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SUBJECT: A TEST OF THE AGE THEORY (PART I)

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ABSTRACT

The Selengut-Goertzel and the P_1-approximations to the Boltzmann equation for the neutron slowing down process in a medium of finite extent are rewritten as integral equations over Placzek functions. In this form, the equations can be solved by iteration; also, in the asymptotic case, they permit the corresponding age equations to be read off directly. The first order non-asymptotic deviations of the flux from the age theoretical expressions are calculated. Their effect on the critical size of a reactor which consists of a material with many closely spaced resonances will be given in a later report.

DISCLAIMER

A TEST OF THE AGE THEORY

PART I

1. Because of the approximations made in the derivation of the age theory it seems unlikely that this method would give accurate results for media with strong and closely spaced resonances. Furthermore, there has been some argument in the past as to the most appropriate expression for the slowing down density in this case. In order to assess the value of the age approximation for reactor analysis, we propose to carry out criticality calculations, employing an approximation to the Boltzmann equation which is not as sweeping as age theory on one hand, and on the other the age theoretical expression which can be derived from the more rigorous theory by further approximations.

2. This comparison is greatly facilitated if the Boltzmann equation for a capturing medium is first rewritten in such a way that it indicates directly how the effects of sources and sinks on the neutron flux are propagated upward on the lethargy scale.[1,2) For an infinite homogeneous medium with uniformly distributed isotropic sources the Boltzmann equation is[3)

$$F(u) = \int_0^u du' \, K_0(u-u') \, \frac{\Sigma_s(u')}{\Sigma(u')} \, F(u') + S(u) , \tag{1}$$

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1) Weinberg, A. M., and E. P. Wigner, Theory of Neutron Chain Reactions, Chs. 10 and 11, Chicago U. Press, 1958. Our treatment of the infinite medium case, while somewhat more elaborate, follows the presentation in this book quite closely.

2) Corngold, N., Proc. Phys. Soc. A 70, 793 (1957).

3) Marshak, R. E., Rev. Mod. Phys. 19, 186 (1947).

where $\frac{\Sigma_s(u)}{\Sigma(u)} = c(u)$ is the average number of secondaries; $F(u) = \Sigma(u)\,\phi(u)$ the collision density; $S(u)$ the source density; and

$$(2)\qquad K_0(u) = \begin{cases} \alpha^{-1} e^{-u}, & 0 \le u \le \xi \\ 0, & \xi < u < \infty \end{cases}; \qquad \alpha = 4A/(A+1)^2, \quad \xi = \ln\frac{1}{1-\alpha}.$$

It is convenient to separate the contributions of the collided and the virgin neutrons to the collision density:

$$(3)\qquad F(u) = \psi(u) + S(u).$$

Then eq. (1) takes the form

$$(1a)\qquad \psi(u) = \int_0^u du'\, K_0(u-u')\,\psi(u') + \int_0^u du'\, K_0(u-u')\left[-(1-c(u'))\psi(u') + c(u')\,S(u')\right].$$

This equation will be solved in two steps. First we assume a non-capturing medium, $c=1$, and a δ-function source in energy at $u=u''$; then (1a) becomes

$$(4)\qquad X_0(u-u'') = \int_0^u du'\, K_0(u-u')\, X_0(u'-u'') + K_0(u-u'').$$

The argument of X_0 in this equation was written in the displacement form to indicate explicitly that in a medium without capture the value of the collision density depends only on the distance in lethargy from the source, and not on the absolute value of the lethargy.

$X_0(u-u'')$ can be calculated from the integral equation (4) by a series of successive elementary integrations.[4] It is found to vary quite rapidly in the interval directly above the source lethargy u'', but then it smoothes out very quickly, assuming its constant asymptotic value $X_0 \sim \frac{1}{\xi_0} = \left[1 + \alpha^{-1}(1-\alpha)\ln(1-\alpha)\right]^{-1}$ about 3ξ above source lethargy.

