_{\parallel}$, as a direct result of the vanishing of $\nabla \cdot j_{\perp}$ when integrated over the volume between two magnetic surfaces. The solution of equation (29) will lead to a constant of integration in $j_{\parallel}$. This constant corresponds to the mean value of $j_{\parallel}$ integrated over the entire volume between two magnetic surfaces, weighting each volume element by the local value of B_o. When the effect of a very small resistivity is considered, this mean value of $j_{\parallel}$ must approach zero, since an electromotive force around the magnetic axis cannot be maintained for an indefinite period. Hence we may set this mean value of $j_{\parallel}$ equal to zero everywhere, and eliminate the arbitrary constant. The total current j_1 has now been determined uniquely.

When j_1 has been determined, the solution of equations (26) and (27) for B_1 is a well known problem and is, in principle, not difficult. The nature of this solution will depend on the boundary conditions. For example, the solution obtained if an infinitely conducting wall surrounds the plasma, with B_1 assumed zero outside, will differ from the solution obtained if no such wall is assumed, and B_1 is assumed to approach zero at infinity. We assume that B_1, similarly to B_o, possesses magnetic surfaces.

We have seen that j_1 and B_1 can be computed directly once p_1 and B_o are known. Similarly j_n and B_n can be computed if B_{n-1} and p_n are known. Thus we can solve equations (24) through (27) by successive iteration if p_n can be specified for each n. The chief condition on p_n is that each $\underset{\sim}{B}_{n-1} \cdot \underset{\sim}{\nabla} p_n$ vanish, and that hence p_n is constant on the magnetic surfaces produced by $\underset{\sim}{B}_{n-1}$. Otherwise p_n is arbitrary, as we have already seen for p_1. Since $\underset{\sim}{B}_{n-1}$ differs from $\underset{\sim}{B}_{n-2}$, and will have different magnetic surfaces, in general, p_n must differ from p_{n-1}. However, if the difference between $\underset{\sim}{B}_{n-1}$ and $\underset{\sim}{B}_{n-2}$ is small, the difference between p_n and p_{n-1} may also be kept small. For example, if the topology of the magnetic surfaces remain unchanged in successive iteration, each magnetic surface may be labelled by the total flux, Ψ , which it encloses, and p may be assumed the same function of Ψ in all iterations.

To establish that a solution of equations (1) through (4) can be obtained in this way, it remains to establish that the successive iterations do, in fact, converge. It seems physically clear that for sufficiently small p the successive values of j_n and B_n will in fact converge to a limit. As p_1 decreases, j_1 will also decrease, and the difference between B_1 and the vacuum field, $\underset{\sim}{B}_0$, will also diminish. The deviations of the magnetic surfaces of $\underset{\sim}{B}_1$ from those of $\underset{\sim}{B}_0$ will also decrease, and $(p_2-p_1)/p_1$, the relative modifications of p_1 required in the next iteration, will decrease. Hence $(\underset{\sim}{B}_2-\underset{\sim}{B}_1)/B_0$ should become smaller with decreasing p. Similarly $(\underset{\sim}{B}_{n-1}-\underset{\sim}{B}_n)/B_0$ decreases with decreasing p, and if for some p this ratio is assumed bounded for all n, convergence of this iterative process is assured for sufficiently small p.

Let us discuss the physical meaning of this iterative process. In the first iteration we take the magnetic surfaces of the vacuum field, $\underset{\sim}{B}_0$, and make the pressure constant on each such surface, and a smoothly varying function from one surface to the next. Exactly as in the torus the current transverse to $\underset{\sim}{B}_0$ will show a divergence. Unlike the torus, this divergence can be eliminated by "secondary currents", parallel to $\underset{\sim}{B}_0$, whose divergence cancels the divergence of the transverse current. However, these currents within the plasma will deform the magnetic field. The transverse component of j_1 simply reduces the confining field. However, the longitudinal component produces a secondary magnetic field perpendicular to B_0. The total magnetic field B_1, computed with the plasma current j_1, as well as the external field-producing current j_0, may have quite different properties from B_0. If p and j_1 are both small, the magnetic surfaces of B_1 will be displaced only slightly from those of B_0. A small change in the pressure distribution will then make p_2 constant along the magnetic surfaces of B_1, and successive iterations should converge rapidly to a solution of the exact equations.

