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General Electric Company
KNOLLS ATOMIC POWER LABORATORY*
Schenectady, New York

SHOCK AND VIBRATION IN NAVAL REACTORS

by

R. M. Mains

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SHOCK AND VIBRATION IN NAVAL REACTORS

(Originally prepared for Reactor Handbook)

by

Robert M. Mains

I. INTRODUCTION

The problem of design of a reactor to withstand the shocks and vibration incident to shipboard service in Naval craft consists of three principal parts. First the loads must be defined in some rational manner, then the responses of the various reactor structures to these loads must be predicted, and finally the responses must be compared to some damage criteria to determine whether the design is adequate or not. A brief discussion of each of these major parts will be undertaken, and then a specific example will be worked through to illustrate what is involved in the whole process.

II. DEFINITION OF LOADS

If one were starting from scratch on a new problem, it would be necessary to conduct a series of service measurements on equipment similar to the intended design and then by proper statistical studies and the application of judgement to establish the design loads. In the absence of service measurements, an operational analysis could be made, which could then be tempered with judgement to produce design loads. In our case, service measurements have been made and from these the Navy has established the following loads:

9-5-2

A. Vibration (according to letter P. R. Clark to Mgr. SOO, June 19, 1956, RD:NRB:NMCole I-2865 and letter Sweek to Mgr. SOO, February 17, 1954 SA #2814)*

* These requirements are for the reactor compartment of a particular submarine - they do not necessarily apply to all Naval vessels.

1. The vibration environment shall be considered to consist of a single-frequency excitation lasting for the full design life of the reactor.

2. The amplitudes of vibration shall be: **

** Amplitude = 1/2 of total excursion

0.025 in from 0 to 6 cps.

0.010 in from 6 to 16 cps.

0.005 in from 16 to 32 cps.

3. The frequency of vibration shall be taken as that frequency between 0 and 32 cps which is most damaging to the structure.

4. Design to keep the natural frequency higher than 32 cps, and the stresses below the fatigue limit by an appropriate safety factor.

5. If the natural frequency cannot be kept above 32 cps, stresses must be kept below the fatigue limit by an appropriate safety factor. In calculating stresses, a magnification factor of 10 should be assumed if the actual magnification factor cannot be calculated or determined.

B. Shock (according to KAPL Specification KPE-1094 as modified by comments from the Bureau of Ships, and KAPL letter SAR 72-110 as modified by comments from the Bureau of Ships)

1. A dynamic analysis of the reactor components and support shall be made, with a velocity shock input to the hull of 5 ft/sec vertical or athwartship, or 2.5 ft/sec fore and aft, for equipment weighing less than 10,000 lbs and mounted on the hull or bulkheads. For equipment weighing more than 10,000 lbs, these velocity shocks shall be 4 ft/sec and 2 ft/sec, respectively.
2. For this dynamic analysis, the hull shall be assumed to be rigid for purposes of determining magnitudes of response (or load factors). For considerations other than magnitude of response in which frequency is important, the hull shall be assumed to be flexible between reactor compartment bulkheads. (Excitation frequency for control mechanisms, for example.)
3. For equipment not mounted on the hull or bulkheads, either it will be heavy enough that it will need to be considered in the dynamic analysis of Part 1 or it will be light enough that the magnitude and frequency of response from Part 2 may be used as the excitation function.

III. PREDICTION OF STRUCTURAL RESPONSES TO THE LOADS

For the prediction of structural responses to the loads, it is probably simplest to determine the various normal modes of the structure, compute the responses of each normal mode to the load, and then superpose these responses to get the total response. This procedure is sound so long as structural behavior is elastic, but a proper analysis to account for plastic behavior is much more difficult, and may often be too difficult to attempt. The normal modes may be determined entirely by calculation, entirely by measurements on a model, or by combinations of model measurements and calculation. For the elementary theory of normal mode calculation, a variety of texts are available, such as Den Hartog, "Mechanical Vibrations" and Timoshenko, "Vibration Problems in Engineering". For the principles involved in model design and measurement, the "Handbook of Experimental Stress Analysis", edited by Hetenyi, is an excellent source.

The most important problem in response prediction, however, is that of deciding what structural elements and couplings are significant and setting up the analyses or experiments so that the resulting answers are meaningful. To illustrate what is involved and provide some feeling for the effects of various parts of a reactor support structure, the illustrative calculations will show a number of variations in degrees of freedom (number of normal modes considered) and parameters (particularly stiffness or flexibility).

IV. DAMAGE CRITERIA

The previous two major parts of the problem are in relatively good shape - the loads have been defined and reliable means for predicting elastic responses to the loads are available. When the question of criteria for evaluating damage is raised, however, the situation is quite different. Ordinary concepts such as a limit of elastic behavior as defined by a static yield strength are no longer significant, since ductile materials may sometimes sustain stresses of more than the static yield stress under shock with no other damage than a very minor amount of permanent set. Fatigue strength may be of significance only if resonant vibration is involved, in which case failure may develop rapidly. Without resonance, vibration is unlikely to lead to structural damage (electronic control equipment may still malfunction under non-resonant vibration, however).

To summarize what little is known of dynamic damage criteria, perhaps the best that can be done is to list some do's and don'ts which the author has found to be good design practice, as follows:

A. Vibration

1. Avoid resonant vibration wherever possible.
2. If resonance cannot be avoided, build in structural damping to keep down magnification factors and try to keep resonance stresses below the endurance limit.
3. Adding mass to the structure in such a way that it does not store potential energy will reduce magnification factors, but also will lower resonant frequencies.
4. Adding mass to the structure in such a way that it does store potential energy results from efforts to strengthen the structure. This usually stiffens the structure, may increase the resonant frequencies, and may also increase the magnification factors. This may hurt more than it helps.
5. Avoid stress raisers and joints that can come loose and rattle.
6. Avoid collision of parts.

B. Shock

1. Avoid hanging a small tail on a large dog in such a way that the frequency of the tail is close to that of the dog. This leads to resonance excitation again.

2. Avoid brittle materials.

3. Don't be too disturbed if calculated stresses (on an elastic basis) come out to be more than* the static yield stress unless the first mode frequency is very low (under 30 or 40 cps). The chances are that the structure won't be aware of your calculation and will perform in a satisfactory manner.

4. Watch out for excessive deformations of parts and avoid collision. Calculated deformations can usually be two or three times the elastic limit deformation with no large permanent distortion.

5. Make sure joints are sound and avoid unnecessary stress raisers.

If the above precepts are followed, designs will probably be safe, but there is no guarantee that this will be so. In this area at present, good judgement is the best insurance against failure - data and facts are scarce.

V. ILLUSTRATIVE CALCULATIONS OF FREQUENCY

The determination of shock and vibration responses of a structure requires first that the geometric configuration of the structure and the masses which load it be known. At the beginning of a design, some means of getting a structure to analyze must be used. In the case of

* See "Delayed-Yield Time Effect in Mild Steel Under Oscillatory Axial Loads", R. O. Belsheim, NRL Report 4312 (1954).

the SAR reactor and support, an equivalent static load factor of 100 was used to establish general configurations and sizes in conjunction with other design requirements such as pressure, temperature, and the like. The results of this preliminary design effort are shown schematically in Fig. 1, insofar as the pertinent structural elements are concerned. Table I lists the various separate elements together with weights, dimensions, and other information.

Whether the problem is determination of vibration stresses or shock stresses, the first thing that must be done is to determine the resonant frequencies of the structure under consideration. As a first approximation, the reactor vessel and its immediately supporting structures could be considered as attached to a rigid hull as in Fig. 2. The mathematical model for this structure is also shown in Fig. 2. Equations of motion for this model are given on calculation sheet 3, and by substitution of values for mass and stiffness, the frequency can be calculated as:

$$f = \frac{1}{2\pi}\sqrt{\frac{5.97 \times 10^7}{403}} = 61 \text{ cps.} \quad * \qquad (5.1.4.1)$$

---

* Stiffness = $5.97 \times 10^7 = (\Delta/V_2 + \Delta/V_3 + \Delta/V_4)^{-1}$ from Table I with Δ/V_2 taken as 1.00×10^8.

Mass = 403 = 155200/386 from Table I.

