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TRANSIENT HEAT TRANSFER IN REACTOR  
COOLANT CHANNELS

By  
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October 31, 1957

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**Transient Heat Transfer in Reactor Coolant Channels**

by

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Task IX of Contract AT(30-3)-187

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## ABSTRACT

4 An analysis is presented of the transient behavior of a generalized coolant channel neglecting temperature dependent reactivity changes. The analysis is applicable to forced convection cooling of heterogeneous reactor fuel elements or electrically heated simulation thereof. Derivations are given for cases of variation of coolant inlet temperature and of heat generation. An approximation is also developed applicable to thin fuel elements. From this, solutions are obtained for cases of impulsive, step, linear, and step-exponential variations of inlet temperature, and, of impulsive and uniform variations of heat generation. The solutions presented will be of use during preliminary stages of design of new heterogeneous reactor concepts (when the use of computing machines may not be warranted), and, in the design and interpretation of transient experiments simulating reactor fuel channels. ? *reference / part*

## 1. Introduction

This report describes the results of a general analysis of heat transfer transients in heat generating coolant channels initiated by time variations of coolant inlet temperature and of heat generation rate. The general heat generating channel considered can be interpreted as a forced convection fluid cooled fuel channel in a heterogeneous nuclear reactor or as an electrical heat generating simulation of such a channel. It was for application to the latter that this analysis was originally undertaken, but it was soon realized that with the various parameters given general interpretation the results obtained would also be of use to the nuclear engineer even though temperature dependent reactivity changes are neglected -- especially during the preliminary or conceptual stages of design when the use of computing machines may not be warranted.

During the early stages of design the nuclear engineer is faced with many problems concerning transient responses of fluid and fuel temperatures to possible variation of inlet temperature and power level. An example of the former occurs frequently with many power reactors as result of sudden changes in the power demand at the electrical generating station. Examples of the latter are more common and include accidental power excursions, scrams, and normal changes of power level during operation. The engineer is usually forced to use the results of crude "lump" analyses for the preliminary examination of such transients and the accuracy of such estimates are always in doubt. The analysis described in this report will cover many of these possible cases with sufficient accuracy when temperature dependent reactivity changes can be ignored so that improved estimates of transients can be made relatively quickly without the need of computing machines.

For the simulation of reactor transients with electrical heat generation in simulated coolant channels the results presented can be used to plan experiments with respect to similitude; analyze the measured data with respect to temperature-time relation for the prototype; and indicate methods of extracting from experimental data effective values of fuel element thickness, heat transfer coefficients, and heat transfer areas to be later used with machine computations of complicated reactor transients which include temperature dependent reactivity changes.

## 2. A Generalized Coolant Channel

A typical reactor or simulated reactor coolant channel is illustrated in Figure 1. The simple configuration shown will apply to a large variety of actual configurations when the geometrical quantities (i.e. A, b,  $\ell$ , and S) are suitably interpreted. Applied to a reactor, however, the generalized coolant channel neglects the possible effect of cladding on heat transfer from the fuel element. In most cases this will not introduce serious errors in the final results obtained if appropriate average values of the important physical properties are used. Further generalization is possible if the usual steady-state convection heat transfer equation is assumed to apply, and the appropriate differential equations are written in convenient dimensionless form.

### The Differential Equation

The one dimensional heat conduction equation is assumed for the solid as follows:

$$\frac{\partial^2 T}{\partial y^2} + \frac{Q}{bAk} = \frac{1}{\alpha} \frac{\partial T}{\partial \tau} \quad (2.1)$$

The term  $Q/bA$  represents the rate of heat generated in the solid per unit volume,  $k$  is the thermal conductivity of the solid, and  $\alpha$  its thermal diffusivity. With the usual steady-state convection heat transfer equation assumed applicable, the boundary condition for  $y = b$  is

$$-k \frac{\partial T(b, x, \tau)}{\partial y} = h[T_s(x, \tau) - t(x, \tau)] \quad (2.2)$$

where  $T_s = T(b, x, \tau)$ , the temperature of the surface of the solid in contact with the fluid

$h$  = the heat transfer coefficient, assumed constant

The boundary condition for  $y = 0$  is

$$\frac{\partial T(0, x, \tau)}{\partial y} = 0 \quad (2.3)$$

The initial condition for equation (2.1) is determined by specifying that for  $\tau \leq 0$ , both  $T$  and  $Q$  are independent of time.

Then from equation (2.1),

$$\frac{d^2 T(y,x,0)}{dy^2} = - \frac{Q_0(x)}{bAk} \quad (2.4)$$

where  $Q_0 = Q(x,0)$

Integration of equation (2.4) results in the familiar steady-state temperature distribution which can be used as an initial condition for equation (2.1).

$$T(y,x,0) = T_s(x,0) + \frac{Q_0(x)}{2bAk} (b^2 - y^2) \quad (2.5)$$

A heat balance on a differential element of the coolant gives

$$\frac{\partial t}{\partial x} + \frac{1}{V} \frac{\partial t}{\partial \tau} = \frac{hA}{C_p W} (T_s - t) \quad (2.6)$$

with the boundary condition

$$t(0,\tau) = t_0(\tau) \quad (\text{inlet temperature}) \quad (2.7)$$

and the initial condition

$$t(x,0) = t_0(0) + \frac{1}{C_p W} \int_0^x Q_0(\lambda) d\lambda \quad (2.8)$$

The initial condition, equations (2.5) and (2.8), are related by

$$Q_0(x) = hA[T_s(x,0) - t(x,0)] \quad (2.9)$$

$$= C_p W \frac{\partial t(x,0)}{\partial x} \quad (2.10)$$

## 2.1. The Differential Equations in Dimensionless Form

For the purposes of generalization, a dimensionless solid temperature  $\psi$ , and dimensionless coolant temperature  $\xi$  are introduced. These are defined as follows:

$$\psi = \frac{hA}{Q_T} [T(y,x,\tau) - T(y,x,0)] \quad (2.11)$$

$$\xi = \frac{hA}{Q_r} [t(x, \tau) - t(x, 0)] \quad (2.12)$$

where  $Q_r$  is a constant reference value of the heat generation rate per unit length. The value assigned to  $Q_r$  will depend on the particular problem to be solved. In order to change to dimensionless form the following relations are needed:

$$T(y, x, \tau) = \frac{Q_r}{hA} \psi(y, x, \tau) + T(y, x, 0) \quad (2.13)$$

$$t(x, \tau) = \frac{Q_r}{hA} \xi(y, x, \tau) + t(x, 0) \quad (2.14)$$

$$\frac{\partial T}{\partial y} = \frac{Q_r}{hA} \frac{\partial \psi}{\partial y} - \frac{Q_o(x)}{bAk} y \quad (2.15)$$

(using equation (2.5))

$$\frac{\partial^2 T}{\partial y^2} = \frac{Q_r}{hA} \frac{\partial^2 \psi}{\partial y^2} - \frac{Q_o(x)}{bAk} \quad (2.16)$$

$$\frac{\partial t}{\partial x} = \frac{Q_r}{hA} \frac{\partial \xi}{\partial x} + \frac{Q_o(x)}{C_r W} \quad (2.17)$$

(using equation (2.10))

$$\frac{\partial T}{\partial \tau} = \frac{Q_r}{hA} \frac{\partial \psi}{\partial \tau} \quad (2.18)$$

$$\frac{\partial t}{\partial \tau} = \frac{Q_r}{hA} \frac{\partial \xi}{\partial \tau} \quad (2.19)$$

$$T_s - t = \frac{Q_r}{hA} [\psi_s - \xi] + \frac{Q_o}{hA} \quad (2.20)$$

(using equation (2.9))

where  $\psi_s = \psi(b, x, \tau)$ , a dimensionless surface temperature

Substitution of equations (2.13) to (2.20) into equations (2.1) and (2.6) results in

$$\frac{bk}{h} \frac{\partial^2 \psi}{\partial y^2} + \frac{Q - Q_o}{Q_r} = \frac{bk}{ah} \frac{\partial \psi}{\partial \tau} \quad (2.21)$$

and

$$\frac{C_r W}{hA} \frac{\partial \xi}{\partial x} + \frac{C_r W}{VhA} \frac{\partial \xi}{\partial \tau} = (\psi_s - \xi) \quad (2.22)$$

The boundary condition, equation (2.2) becomes

$$\frac{k}{h} \frac{\partial \psi_s}{\partial y} = - (\psi_s - \xi) \quad (2.23)$$

Examination of these equations suggests convenient forms for dimensionless solid thickness, channel length, and time. The following were chosen:

$$B = \sqrt{\frac{hb}{k}} \quad , \quad \text{dimensionless solid thickness} \quad (2.24)$$

$$L = \frac{hA\ell}{C_f W} \quad , \quad \text{dimensionless channel length} \quad (2.25)$$

$$\begin{aligned} \theta &= \frac{ah}{bk} \tau \\ &= \frac{h}{bC_{m^o_m}} \tau \quad , \quad \text{dimensionless time} \quad (2.26) \end{aligned}$$

In addition substitution of (2.25) and (2.27) into equation (2.22) introduces the parameter

$$\begin{aligned} \sigma &= \frac{C_f W}{\sqrt{Ab} C_{m^o_m}} \\ &= \frac{C_f \rho_f (S)}{C_{m^o_m} (Ab)} \quad (2.27) \end{aligned}$$

which is the ratio of the total heat storage capacity of the fluid contained in the coolant channel to that of the solid. With these dimensionless quantities, equations (2.21) and (2.22) can be written as

$$\frac{\partial^2 \psi}{\partial y^2} + v(x, \theta) = \frac{\partial \psi}{\partial \theta} \quad (2.28)$$

and 
$$\frac{\partial \xi}{\partial x} + \sigma \frac{\partial \xi}{\partial \theta} = (\psi_s - \xi) \quad (2.29)$$

where

$$0 \leq y \leq B,$$

$$0 \leq x \leq L,$$

and

$$v(x, \theta) = \frac{Q(x, \theta) - Q_0(x)}{Q_T} \quad (2.31)$$

The boundary conditions become:

for  $\psi(y, x, \theta)$

$$\frac{\partial \psi(0, x, \theta)}{\partial y} = 0 \quad (2.32)$$

$$\frac{\partial \psi(B, x, \theta)}{\partial y} = -B(\psi_s - \xi) \quad (2.33)$$

for  $\xi(x, \theta)$

$$\xi(0, \theta) = \xi_0(\theta) \quad (\text{dimensionless inlet temperature}) \quad (2.34)$$

and the initial conditions are

$$\psi(y, x, 0) = \xi(x, 0) = 0 \quad (2.35)$$

## 2.2. Representation of Variations Initiating Transients

The transient behavior of the generalized reactor coolant channel represented by the above differential equations are initiated by time variations of either coolant inlet temperature, heat generation in the fuel element, or combinations of both. These variations are represented in dimensional form by the functions  $\xi_0(\theta)$  and  $v(x, \theta)$ . Some clarification of these representations is warranted at this point in the presentation especially with respect to the value assigned to the reference heat generation rate,  $Q_r$ .