An elegant treatment of the magnetostatic equations (1), (3) and (4) has been given by Kruskal and Kulsrud (1957) in terms of the invariants of a perfectly conducting gas, undergoing deformations in a stellarator. Since the lines of force are frozen into the ionized gas, the magnetic surfaces move with the gas. The total flux Ψ through a given magnetic surface must be constant during any deformation; moreover for a particular surface Ψ must obviously be the same for any cross-section plane intersecting the surface. Another invariant must be M, the total mass of material in the volume V enclosed by the magnetic surface. A third invariant is the total transverse flux χ. This quantity is defined as the total flux through a closed loop of ribbon, one edge of which is the magnetic axis, and the other edge of which lies on the given magnetic surface. In the absence of a rotational transform, the outer edge of the ribbon can follow a line of force, in which case χ vanishes; if the ribbon is twisted n times, χ then equals $n\Psi$. For a particular choice of ribbon χ is then invariant during a perturbation of the gas. Kruskal and Kulsrud show that for any given M (Ψ) and

$\chi(\psi)$, the values of p and $\underset{\sim}{B}$ which minimize the total energy W provide a solution to the magnetostatic equations. It is suggested that this formulation of the problem might be useful for numerical work. While this method has not as yet been used to obtain detailed solutions of equations (1), (3) and (4), it is physically instructive to examine the basic invariants of the problem, which are also important in stability analyses.

3.2 Limiting pressure; scallops

The analysis above may be applied to compute the limiting pressure in an idealized case of the figure-eight stellarator. The plasma radius r_1 will be assumed very small compared to the major radius R of the curving sections and to the length L measured once around the magnetic axis. The vacuum magnetic field $\underset{\sim}{B}_o$ will be assumed to have the same magnitude throughout the plasma region, except for the small variation required in the curving end sections. In an end section we use the same coordinates R, θ and z introduced in the torus. We omit the subscript 1 from j_1 and p_1. The longitudinal current $j_\perp$ is then j_φ and in the curving sections it is readily shown that to first order in r/R,

$$\frac{\partial j_{\parallel}}{R\partial\varphi} = -\underset{\sim}{\nabla}\cdot\underset{\sim}{j}_\perp = -\frac{2}{RB_o} \qquad (30)$$

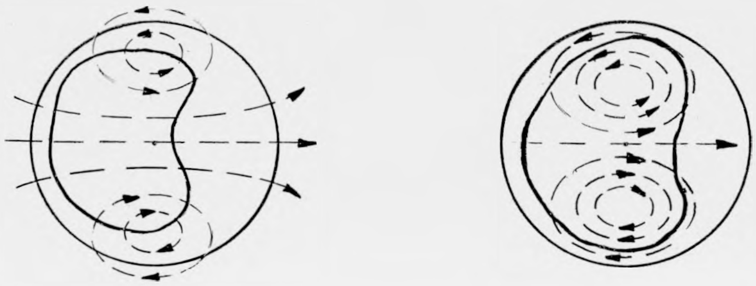
In the straight section $j_\perp$ is constant and equal to its value at the end of the curving end section, which will here be assumed a semicircle. If $j_{\parallel}$ is taken to be zero in the middle of the end section, then in the straight section, where we introduce coordinates r, θ and ℓ, we have

$$j_{\parallel} = \frac{\pi}{B_o}\frac{dp}{dr}\cos\theta. \qquad (31)$$

In equation (31) we assume that the magnetic surfaces are circular, and hence that p is a function only of r, the distance from the magnetic axis. The assumption that $j_{\parallel}$ is zero at the center of the end sections is approximately valid if the rotational angle ι is about 180°, in which case the current divergence in one end section is approximately cancelled in the other. In other cases $j_{\perp}$ will exceed the value found from equation (31). The fact that the end sections must curve through more than 180° in a figure-8 also increases $j_{\perp}$ somewhat above the value in equation (31).