The Placzek function expresses the effect of a δ-function disturbance on the collision density at higher lethargies. The effect of sources or sinks which have

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4) Placzek, G., Phys. Rev. 69, 423 (1946).

a continuous distribution in lethargy, as in equation (1a), can therefore be written as a superposition of solutions of (4), so that $\overline{X}_0(u-u'')$ is the Green's function of the integral equation (1a). There, at $u=u''$ the effective loss is $(1-c(u''))\,\psi(u'')$, the effective gain $c(u'')\,S(u'')$; so if we multiply (4) by the difference of the two and integrate over u'', we find by comparison with (1a) that

$$(5) \qquad \psi(u) = -\int_0^u du''\,\overline{X}_0(u-u'')\,(1-c(u''))\,\psi(u'') + \int_0^u du''\,\overline{X}_0(u-u'')\,c(u'')\,S(u'') \,.$$

This is again an integral equation for $\psi(u)$. It is, however, superior for computational purposes to (1a), for there the kernel vanished in most of the interval of integration, whereas in (5) the kernel is non-zero, and rather constant over most of the range, so that all values of $\psi(u'')$ between 0 and u contribute almost evenly to the integral: therefore, the form (5) is particularly suited for evaluation by iterative techniques.

The age equation is obtained from (5) if $\overline{X}_0$ is approximated throughout its entire range of integration by its constant asymptotic value. Then the integral equation (5) can be differentiated, yielding a differential equation:

$$(6) \qquad \frac{d\psi_A(u)}{du} = -\frac{1}{\xi_0}(1-c(u))\,\psi_A(u) + \frac{1}{\xi_0}\,c(u)\,S(u) \,,$$

and for the collision density

$$(6a) \qquad \frac{dq_A}{du} + (1-c(u))\,F_A(u) = S(u)$$

or

$$(7) \qquad \frac{dq_A}{du} + \Sigma_a(u)\,\phi_A(u) = S(u)$$

with a slowing down density

$$(7a) \qquad q_A(u) = \xi_0\,\psi_A(u) = \xi_0\,(\Sigma(u)\,\phi_A(u) - S(u)) \,.$$

It is instructive to compare expressions (7) and (7a) with another one which is often seen in the literature, and which we shall call the Fermi age equation:

$$\frac{d}{du}\left(\xi\Sigma_s\phi\right) + \Sigma_a\phi = S \; . \tag{8}$$

This can be derived in the following way: in an equilibrium state, the number of neutrons removed at lethargy u by scattering and absorption, $(\Sigma\phi)_u$, must be equal to the neutrons born at that lethargy, $(S)_u$, and those scattered at lethargy $u-\xi$, $(\Sigma_s\phi)_{u-\xi}$, where ξ is the average lethargy gain per collision. The balance equation

$$(\Sigma\phi)_u - (\Sigma_s\phi)_{u-\xi} - (S)_u = 0$$

can be expanded about u, up to first order in ξ, to yield eq. (8); if, on the other hand, the expansion is performed about $u-\xi$, we find (7) and (7a). Our more elaborate derivation indicates, however, that although (7) and (8) are equal to first order in ξ, the form (7) is to be preferred.

It is also possible to write down a differential equation for the first-order deviation $\delta\psi(u)$ of the age solution $\psi_A(u)$ from the exact $\psi(u)$. With $\delta\overline{X}_0 = \overline{X}_0 - \frac{1}{\xi_0}$ we find from (5)

$$\begin{aligned}\delta\psi(u) &= -\frac{1}{\xi_0}\int_0^u du'\,(1-c(u'))\,\delta\psi(u') \\ &\quad + \int_0^u du'\,\delta\overline{X}_0(u-u')\Big[-(1-c(u'))\,\psi_A(u') + c(u')\,S(u')\Big],\end{aligned} \tag{9}$$

where a term in $\delta\overline{X}_0\cdot\delta\psi$ has been neglected. The last integral is completely known:

$$T(u) = \int_0^u du'\,\delta\overline{X}_0(u-u')\Big[-(1-c(u'))\psi_A(u') + c(u')\,S(u')\Big] = \xi_0\int_0^u du'\,\delta\overline{X}_0(u-u')\,\frac{d}{du'}\psi_A(u'). \tag{10}$$