As a next approximation to the problem, the structure of Fig. 3 could be analyzed. The equations of motion for this solution are given on calculation sheet 2, and the matrix form of solution is shown on calculation sheet 5. To solve for frequencies, stiffness values from Table I (the inverse of the flexibility values given in the table) and mass (or weight) ratios from the same table can be substituted in the determinant. The solution of the determinant for values of ω_1/ω is then carried through by standard procedures. For this operation, it is helpful to make additional substitutions such as:

$$\frac{\omega_2}{\omega} = \frac{\omega_1}{\omega} \cdot \frac{\omega_2}{\omega_1} \quad , \quad \frac{\omega_3}{\omega} = \frac{\omega_1}{\omega} \cdot \frac{\omega_3}{\omega_1} \qquad \text{etc.}$$

The frequencies which result from such a solution are shown in Table II, together with the mode shapes (relative magnitudes of the various X's). These mode shapes are found by substituting the three resonant frequencies into the motion equations, one at a time, and obtaining three sets of simultaneous equations in X which can then be solved.

In this calculation, as in all others discussed herein, the weights or masses used in the computations have been divided up and prorated between the elements of the mathematical model by judgement in order to approximate more closely the true solution. For example, M_3 of Fig. 3 is calculated thus:

$$M_3 = \left[\underset{③}{\frac{21400}{2}} + \underset{④}{\frac{35000}{3}} + \underset{④}{\frac{40000}{3}} + \underset{⑥}{21000 \times \frac{2}{3}} + \underset{⑦}{42000 \times \frac{2}{3}}\right]\frac{1}{386} \qquad (5.1.4.2)$$

← Elements in Table I.

In addition, stiffness values for use in equations of motion are the inverse of Δ/V values in the table, while influence coefficients for use in the flexibility matrix must be calculated. For example, the influence coefficient for X_1 with load at X_1 is found by:

$$\underset{④}{5.55 \times 10^{-9}} + \underset{③}{1.00 \times 10^{-8}} + \underset{②}{1.20 \times 10^{-9}} = 16.75 \times 10^{-9} \qquad (5.1.4.3)$$

← Elements in Table I.

As may be seen, the foregoing operation for a 3-degree-of-freedom system is a considerable amount of work, especially if automatic computing equipment is not available. For this reason, the matrix form of solution shown on calculation sheet 5 is preferred because of its greater simplicity for numerical computation and its ready adaptability to automatic computation. The answers produced by the two methods are the same.

To show the effect of further subdivision of the structure, the model of Fig. 4 with 5 degrees of freedom is offered. The equations of motion are given on calculation sheets 1 and 2, and the matrix form is given on calculation sheet 5. It may be observed that equations and matrices for Figs. 3 and 2 were obtained by elimination of the appropriate items from the equations and matrix for Fig. 4, with an adjustment in the mass distribution each time. Equations and matrices could have been derived in each case, but it was desirable to show the relationships involved. Results for the solutions to Fig. 4 are given in Table II.

In Fig. 5 is shown the simplified structure for the lateral modes of the reactor vessel. Since the lower frequencies of Figs. 3 and 4 were not greatly different, it seemed reasonable to combine masses 6 and 7 into a single mass. In addition, a comparison of flexibility values that affect the lateral mode for items 2 and 3 in Table I shows that the support skirt, item 2, is considerably stiffer than the shield tank flange, item 3. It is therefore reasonable to lump mass 2 in with mass 1 and achieve a 6-coordinate lateral mode instead of the 10-coordinate lateral mode that would go with Fig. 4. This amounts to a considerable simplification which is justifiable for this particular problem only - it is not necessarily a general proposition. Notice, too, that in Fig. 5 the deck is assumed to be rigid which would mean about 10 percent error in the Δ/H flexibility of mass 3, as shown by the Δ/H values for items 4, 5, and 8 in Table I. The overall effect of this amount of error in mass 3 flexibility on the resulting frequencies and mode shapes will be less than 5 percent because of the large Δ/H flexibility of mass 1.

The motion equations for Fig. 5 are possible to derive, but at the cost of a huge amount of algebra which automatic computers do not yet handle. Consequently, only the matrix form of solution is shown. It is worth noting that the various kinds of flexibility in the matrix have different units, as do the terms for mass and moment of inertia. These differences combine to leave the final solution in terms of sec^2 only, as it should be. The results of the matrix solution are shown in Table III.

In order to evaluate the effect of the deck and hull on the vertical mode (where the effect would be more pronounced) the model of Fig. 6 was analyzed. Again, the solution is presented as a matrix for simplicity, and the results are given in Table IV. It is of particular interest to note that whereas all of the lowest rigid-hull frequencies calculated for the three preceding cases were 50 cps or greater, the lowest frequency for Fig. 6, the flexible-hull case, was in the neighborhood of 17 cps.

The various lowest-mode frequencies for all the above styles of calculations are compared in Table V. Included in the table are several additional values computed to determine the effect of varying the stiffness of the shield tank flange. A study of this table affords a scale of judgement on the significance of different model configurations for analysis, and the effect of stiffness changes.

VI. MODEL MEASUREMENTS OF FLEXIBILITY

Because of the complexity of some elements of the structure, particularly the shield tank flange, the deck, and the hull, a plastic (Plexiglas) model in 1/6 scale was built to conform to Fig. 1. This model was then used to measure flexibilities of the various elements, separately and in combination. Table VI shows a comparison of flexibilities scaled up from the model measurements with those calculated or estimated for the prototype several months before. For the most part, the scaled-up and predicted values of flexibility agree within quite reasonable limits, and two purposes have been served:

1. The confidence one can have in both measured and predicted values is greatly enhanced by the agreement of the values.
2. That good structural models can be relied upon to give flexibility values which are quite useful for frequency calculations has been demonstrated.

At least three different situations occur in flexibility calculations; the structural elements may be so simple that calculations are short and reliable, in which event no models are justifiable; a few of the elements are complex or their interactions are complex, in which event models to evaluate flexibility may well be more economical than calculations for these particular elements or interactions; the whole structure may be too complex for analysis to be economical, in which event models afford a happy solution. It is usually feasible to build a flexibility model of most structural elements, and these models can be used also for stress determinations if the model is made with this in mind. It is not, in general, feasible to use these models for experimental determination of frequency, however, in the author's opinion. It is better to calculate those flexibilities and stresses which are well understood, measure models for those which are not, and then compute frequencies for use in vibration and shock analysis. Whenever a true prototype is available for measurement, all of these processes can be replaced by direct measurements on the prototype.

In the model results shown in Table VI, the prediction of flexibility for the reactor vessel support skirt shows the need for a more careful analysis and a check of the thickness of the model material to bring the predicted and measured results more into line. The estimation of flexibility of the hull did not include the contribution of the outer hull, while the model included the outer hull. Consideration of the outer hull in the estimate would give a value much nearer to the measured flexibility. The deck skirt flexibility measurements suffered in accuracy because the magnitudes involved were of the same order as the least count of the measuring devices, and because some values could not be measured directly. For this reason, the calculations are better than the measurements for the deck skirt.

VII. VIBRATION AND SHOCK STRESS CHECK

Once the frequencies and mode shapes have been determined, as in the foregoing illustrations, the next logical step consists of estimating stresses. For an example, consider the shield tank outer wall, element k_4 in the model of Fig. 6. The frequency of the first mode has been found to be 17 cps, and the vibration excitation at this frequency is 0.005 in. amplitude.* With an assumed magnification factor of 10 in** this mode, the computations shown on calculation sheet 6 are the result. The stress of 2800 psi is low enough to be of no concern regardless of the number of cycles of loading. Had this stress been above the endurance limit for the material, then a comparison of fatigue life with required life would have had to be made.

* See item II A (2)

** See item II A (5)

Other vibration stresses may be estimated in a similar fashion from the deformations of the various spring elements, and the lateral mode vibration stresses can be estimated in similar fashion from the results for the model of Fig. 5. It should be noted that the word estimate is used in conjunction with these stress calculations for the following reasons:

1. The vibration excitation is by no means an exact item.
2. The mathematical model from which frequencies and mode shapes are determined is a simplified approximation to the structure, so that the frequencies and mode shapes are approximate.
3. In most practical cases, the calculation of stress is inexact, despite a widespread tendency to carry stress calculations to five "significant" figures.
4. Even if the stress could be established within ± 10 percent, the failure criteria may be no better than ± 50 percent or not understood at all.

In brief, a calculated vibration stress of 2800 psi in MIL-S-16113-B Grade HT* steel is clearly no problem, 28000 psi would require a check of endurance limit and other physical properties, and 280,000 psi would almost surely indicate trouble. A calculated stress of 56,000 psi, on the other hand, would not necessarily be cause for alarm.

* Yield strength of 16113-B grade HT steel is 42,000 psi.