The longitudinal current density given in equation (31) exceeds by a factor π the transverse current density, $j_{\perp}$. In the infinite cylinder the transverse current reduces $\underset{\sim}{B}$ below $\underset{\sim}{B}_o$, and limits the gas pressure that can be confined to less than $B_o^2/8\pi$. In the present curved system the longitudinal currents are greater than the transverse ones. Moreover, they produce a secondary magnetic field whose direction is different from the vacuum field, with the result that the magnetic surfaces become distorted when plasma is present. For values of p_m, the maximum pressure, comparable with $B_o^2/8\pi$ this distortion becomes so great that the sequence $\underset{\sim}{B}_o$, $\underset{\sim}{B}_1$, and $\underset{\sim}{B}_n$ no longer converges, and apparently no simple solution exists.

In the straight section the secondary field $\underset{\sim}{B}_1 - \underset{\sim}{B}_o$ may be computed exactly from the longitudinal current density given in equation (31). This secondary field is entirely transverse to $\underset{\sim}{B}_o$. The direction of the lines of force of this secondary field are shown schematically in Figure 6, drawn for the idealized case where the density distribution is parabolic; i.e., where $p_1(r)$ equals $p_m(1-r^2/r_1^2)$. If a cold gas is first placed in the vacuum field $\underset{\sim}{B}_o$, and the plasma then ionized and heated, an infinitely conducting wall will prevent the secondary magnetic field from leaking through; the same effect could also be produced, in principle, by passing sheet currents of the required strength along the tube wall. The field pattern in such a case is shown by the diagram on the right–hand side of Figure 6.



a–Non–conducting wall   b–Infinitely conducting wall

Figure 6
Lines of force of secondary magnetic field

The vacuum magnetic surfaces, which may be assumed circular in the middle of each straight section, will become distorted at each end of the stellarator by these secondary fields. In Figure 6 a cross–section which is initially circular ~~at r equal to r_p~~ becomes distorted to the two solid lines shown in the figures. The greater the length of the straight section, the more distorted the magnetic surface becomes.

To measure the distortion we may compute the secondary magnetic field $B_1 - B_o$ on the axis; we denote this quantity by ΔB_a. If ΔB_a is expressed as an integral over $j_{\parallel}(r, \theta)$, assuming no change with ℓ, then we find (Spitzer, 1952)

$$\Delta B_a = \frac{2\pi^2 p_m}{B_o} , \tag{32}$$

The inclination of the line of force of the B_1 field to the cylinder axis is evidently $\Delta B_a/B_o$; the maximum displacement, d, of the magnetic axis, at the end of the straight section, equals this inclination, multiplied by half the length of the straight section. The additional displacement produced in the end section may be taken into account approximately by assuming that the inclination remains the same all the way to the end of the machine, a distance $L/4$ from the center of the straight section to the center of one of the curving end sections. The total displacement then becomes

$$d = \frac{\pi^2 p_m L}{2B_o^2} , \tag{33}$$

This computation of the displacement is valid only for d small compared with r_1. Evidently when equation (33) gives a value of d large compared to r_1, the magnetic surfaces of $\underset{\sim}{B}_1$ are so distorted from those of $\underset{\sim}{B}_o$ that p_2 will differ greatly from p_1, and no convergence of the sequence may be expected. Hence the criterion for a solution of the type described here is that d is less than r_1, which yields for β the inequality.

$$\frac{8\pi p_m}{B_o^2} = \beta < \frac{16}{\pi} \frac{r_1}{L} \approx 5 \frac{r_1}{L} . \tag{34}$$

Since L/r_1 cannot readily be decreased much below 50, if $\underset{\sim}{B}_o$ is produced by external coils with appreciable winding depth, β in a simple figure-8 system must be small compared to 0.1 if a solution of the type described here is to exist. It is readily shown that this limitation applies only to the value of β in the curving sections. A weaker field in the straight sections, with a higher β, is possible without increasing d/r_p, provided that the total length L and the values of r_1 and β in the curved sections satisfy relation (34). This general result is a very serious limitation on the amount of gas that can be confined, and thus on the rate of generation of thermonuclear power in a practical device.