Differentiating (9), we obtain an age equation for $\delta\psi(u)$ which is very similar to (6):

$$\frac{d}{du}\left(\xi_0\,\delta\psi(u)\right) + (1-c(u))\,\delta\psi(u) = \xi_0\,\frac{dT(u)}{du}\;, \tag{11}$$

which indicates that a good approximation to $\psi(u)$ will be given by the solution of the equation

$$\frac{d}{du}\left(\xi_0\psi(u)\right) + (1-c(u))\,\psi(u) = c(u)\,S(u) + \xi_0\,\frac{d}{du}\int_0^u du'\,\delta\overline{X}_0(u-u')\,\xi_0\,\frac{d}{du'}\psi_A(u'). \tag{12}$$

3. Next, we turn to the problem of a finite bare reactor with spacially constant cross sections. If it consists of atoms of one type only, the P_1-approximation to the lethargy part of the Boltzmann equation reads

$$
(13)\qquad \left.\begin{aligned}
\frac{iB}{\Sigma(u)}\, G(u) &= \int_0^u du'\, K_0(u-u')\, \frac{\Sigma_s(u')}{\Sigma(u')}\, F(u') - F(u) + S(u)\,, \\
\frac{1}{3}\frac{iB}{\Sigma(u)}\, F(u) &= \int_0^u du'\, K_1(u-u')\, \frac{\bar{\mu}\,\Sigma_s(u')}{\Sigma(u')}\, G(u') - G(u)\,,
\end{aligned}\right\}
$$

with $F(u) = \Sigma(u)\,\phi(u)$, $G(u) = \Sigma(u)\,J(u)$, $\bar{\mu} = \frac{2}{3A}$, and

$$
k_1(u) = \begin{cases} \frac{1}{\alpha\bar{\mu}}\left(\frac{A+1}{2}\, e^{-\frac{3}{2}u} - \frac{A-1}{2}\, e^{-\frac{1}{2}u}\right), & u = 0..\xi\,; \\ 0 & u > \xi\,. \end{cases}
$$

Equations (13) will be treated rigorously in the next paragraph; for the present, we will let the first equation of (13) stand, but in the second equation we set approximately $\frac{\bar{\mu}\,\Sigma_s(u')}{\Sigma(u')}\,G(u') = \frac{\bar{\mu}\,\Sigma_s(u)}{\Sigma(u)}\,G(u)$. This is the Selengut-Goertzel approximation; the integral over k_1 can now be evaluated, and one obtains Fick's relation between the flux and the current:

$$
(14)\qquad \frac{G(u)}{\Sigma(u)} = -iBD\,\frac{F(u)}{\Sigma(u)}
$$

with

$$
D = \frac{1}{3(\Sigma - \bar{\mu}\,\Sigma_s)}
$$

In this approximation, it will turn out that we shall be lead back to eqs. (5) and (6a), with an appropriate expression for $\epsilon(u)$, the average number of secondaries after a collision that are available for the next collision.

We can eliminate the current now, so that (13) goes over into

$$
(15)\qquad -B^2\,\frac{F(u)}{\Sigma(u)} = \int_0^u du'\, K_0(u-u')\, \frac{\Sigma_s(u')}{\Sigma(u')}\, F(u') - F(u) + S(u)\,.
$$

Again, it is convenient to write $F(u)$ as a sum of two terms:

$$
(16)\qquad F(u) = P(u)\left[\psi(u) + S(u)\right],
$$

where $P(u)$, the non-leakage probability, is found by direct substitution into (15) to be

(17) $$P(u) = \frac{1}{\frac{D(u)}{\Sigma(u)} B^2 + 1} ,$$

while $\phi(u)$ is seen to satisfy our fundamental integral equations (1a) and (5), with

(18) $$c(u) = \frac{\Sigma_s(u)}{\Sigma(u)} P(u) .$$

Substituting this $c(u)$ into the age equation (6a), we obtain

(19) $$\frac{dq}{du} + B^2 D(u)\,\phi(u) + \Sigma_a(u)\,\phi(u) = S(u) ,$$

with

(19a) $$q(u) = \xi_0 \left[\left(\Sigma(u) + B^2 D(u) \right) \phi(u) - S(u) \right] .$$

This expression for $q(u)$ is to be compared with the Fermi slowing down density $q(u) = \xi \Sigma_s \phi$. (19a) prescribes that instead of just the scattering cross section, we have to use the total cross section, plus a leakage term, but minus the effect of the virgin neutrons.