For a check of vertical shock stresses, the rigid-hull model of Fig. 4 is used for the determination of the load factor. Once the load factor is obtained, the checks of vertical clearance and stresses are carried through on the flexible-hull basis, the same as for vibration stresses. These numbers are worked out on calculation sheet 7, with a resulting calculated stress of 80,000 psi. Inasmuch as less than 100 psi of this stress is direct stress while 80,000 psi of this is a bending stress which exists only over a very short distance at the top of the shield tank cylinder, it is concluded that no serious damage is likely to occur under shock and only a very minor amount of permanent set in bending at the top of the shield tank.

It should be noted that the application of the load factor of 42 to each of the masses of the model of Fig. 4 leads to a calculated stress of 11,500 psi, which is much too high to be a measure of the direct stress and too low to be measure of the bending stress. Such a crude use of an equivalent static load factor is not particularly useful in judging the adequacy of a structure under shock. The simplified approach shown on the upper part of calculation sheet 7, however, should be sufficiently accurate for the purpose of judging the adequacy of a structure under shock, especially since the same comments with regard to stress estimates apply to shock as to vibration.

VIII. SMALLER COMPONENTS ATTACHED TO THE REACTOR VESSEL

Consider next the problems of shock and vibration from the standpoint of a control mechanism attached to the top head of the reactor pressure vessel. In this case, a mechanism weighs less than 1000 lbs, while the reactor vessel and the things it carries weigh about 155,000 lbs, and all of the mechanisms together amount to about 10 percent of the reactor vessel weight. For this reason, the motion response of the reactor vessel to shock or vibration is very little modified by the response characteristics of the control mechanisms. If the weight ratio were not so small, then the control mechanisms would have had to be considered in the analyses of Fig. 4 or 6. As it is, quite acceptable results can be obtained by using the reactor vessel response to shock and vibration as the excitation for the control mechanism.

It is now necessary to determine the frequencies and mode shapes for the important elements of the control mechanism, just as was the case with the reactor vessel and its support structure. A mathematical model could be devised and analyzed in exactly the same fashion as before. It hardly seems profitable to do this again, so experimentally determined frequencies will be used for this illustration.

In a shock test of a control mechanism, the frequency of the lead screw was measured as 120 cps. In addition, the hull flexibility measured on a scale model turned out to be 13 times less than estimated* as shown in Table VI.

* See Item IV for explanation of this discrepancy.

This means that if the solution of Fig. 6 were corrected to account for this much smaller hull flexibility, the first mode vertical frequency would come out near the 54 cps obtained with Fig. 4. (The lower frequency solution for Fig. 6 was used for illustration of methods, but the higher frequency corrected value will now be used since it provides a more severe case.)

So the problem reduces to this:

1. Vibration -

The highest frequency vibration excitation is 32 cps at 0.005 in. amplitude.

A 54 cps reactor vessel structure would respond to this excitation with

$$0.005 \times \left| \frac{1}{1-\left(\frac{54}{32}\right)^2} \right| = 0.0027 \text{ in. at 32 cps.} \qquad (5.1.4.4)$$

A 120 cps control mechanism lead screw would respond to this with

$$0.0027 \times \left| \frac{1}{1-\left(\frac{120}{32}\right)^2} \right| = 0.00021 \text{ in. at 32 cps.} \qquad (5.1.4.5)$$

And this is

$$\frac{1}{10} \times 32^2 \times 0.00021 = 0.022 \text{ g's.} \qquad (5.1.4.6)$$

Since the lead screw is never less than about 30 in. effective length, the strain is 0.000007 or about 200 psi. The 0.022 g's times the effective weight of the material supported by the lead screw, divided by the area of the lead screw gives a stress of about 5 psi. Consequently on either of these bases vibration is no problem as far as the lead screw is concerned.

2. Shock -

The 54 cps reactor vessel structure was calculated to have a vertical shock response of

$$N = \frac{4 \times 2\pi \times 54}{32.2} = 42 \text{ g's at 54 cps.} \quad (5.1.4.7)$$

For a frequency ratio, $r = \frac{120}{54} = 2.2$ and a mass ratio, $\mu = 0.006$, the load factor chart of the Appendix gives L = 2, and

$$\sigma = 2 \times 42 \times \frac{\text{weight supported by lead screw}}{\text{area of lead screw}} \quad (5.1.4.8)$$

$$\approx 8500 \text{ psi.}$$

This order of stress in 17-4 PH* stainless is no problem.

---

* Yield strength of 17-4 PH stainless is 125,000 psi.

As a further check on the acceptability of the lead screw, the amplitude at 120 cps would be

$$\frac{10 \times 2 \times 42}{(120)^2} = 0.058'' \div 30'' \approx 0.002 \text{ in/in} \quad (5.1.4.9)$$

and this magnitude of strain in 17-4 PH is not a problem, even for several applications of the shock.

TABLE II.

COMPARISON OF FREQUENCIES & MODE SHAPES FOR MODELS OF FIGS. 2, 3, 4

		FIG. 2	FIG. 3			FIG. 4				
FREQUENCY - CPS.		61	50	214	407	54	86	191	452	554
RELATIVE AMPLITUDE	X_3	—	+.359	+1.000	-.135	+.381	+.045	+.680	+1.000	-.196
	X_2	—	+.952	-.016	+1.000	+.944	-.070	-.036	+.213	+1.000
	X_1	+1.000	+1.000	-.131	-.458	+1.000	-.082	-.116	-.073	-.206
	X_6	—	—	—	—	+.407	+.106	+1.000	-.854	+.086
	X_7	—	—	—	—	+.628	+1.000	-.294	+.036	-.002

TABLE III

FREQUENCIES & MODE SHAPES FOR FIG. 5

FREQUENCY - CPS		41	148	316	443	823	1200
RELATIVE AMPLITUDE	X_3	+.0089	+.1591	+.1577	+.3125	+1.000	-.7934
	X_6	-.0178	+.2596	+1.000	+1.000	-.2555	+1.000
	X_1	+1.000	+1.000	-.0168	-.0146	-.0130	+.00425
	Θ_3	-.00053	+.00171	+.01065	+.00366	-.0163	-.0446
	Θ_6	-.00054	+.00202	+.0320	-.01664	-.00503	+.00452
	Θ_1	-.0174	+.0394	-.00108	-.00056	-.00029	+.00028

TABLE IV

FREQUENCIES & MODE SHAPES FOR FIG. 6.

FREQUENCY - CPS		17.3	80	100	207	371	537	918
RELATIVE AMPLITUDE	X_3	+.9296	+.1477	-.2457	+.6215	-.3882	-.0125	+1.000
	X_2	+.9944	-.2434	+.1551	-.0164	-.0378	+1.000	-.0397
	X_1	+1.000	-.2778	+.1908	-.0815	+.0240	-.2297	+.0027
	X_6	+.9345	+.2079	-.1973	+1.000	+1.000	+.0064	-.1301
	X_7	+.9690	+1.000	+1.000	-.2433	-.0647	-.0002	+.0013
	X_9	+.9295	+.2015	-.4175	-.7289	+.0649	-.0004	-.0426
	X_8	+.9241	+.1472	-.2439	+.7702	-.6754	-.0783	-.7323

TABLE V

EFFECT ON LOWEST MODE FREQUENCY OF COMPLEXITY OF ANALYSIS AND VARIATION OF FLEXIBILITIES

	Δ/V #/in Items 3 & 6	f_n cps.
VERTICAL MODES		
1 - degree system - rigid hull - no deck participation		
Flexibility of item (3)	10.0×10^{-9}	61
with no help from item (6)	4.55×10^{-9}	75
Flexibility of item (3)	3.22×10^{-9}	79
with full help from item (6)	1.25×10^{-9}	89
3 - degree system - rigid hull - no deck participation		
Flexibility of item (3)	10.0×10^{-9}	52
with no help from item (6)	4.55×10^{-9}	62
Flexibility of item (3)	3.22×10^{-9}	65
with full help from item (6)	1.25×10^{-9}	71
5 - degree system - rigid hull - no deck participation		
Flexibility of item (3)	10.0×10^{-9}	53
with no help from item (6)	4.55×10^{-9}	62
Flexibility of item (3)	3.22×10^{-9}	65
with full help from item (6)	1.25×10^{-9}	69
7 - degree system - flexible hull - deck participating		
Flexibility of item (3) with no help from item (6)	10.0×10^{-9}	17.3
Flexibility of item (3) with full help from item (6)	1.25×10^{-9}	17.5
LATERAL MODE		
6 - degree system - rigid hull - rigid deck		
Item (3) gets no help from item (6)	10.0×10^{-9}	40.7
Item (3) gets full help from item (6)	1.25×10^{-9}	58.4