This upper limit on β can be raised if one can reduce the distance over which the secondary currents must travel before neutralization. This objective may be achieved naturally by the transverse helically invariant fields described above. Let us suppose that these fields produce a rotation, ι, of π radians in an axial distance ℓ_c. In a toroidal system, the longitudinal currents need travel only a distance ℓ_c before cancellation; we shall refer to ℓ_c as the "cancellation distance". Moreover, if ℓ_c is small compared to R, the major radius of the toroid, the maximum value of the cancellation current is less by a factor $2\ell_c/\pi^2 R$ than the value found from equation (31). Evidently the maximum value of β for which an equilibrium solution exists in the toroid with transverse fields may be made close to unity if R much exceeds ℓ_c. In such a toroidal system the maximum β that may actually be used is presumably limited by instabilities rather than by field deformation produced by secondary currents.

To keep the thermonuclear power high in a system where instability requires that β be kept low in the curving end sections, it may be desirable to include long straight sections, with a relatively low field and a high value of β. A straight section with a weaker field and a larger cross-section will have a more favourable ratio of thermonuclear power to power dissipated in the external field. A problem arises when a curved section, with transverse fields of multiplicity ℓ equal to 3, is to be joined to such a long straight section. According to equation (23), the rotational angle for ℓ equal to 3 increases with r, the distance from the magnetic axis, and in one curving section the cancellation of the secondary currents cannot be exact. Hence there will be some residual currents passing along the magnetic lines of force in the straight section, and the maximum length of the straight sections will be limited by a relationship similar to (34).

In principle it is possible to obtain a more exact cancellation of the secondary currents in a single curving section. If a transverse field with $\ell = 2$ is used, and if the helical length $2\pi/k$ is much greater than the plasma radius, r_1, the rotational angle ι per unit axial length is about the same for all radii, according to equation (23). Suppose now that we choose parameters such that the total rotation produced in a single curving section is $2\pi n$, where n is any integer. Then each line of force encircles the magnetic axis exactly n times in the curving section, and the integral of $\underset{\sim}{\nabla} \cdot \underset{\sim}{j}_{\perp}$ along each line vanishes at least to first order by symmetry.

A similar method of achieving this result is to juxtapose sections of opposite curvature, as shown in Figure 7. To obtain a net bending of the lines of force, the sections of positive curvature

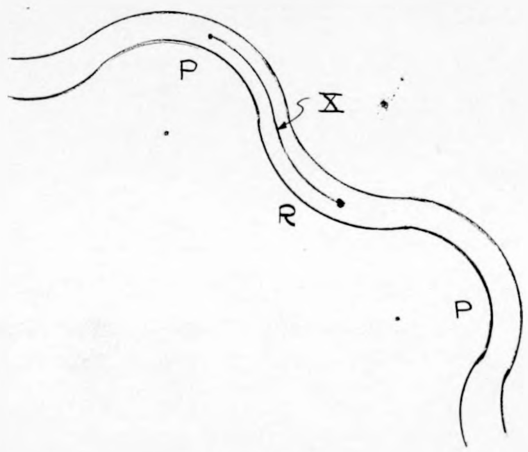


Figure 7
Scalloped Curving Section

(the P sections in Figure 7) must be longer than the sections in which the curvature is reversed in direction but equal in magnitude (the R sections). Such a magnetic configuration, called a "scalloped section", has been analyzed by Grove (1954); the magnetic design of the necessary external coils has been studied by Koenig (1956a). The integral of $\underset{\sim}{\nabla} \cdot \underset{\sim}{j}_{\perp}$ along a line of force between the centers of two of these sections (line X in Figure 7) may be made to vanish approximately if the magnetic field in the R section is made appropriately weaker than in the P section. The cancellation is exact to first order both in r/R and in β.