Let us treat now the case of a mixture of a light (I) and an infinitely heavy (II) scatterer and absorber. If Σ denotes the combined total cross section, the lethargy part of the Boltzmann equation becomes, in our approximation,

(17) $$F(u) + B^2 D(u) \frac{F(u)}{\Sigma(u)} = \int_{u-\xi^{I}}^{u} du'\, K_0^{I}(u-u') \frac{\Sigma_s^{I}(u')}{\Sigma(u')} F(u') + \int_{u-\xi^{II}}^{u} du'\, K_0^{II}(u-u') \frac{\Sigma_s^{II}(u')}{\Sigma(u')} F(u') + S(u) ,$$

with

(17a) $$D(u) = \frac{1}{3\left(\Sigma - \frac{2}{3A^{I}} \Sigma_s^{I}\right)} .$$

Since for an infinitely heavy nucleus $\alpha^{II} = 0$, $\xi^{II} \approx \alpha^{II}$, the second integral can be evaluated immediately and gives $\frac{\Sigma_s^{II}(u)}{\Sigma(u)} F(u)$. Eq. (17) can again be brought into the form (1a), where now

(18) $$c(u) = \frac{\Sigma_s^{I}}{\Sigma - \Sigma_s^{II} + B^2 D} .$$

The age equation (6a) becomes

(19) $$\frac{dq}{du} = B^2 D(u)\phi(u) + \left[\Sigma_a^{I}(u) + \Sigma_a^{II}(u)\right]\phi(u) = S(u) ,$$

with

(19a) $$q = \xi_0^{I}\left[(\Sigma - \Sigma_s^{II} + B^2 D)\phi - S\right],$$

contrasted with the Fermi slowing down density $q = \xi^{I}\Sigma^{I}\phi$. Since no slowing down takes place in scattering on the infinitely heavy nucleus, the scattering cross section Σ_s^{II} does not appear directly in either expression for the slowing down density.

4. In the consistent P_1-approximation, no approximations at all are made in the system of integral equations (13). Again we shall try to transform the equations into equivalent integral equations with Placzek-function kernels.

As a first step, we perform a linear transformation which separates out the virgin neutrons, and at the same time casts the system of equations into a standard form:

(20) $$\left.\begin{aligned} F &= P_1\psi_1 - 3P_2\psi_2 + P_1 S \\ iG &= P_2\psi_1 + P_1\psi_2 + P_2 S \end{aligned}\right\} ,$$

with

(20a) $$P_1 = \frac{1}{1 + \frac{1}{3}\frac{B^2}{\Sigma^2}} , \qquad P_2 = \frac{\frac{1}{3}\frac{B}{\Sigma}}{1 + \frac{1}{3}\frac{B^2}{\Sigma^2}} .$$

The system of integral equations (13) reads now

(21) $$\left.\begin{aligned} \psi_1 &= \int_0^u du'\, K_0(u-u')\,\frac{\Sigma_s(u')}{\Sigma(u')}\cdot(P_1\psi_1 - 3P_2\psi_2 + P_1 S) \\ \psi_2 &= \int_0^u du'\, K_1(u-u')\,\frac{\bar{\mu}\,\Sigma_s(u')}{\Sigma(u')}\cdot(P_2\psi_1 + P_1\psi_2 + P_2 S) \end{aligned}\right\} .$$

The first equation, with a kernel K_0, can be transformed as before by the use of the Placzek function χ_0. In an entirely analogous way we define a Placzek function χ_1 for the kernel K_1 as the solution of the equation

(22) $$\hat{X}_1(u) = \int_0^u du'\, K_1(u-u')\, \hat{X}_1(u') + k_1(u).$$

As we shall show in the Appendix, it is important to include the factor $1/\bar{\mu}$ in the kernel K_1, for only then will $\hat{X}_1$ be asymptotically constant:

(22a) $$\hat{X}_1(u) \sim \frac{1}{\xi_1} = \left[\int_0^\infty du\, u\, K_1(u) \right]^{-1} .$$

Using these functions and employing the same argument that led to eq. (5), we obtain the following system of integral equations which are equivalent to (21):

(23) $$\begin{aligned} \psi_1(u) &= - \int_0^u du'\, \hat{X}_0(u-u') \left[\psi_1 - \frac{\Sigma_s}{\Sigma}(P_1\psi_1 - 3P_2\psi_2) \right] + \int_0^u du'\, \hat{X}_0(u-u') \frac{\Sigma_s}{\Sigma} P_1 S, \\ \psi_2(u) &= - \int_0^u du'\, \hat{X}_1(u-u') \left[\psi_2 - \frac{\bar{\mu}\Sigma_s}{\Sigma}(P_2\psi_1 + P_1\psi_2) \right] + \int_0^u du'\, \hat{X}_1(u-u') \frac{\bar{\mu}\Sigma_s}{\Sigma} P_2 S. \end{aligned}$$

These equations can now be solved by iteration. Finally, an improved age equation is obtained if both $\hat{X}_0$ and $\hat{X}_1$ are approximated by their asymptotic values over the whole range of integration; (23) can then be differentiated, and a system of first-order differential equations results.

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Appendix

The Placzek function belonging to the kernel $K_0(u)$ is constant for large values of the argument:

$$\Xi_0(u) \sim \xi_0^{-1} = \left[\int_0^\infty du\, u\, K_0(u) \right]^{-1} = \langle u \rangle^{-1} .$$

We ask for the conditions that a kernel $K_m(u)$ must satisfy so that $\Xi_m(u) \sim$ const., and endeavor to find the value of that constant.

$\Xi_m(u)$ satisfies the integral equation

$$\Xi_m(u) = \int_0^u du'\, K_m(u-u')\, \Xi_m(u') + K_m(u) .$$

Taking Laplace transforms, we get

$$\tilde{\Xi}_m = \tilde{K}_m \tilde{\Xi}_m + \tilde{K}_m , \qquad \tilde{\Xi}_m = \frac{\tilde{K}_m}{1 - \tilde{K}_m} ,$$

and reinverting

$$\Xi_m(u) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dv\, e^{uv} \frac{\tilde{K}_m(v)}{1 - \tilde{K}_m(v)} .$$

The asymptotic behavior of $\Xi_m(u)$ is determined by that root of the equation $1 - \tilde{K}_m(v)$ which has the largest real part. This must be zero if $\Xi_m(u)$ is to be asymptotically constant. We therefore expand

$$1 - \tilde{K}_m(v) = 1 - \int_0^\infty du\, K_m(u) + \frac{v}{1!} \int_0^\infty du\, u\, K_m(u) - + \ldots ,$$

and see that we must have

$$\int_0^\infty du\, K_m(u) = 1 ;$$

then

$$\Xi_m(u) \sim \frac{1}{\xi_m} = \left[\int_0^\infty du\, u\, K_m(u) \right]^{-1} .$$

In particular, for

$$K_1(u) = \begin{cases} \dfrac{1}{\alpha \mu} \left(\dfrac{A+1}{2} e^{-\frac{3}{2}u} - \dfrac{A-1}{2} e^{-\frac{1}{2}u} \right) , & u = 0 \ldots \xi \\ 0 , & u > \xi \end{cases}$$

we have $\int_0^{u} du\, K_1(u) = 1$, and

$$\xi_1 = \int_0^\infty du\, u\, K_1(u) = -A^2 + \frac{5}{3} + \frac{1}{4}\left(A^3 - 3A + 2\right) \ln\left(\frac{A-1}{A+1}\right)^2 .$$

For hydrogen, $A = 1$, the Placzek function $X_1(u)$ is constant for all values of u.
Graphs of $X_1(u)$ for different values of A will be given in a forthcoming report.

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