For inlet temperature variations,  $Q_r$  is chosen in terms of a reference inlet temperature different from the initial steady state value. For example, suppose

$$t_0(\tau) = t_0(0) + \Delta t(1 - e^{-a\theta})$$

where  $\Delta t$ , and "a" are constants.

This represents a variation during which the inlet temperature of the coolant rises (or decreases) in temperature from its initial steady value  $t_0(0)$  to a new value,  $t_0(0) + \Delta t$ , at an exponentially decreasing rate. From equation (2.12)

$$\begin{aligned}\xi_0 &= \frac{hA}{Q_r} [t_0(\tau) - t_0(0)] \\ &= \frac{hA\Delta t}{Q_r} (1 - e^{-a\theta})\end{aligned}$$

and  $Q_r$  is chosen so that

$$Q_r = hA\Delta t$$

Thus

$$\xi_0(\theta) = 1 - e^{-a\theta}$$

For heat generation variations, changes in both power level and heat flux distribution must be considered. For convenience we define the cases of (1) uniform variations, (2) simple (or separable) non-uniform variations, and (3) compound (or non separable) non-uniform variations.

We define a heat generation variation as uniform when it can be represented by a function of the form

$$Q(x, \theta) = \Delta Q(\theta) + Q_0(x)$$

Thus, for this case equation (2.31) gives

$$v = \frac{\Delta Q(\theta)}{Q_r}$$

and is a function of  $\theta$  only although the actual flux distribution is not necessarily uniform. The reference heat generation rate is most conveniently chosen as either the maximum or average value of the overall heat generation change,  $\Delta Q$ . This case should apply to many reactor flow channels with nonuniform flux distribution when the overall power level change is not too great.

We define a heat generation variation as simple non-uniform when it can be represented by a function of the form

$$Q(x, \theta) = Q_1(\theta)v_2(x)$$

For this case equation (2.31) gives

$$\begin{aligned}v &= \frac{Q_1(\theta) - Q_1(0)}{Q_r} v_2(x) \\ &= \frac{\Delta Q_1(\theta)v_2(x)}{Q_r}\end{aligned}$$

and the reference heat generation rate is chosen with respect to a maximum or average value of the variation  $\Delta Q_1$ .

A simple non-uniform variation case applicable to many reactors is represented by

$$v_2(x) = \sin \frac{\pi}{L}x$$

with  $\Delta Q_1$  describing the variation of maximum or average flux with time. Another example is the important problem of transients initiated by a reactor scram. An approximation for this case is represented by

$$\Delta Q_1(\theta) = - (1 - e^{-a\theta})Q_1(0)$$

where  $1/a$  represents a dimensionless period for the power decay after a scram. Here,  $Q_r$  would be chosen equal to  $-Q_1(0)$ , and  $v_2(x)$  represents the flux distribution before the scram.

We define a heat generation variation as compound non-uniform when it can not be represented by a function in which the dependence on  $\theta$  and  $x$  can be separated as in the previous cases. It appears unlikely that such cases will be of importance, and is mentioned here for completeness only.

### 3. Laplace Transform Solution for the Generalized Coolant Channel

It is relatively simple to obtain solutions of the generalized equations in the Laplace Transform time domain for transients caused by both heat generation and inlet temperature variations. As is usual with such problems, the difficult part of the solution comes about when we attempt to return to the real time domain (inversion).

$$\text{Let } \mathcal{L} \psi(y, x, \tau) = \Psi(y, x, s)$$

$$\mathcal{L} \xi(x, \tau) = \Xi(x, s)$$

$$\mathcal{L} v(x, \tau) = V(x, s)$$

$$\mathcal{L} \xi_0(\tau) = U(s)$$

The transform of equation (2.28) is

$$\frac{d^2 \Psi}{dy^2} + V = s\Psi \quad (3.1)$$

The transform of equation (2.29) is

$$\frac{d\xi}{dx} + \cos \xi = (\Psi_s - \xi) \quad (3.2)$$

The transformed boundary conditions are

$$\frac{d\Psi}{dy}(0, x, s) = 0 \quad (3.3)$$

$$\frac{d\Psi}{dy}(B, x, s) = -B(\Psi_s - \xi) \quad (3.4)$$

$$\xi(0, s) = \xi_0(s) \quad (3.5)$$

A solution of equation (3.1), satisfying the first boundary condition is

$$\Psi = C_1 \cosh \sqrt{s} y + \frac{V}{s} \quad (3.6)$$

where  $C_1$  is an arbitrary constant. Differentiation of equation (3.6) with respect to  $y$ , and use of the second boundary condition results in

$$C_1 = \frac{s\xi - V}{s\left(\frac{\sqrt{s}}{B} \sinh B\sqrt{s} + \cosh B\sqrt{s}\right)} \quad (3.7)$$

Substitution for  $C_1$  in equation (3.6) and evaluation at  $y = B$  results in

$$\Psi_s = \frac{(s\xi - V) \cosh B\sqrt{s}}{s\left(\frac{\sqrt{s}}{B} \sinh B\sqrt{s} + \cosh B\sqrt{s}\right)} + \frac{V}{s} \quad (3.8)$$

from which

$$\Psi_s - \xi = (V - s\xi)\phi(B, s) \quad (3.9)$$

where

$$\phi = \frac{\sinh B\sqrt{s}}{s \sinh B\sqrt{s} + B\sqrt{s} \cosh B\sqrt{s}} \quad (3.10)$$

Substitution of equation (3.9) into (3.2) results in

$$\frac{d\mathcal{E}}{dx} + (\sigma + \phi)s\mathcal{E} = V(x,s)\phi \quad (3.11)$$

This is a first order linear differential equation with the integrating factor

$$e^{(\sigma + \phi)sx}$$

such that

$$\frac{d}{dx}[e^{(\sigma + \phi)sx} \mathcal{E}] = V\phi e^{(\sigma + \phi)sx} \quad (3.12)$$

Integration from  $x = 0$ ,  $\mathcal{E} = \mathcal{E}_0(s)$ , to any value of  $x$  and  $\mathcal{E}$  results in

$$\mathcal{E}(x,s) = \mathcal{E}_0(s)G(x,s) + \phi \int_0^x V(\lambda,s)G(x-\lambda,s)d\lambda \quad (3.13)$$

where  $G(x,s) = e^{-(\sigma + \phi)sx} \quad (3.14)$

For the case of uniform variation of heat generation -- i.e.  $v$  a function  $\theta$  only -- equation (3.13) can be integrated to give

$$\mathcal{E}(x,s) = \mathcal{E}_0(s)G(x,s) + V(s) \frac{\phi[1 - G(x,s)]}{(\sigma + \phi)s} \quad (3.13a)$$

Equation (3.13) is a general solution for the transformed generalized fluid temperature. An equation for the transformed generalized solid temperature may also be obtained by combining equations (3.6), (3.7) and (3.13).

Inversion to the real time domain is difficult mainly because of the cumbersome form of the term represented by  $\phi$  (equation (3.10)). For small  $B$ , however,  $\phi$  can be approximated by a simple expression so that inversion of equation (3.13) becomes possible for a variety of inlet fluid temperature and heat generation transients.

### 3.1. Thin Fuel Element Approximation

When the dimensionless solid thickness,  $B$ , is small this corresponds to either a thin heat generating element, or a large solid thermal conductivity. Taking  $B$  equal to zero is equivalent to assuming solid temperatures uniform in the direction of heat flow, and is the assumption upon which the analytical solutions reported in the literature for similar problems are based.<sup>(2, 3, 5, 6)</sup> In one case<sup>(4)</sup> solutions for steady state responses to harmonic variations of inlet temperature were obtained for the opposite extreme of representing the solid as semi-infinite in extent.

In the next section are presented solutions for steady state responses to harmonic variations of both inlet temperature and uniform variation of heat generation for any value of  $B$ . For other transients, however, solutions have been obtained to date only for the case of  $B$  small but not necessarily zero. These solutions are based on the following treatment.

A simple application of l'Hospital's rule to equation (3.10) will show that

$$\lim_{B \rightarrow 0} \phi(B, s) = \frac{1}{1 + s} \quad (3.15)$$

Thus, for the case of  $B = 0$ , the function  $\phi$  can be replaced by  $\frac{1}{1 + s}$  in the preceding transformed equations. Inversion of many forms of equation (3.13) then becomes relatively simple.