As yet there has been no detailed comparison of these two methods for cancelling out secondary currents in a single curving sections. If a strong transverse field with $\ell = 3$ is required for stability, it is not clear that a closer cancellation of secondary currents is needed, nor has it been shown that either of these two alternate methods is compatible with such a stabilizing field.

4. Confinement of Single Particles in the Stellarator

We now focus attention on the separate charged particles in an ionized gas and examine their confinement in the magnetic field of a stellarator. In the absence of collisions the confinement will be shown to follow quite generally from the existence of a rotational transform and from the adiabatic invariance of the magnetic moment of a gyrating particle. In this respect the present analysis differs greatly from earlier treatments (see Spitzer, 1951). The discussion will be devoted first to particles which perform successive revolutions about the stellarator, moving parallel to the magnetic axis. Second, particles will be considered which remain in one section of the machine, either because they are trapped in a region of weaker field, or because their velocity parallel to the axis is essentially zero.

Throughout the ensuing discussion the only point considered is whether or not the charged particles remain within the gas, without striking the wall. No attempt will be made to demonstrate that the particle trajectories, obtained on the free-particle picture, are consistent with the macroscopic electric fields in which the particles are assumed to move.

This procedure seems relatively safe; if the macroscopic equations can be satisfied, with particular fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$, it seems likely that the exact consideration of single particles would yield a quasi-stationary solution with electric and magnetic fields not differing greatly from the functions found on the macroscopic picture.

4.1 Particles performing revolutions around the stellarator

A particle whose velocity has a component parallel to the magnetic axis, and which is never reflected from any magnetic gradients in the machine, performs successive revolutions about the stellarator. If followed in time, such a particle will intersect any cross-sectional plane, extending across the stellarator tube, an indefinite number of times. We shall now show that the successive intersections of a group of similar particles with a cross-sectional plane may be used to generate a transformation of this plane into itself, satisfying the same conditions as the H-transform generated by the lines of magnetic force, discussed in section 2.1. Hence single particles are confined by the magnetic field to exactly the same approximation as the lines of magnetic force remain within the stellarator tube.

To demonstrate this result, we consider a particle with a particular energy, W, which is clearly a constant of the motion in the absence of collisions. Let the magnetic moment of the charged particle be μ; while this is not rigorously constant, we may treat it as a constant* of the motion, in view of the result by Kruskal (1957) that μ is constant to all orders of ak where a is the Larmor radius, and k is about $|\nabla(\ell nB)|$. The position

* The theory does not yield a simple definition of μ which is accurate to all orders in ak, and is constant to all orders, but shows that such a definition must exist, and how to construct it.

of the "guiding center" of the particle is also a well defined quantity. When this guiding center passes through some particular cross-sectional plane, the point of intersection generates a point P, which we shall call an "intersection point".

Now let us fill the stellarator tube with particles within ranges ΔW and $\Delta \mu$ about the same energy, W, and the same magnetic moment, μ, and let the density in phase space, within these narrow ranges of W and μ, be constant everywhere. The density everywhere will now be constant in time. By assumption, the value of μ/W is low enough so that the parallel velocity, $w_{\parallel}$, can never go to zero. Hence the particles cannot be reflected, and those with positive $w_{\parallel}$ form a separate class from those with negative $w_{\parallel}$. Here we shall consider only the particles with positive $w_{\parallel}$. Per unit time these particles will generate a large number of intersection points in the cross-sectional plane. The number of such points generated per unit area will be called the "density of intersection points". Evidently this density will be constant in time for the ensemble assumed here.

A particular intersection point P may be generated by a particle at any phase of gyration in its Larmor circle. However, Kruskal (1957) has shown that the path followed by a guiding center is independent of the initial phase of gyration to all orders in ak, where a and k signify, as before, the Larmor radius and $|\nabla(\ell n B)|$. Hence if the guiding centers of two particles pass through the same point P_1, in one intersection, they will both pass through the same point P_2 in the next intersection. Thus each point P_1 in the cross-sectional plane may be assumed to generate one and only one point P_2 in the same plane after the particles have performed one revolution about the stellarator.