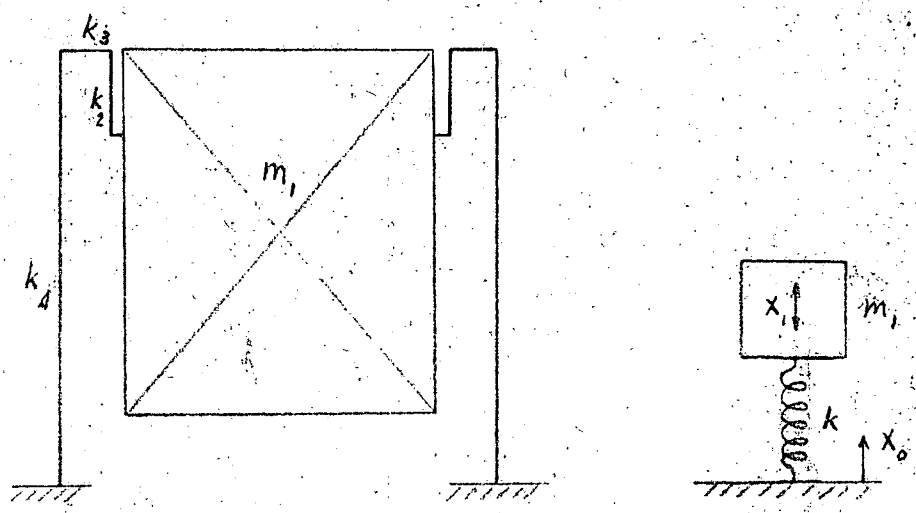


FIG. 2 VERTICAL MODE

1 Degree of Freedom

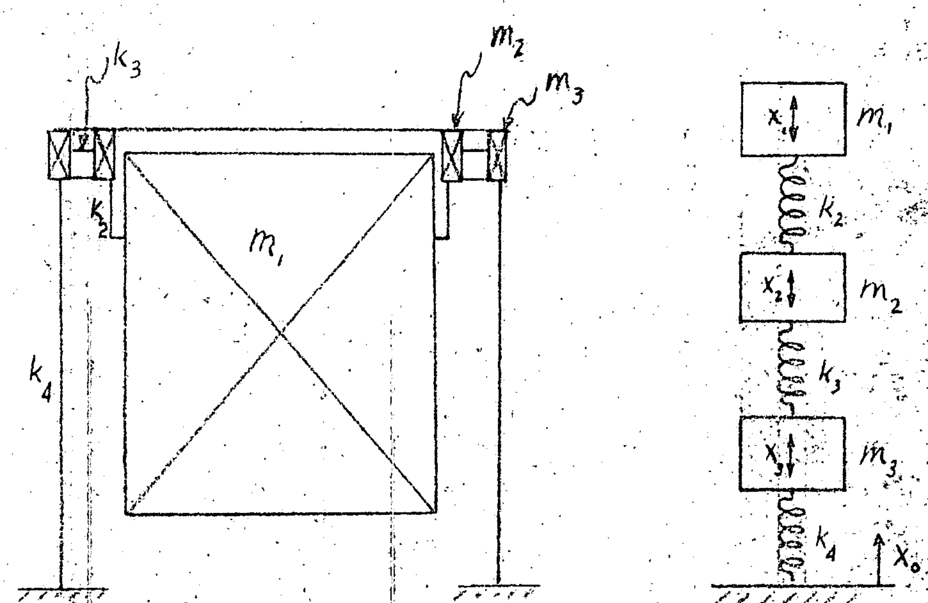


FIG. 3 VERTICAL MODE

3 Degrees of Freedom

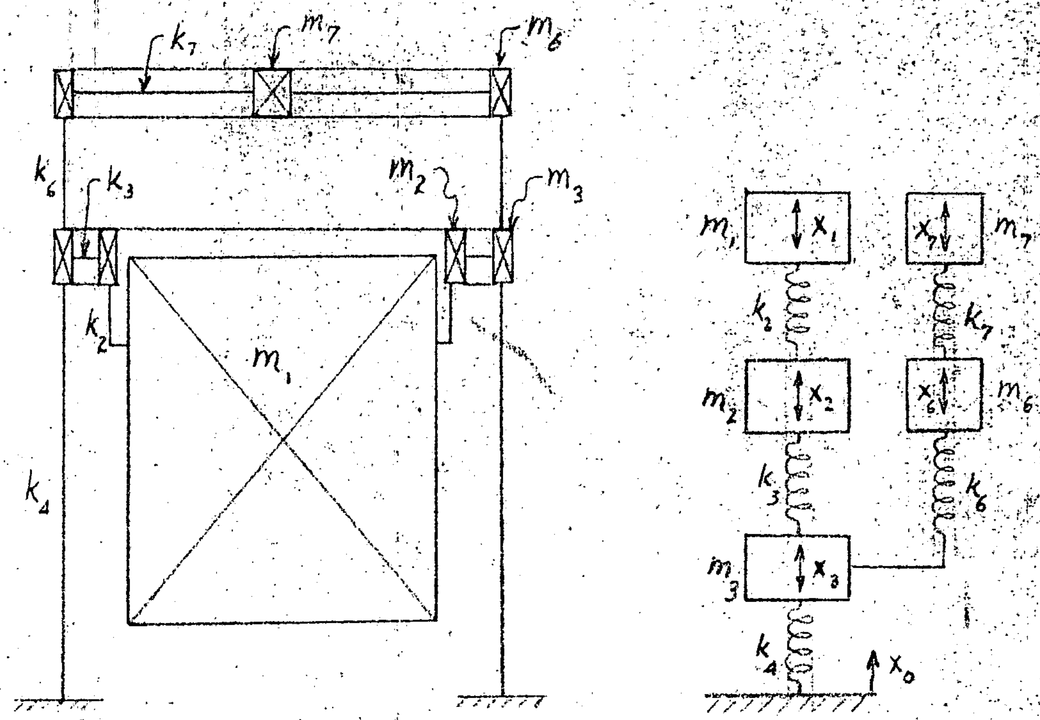


FIG. 4 VERTICAL MODE

5 Degrees of Freedom

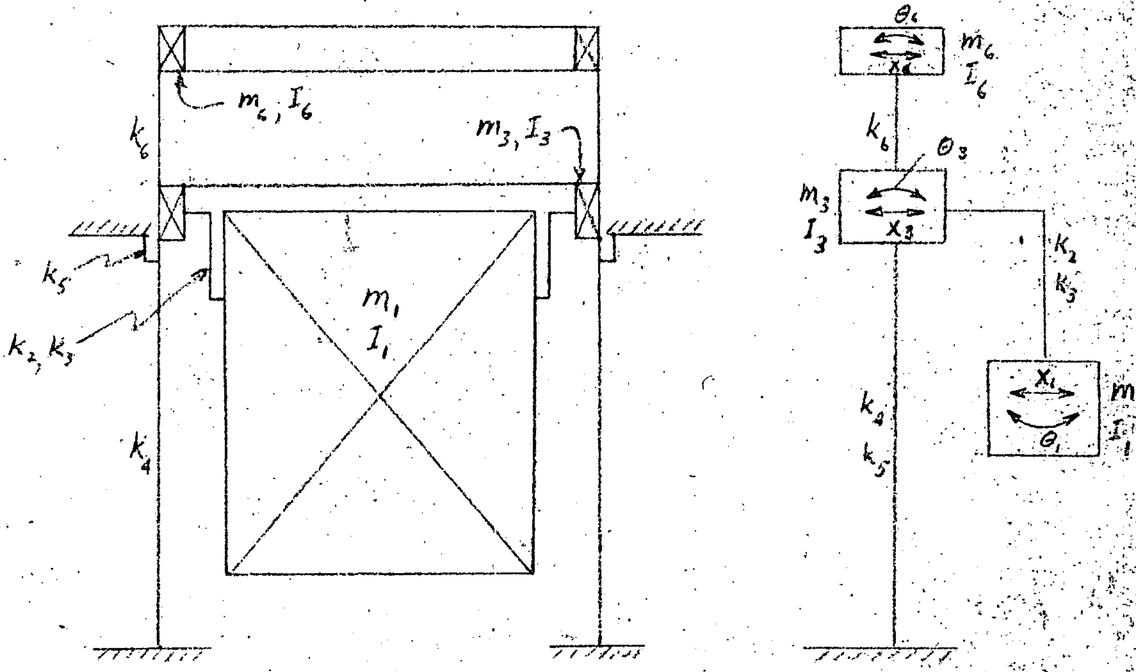


FIG. 5 LATERAL MODE

6 Degrees of Freedom

815 027

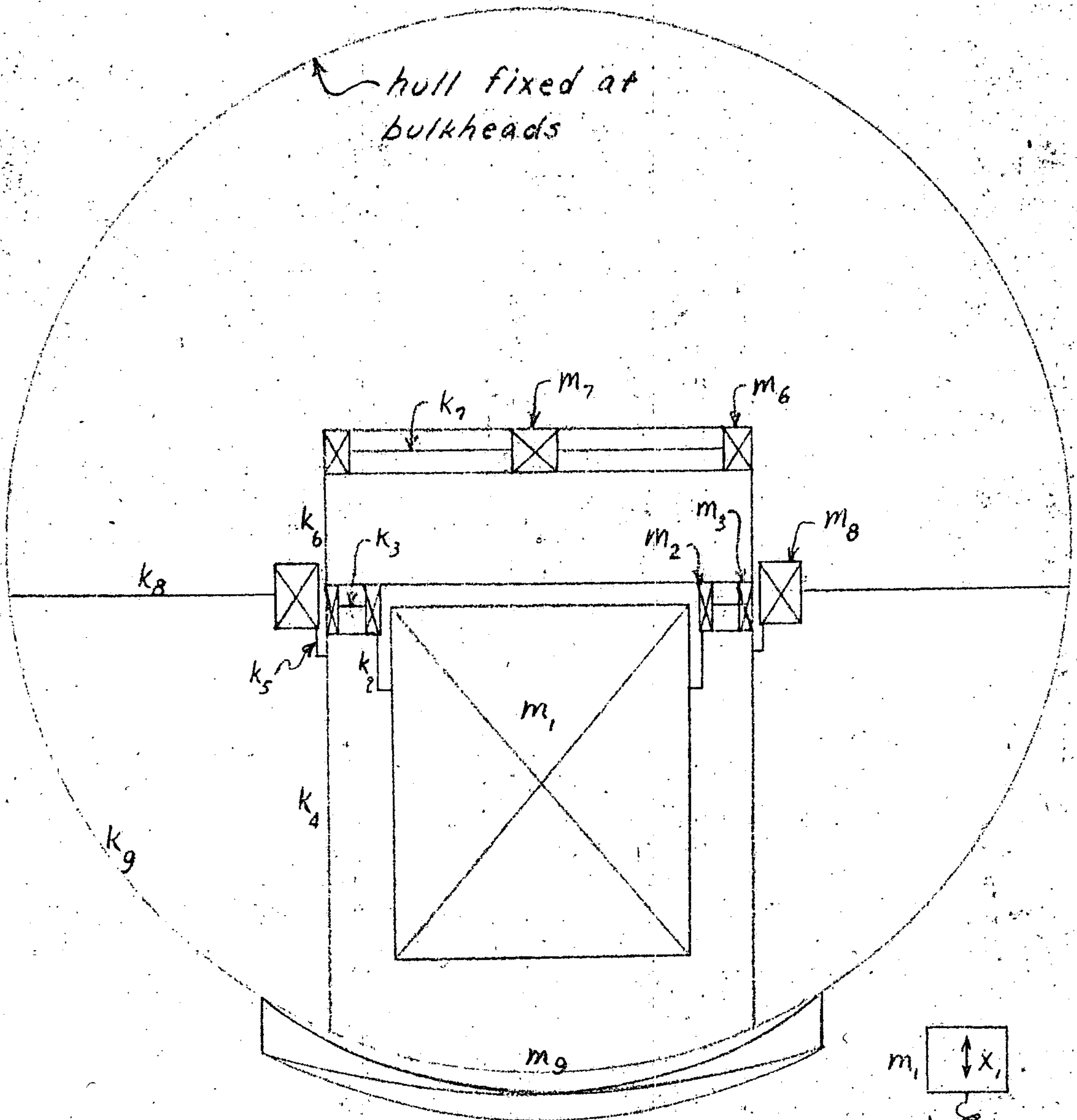
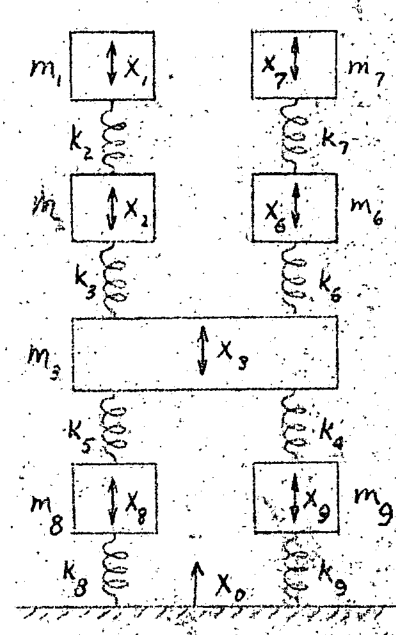


FIG. 6 VERTICAL MODE

7 Degrees of Freedom



EQUATIONS OF MOTION FOR FIG. 4.

$$\left.\begin{aligned}
&m_1 \ddot{x}_1 + k_2(x_1 - x_2) = 0 \\
&m_2 \ddot{x}_2 + k_3(x_2 - x_3) - k_2(x_1 - x_2) = 0 \\
&m_3 \ddot{x}_3 + k_4 x_3 - k_3(x_2 - x_3) - k_6(x_6 - x_3) = k_4 x_0 \\
&m_6 \ddot{x}_6 + k_6(x_6 - x_3) - k_7(x_7 - x_6) = 0 \\
&m_7 \ddot{x}_7 + k_7(x_7 - x_6) = 0
\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}
&m_1 \ddot{x}_1 + k_2(x_1 - x_2) = 0 \\
&m_2 \ddot{x}_2 + k_3(x_2 - x_3) + m_1 \ddot{x}_1 = 0 \\
&m_3 \ddot{x}_3 + k_4 x_3 + m_1 \ddot{x}_1 + m_2 \ddot{x}_2 + m_6 \ddot{x}_6 + m_7 \ddot{x}_7 = k_4 x_0 \\