An approximate expression for  $\phi(B, s)$  can be obtained by first expanding equation (3.10) into powers of  $B$ , as shown in Appendix "A". The following result is obtained.

$$[\phi(B, s)]^{-1} = (1 + s) + \frac{s}{3}B^2 - \frac{s^2}{45}B^4 + \frac{2s^3}{945}B^6 - \dots \quad (3.16)$$

This series appears to converge quite rapidly for moderate values of  $s$  (a complex variable) and  $B \leq 1$ , although no rigorous mathematical proof of this has been obtained as yet. It can be shown, however, that neglecting  $B^4$  and higher exponent terms is equivalent to the assumption that during a transient the average solid temperature is related to the surface temperature and surface heat flux as in the steady state (parabolic temperature distribution). This is proved in Appendix "B".

Thus we obtain the following approximate form for  $\phi$  when  $B$  is small.

$$\phi(B, s) \approx \frac{1}{1 + (1 + \frac{B^2}{3})s}, \quad B \leq 1 \quad (3.17)$$

Since large values of  $s$  correspond to small values of  $\theta$ , it is expected that use of this approximation will produce errors that may be significant at the very beginning of a transient. The probable magnitude of these errors is not presently known.

We note that the above expression differs from the form for  $B = 0$  (equation (3.15)) only by a real positive constant multiplying the transformed time variable  $s$ . But multiplication of the transformed time variable by a real positive number corresponds to division of the real time variable by the same number according to the following theorem\*

If  $F(s) = \mathcal{L} f(\theta)$ , and if  $n$  is any real positive number

then 
$$F(ns) = \frac{1}{n} \mathcal{L} f\left(\frac{\theta}{n}\right)$$

For  $B \leq 1$  then, we can replace  $\phi$  by its limiting value for  $B = 0$ , change the real time variable by dividing it by  $1 + \frac{B^2}{3}$ , and write equations (3.13) and (3.14) in accordance with the above theorem. This results in the following forms of solution for the transformed generalized fluid temperature for the case of small  $B$ .

$$\mathcal{E}(x, s) = \gamma \mathcal{E}_0(s) G(x, s) + \frac{1}{1 + \frac{B^2}{3}} \int_0^x V(\lambda, s) G(x - \lambda, s) d\lambda \quad (3.18)$$

where

$$G(x, s) = e^{-(\sigma + \frac{1}{1 + \frac{B^2}{3}})sx} \quad (3.19)$$

and

$$\gamma = 1 + \frac{B^2}{3} \quad (3.20)$$

For this case the real time variable corresponding to  $s$  is now

$$\theta = \frac{gh\tau}{bk\gamma} \quad (3.21)$$

---

\* This theorem is easily proved directly from the formal definition of the Laplace Transform.

and the dimensionless channel length has been changed to

$$L = \frac{hA\ell}{C_p W \gamma} \quad (3.22)$$

For the case of uniform variation of heat generation, the above equation becomes

$$\Xi(x,s) = \gamma \Xi_0(s) G(x,s) + \frac{V(s)[1 - G(x,s)]}{(1 + \sigma + \sigma s)s} \quad (3.18a)$$

#### 4. Steady-state Responses to Harmonic Variation

##### 4.1. Inlet Temperature Variations, Fluid Temperature Response

The transformed fluid temperature response to an inlet temperature variation only is given by equation (3.13) as

$$\Xi(x,s) = \Xi_0(s) G(x,s).$$

If the inlet temperature variation is represented by the general harmonic

$$\xi_0(\theta) = e^{i\omega\theta} \quad (4.1.1)$$

where  $\omega$  is its frequency of variation in radians per unit of dimensionless time, then according to Laplace transform theory, the steady state fluid temperature response can be written as follows.

$$\xi(x,\theta) = e^{i\omega\theta} G(x,i\omega) \quad (4.1.2)$$

From equation (3.14)

$$G(x,i\omega) = \exp\left\{-[\sigma + \phi(B,i\omega)]i\omega x\right\} \quad (4.1.3)$$

where  $\phi$  is given by equation (3.10). The term  $\phi(B,i\omega)$  can be represented by

$$\phi(B,i\omega) = K_1(B,\omega) - iK_2(B,\omega) \quad (4.1.4)$$

where  $K_1$  and  $K_2$  are complicated functions of  $B$  and  $\omega$  and are given

in Appendix "C". Substitution of equations (4.1.4) and (4.1.3) into equation (4.1.2) gives

$$\xi_{\infty}(x, \theta) = \exp.(-K_2 \omega x) \exp. \{ [\omega \theta - (\sigma + K_1)x] i \} \quad (4.1.5)$$

Thus, the steady state fluid temperature response to a harmonic variation of inlet temperature, will be reduced in amplitude by the factor

$$v_1 = \exp.(-K_2 \omega x) \quad (4.1.6)$$

and will have a phase lag equal to

$$\eta_1 = (\sigma + K_1)x \quad (4.1.7)$$

For the case of small B, equations (3.15) and (3.18) are used. This results in

$$v_1 = \gamma \exp. \left[ - \frac{\omega^2 x}{1 + \omega^2} \right] \quad (4.1.8)$$

and

$$\eta_1 = \left( \sigma + \frac{1}{1 + \omega^2} \right) x \quad (4.1.9)$$

where the dimensionless distance and time scales are given by equations (3.21) and (3.22). For the case of  $B = 0$  ( $\gamma = 1$ ), these results are equivalent to those reported in Carslaw & Jaeger.<sup>(2)</sup>

#### 4.2. Inlet Temperature Variation, Surface Temperature Response

The transformed surface temperature response to an inlet temperature variation only is obtained by combining equations (3.19) and (3.13). This results in

$$\Psi_s = \Sigma_0 [1 - s\phi(B, s)] G(x, s) \quad (4.2.1)$$

As in the preceding section, the inlet temperature variation is represented by the general harmonic given by equation (4.1.1) and the steady state surface temperature response is then obtained from

$$\psi_{s\infty}(x, \theta) = e^{\omega \theta i} [1 - i\omega \phi(B, i\omega)] G(x, i\omega) \quad (4.2.2)$$

Evaluation is straight forward and results in the following:

$$Y_{s\omega} = v_1 v_2 \exp. [\omega\theta - \eta_1 - \eta_2] i \quad (4.2.3)$$

where

$$v_2 = [(1 - \omega K_2)^2 + \omega^2 K_1^2]^{\frac{1}{2}} \quad (4.2.4)$$

$$\tan \eta_2 = \frac{\omega K_1}{1 - \omega K_2} \quad (4.2.5)$$

and  $v_1$  and  $\eta_1$  are given by equations (4.1.6) and (4.1.7).

#### 4.3. Uniform Heat Generation Variations, Fluid Temperature Response

The transformed fluid temperature response to a uniform heat generation variation only is given by equation (3.13a) as

$$\Xi(x,s) = \frac{V(s)\Phi(B,s)[1 - G(x,s)]}{[\sigma + \Phi(B,s)]s}$$

The procedure to find the steady state response to a harmonic variation of  $v(\theta)$  (i.e.  $v(\theta) = e^{i\omega\theta}$ ) is the same as in the preceding sections, but the complex algebra, although straight forward, is cumbersome. Therefore only the final result is given here.

It is found that

$$E_{\omega}(x,\theta) = v_3 e^{(\omega\theta - \eta_3)} i$$

where

$$v_3 = \frac{\left\{ [K_2 K_3 - (\sigma + K_1) K_4]^2 + [K_2 K_4 + (\sigma + K_1) K_3]^2 \right\}^{\frac{1}{2}}}{\omega [K_2^2 + (\sigma + K_1)^2]}$$

$$\tan \eta_3 = \frac{K_2 K_4 + (\sigma + K_1) K_3}{K_2 K_3 - (\sigma + K_1) K_4}$$

$$K_3 = K_1 (1 - v_1 \cos \eta_1) - K_2 v_1 \sin \eta_1$$

$$K_4 = K_2 (1 - v_1 \cos \eta_1) + K_1 v_1 \sin \eta_1$$

and  $v_1$  and  $\eta_1$  are given by equations (4.1.6) and (4.1.7).

## 5. Transient Responses to Inlet Temperature Variations

In this section solutions for the fluid temperature response to various coolant inlet temperature variations are presented. The solutions are based on the thin fuel element approximation which, according to equation (3.18), are given by inversion of

$$E(x,s) = \gamma \xi_0(s) G(x,s)$$

where the real time variable corresponding to  $s$  is given by

$$\theta = \frac{cht}{bk\gamma},$$

the dimensionless channel length by

$$L = \frac{hA\ell}{C_p W \gamma},$$

and

$$\gamma = 1 + \frac{B^2}{3}$$

### 5.1. Impulsive Variation

An impulsive variation of inlet temperature is represented by the function

$$\begin{aligned} \xi_0(\theta) &= \delta(\theta)^* \\ &= 0 && \text{for } \theta < 0 \\ &= 1 && \text{for } \theta = 1 \\ &= 0 && \text{for } \theta > 0 \end{aligned} \quad (5.1.1)$$

---

\* This function is the Dirac delta function  $\delta(\theta)$  which has the further properties that

$$\int_{-\infty}^{\infty} \delta(\lambda) d\lambda = 1$$

and

$$\int_{-\infty}^{\infty} \delta(\lambda-a) f(\lambda) d\lambda = f(a)$$

This could represent a small slug of hot or cold coolant suddenly appearing at the channel inlet. The response to such a variation is also the usual definition of the inverse transfer function for the system for inlet temperature variations and is useful for obtaining solutions for any variation by application of the convolution integral. This is its main purpose here.