We digress briefly to examine this result from a somewhat different viewpoint. In general, if a particle passes through a plane at a particular point at a particular time, the particle is not specified uniquely, since the three components of velocity are arbitrary. In the present problem, however, we have two constants of the motion, W and μ, and the remaining arbitrariness, the phase of gyration in the Larmor circle, has a negligible effect on the motion of the guiding center and may be ignored. Thus specifying the intersection point P of the particle at a particular time specifies the subsequent trajectory of the guiding center for all time, to a high approximation.

We have now proved that revolution of a particle about the stellarator generates a transform of a cross-sectional plane, each point going into some other point. Moreover, it is obvious that the transformed plane is a transform of itself, in that the density of intersection points generated within a time Δt is the same function of position in the two planes. Hence it follows that the transform generated by the successive particle intersections, which we shall call a P-transform, obeys the same laws as the H-transform generated by the magnetic lines of force. Hence the particles will be confined to a very high order if the P-transform is primarily rotational. Since the particle drift in a single revolution is at most a few times the Larmor radius, a, and the plasma diameter is many times a for all but the most energetic particles, it follows that the P-transform, like the H-transform, is, in general, primarily rotational. If the secondary longitudinal currents are largely cancelled out in each curving section, by means of transverse fields or scalloped sections, the net drift across the lines of force for a particle passing through the section will also be largely eliminated, and the P-transform will become nearly identical with the H-transform, except for rotation about the magnetic axis, for even the most energetic particles. We conclude that the confinement of particles which revolve about the stellarator should be nearly as complete as the confinement of the lines of force.

In view of the importance of this result, it is well to review the assumptions on which it is based. These are:

(a) The magnetic and electric fields are independent of time

(b) The change of magnetic moment, μ, and the dependence of the guiding-center trajectory on phase of gyration, are both negligible

Assumption (a) is a basic assumption, whose validity must be checked experimentally. Assumption (b) has been proved to all orders of ak, and seems unlikely to be a source of serious error.

4.2 Particles rotating about the magnetic axis

For particles which do not perform successive revolutions about the stellarator, the situation may be complicated. We shall consider only two special classes of such particles, – first, those which are definitely trapped in one particular section of the stellarator, and second, – those which are moving through a curved section so slowly that their velocity parallel to the magnetic axis may be set equal to zero. For each of these two classes we shall find that confinement is again assured to a high approximation, thanks to the rotation of the particles about the magnetic axis. This rotation is produced partly by the radial electric field, required by equation (2), and partly by the gradient in magnetic field required by the diamagnetic effect of the plasma, – see equation (12).

Let us consider first the particles which are definitely trapped within some section of the stellarator. For these the ratio μ/W is so large that the velocity component $w_{\parallel}$, parallel to the magnetic field, vanishes on two cross-sectional surfaces bounding a section of the machine, and the motion of particle is necessarily restricted to the lines of force between these two surfaces. As before, we introduce an assembly of particles with about the same μ and W, and with constant density in phase space. Let us introduce a cross-sectional plane between the two bounding surfaces. Each particle will pass through the plane between successive reflections. An "intersection point" P in this plane is now generated by the intersection with this plane of a guiding center of any particle within this assembly; only passage in one direction will be considered to generate intersection points, the passages in the reverse direction being ignored. The density of intersection points in the plane is again defined as the number of such points generated per unit area per unit time. Exactly as before, the successive intersection points generated by these particles define a P-transform of the cross-sectional plane into itself, and as before confinement is assured if the P-transform is primarily rotational.