&m_6 \ddot{x}_6 + k_6(x_6 - x_3) + m_7 \ddot{x}_7 = 0 \\
&m_7 \ddot{x}_7 + k_7(x_7 - x_6) = 0
\end{aligned}\right\} \quad (2)$$

Let $x_n = a_n(\cos wt + i \sin wt)$

then $\ddot{x}_n = -w^2 x_n$

Also define:

$$w_1 = \sqrt{\frac{k_2}{m_1}}, \quad w_2 = \sqrt{\frac{k_3}{m_2}}, \quad w_3 = \sqrt{\frac{k_4}{m_3}}, \quad w_6 = \sqrt{\frac{k_6}{m_6}}, \quad w_7 = \sqrt{\frac{k_7}{m_7}}$$

Now equations (2) become:

$$\left.\begin{aligned}
&-x_1 + \left(\frac{w_1}{w}\right)^2 (x_1 - x_2) = 0 \\
&-x_2 + \left(\frac{w_2}{w}\right)^2 (x_2 - x_3) - \frac{m_1}{m_2} x_1 = 0 \\
&-x_3 + \left(\frac{w_3}{w}\right)^2 x_3 - \frac{m_1}{m_3} x_1 - \frac{m_2}{m_3} x_2 - \frac{m_6}{m_3} x_6 - \frac{m_7}{m_3} x_7 = \left(\frac{w_3}{w}\right)^2 x_0 \\
&-x_6 + \left(\frac{w_6}{w}\right)^2 (x_6 - x_3) - \frac{m_7}{m_6} x_7 = 0 \\
&-x_7 + \left(\frac{w_7}{w}\right)^2 (x_7 - x_6) = 0
\end{aligned}\right\} \quad (3)$$

The various natural frequencies of the system are found by solving the determinant of equations (3).

