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REACTOR KINETICS
QUARTERLY PROGRESS REPORT
OCTOBER – DECEMBER 1956

ATOMICS INTERNATIONAL
A DIVISION OF NORTH AMERICAN AVIATION, INC.

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QUARTERLY PROGRESS REPORT

OCTOBER – DECEMBER 1956

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ABSTRACT

The KEWB I reactor was operated at its design power of 50 kilowatts. Investigation of the energy coefficient of radiolytic-gas production shows the rate of gas evolution to be linearly decreasing with increasing reactor core pressure. Experimental transient studies with initial core temperature and initial core pressure as parameters result in lower peak reactor powers with both increased core temperature and decreased core pressure. The radiolytic-gas evolution is found to play a dominant role in reactor shutdown, but the effectiveness of the gas as a shutdown mechanism cannot be predicted by assuming that gas bubbles instantaneously contribute to reactivity compensation.

The power and temperature transfer functions for a water boiler reactor are obtained for the case of m groups of delayed neutrons including the effects of temperature and radiolytic-gas generation. The nature of the reactor transfer function is investigated, and a number of asymptotic and approximate expressions for the function are given. The derivation and limitations of the space-independent reactor kinetic equations are discussed briefly.

I. THE KEWB I PROGRAM

(J. W. Flora, D. P. Gamble, E. L. Gardner, E. B. Hecker, D. L. Hetrick, D. R. Muller, M. E. Remley, G. Schumann, R. K. Stitt, and R. E. Wimmer)

A. INTRODUCTION

The objectives of the KEWB Program – Kinetic Experiments on Water Boilers – include the experimental study of the kinetic behavior of homogeneous reactors to develop the knowledge and understanding of the effects of some of the parameters which play dominant roles in the dynamic characteristics and safety of the reactors. The KEWB I facility, which is a prototype 50-kw water boiler type reactor described in an earlier report,[1] was placed in operation in July, 1956. Initial experimental results were reported in the first of this series of quarterly progress reports.[2]

During the current report period loading of excess reactivity to nearly 4.0 per cent was completed. Measurements of several reactor constants were made, and more than 50 control rod induced transient tests were conducted. Reactor characteristics at rated power of 50 kilowatts were observed during three separate tests totaling 3-1/2 operating-hours. Detailed investigation of the energy coefficient of hydrogen and oxygen production was undertaken and is being continued. The fuel solution, in final form, has been analyzed for physical and chemical properties which are presented in an appendix to this report. Power calibration was finalized by two techniques; heat transfer by the cooling system, and temperature rise in the core assembly of known heat capacity.

Experimental findings of particular interest are discussed below; the transient tests have been divided into an alphabetical series for clarity of reference.

B. REACTIVITY EFFECTS OF COOLING WATER

The reactivity furnished by the water in the cooling coils was measured by the rod compensation technique and found to be 1.38% Δ k/k. This figure represents the effects of 520 cm^3 of water, or about 0.0025%/cm^3. Based on a preliminary

determination of mass coefficient of reactivity, the fuel solution is worth approximately 0.0025%/cm^3. The effect of water added as a fuel diluent is also about 0.002%/cm^3.

C. EQUILIBRIUM POWER OPERATION

The purpose of these tests was to observe the integral reactor system under maximum steady-state conditions. Operation at full rated power of 50 kilowatts was carried out during three runs totaling approximately 3-1/2 hours. The cooling system, a once-through tap water arrangement, holds the core temperature to 70° C with a 12° C rise in the coolant supplied from a source at 25° C. Coolant flow rate under these conditions is approximately 15 gpm. The highest temperature sensed in the recombiner catalyst bed was approximately 300° C, which is well below the $H_2 - O_2$ ignition temperature. No gas inflammation was experienced during rise-to-power or equilibrium operation. Short period stability of the reactor appears to be excellent, and little difficulty is expected in establishing initial conditions for transient tests involving starting powers at the kilowatt level.