The Laplace transform of the impulsive function is unity. Therefore

$$\Xi(x,s) = \gamma G(x,s), \quad (5.1.2)$$

the transfer function of the system. Inversion can be represented by

$$\xi(x,s) = \gamma g(x,\theta) \quad (5.1.3)$$

where

$$\mathcal{L} g(x,\theta) = G(x,s) \quad (5.1.4)$$

Equation (5.1.3) represents the inverse transfer function.

For the thin element approximation, equation (3.19) gives

$$\begin{aligned} G(x,s) &= e^{-(\sigma + \frac{1}{1+s})sx} \\ &= e^{-\sigma sx} e^{-\frac{sx}{1+s}} \end{aligned} \quad (5.1.5)$$

$$= e^{-x} e^{-\sigma sx} e^{\frac{x}{1+s}} \quad (5.1.6)$$

(The last step makes use of the algebraic identity

$$-\frac{sx}{1+s} = \frac{x}{1+s} - x \quad )$$

From Campbell and Foster,<sup>(1)</sup> formula number 654.2

$$\mathcal{L}^{-1} \left[ e^{\frac{x}{1+s}} - 1 \right] = \sqrt{\frac{x}{\theta}} e^{-\theta} I_1(2\sqrt{x\theta}) \quad (5.1.7)$$

and

$$\mathcal{L}^{-1}[1] = \delta(\theta), \text{ the impulsive function or Dirac's delta function defined by equation (5.1.1)}$$

Thus

$$\mathcal{L}^{-1}\left[e^{\frac{x}{1+s}}\right] = \delta(\theta) + \sqrt{\frac{x}{\theta}} e^{-\theta} I_1(2\sqrt{x\theta}) \quad (5.1.8)$$

also

$$\mathcal{L}^{-1}\left[e^{-\sigma s x}\right] = \delta(\theta - \sigma x) \quad (5.1.9)$$

Application of the convolution theorem\* to equation (5.1.6) then gives

$$g(x, \theta) = e^{-x} \int_0^{\theta} \delta(\theta - \sigma x - \lambda) \left[ \delta(\lambda) + \sqrt{\frac{x}{\lambda}} e^{-\lambda} I_1(2\sqrt{x\lambda}) \right] d\lambda \quad (5.1.10)$$

$$= 0 \quad \text{for } \theta^+ < 0$$

$$= e^{-x} \left[ \delta(\theta^+) + \sqrt{\frac{x}{\theta^+}} e^{-\theta^+} I_1(2\sqrt{x\theta^+}) \right] \quad \text{for } \theta^+ \geq 0 \quad (5.1.11)$$

$$\text{where } \theta^+ = \theta - \sigma x \quad (5.1.12)$$

(The integration of equation (5.1.10) follows from the property of the Dirac delta function given in the footnote on p. 16.)

---

\* See any text on Laplace Transform Theory for a discussion of this theorem. It states that

$$\text{if } G(s) = G_1(s)G_2(s)$$

$$\text{and if } \mathcal{L} g(\theta) = G(s)$$

$$\mathcal{L} g_1(\theta) = G_1(s)$$

$$\mathcal{L} g_2(\theta) = G_2(s)$$

then

$$g(\theta) = \int_0^{\theta} g_1(\theta - \lambda) g_2(\lambda) d\lambda$$

Thus the fluid temperature response to an impulsive variation of inlet temperature is given by

$$\xi(x, \theta) = \gamma e^{-(\theta^+ + x)} \sqrt{\frac{x}{\theta^+}} I_1(2\sqrt{x\theta^+}) \quad (5.1.13)$$

for  $\theta^+ > 0$

Note that the quantity  $\alpha x$  represents the dimensionless time required for the fluid to travel from the inlet to the point along the channel under consideration.

## 5.2. Step Variation

A step variation of inlet temperature is represented by the function

$$\begin{aligned} \xi_0(\theta) &= 0 && \text{for } \theta < 0 \\ &= 1 && \text{for } \theta \geq 0 \end{aligned}$$

The solution for fluid temperature response can be written immediately by convolution using the results of the preceding section. Thus

$$\xi(x, \theta) = \gamma \int_0^\theta \xi_0(\theta - \lambda) g(x, \lambda) d\lambda \quad (5.2.1)$$

where  $g(x, \lambda)$  is given by equation (5.1.11). This results in

$$\begin{aligned} \xi(x, \theta) &= \gamma e^{-x} \left[ 1 + \int_0^{\theta^+} \sqrt{\frac{x}{\lambda}} e^{-\lambda} I_1(2\sqrt{x\lambda}) d\lambda \right] \\ &= \gamma f_1(\theta^+, x) && \text{for } \theta^+ \geq 0 \end{aligned} \quad (5.2.2)$$

Evaluation of the function  $f_1(\theta^+, x)$  is discussed in Section 7.2. For the case of  $\gamma = 1$  ( $B = 0$ ), the result is equivalent to a solution reported in Carslaw and Jaeger.<sup>(2)</sup>

### 5.3. Linear Variation

A linear variation of inlet temperature is represented by the function

$$\xi_0(\theta) = \theta, \quad \theta \leq 0 \quad (5.3.1)$$

Application of the convolution theorem as in the preceding section gives

$$\xi(x, \theta) = \int_0^{\theta^+} (\theta^+ - \lambda) g(x, \lambda) d\lambda \quad (5.3.2)$$

$$= \theta^+ f_2(\theta^+, x) - e^{-x} \int_0^{\theta^+} \sqrt{x\lambda} e^{-\lambda} I_1(2\sqrt{x\lambda}) d\lambda \quad (5.3.3)$$

$$= \theta^+ f_1(\theta^+, x) - f_2(\theta^+, x), \quad \theta^+ \geq 0 \quad (5.3.4)$$

Evaluation of  $f_2(\theta^+, x)$  is discussed in section 7.3.

### 5.4. Step-Exponential Variation

This variation of inlet temperature is represented by the function

$$\begin{aligned} \xi_0(\theta) &= 0, & \theta < 0 \\ &= e^{-a\theta}, & \theta > 0 \end{aligned} \quad (5.4.1)$$

5.5.

As in the preceding sections, we find for  $\theta^+ > 0$

$$\xi(x, \theta) = \int_0^{\theta^+} e^{-a(\theta^+ - \lambda)} g(x, \lambda) d\lambda \quad (5.5.1)$$

$$= \gamma e^{-(a\theta^+ + 1)} \left[ 1 + \int_0^{\theta^+} e^{-(1-a)\lambda} \sqrt{\frac{x}{\lambda}} I_1(2\sqrt{x\lambda}) d\lambda \right] \quad (5.5.2)$$

For a < 1 write equation (5.5.2) as

$$\begin{aligned} \xi &= \gamma e^{-(a\theta^+ + x)} \left[ 1 + \int_0^{(1-a)\theta^+} e^{-\lambda \sqrt{\frac{x}{(1-a)\lambda}}} I_1 \left( 2\sqrt{\frac{x\lambda}{1-a}} \right) d\lambda \right] \\ &= \gamma \exp. \left( \frac{ax}{1-a} - a\theta^+ \right) f_1 \left[ (1-a)\theta^+, \frac{x}{1-a} \right] \end{aligned} \quad (5.5.3)$$

For a > 1 write equation (5.5.2) as

$$\begin{aligned} \xi &= \gamma e^{-(a\theta^+ + x)} \left[ 1 + \int_0^{(a-1)\theta^+} e^{\lambda \sqrt{\frac{x}{(a-1)\lambda}}} I_1 \left( 2\sqrt{\frac{x\lambda}{a-1}} \right) d\lambda \right] \\ &= \gamma e^{-(a\theta^+ + x)} f_2 \left[ (a-1)\theta^+, \frac{x}{a-1} \right] \end{aligned} \quad (5.5.4)$$

Consideration of the above and the definition of  $f_1$  will show that

$$f_2(\theta^+, x) = e^{-x} f_1(-\theta^+, -x)$$

This function has been evaluated by Clark, Arpaçi, and Treadwell<sup>(e)</sup> (see section 7.4).

For a = 1 equation (5.5.2) becomes

$$\begin{aligned} \xi &= \gamma e^{-(\theta^+ + x)} \left[ 1 + \int_0^{\theta^+} \sqrt{\frac{x}{\lambda}} I_1(2\sqrt{x\lambda}) d\lambda \right] \\ &= \gamma e^{-(\theta^+ + x)} I_0(2\sqrt{x\theta^+}) \end{aligned} \quad (5.5.5)$$

Note that the results of this and the previous sections can be used to write immediately solutions for the transient response to inlet temperature variations represented by such functions as

$$\xi_0(\theta) = a_1 + a_2\theta + a_3e^{a_4\theta}$$

where the  $a_i$  are constants.

## 6. Transient Responses to Heat Generation Variations

In this section solutions for the fluid temperature response to heat generation variations are presented. The solutions obtained to date are based on the thin fuel element approximation and are for uniform variations only. (The writer expects to have solutions available for various simple non-uniform variations in the near future.\*) According to equation (3.18a), the transformed fluid temperature response for these cases is given by

$$R(x,s) = \frac{V(s)[1 - G(x,s)]}{(1 + \sigma + \sigma s)s}$$

where the real time variable corresponding to  $s$  is given by

$$\theta = \frac{ah\tau}{bk\gamma}$$

the dimensionless channel length by

$$L = \frac{hA\ell}{C_p W \gamma}$$

and

$$\gamma = 1 + \frac{B^2}{3}$$

### 6.1. Impulsive Uniform Variation

As in section 5.1, an impulsive uniform variation of heat generation is represented by the function

$$\begin{aligned} v(\theta) &= \delta(\theta) \\ &= 0 && \text{for } \theta < 0 \\ &= \infty && \text{for } \theta = 0 \\ &= 0 && \text{for } \theta > 0 \end{aligned}$$

and its Laplace transform is unity. Thus for this case

$$\begin{aligned} R_0(s) &= \frac{[1 - G(x,s)]}{(1 + \sigma + \sigma s)s} \\ &= R(x,s) \end{aligned} \tag{6.1.1}$$

---

\* It is unlikely, however, that these solutions will be reported under the same sponsorship. Those interested may contact the writer directly.