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{vmatrix} x_1 & x_2 & x_3 & x_6 & x_7 \\ \hline -1+\left(\frac{\omega_1}{\omega}\right)^2 & -\left(\frac{\omega_1}{\omega}\right)^2 & 0 & 0 & 0 \\ -\frac{m_1}{m_2} & -1+\left(\frac{\omega_2}{\omega}\right)^2 & -\left(\frac{\omega_2}{\omega}\right)^2 & 0 & 0 \\ -\frac{m_1}{m_3} & -\frac{m_2}{m_3} & -1+\left(\frac{\omega_3}{\omega}\right)^2 & -\frac{m_6}{m_3} & -\frac{m_7}{m_3} \\ 0 & 0 & -\left(\frac{\omega_6}{\omega}\right)^2 & -1+\left(\frac{\omega_6}{\omega}\right)^2 & -\frac{m_7}{m_6} \\ 0 & 0 & 0 & -\left(\frac{\omega_7}{\omega}\right)^2 & -1+\left(\frac{\omega_7}{\omega}\right)^2 \end{vmatrix} = 0 \qquad (4)$$

Once the frequencies have been found, they can be substituted in equations (3) to get mode shapes (displacement functions).

EQUATIONS FOR FIG. 3.

By dropping the x_6 and x_7 terms from eq. (3) and determinant (4), the equations for Fig. 3 are obtained:

$$\left.\begin{aligned} -x_1 + \left(\frac{\omega_1}{\omega}\right)^2 (x_1 - x_2) &= 0 \\ -x_2 + \left(\frac{\omega_3}{\omega}\right)^2 (x_2 - x_3) - \frac{m_1}{m_2} x_1 &= 0 \\ -x_3 + \left(\frac{\omega_3}{\omega}\right)^2 x_3 - \frac{m_1}{m_3} x_2 - \frac{m_2}{m_3} x_2 &= \left(\frac{\omega_3}{\omega}\right)^2 x_0 \end{aligned}\right\} \qquad (5)$$

$$\begin{vmatrix} -1+\left(\frac{\omega_1}{\omega}\right)^2 & -\left(\frac{\omega_1}{\omega}\right)^2 & 0 \\ -\frac{m_1}{m_2} & -1+\left(\frac{\omega_2}{\omega}\right)^2 & -\left(\frac{\omega_2}{\omega}\right)^2 \\ -\frac{m_1}{m_3} & -\frac{m_2}{m_3} & -1+\left(\frac{\omega_3}{\omega}\right)^2 \end{vmatrix} = 0 \qquad (6)$$

EQUATIONS FOR FIG. 2.

By dropping the x_2 and x_3 terms from eq. (5) and determinant (6), the equations for Fig. 1 are obtained; when the driving displacement, x_0, is properly applied:

$$-x_1 + \left(\frac{\omega_1}{\omega}\right)^2 x_1 = \left(\frac{\omega_1}{\omega}\right)^2 x_0 \qquad (7)$$

$$\left.\begin{aligned} -1 + \left(\frac{\omega_1}{\omega}\right)^2 &= 0 \\ \omega = \omega_1 &= \sqrt{\frac{k}{m_1}} \end{aligned}\right\} \qquad (8)$$

For Fig. 6, equations similar to (3), (5) and (7) become too involved to be practical, and a change to matrix manipulation is to be preferred. For this, equation (3) can be written as:

$$(-M\omega^2 + K)x = 0 \qquad (9)$$

in which M, K, and x are matrix terms.
Transpose equation (9) to get

$$+M\omega^2 x = +Kx \qquad (10)$$

premultiply by K^{-1} and get

$$\omega^2 K^{-1} M x = K^{-1} K x = I x \qquad (11)$$

in which I is the diagonal unity matrix. The result is then:

$$K^{-1} M x = \frac{x}{\omega^2} \qquad (12)$$

Fortunately K^{-1} is the deflection influence matrix, in which each term is the deflection of each mass in the x direction caused by a unit load on the mass in the x direction.

MATRIX FOR FIG. 6.

Load at	X_3	X_2	X_1	X_6	X_7	X_9	X_8	*	Mass no.
3	7.98	7.98	7.98	7.98	7.98	7.88	7.93	90.9	3
2	7.98	9.10	9.10	7.98	7.98	7.88	7.93	98.5	2
1	7.98	9.10	9.22	7.98	7.98	7.88	7.93	394.	1
6	7.98	7.98	7.98	8.31	8.31	7.88	7.93	80.8	6
7	7.98	7.98	7.98	8.31	14.02	7.88	7.93	53.0	7
9	7.88	7.88	7.88	7.88	7.88	8.27	7.81	219.	9
8	7.93	7.93	7.93	7.93	7.93	7.81	7.96	93.5	8

all $\times 10^{-8}$ in/lb. (matrix); lb. sec²/in. (* column)

MATRIX FOR FIG. 5.

	x_3	x_6	x_1	θ_3	θ_6	θ_1	*	
H at 3	+ 3170.	+ 3540.	+ 2820.	+ 7.39	+ 7.39	+ 7.39	153	M_3
H at 6	+ 3540.	+ 17900.	− 2150.	+ 119.4	+ 134.	+ 119.4	83	M_6
H at 1	+ 2820.	− 2150.	+ 212000.	− 98.9	− 98.9	− 3300.	518	M_1
M at 3	+ 7.39	+ 119.4	− 98.9	+ 2.24	+ 2.24	+ 2.24	190,000	I_3
M at 6	+ 7.39	+ 134.	− 98.9	+ 2.24	+ 11.14	+ 2.24	185,000	I_6
M at 1	+ 7.39	+ 119.4	− 3300.	+ 2.24	+ 2.24	+ 72.6	755,000	I_1

all $\times 10^{-13}$

$\frac{\Delta}{H}$ - in/lb $\frac{\theta}{H}$ - rad/lb. Mass - lb. sec²/in

$\frac{\Delta}{M}$ - in/in-lb $\frac{\theta}{M}$ - rad/in-lb I - in. lb. sec²

* These columns of numbers are the elements of a diagonal matrix, written this way to save space. Do not mistake them for a column matrix.

MATRIX FOR FIG. 4.

Load at	X_3	X_2	X_1	X_6	X_7
3	5.55	5.55	5.55	5.55	5.55
2	5.55	15.55	15.55	5.55	5.55
1	5.55	15.55	16.75	5.55	5.55
6	5.55	5.55	5.55	8.88	8.88
7	5.55	5.55	5.55	8.88	65.8

all $\times 10^{-9}$

m *
101
93
403
83
54

lb. sec²/in

MATRIX FOR FIG. 3.

Load at	X_3	X_2	X_1
3	5.55	5.55	5.55
2	5.55	15.55	15.55
1	5.55	15.55	16.75

all $\times 10^{-9}$

m *
155
202
403

lb. sec²/in

* These columns of numbers are the elements of a diagonal matrix, written this way to save space. Do not mistake them for a column matrix.

VIBRATION STRESS CHECK

1^{st} mode frequency from Fig. 6, 17 cps.

Excitation at 17cps = 0.005 in.

X magnification factor of 10 = 0.05 in

Mode shape - from TABLE IV.

X_3	0.9296	× 0.05 =	0.04648
X_2	0.9944		0.04972
X_1	1.0000		0.05000
X_6	0.9345		0.04673
X_7	0.9690		0.04845
X_9	0.9295		0.04647
X_8	0.9241		0.04620

Stretch in Shield Tank Outer Wall = $X_3 - X_9$ = 0.00001

$$\sigma = 0.00001 \times 3\times10^7 / 138 = \underline{2} \text{ psi} \text{ - negligible}$$

$$\frac{X_2 - X_3}{15.25} = \Delta\theta \text{ of S.T. flange} = \frac{.00324}{15.25} = .000212 \text{ rad}$$

$$\Delta r = 7.74 \times .000212 = .00164 \text{ in}$$

$$M_o = 2\beta^2 D\,\Delta r + 2\beta D\,\Delta\theta = 5.37\times10^5 \times 1.64\times10^{-3} + 4.86\times10^6 \times 2.12\times10^{-4}$$

$$M_o = 2010 \text{ in-lb/in}$$

$$\sigma = \frac{M_o c}{I} = \frac{2010 \times 1}{\frac{1}{12}\times\frac{8}{.91}} = \underline{2750} \text{ psi}$$

2750 psi in MIL-S-16113-B gr HT is no problem - well below endurance limit.

SHOCK STRESS CHECK

1st mode frequency from Fig. 4, 54 cps.