Thus the condition for confinement is simply that the rate of rotation be greater than any systematic unidirectional drift. We may apply this criterion to a scalloped end section, where particles will be trapped in the weak-field regions of reverse curvature (region R in Figure 7). The velocity of drift due to the curvature of the magnetic axis is given by the sum of equations (9) and (11). The rotational velocity associated with the electric field is given by combining equations (7) and (2). The condition that the rotational velocity exceed the unidirectional drift becomes simply

$$\frac{R}{3}\,\frac{d(\ell n\, n_i)}{dr} > \frac{w^2}{\overline{w^2}} , \qquad (35)$$

where w is the particle velocity and $\overline{w^2}$ is the mean-square value of the thermal velocity for all particles of the same mass. Since the logarithmic gradient of n_i in the outer regions of the tube exceeds $1/r_1$, where r_1 is the plasma radius, and since r_1/R_1 is generally less than 1/3, it is evident that the P-transform in this case will in general be primarily rotational sufficiently far from the magnetic axis. Particles of relatively large w provide an important exception to this conclusion.

A rotational velocity is given also by the radial variation of B, produced by the diamagnetic effect of the plasma. The magnitude of this effect depends on β, the ratio of the gas pressure on the magnetic axis to the magnetic energy density outside the plasma. Combining equations (9), (11) and (12), assuming that p is of the form $p_o(1-r^2/r_1^2)$, and considering values of r nearly equal to r_1, we find that the P-transform will be primarily rotational provided that

$$\beta > \left(1 + \frac{w_{\parallel}^2}{w_{\perp}^2}\right) \frac{r_1}{R} \quad ; \qquad (36)$$

as before, $w_{\parallel}$ and $w_{\perp}$ are the longitudinal (parallel to B) and transverse (perpendicular to B) components of the particle velocity. For trapped particles $w_{\parallel}$ will not much exceed $w_{\perp}$. Evidently if β much exceeds r_1/R confinement of single trapped particles in a curving section seems generally assured, regardless of the particle velocity. We shall consider later the special case in which the rotation produced by the electric field is equal but opposite to the rotation produced by the diamagnetic effect.

We consider next the particles for which $w_{\parallel}$ is so small that it may be ignored. In such a case, a particle will rotate around the magnetic axis, remaining always in the same cross-sectional plane. Confinement is established in exactly the same manner as for the trapped particles, the one difference being that a particle remains always in a particular cross-sectional plane, rather than intersecting it at intervals. To generate a P-transform we now take the location of guiding centers at successive times, separated by some interval Δt, large compared to $2\pi/\omega_c$, the period of gyration. With this change the analysis goes through exactly as before; the criteria that the

P–transform be primarily rotational are again equations (35) and (36), with now $w_{//}$ set equal to zero.

We see that single particles should be confined to a relatively high approximation in these idealized situations, provided that the ratio of $r_\perp$ to R is not too great. Even if β is very small, so that inequality (36) is not satisfied, the rotation produced by the electric field will assure confinement for all but the most energetic particles, provided again that $r_\perp / R$ is small.

In view of the importance of this radial electric field it is desirable to review the physical reason for its appearance. From the standpoint of the macroscopic equations, this field is exactly that which is required to keep the angular momentum of the plasma about the magnetic axis very small. The only torque tending to produce rotation about the magnetic axis corresponds to the pondermotive force on the radial current. Since the charge separation must be small in a plasma of high density, the radial current must also be small, and the angular momentum nearly zero. If the angular momentum is to be negligible, so must be the macroscopic rotational velocity of the positive ions, and the radial electric field is exactly that required to produce this result. This radial field produces a rotational drift velocity which is equal and opposite to the macroscopic velocity resulting from the pressure gradient, and the net macroscopic velocity essentially vanishes.

The radial electric field may also be viewed from a microscopic standpoint. As charged particles are heated in a cylindrical geometry, their Larmor radii will increase and there will be a net outward motion of the particles. Not only will the guiding centers move outward, as a result of random walk in successive collisions, but the mean square distance of the particles from the axis will increase even further because of the increased Larmor radius. The positive ions because of their greater mass will move outwards much more than the electrons, resulting in some separation of charge. When account is taken of the dielectric constant of the plasma, for steady electric fields transverse to $\underset{\sim}{B}$, the electric field resulting from this charge separation is exactly that found from the macroscopic equation (2). It is evident from this equation that the difference of electrostatic potential from the magnetic axis to the outer regions of the plasma, in volts, is of order k T, in electron volts.