This function represents the transfer function for the system response to uniform heat generation variations. Its inverse,  $r(x, \theta)$  -- the inverse transfer function -- can be used to write immediately the response to any variation by application of the convolution theorem.

To find  $\mathcal{L}^{-1}R(x, s) = r(x, \theta)$  we note the following:

$$\mathcal{L}^{-1}\left[\frac{1}{(1 + \sigma + \sigma s)s}\right] = \frac{1}{1 + \sigma} \left(1 - e^{-\frac{1+\sigma}{\sigma}\theta}\right) \quad (6.1.2)$$

and  $\mathcal{L}^{-1}G(x, s) = g(x, \theta)$

where  $g(x, \theta)$  is given by equation (5.1.11). Then convolution gives

$$r(x, \theta) = \frac{1 - e^{-m\theta}}{1 + \sigma} - \frac{1}{1 + \sigma} \int_0^\theta (1 - e^{-m(\theta-\lambda)}) g(x, \lambda) d\lambda \quad (6.1.3)$$

where  $m = \frac{1 + \sigma}{\sigma}$  for convenience. Substitution of equation (5.1.11) and the use of equations (5.2.2) and (5.5.4) for definitions of  $f_1$  and  $f_2$ , results in the following

$$f(x, \theta) = \frac{1 - e^{-m\theta}}{1 + \sigma} \quad \text{for } \theta < \sigma x \quad (6.1.4)$$

$$= \frac{1 - e^{-m\theta}}{1 + \sigma} - \frac{1}{1 + \sigma} [f_1(\theta^+, s) - e^{-(m\theta^+ + x)} f_2\left(\frac{\theta^+}{\sigma}, \sigma x\right)] \quad (6.1.5)$$

$$\text{for } \theta^+ \geq 0$$

where  $\theta^+ = \theta - \sigma x$

Evaluation of the functions  $f_1$  and  $f_2$  are discussed in sections 7.2 and 7.4.

## 6.2. Step Uniform Variation

A step uniform variation of heat generation is represented by the function

$$\begin{aligned} v(\theta) &= 0, & \text{for } \theta < 0 \\ &= 1, & \text{for } \theta \geq 0 \end{aligned}$$

The solution for fluid temperature response can be written immediately by convolution using the results of the preceding section. For this case, however, it is simpler to find the transform of  $\xi(x, \theta)$  and then its inverse. Since  $\mathcal{L} v(\theta) = 1/s$ , equation (3.18a) becomes

$$\Xi(x, s) = \frac{[1 - G(x, s)]}{(1 + \sigma + \sigma s)s^2} \quad (6.2.1)$$

The procedure for finding the inverse is the same as in the preceding section. First it is noted that

$$\mathcal{L}^{-1}\left[\frac{1}{(1 + \sigma + \sigma s)s^2}\right] = \frac{\theta}{m} - \frac{1}{m^2}(1 - e^{-m\theta}) \quad (6.2.2)$$

where  $m = \frac{1 + \sigma}{\sigma}$  as before. Then the convolution theorem is applied using  $\mathcal{L}^{-1}[G(x, s)] = g(x, \theta)$  from equation (5.1.11). The integration is written in terms of the defined functions  $f_1$ ,  $f_2$ , and  $f_3$ . The following results:

$$\sigma m^2 \xi(x, \theta) = (m\theta - 1) + e^{-m\theta} \quad \text{for } \theta < \sigma x \quad (6.2.3)$$

$$\begin{aligned} &= (m\theta - 1) + e^{-m\theta} - (m\theta^+ - 1)f_1(\theta^+, x) \\ &\quad + mf_2(\theta^+, x) - e^{-(m\theta^+ + x)}f_3\left(\frac{\theta^+}{\sigma}, \sigma x\right) \end{aligned} \quad (6.2.4)$$

$$\text{for } \theta^+ \geq 0$$

where  $\theta^+ = \theta - \sigma x$

For the case of  $\gamma = 1$  ( $B = 0$ ), the above is equivalent to the solution obtained by Clark, Arpaci, and Treadwell,<sup>(3)</sup> although their result was obtained for a step increase of uniform heat generation from an initially unheated condition.

## 7. The Functions $f_1(x, y)$ \*

In this section we briefly describe relations for the functions obtained in section 5 in terms of functions for which graphical or tabular representations are available.

---

\* In this section  $x$  and  $y$  represent arbitrary variables.

### 7.1. $f_0(x,y)$

Define

$$f_0(x,y) = e^{-y} \int_0^x e^{-\lambda} I_0(2\sqrt{y\lambda}) d\lambda \quad (7.1.1)$$

This function can be related to the functions of section 5.

Values of the equivalent of this function for positive  $x$  and  $y$  were computed by Schumann<sup>(7)</sup> and later extended and graphed by Furnas.<sup>(8)</sup> The equivalent of Furnas' graphs have been presented in the publications of several authors but, of the several we have seen, are either difficult to read or have ranges of the variable  $y$  too large for the type of transients under consideration. Evidently because of this difficulty, Rizika<sup>(8)</sup> recomputed the equivalent of  $e^y f_0(x,y)$ . His graph is reproduced in Figure 2. The function is identified on the graph as an infinite series. Its equivalence to  $e^y f_0(x,y)$  is easily shown by using

$$I_0(2\sqrt{y\lambda}) = \sum_{n=0}^{\infty} \frac{(y\lambda)^n}{(n!)^2} \quad (7.1.2)$$

### 7.2. $f_1(x,y)$

From section 5.2

$$f_1(x,y) = e^{-y} \left[ 1 + \int_0^x e^{-\lambda} \sqrt{\frac{y}{\lambda}} I_1(2\sqrt{y\lambda}) d\lambda \right] \quad (7.2.1)$$

Integrate by parts using

$$\sqrt{\frac{y}{\lambda}} I_1(2\sqrt{y\lambda}) d\lambda = dI_0(2\sqrt{y\lambda})$$

This results in

$$f_1(x,y) = f_0(x,y) + e^{-(x+y)} I_0(2\sqrt{yx}) \quad (7.2.2)$$

A graph of  $e^y f_1(x, y)$  has been prepared by Rizika<sup>(e)</sup> and is reproduced in Figure 3. Rizika identifies the function as an infinite series. The equivalence of  $e^y f_1$  and Rizika's function is easily shown by using

$$I_1(2\sqrt{y\lambda}) = \sum_{n=0}^{\infty} \frac{(y\lambda)^{\frac{1}{2}+n}}{(n!)^2(n+1)} \quad (7.2.3)$$

in equation (7.2.1).

### 7.3. $f_2(x, y)$

From section 5.3

$$f_2(x, y) = e^{-y} \int_0^x e^{-\lambda\sqrt{y\lambda}} I_1(2\sqrt{y\lambda}) d\lambda \quad (7.3.1)$$

Integrate by parts using

$$d[\sqrt{y\lambda} I_1(2\sqrt{y\lambda})] = y I_0(2\sqrt{y\lambda}) d\lambda$$

This results in

$$f_2(x, y) = y f_0(x, y) - e^{-(x+y)} \sqrt{yx} I_1(2\sqrt{yx}) \quad (7.3.2)$$

A graph of

$$e^{-x} \sqrt{yx} I_1(2\sqrt{yx})$$

has been prepared by Clark, Arpacı, and Treadwell<sup>(s)</sup> and is reproduced in Figure 4. In their figure the function is identified as an infinite series. Its equivalence to the above is easily shown by using equation (7.2.3).

#### 7.4. $f_3(x,y)$

In section 5.5 the function  $f_1$  with both arguments negative was introduced as

$$\begin{aligned} f_3(x,y) &= e^{-y} f_1(-x,-y) \\ &= \left[ 1 + \int_0^x e^{-\lambda} \sqrt{\frac{y}{\lambda}} I_1(2\sqrt{y\lambda}) d\lambda \right] \end{aligned} \quad (7.4.1)$$

A graph of  $f_3(x,y)$  has been prepared by Clark, Arpacı, and Treadwell and is reproduced in Figure 5. In their figure the function is identified as an infinite series. Its equivalence to  $f_3(x,y)$  is easily shown by using equation (7.2.3).