Shock Load Factor $N = \frac{4 \times 2\pi \times 54}{32.2} = 42.$

Since acceleration in simple harmonic motion is

$$a = x\,\omega^2$$

and since response to shock of any single mode is simple harmonic motion (with damping ignored), then the amplitude required to produce 42 g's at 54 cps is:

$$\text{amp. } X_1 = \frac{42 \times 386}{(2\pi \times 54)^2} = 0.141''$$

(for practical purposes $\frac{386}{(2\pi)^2} \approx 10$)

From the mode shape for mode 1, Table II,

when $X_1 = 0.143''$, $X_3 = \frac{.381}{1.000} \times .141 = .054''$

The direct stress is then

$$\frac{3 \times 10^7 \times .054}{138} = 11{,}800\text{ psi}$$

To use the "load factor" more directly, each weight contributing to stress in spring k_4 must be multiplied by the load factor and by the amplitude ratio in the mode (since force = $\frac{\text{weight}}{g}$ × acceleration, and acceleration is proportional to amplitude, which is defined relatively by the mode shape).

$$42 \times 1.000 \times 156{,}000 = 6{,}560{,}000$$
$$42 \times .944 \times 36{,}000 = 1{,}430{,}000$$
$$42 \times .381 \times 39{,}000 = 626{,}000$$
$$42 \times .407 \times 32{,}000 = 550{,}000$$
$$42 \times .628 \times 21{,}000 = 556{,}000$$
$$9{,}720{,}000$$

Load Factor | Mode shape factor | Weight

$$\div\ \pi \times 135 \times 2 = 11{,}500 \text{ psi}$$

This style of calculation must always give the same answer as the preceding calculation since the assumptions involved are identical.

Suppose now we wish to calculate the stress in k_4 as a part of Fig. 6, but with the same load factor. Then the amplitude of x_1 (for the 17 cps structure) is

$$x_1 = \frac{10 \times 42}{(17)^2} = 1.45''$$

(4" clearance available, so OK)

by ratio from sheet 6

$$\frac{1.45}{.05} \times 2 = 58 \text{ psi direct stress}$$

$$\times 2730 = 80{,}000 \text{ psi bending}$$

OK in MIL-S-16113-B gr. HT with minor permanent set at top of shield tank.

Shock Stress Check

A summation of forces as before produces

11,700,000 lbs, of which $\frac{.0296}{.1252}$ is carried by k_4. This gives a stress of 3250 psi to compare with the previous 58 psi. The discrepancy is undoubtedly due to the fact that x_3 and x_9 are so nearly equal that small errors in either show up as large errors in the spring force, while the force computation is not similarly affected.

For the sake of comparison, equation 37 of NRL report 4420* was used to compute responses in the first three modes, with a step velocity shock large enough to give 42 g on mass no. 1. The stress in k_4 was +520 psi in the first mode, -2760 psi in the second mode, and -7180 psi in the third mode.

From all of this we can conclude that, as usual, when a stress computation depends upon the small difference of large quantities, as for k_4 in mode 1 of Fig. 6,

*"Dynamics of Linear Elastic Structures" - Blake & Swick NRL Report 4420

the accuracy of the computation cannot be high. This is true whatever the style of computation, since the mode shape does not change. It has been the Author's experience that, if the mathematical model of the structure is reasonably valid, and if the calculations of model behavior suffer in accuracy by reason of small differences of large quantities, then later measurements on the actual structure will show considerable variation, almost as if the structure were in a condition of meta stable equilibrium. K_1 and K_2 will show no discrepancy in various styles of computation in this structure (Fig. 6), for example, but k_5 will.

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For preliminary design work and for stress estimates a strong case can be made for using calculations like those on Sheet 7 and the top of Sheet 8. For more refined calculations and for higher modes, see NRL Report 4420. Each individual designer or analyst must judge for himself how far to go in a specific problem.

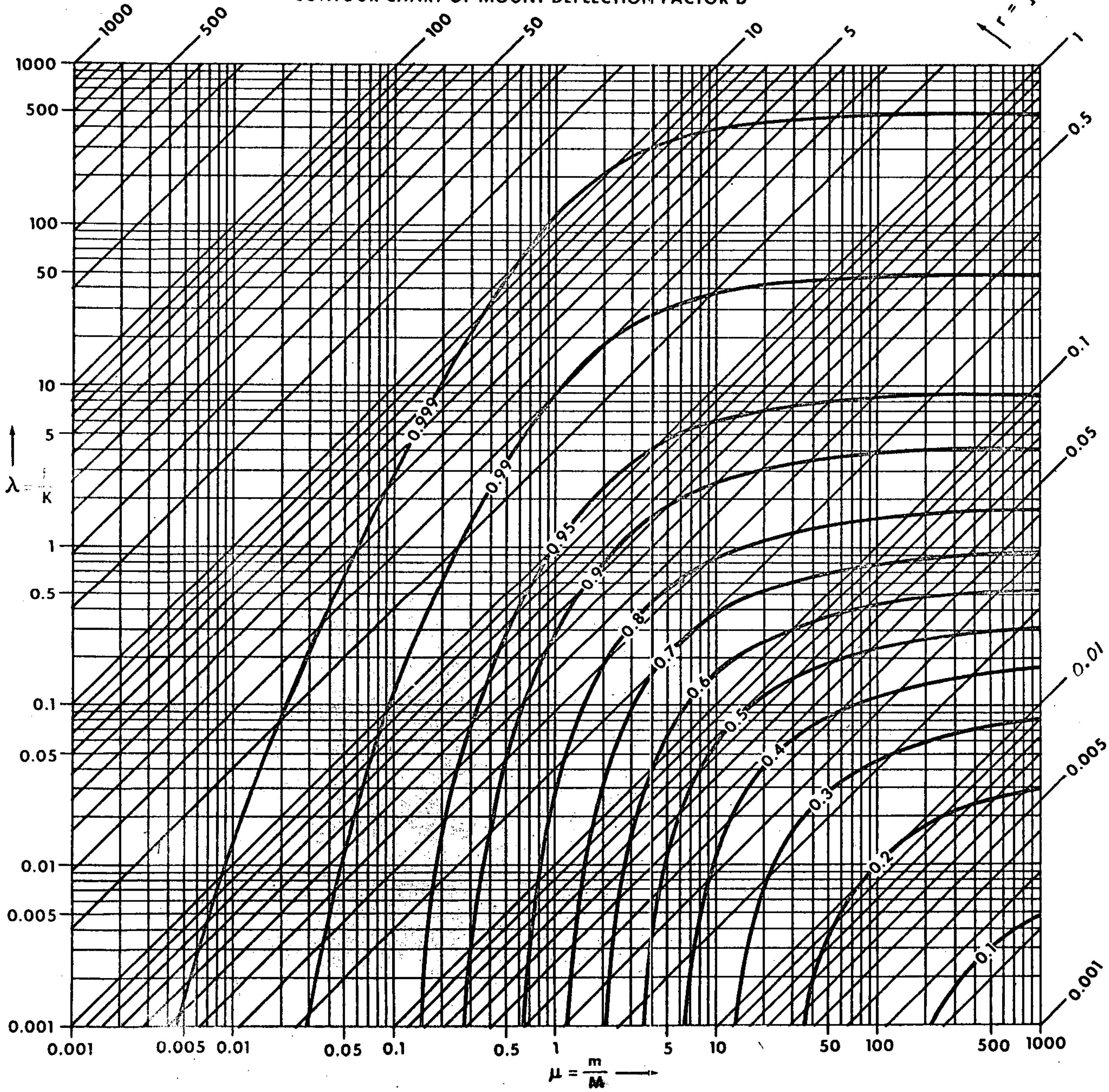
VELOCITY-SHOCK DIAGRAMS FOR TWO-DEGREE VIBRATION SYSTEMS

R. B. McCALLEY, JR., *Engineer, Knolls Atomic Power Laboratory, (operated for the Atomic Energy Commission by General Electric Co.,) Schenectady, N. Y.*

Increased mobility of electrical and electronic equipment is one reason for current interest in the protection of equipment against shock and vibration. Not only are chassis being shock-mounted, but the various components as well. This article is intended to aid designers in calculating the various parameters of two-degree linear systems which are required to specify shock-mount or equipment characteristics.

Two typical problems can be solved with the aid of the charts:

CONTOUR CHART OF MOUNT DEFLECTION FACTOR D



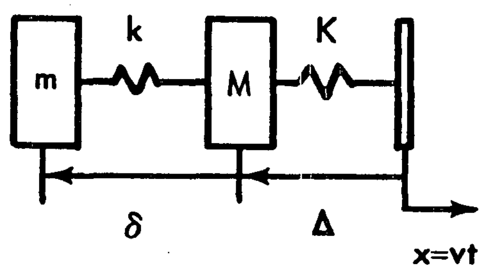
1. Given a piece of equipment, what should be the mount characteristics?