Increase in fission product inventory resulting from these tests has not significantly reduced necessary access to the reactor components.

D. A-SERIES TRANSIENTS

A series of ten transients was carried out at core pressures of 22 and 68 cm Hg absolute, with essentially step inputs of reactivity between 0.3 and 0.7% $\Delta k/k$. The purposes of this series were to test further the recording and measuring instruments under transient conditions, to normalize data of previous transients[2] to the final power calibration, and to indicate acceptable ranges of parameters for the transient program to follow.

E. B-SERIES TRANSIENTS

These 24 transients, the results of which are shown graphically in Fig. 1, were made to determine the dependence of transient peak power upon initial core pressure, and to provide data for further analysis of the transient shutdown mechanisms. Initial core pressures of 15.6, 43.4, and 71.2 cm Hg absolute were selected for convenience, and a series of step reactivity inputs from 0.1 to 0.7 per cent were carried out. In all cases instrumentation scram circuits were overridden

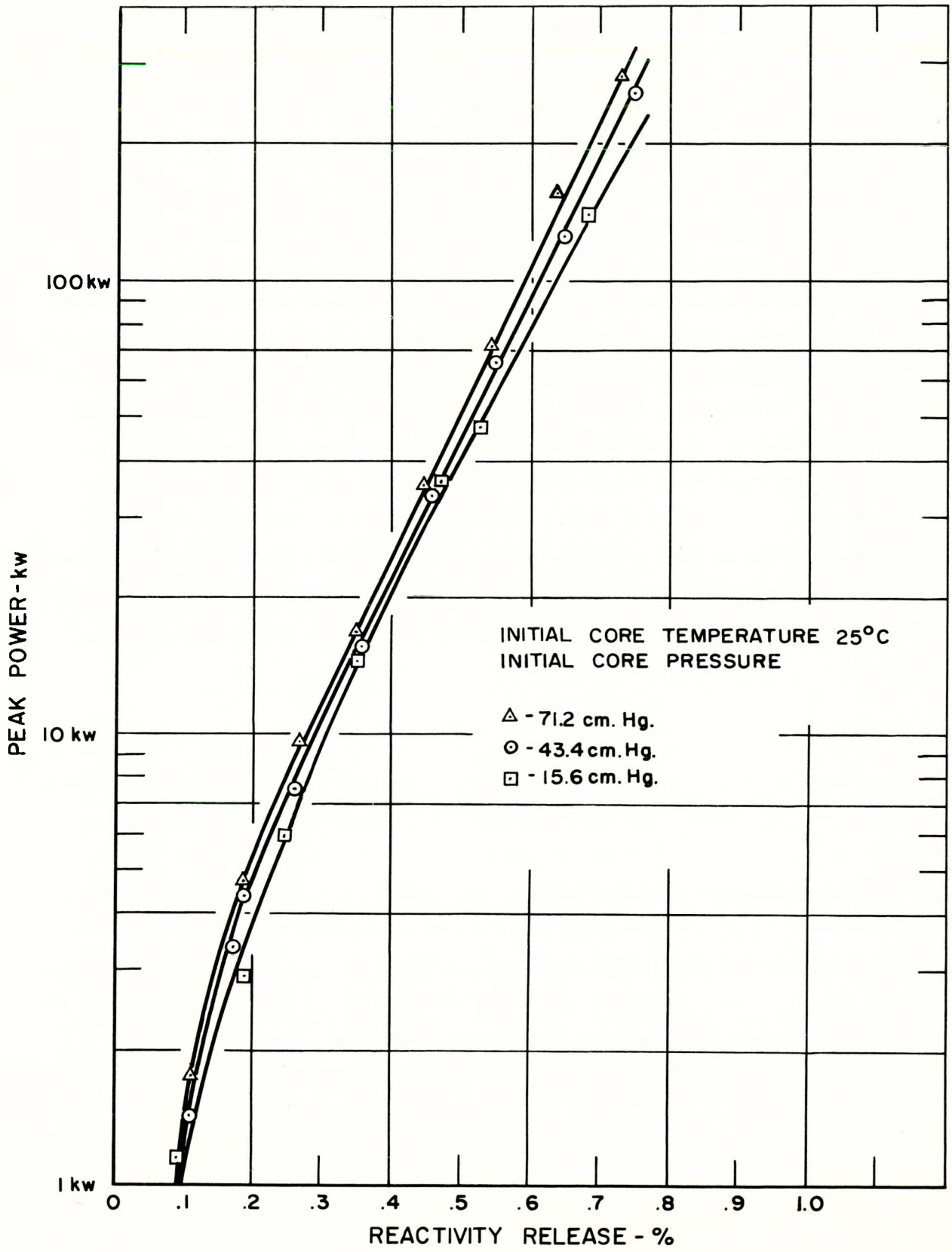

Fig. 1. Peak Power vs Reactivity Step for B-Series Transients

only long enough to observe peak power, and then a manual reactor shutdown was initiated. The range of starting temperatures for these transients was 23.1 to 26.6° C. Subsequent experiments (C-Series, described below) indicate that a 5 per cent spread in the peak power could result from this variation in starting temperature. Therefore, a normalized peak power was computed for each transient; this normalized value is that which would have been experienced for an initial temperature of 25.0° C, the average initial temperature for the series. In the course of the C-Series experiments it was observed that, for a fixed reactivity input of 0.4 per cent, peak power decreases linearly with increasing temperatures in the range of 15 to 50° C. The variation of this function with pressure and reactivity is unknown, and since the B-Series involves both these items as parameters, the direct applicability of information from the C-Series transients as a correction source cannot be conclusively demonstrated. However, it is believed that the technique is justified for the following reasons: (1) the reactivity involved in the C-Series is a good representation of that involved in the B-Series; (2) from theoretical considerations, the sign of the correction cannot be in error; (3) the range of temperature involved is less than 4° C, and the average temperature is far from the boiling point in all tests; and (4) the corrections to the data resulted in smaller deviations in the data.