## Nomenclature

### Dimensional Quantities (Units are illustrative only)

- $A$  = heat transfer area per unit length of coolant channel,  $\text{ft}^2/\text{ft}$
- $b$  = solid (fuel element) thickness, ft (or can be taken as an equivalent solid thickness, total volume of solid divided by total heat transfer area)
- $C_f$  = specific heat of fluid,  $\text{Btu}/(\#)(^\circ\text{F})$
- $C_m$  = specific heat of solid,  $\text{Btu}/(\#)(^\circ\text{F})$
- $h$  = heat transfer coefficient,  $\text{Btu}/(\text{hr})(\text{ft}^2)(^\circ\text{F})$
- $k$  = thermal conductivity of solid,  $\text{Btu}/(\text{hr})(\text{ft})(^\circ\text{F})$
- $l$  = coolant channel length, ft
- $Q$  = local rate of heat generated in solid per unit length of coolant channel,  $\text{Btu}/(\text{hr})(\text{ft})$
- $Q_0$  = value of  $Q$  at time zero
- $Q_r$  = reference value of  $Q$ , a constant
- $S$  = cross-sectional area to fluid flow,  $\text{ft}^2$
- $t$  = local mixed mean fluid temperature,  $^\circ\text{F}$
- $t_0$  = temperature of fluid at coolant channel inlet,  $^\circ\text{F}$
- $T$  = local solid temperature,  $^\circ\text{F}$
- $T_0$  = solid temperature at  $\xi$  of solid or insulated surface,  $^\circ\text{F}$
- $T_s$  = solid temperature at heat transfer surface,  $^\circ\text{F}$
- $V$  = fluid velocity,  $\text{ft}/\text{hr}$
- $W$  = fluid weight rate of flow,  $\#/\text{hr}$
- $x$  = length along channel, measured from inlet, ft
- $y$  = distance in solid, measured from  $\xi$  or insulated surface, ft

- $\alpha$  = thermal diffusivity of solid,  $\text{ft}^2/\text{hr}$   
 $\rho_m$  = density of metal,  $\#/(\text{ft})^3$   
 $\tau$  = time, measured from beginning of transient, hr

Dimensionless Quantities

$$B = \sqrt{\frac{hb}{k}} \quad (\text{solid thickness})$$

$$L = \frac{hA\ell}{C_f W} \quad (\text{channel length})$$

$$= \frac{hA\ell}{C_f W \gamma} \quad (\text{channel length for thin element approximation})$$

$$v = \frac{Q - Q_o}{Q_r}$$

$x$  = dimensionless channel position,  $0 \leq x \leq L$

$$\gamma = 1 + \frac{B^2}{3}$$

$$\sigma = \frac{C_f \rho_f}{C_m \rho_m} \left( \frac{s}{Ab} \right)$$

$$\theta = \frac{ah\tau}{bk} \quad (\text{time})$$

$$= \frac{ah\tau}{bk\gamma} \quad (\text{time for thin element approximation})$$

$$\theta^+ = \theta - \sigma x$$

$$\psi = \frac{hA}{Q_r} [T(y, x, \tau) - T(y, x, 0)] \quad (\text{metal temperature})$$

$$\psi_s = \frac{hA}{Q_r} [T_s(x, \tau) - T(x, 0)] \quad (\text{surface temperature})$$

$$\xi = \frac{hA}{Q_r} [t(x, \tau) - t(x, 0)] \quad (\text{fluid temperature})$$

$$\xi_o = \frac{hA}{Q_r} [t_o(\tau) - t_o(0)] \quad (\text{fluid inlet temperature})$$

### References

- (1) Campbell, G. A., and Foster, R. M., Fourier Integrals for Practical Applications, D. Van Nostrand, New York (1951)
- (2) Carlaw, H. S. and Jaeger, J. C., Conduction of Heat in Solids, Oxford Univ. Press, London (1948), Example Y, section 148
- (3) Clark, J. A., Arpaci, V. S., and Treadwell, K. M., Response of Nuclear Reactor Coolant to Power Transients, presented at Reactor Heat Transfer Conference, Nov. 1956, New York City. To be issued by T.I.S.
- (4) Dayton, R. W., et al., Improved Measurements of Surface Heat Transfer by the Method of Cyclic Temperature Variations, BMI-747, May 28, 1952
- (5) Furnas, C. C., Trans. Am. Inst. Chem. Engrs., 24, 1942-93 (1930) (Original not seen)
- (6) Rizika, J. W., Trans. A.S.M.E., 76, 411 (1954)
- (7) Schumann, T. E. W., J. Franklin Institute, 208, 405-16 (1929)

## APPENDICES

### A. Series Expansion of the Function $\phi(B,s)$

$$\begin{aligned}\phi &= \frac{\sinh B\sqrt{s}}{s \sinh B\sqrt{s} + B\sqrt{s} \cosh B\sqrt{s}} \\ &= \frac{1}{s + \frac{B\sqrt{s}}{\tanh B\sqrt{s}}}\end{aligned}$$

But

$$\frac{\tanh B\sqrt{s}}{B\sqrt{s}} = 1 - \frac{s}{3} B^2 + \frac{2s^2}{15} B^4 - \frac{51s^3}{945} B^6 + \dots$$

Simple algebraic division gives

$$\frac{B\sqrt{s}}{\tanh B\sqrt{s}} = 1 + \frac{s}{3} B^2 - \frac{s^2}{45} B^4 + \frac{2s^3}{945} B^6 - \dots$$

and equation (3.16) follows.

- B. Proof that neglecting  $B^4$  and higher exponent terms in the series expansion for  $\phi(B,s)$  is equivalent to the assumption that during a transient the average solid temperature is related to the surface temperature and surface heat flux as in the steady state.

The average solid temperature is given by

$$T_{av} = \frac{1}{b} \int_0^b T dy$$

For  $\tau < 0$ , equation (2.5) gives

$$T_{av} = T_s(x,0) + \frac{Q_0 b}{3Ak}$$

which is the relationship for the steady state. Assume that during a transient this relationship is a satisfactory approximation when  $h(T_s - t)$ , the transient surface heat flux, is substituted for  $Q_0/A$ .

Thus, the assumption is equivalent to

$$\begin{aligned} T_{av} &\approx T_s + \frac{hb}{3k}(T_s - t) \\ &= \left(1 + \frac{B^2}{3}\right)T_s - \frac{B^2}{3}t \end{aligned}$$

From the definitions of the dimensionless metal temperature  $\psi$  (equation (2.11)), and the dimensionless fluid temperature  $\xi$  (equation (2.12)), the assumption can be expressed in dimensionless form as

$$\psi_{av} = \left(1 + \frac{B^2}{3}\right)\psi_s - \frac{B^2}{3}\xi \quad (B-1)$$

where

$$\psi_{av} = \frac{1}{B} \int_0^B \psi dy$$

Equation (2.28) can be integrated as follows

$$\frac{1}{B} \int_0^B d\left(\frac{\partial \psi}{\partial y}\right) + \frac{1}{B} \int_0^B v dy = \frac{1}{B} \frac{\partial}{\partial \theta} \int_0^B \psi dy$$

$$\frac{1}{B} \frac{\partial \psi}{\partial y} \Big|_{y=B} + v = \frac{\partial \psi_{av}}{\partial \theta}$$

But from equation (2.33)

$$\frac{\partial \psi}{\partial y} \Big|_{y=B} = -B(\psi_s - \xi)$$

Thus

$$-(\psi_s - \xi) + v = \frac{\partial \psi_{av}}{\partial \theta} \quad (B-2)$$

Combine equations (B-1) and (B-2) to eliminate  $\psi_{av}$ , and Laplace transform the result. Thus

$$-(\psi_s - \xi) + v = \left(1 + \frac{B^2}{3}\right) s \psi_s - \frac{B^2}{3} s \xi$$

Now solve for  $(\psi_s - \xi)$ . The following is obtained

$$\psi_s - \xi = \frac{v - s \xi}{1 + \left(1 + \frac{B^2}{3}\right) s}$$

Comparing with equation (3.9), it is noted that the result obtained is equivalent to replacing  $\phi(B, s)$  by

$$\frac{1}{1 + \left(1 + \frac{B^2}{3}\right) s}$$

and that this form for  $\phi$  can be obtained from the series expansion by neglecting  $B^4$  and higher exponent terms.

### C. The Steady-State Harmonic Variation Functions $K_1$ and $K_2$

In section 4 the function  $\Phi(B,s)$  for  $s = \omega i$  is represented by

$$\Phi(B,\omega i) = K_1(B,\omega) - iK_2(B,\omega) \quad (4.1.4)$$

Derivations of expressions for  $K_1$  and  $K_2$  are straight forward but cumbersome requiring the application of complex algebra to the expression for  $\Phi(B,s)$  given by equation (3.10). The results are cited here for reference.

$$K_1 = (\sinh 2\eta + \sin 2\eta) \frac{\eta}{2D_1}$$

$$K_2 = \frac{1}{D_1} [\omega(\sinh^2 \eta + \sin^2 \eta) + \frac{\eta}{2}(\sinh 2\eta - \sin 2\eta)]$$

where

$$\eta = \frac{B\sqrt{2\omega}}{2}$$

and

$$D_1 = 2\eta^2 + (\omega^2 + 2\eta^2)\sinh^2 \eta + (\omega^2 - 2\eta^2)\sin^2 \eta \\ + \eta\omega(\sinh 2\eta - \sin 2\eta)$$

For small values of  $\eta$ , series expressions for these functions are more convenient for computations. The following were obtained, applicable when  $\eta < 1$ :