2. Given the mount, what should be the equipment characteristics?

The charts apply to systems with two degrees of fredom when the base is given a step velocity, V. The system (k, m) is the equipment or element to be protected while the system (K, M) is the cushioning mount.

Nomenclature

D = mount deflection correction factor
g = acceleration of gravity (386 in. $sec.^{-2}$)
k = equipment or element spring constant (lb. $in.^{-1}$)
K = mount spring constant (lb. $in.^{-1}$)
L = load factor ratio
m = equipment or element mass (lb. $sec.^2$ $in.^{-1}$
M = mount mass (lb. $sec.^2$ $in.^{-1}$)
n = load factor for mass m (number of g's)
r = frequency ratio of equipment to mount



t = time (sec.)
T = shock transmissibility
V = step velocity on base (in $sec.^{-1}$)
X = absolute motion of base (in.)
δ = maximum deflection of spring k (in.)
Δ = maximum deflection of spring K (in.)
λ = stiffness ratio of equipment to mount
μ = mass ratio of equipment to mount
ω = uncoupled equipment frequency (rad. $sec.^{-1}$)
Ω = uncoupled mount frequency (rad. $sec.^{-1}$)
cps. = cycles per second
lb. = pounds force
rad. = radians
sec. = seconds

Equations

$$\delta = \frac{V}{\omega} T \qquad \Delta = D \frac{V}{\Omega}\sqrt{1 + \frac{m}{M}}$$

$$n = \frac{V\omega}{g} T = \frac{V\Omega}{g} L$$

$$\omega = \sqrt{\frac{k}{m}} \qquad \Omega = \sqrt{\frac{K}{M}}$$

Values of T, D, and L may be read directly from the charts using any two of the three variables:

$$\lambda = \frac{k}{K} \qquad \mu = \frac{m}{M} \qquad r = \frac{\omega}{\Omega} = \sqrt{\frac{\lambda}{\mu}}$$

The chart for T is particularly useful where the equipment characteristics are fixed and a mount must be selected to reduce the shock transmissibility. On the other hand, the chart for L is convenient where the mount characteristics are fixed and the equipment must be selected to survive the shock loads.

In cases beyond the range of the charts or where higher accuracy is needed, the functions may be computed from the formulas:

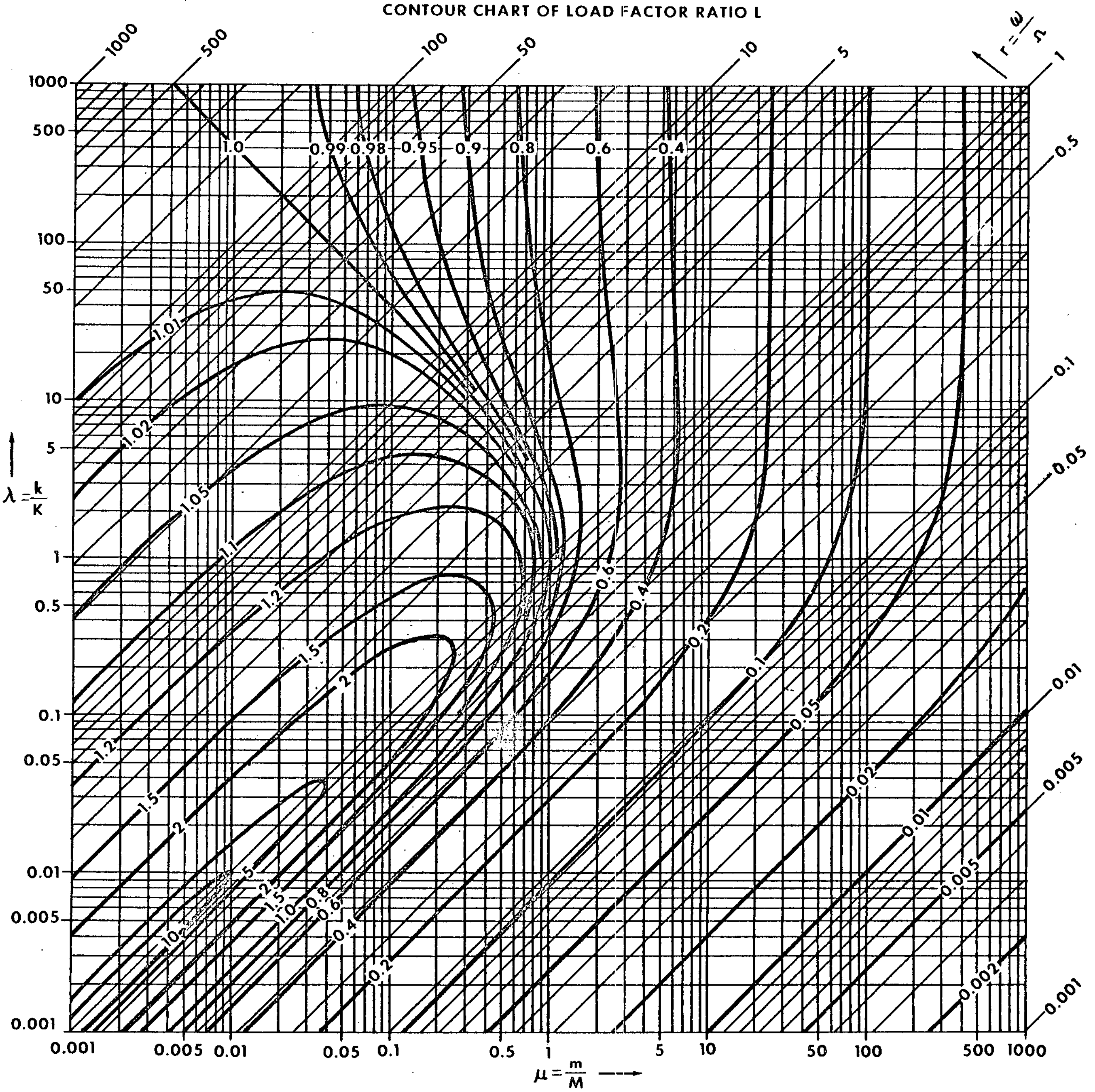
$$D = \frac{r(1+r)+\lambda}{\sqrt{r^2+\lambda}\ \sqrt{(r+1)^2+\lambda}} \qquad L = \frac{r}{\sqrt{(r-1)^2+\lambda}}$$

$$T = \frac{1}{\sqrt{(r-1)^2+\lambda}}$$

Numerical Example

g = 386.1 in. sec.$^{-2}$
k = 1500 lb. in.$^{-1}$
K = 500 lb. in.$^{-1}$
m = 4.00 lb. mass = 0.0104 lb. sec.^{2} in.$^{-1}$
M = 10.00 lb. mass = 0.0259 lb. sec.^{2} in.$^{-1}$
V = 20.0 in. sec.$^{-1}$

CONTOUR CHART OF LOAD FACTOR RATIO L



$$\omega = \sqrt{\frac{k}{m}} = 380 \text{ rad. sec.}^{-1} = 60.5 \text{ cps.}$$

$$\Omega = \sqrt{\frac{K}{M}} = 139 \text{ rad. sec.}^{-1} = 22.1 \text{ cps.}$$

$$\lambda = \frac{k}{K} = 3.00 \quad \mu = \frac{m}{M} = 0.400 \quad r = \frac{\omega}{\Omega} = 2.73$$

$$\frac{V}{\omega} = 0.0525 \text{ in.} \quad \frac{V}{\Omega} = \sqrt{1 + \frac{m}{M}} = 0.170 \text{ in.}$$

$$\frac{V\omega}{g} = 19.7 \quad \frac{V\Omega}{g} = 7.20$$

Using any two of the three coordinates; λ, μ, and r, the values read from the charts are:

T = 0.41 D = 0.99 L = 1.12

so that

$$\delta = \frac{V}{\omega} T = 0.022 \text{ in. } \Delta = D \frac{V}{\Omega}\sqrt{1 + \frac{m}{M}} = 0.17 \text{ in.}$$

$$n = \frac{V\omega}{g} T = \frac{V\Omega}{g} L = 8.1 \text{ (g's)}$$

Acknowledgment:

Two of these charts are taken from the author's paper entitled: "Velocity Shock Transmission in Two Degree Series Mechanical Systems" which appeared in Supplement to Shock and Vibration Bulletin No. 23 (unclassified), published by the Office of the Secretary of Defense, Research and Development, Washington, D. C., June, 1956. The original paper contains the mathematical derivations which are too lengthy to include here.

CONTOUR CHART OF SHOCK TRANSMISSIBILITY T