When ~~the~~ thermonuclear reactions are occurring this result must be modified. The reaction products include positively charged nuclei with relatively high energies and correspondingly large Larmor radii. The outward motion of these particles will increase somewhat the negative potential of the plasma with respect to its outer regions. This increase is less than a factor of two. It may be shown that the charge separation produced by the heating of a group of heavy particles is proportional to a^2, where a is the Larmor radius of the energetic particle. Since the α particles produced in the D–T reaction have an energy of 3.5 Mev. they have an a^2 which is about 100 times that of deuterons at a temperature of 1.5×10^8 degrees. It is usually assumed that only about one per cent of the positive ions interact during the confinement of a gas in a stellarator, and since the reaction products lose their energy in less than 0.1 second, the charge separation resulting from these energetic α particles shall be substantially less than that produced by heated plasma itself. The resultant macroscopic velocity may be expected to be much smaller than the electric drift velocity, which is somewhat less than 10^6 cm/sec for a full-scale reactor. A preliminary examination indicates that the velocity-dependent terms in the basic equations are not very important (Spitzer, 1952, section 2d), but a thorough investigation is lacking.

4.3 Resonances

Another effect which must be considered is the possible interference between the different methods for producing a primarily rotational P–transform. For example, the electric rotational drift velocity may produce a rotation which just cancels that due to the rotational transform of the lines of force. Similarly, for a trapped particle the electric–field rotational drift may cancel out the rotational drift produced by the radial variation of the confining magnetic field. Such an effect will be referred to as "resonance" since the two rotational rates are equal and opposite. An even larger resonance appears when scallops or transverse fields are used to cancel out the secondary currents in a single curving end section. Suppose that a particle is travelling along the scalloped section in Figure 7, and suppose that its rotational drift and longitudinal speed are such that it is on the top of the tube at P, and has rotated to the bottom at R. The inhomogeneity drift in the two sections will change r, the distance from the magnetic axis, in the same way, and total radial drift increases steadily with time, if the resonance is exact. In a full–scale device this type of resonance in a scalloped section will occur only for particles whose longitudinal velocity $w_{//}$, is less by an order of magnitude than the mean thermal velocity.

The effect of resonance is modified by two circumstances. In the first place, resonance usually occurs only for particles at a particular distance, r, from the magnetic axis. As r changes, resonance will disappear, and the transform will become primarily rotational. In the second place, collisions will alter the particle velocities so that resonance is no longer present. As a result of such random encounters, a given particle will remain for only a very short time within the narrow velocity range required for resonance. Encounters between ionized particles will produce a particle deflection of 0.1 radian in only a hundredth of the time required for a deflection of one radian. As a result, these resonant effects are not cumulative for a single particle, and their only result is to increase the effective diffusion coefficient. A rough calculation indicates that this increase in the rate of diffusion is not of practical importance, although it may increase the diffusion rate substantially above the classical value found from electron-ion collisions.

The analysis in these sections is by no means a final investigation of confinement in a stellarator. The basic equations are only approximate, and a more exact treatment would be desirable. The solution of the macroscopic equations should be re-examined with consideration of the macroscopic velocity which may be expected in a thermonuclear reactor. More complicated types of free-particle trajectories remain to be analyzed, and the enhanced rate of diffusion resulting from resonances should be evaluated in fuller detail. Finally, the deviations from closure both of the H-transform and of the P-transform should be taken into account. However, the present analysis does not suggest that any of these effects are likely to be practically important, and the general result is that there do not seem to be any basic obstacles in attempting to confine a plasma within a stellarator magnetic field. If seems probable that if the plasma is in fact quiescent, and that no instabilities appear, confinement should be adequate for a full-scale thermonuclear reactor.

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