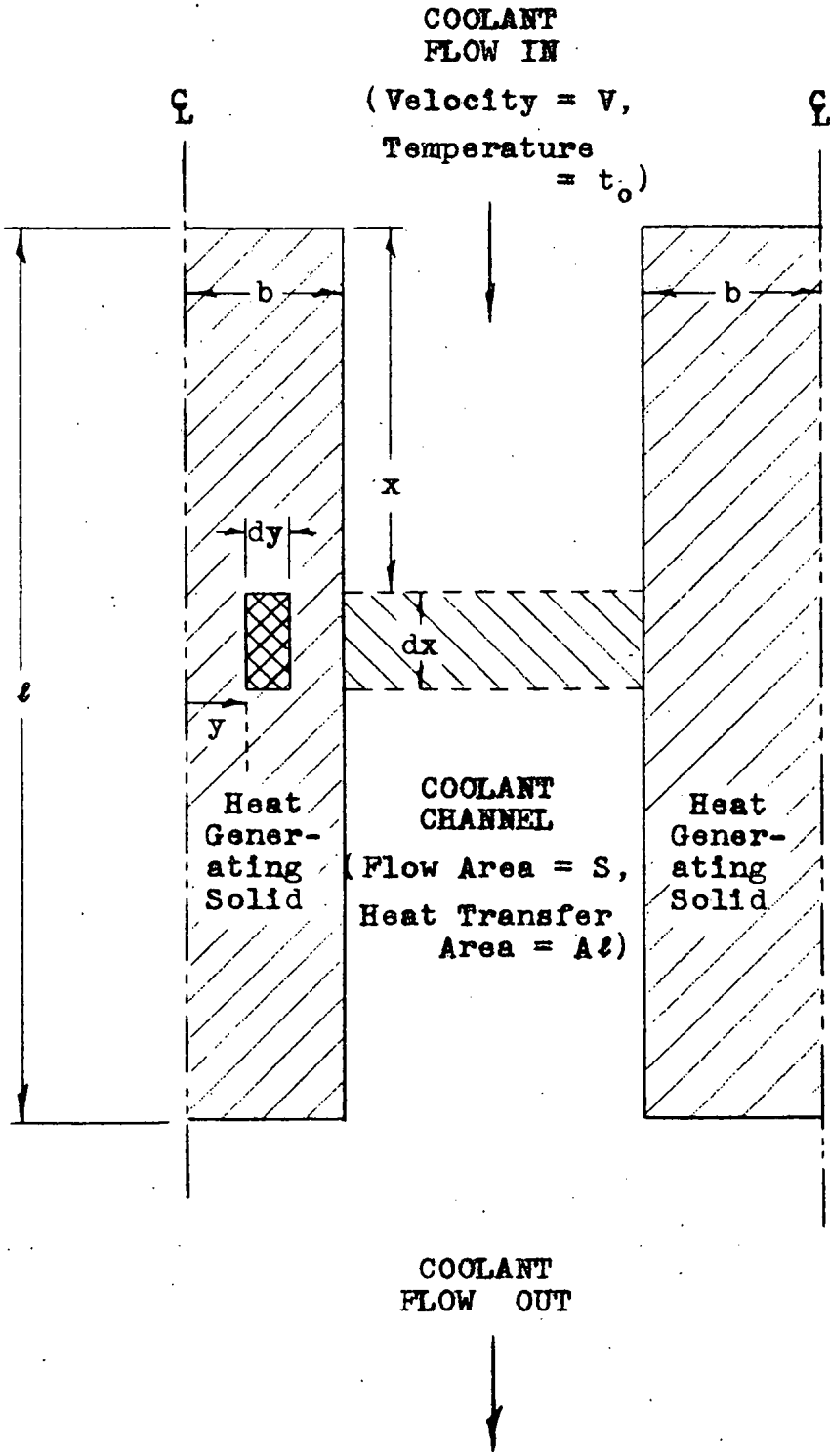
$$K_1 = \frac{1}{D_2} (1 + \frac{2}{15}\eta^4 + \frac{2}{1215}\eta^8 + \dots)$$

$$K_2 = \frac{1}{D_2} (\omega + \frac{2}{3}\eta^2 + \frac{2}{45}\omega\eta^4 + \frac{4}{315}\eta^6 + \dots)$$

where

$$D_2 = (1 + \omega^2) + \frac{4}{3}\omega\eta^2 + \frac{2}{45}(\omega^2 + 15)\eta^4 + \frac{8}{315}\omega\eta^6 + \dots$$

For the case of  $B = 0$ , the above expressions will give the same results as would be obtained by letting  $s = \omega i$  in equation (3.15). The series expressions show that the use of the relations obtained this way will be a satisfactory approximation when  $\omega \gg \eta^2 \ll 1$ , which is equivalent to  $\frac{2}{\omega} \gg B^2 \ll 2$ .



- $\tau$  = time
- $T$  = temperature of solid, a function of  $y$ ,  $x$ , and  $\tau$
- $t$  = mixed mean temperature of coolant, a function of  $x$  and  $\tau$
- $Q$  = rate of heat generation per unit length of channel, a function of  $x$  and  $\tau$

The  $\xi$ s represent boundaries where  $\frac{\partial T}{\partial y} = 0$

Figure 1  
A Typical Reactor Flow Channel

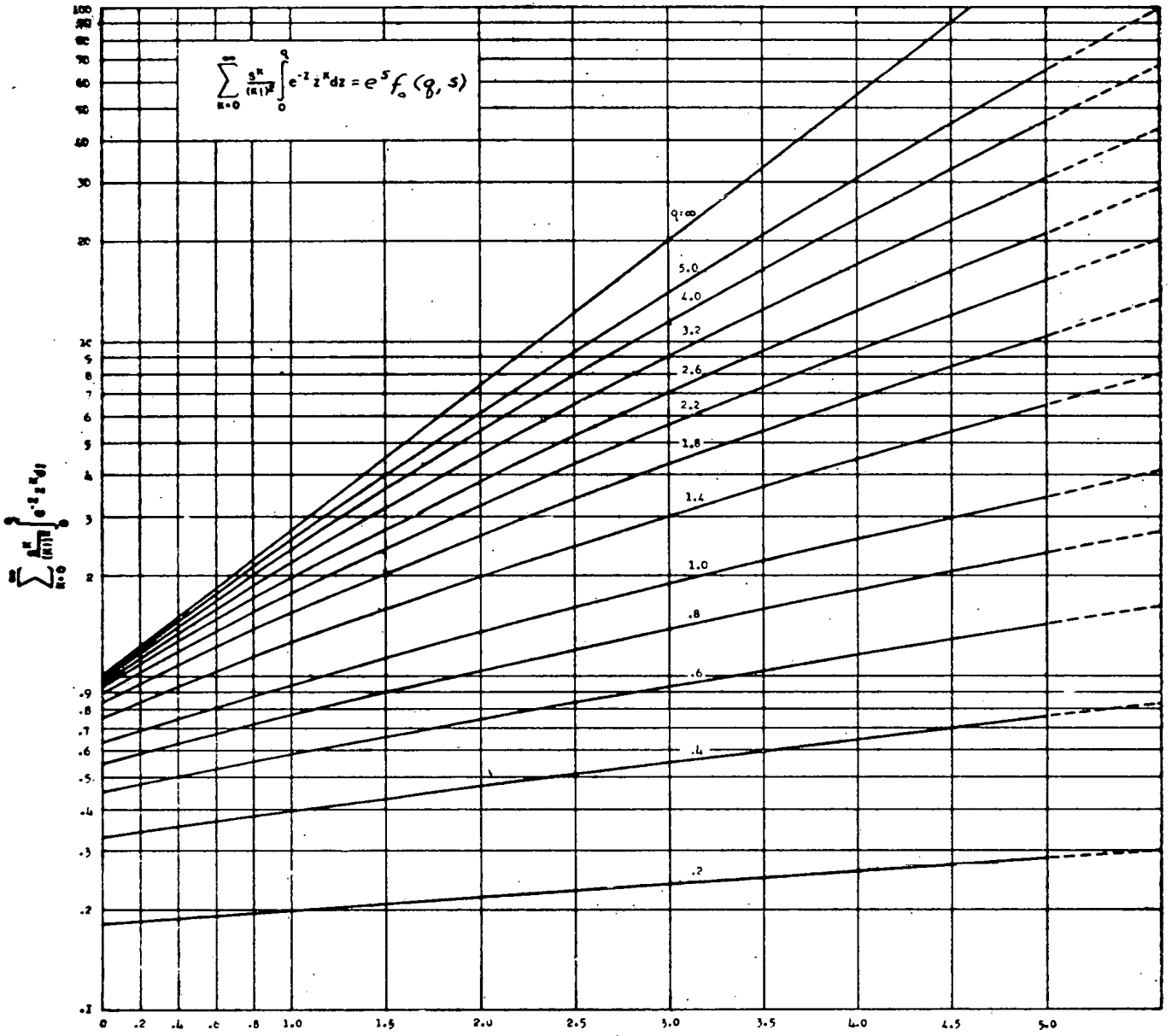


FIGURE 2

$e^s f_0(q, s)$  FROM REFERENCE 6

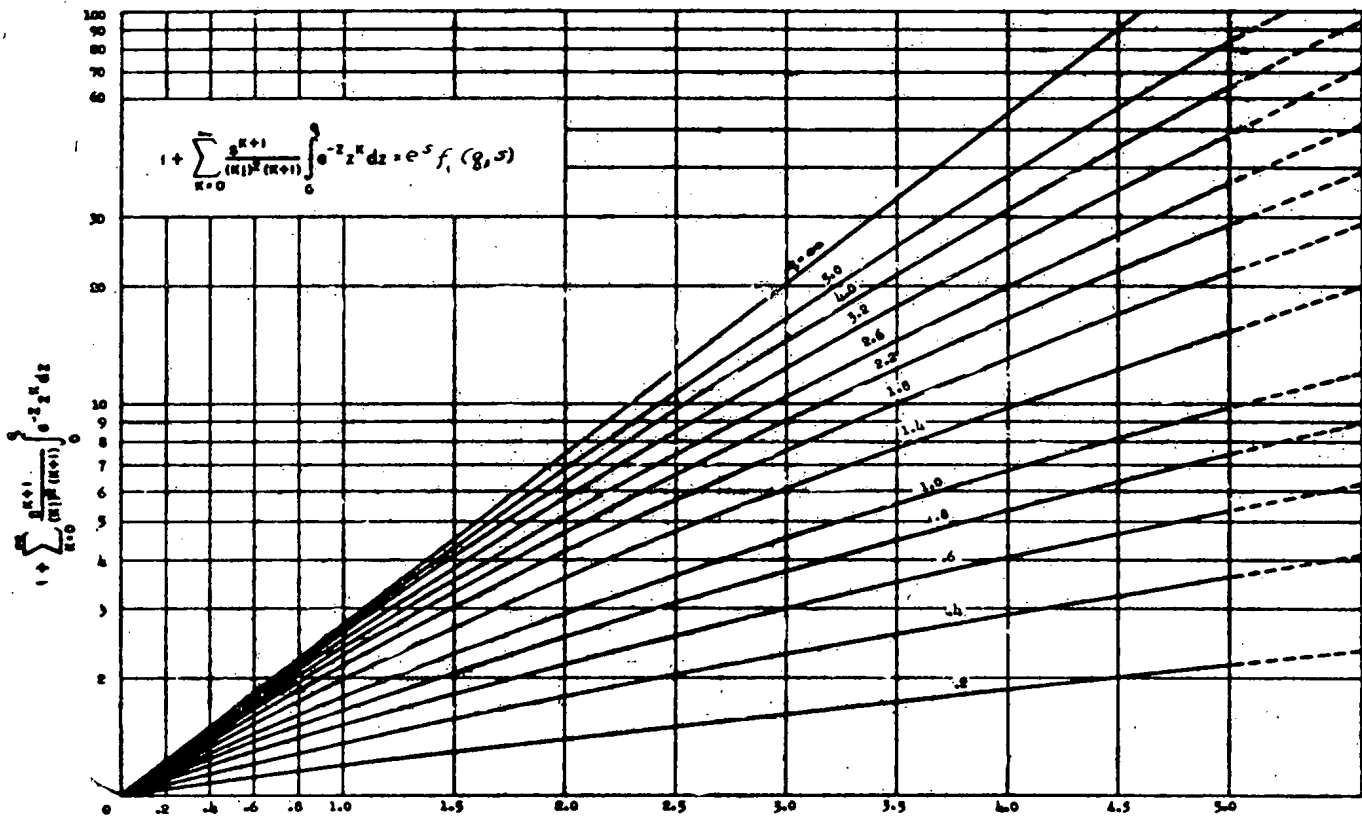


FIGURE 3

$e^s f_1(q, s)$  FROM REFERENCE 6

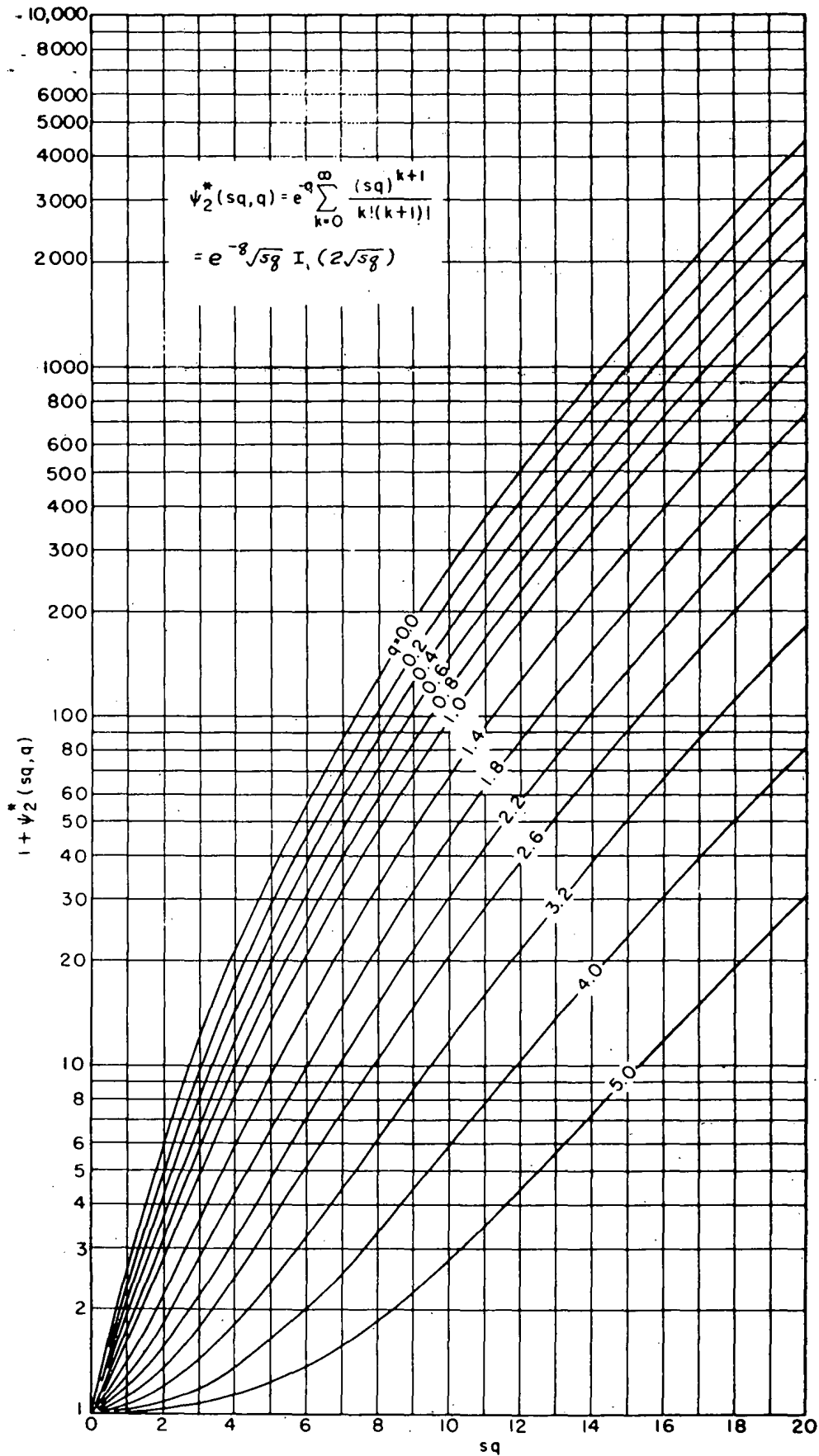


Figure 4

$e^{-8\sqrt{sq}} I_1(2\sqrt{sq})$  FROM REFERENCE 3

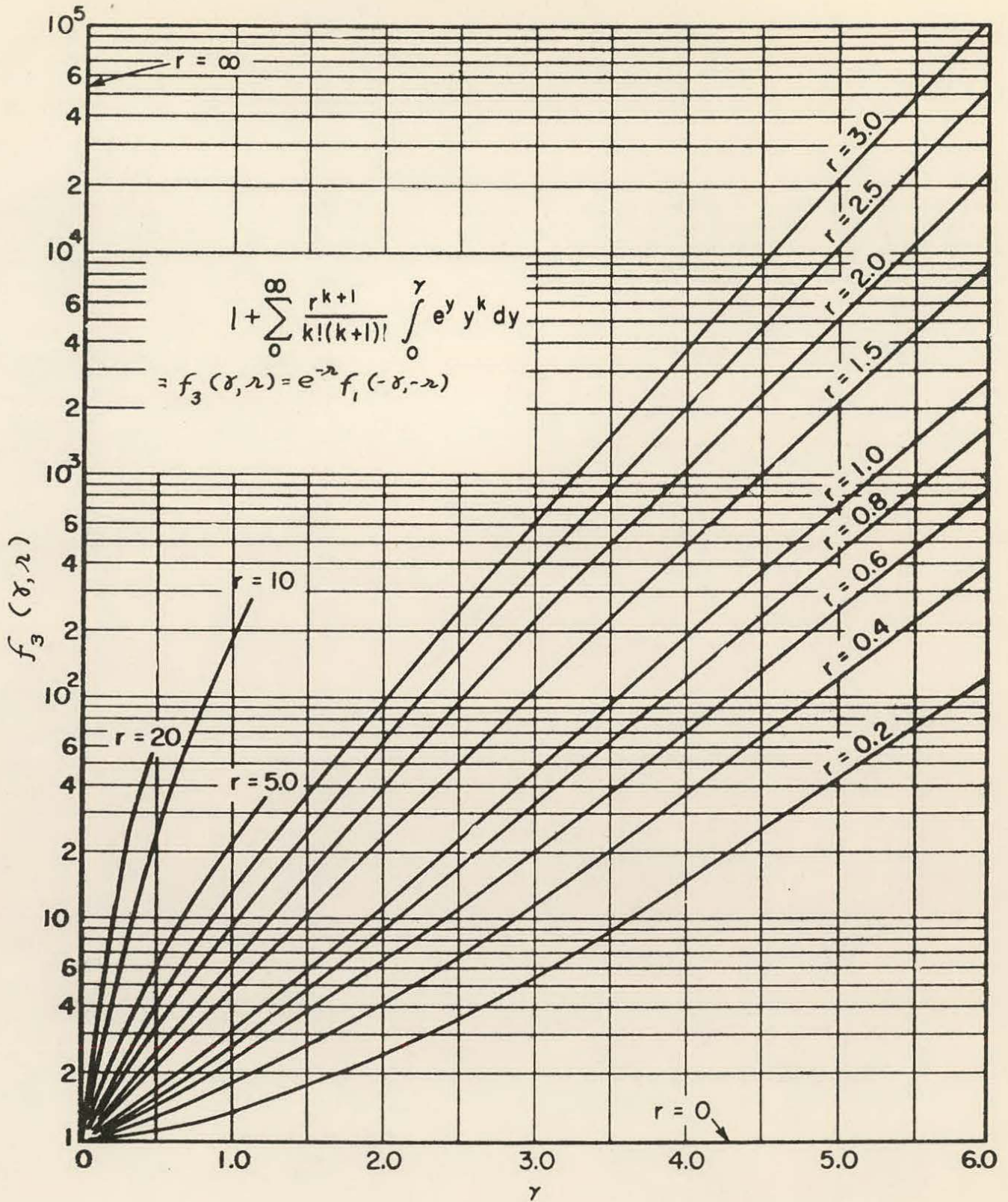


Figure 5

$f_3(\delta, r)$  FROM REFERENCE 3