Note was made as to the temperature rise in the core at the completion of each transient test. The temperature coefficient of reactivity was previously determined as $-0.024\%/°C^2$. In no case does the total temperature rise, *including* a substantial portion of the energy dissipated in the power decay after the transient peak and after the manual shutdown, account for more than 59 per cent of the reactivity input. This last figure is for the slowest transient, where temperature rise would be expected to have its greatest effectiveness in the shutdown mechanism. In transients of greater than 150-kw peak power, temperature rise accounts for less than one-third of the reactivity input.

It is obvious from the data presented in Fig. 1 that, for a given reactivity input, a reduction of peak power may be realized by a reduction in system pressure. In the range of pressures investigated, the peak power reduction is less than 30 per cent. An interesting aspect of the gas production phenomenon was observed for the first time during this series of tests. Sound-monitoring of the core permitted detection of an audible fizzing *just* prior to, during, and for a

considerable time after the larger transient peaks, indicating the average bubble size is indeed small, and that only a small fraction of the gas generated has escaped the fuel solution when the peak power is attained.

F. RELATIVE EFFECTIVENESS OF SHUTDOWN MECHANISMS

Figure 2 shows peak power vs reactivity step for 33 excursions, not corrected for initial temperatures (which varied from 21 to 29° C). The data labeled Livermore[3] and SUPO[4] are of course for different reactors; in particular, the heat capacity of the Livermore reactor is 20 per cent greater, and the pressure at Los Alamos is 59 cm Hg.

The theoretical curve in Fig. 2 was obtained by solving the reactor kinetic equations analytically for the case of an adiabatic transient without radiolytic-gas production. This approach, originally suggested by Mills[5] in connection with another problem, can also be employed to solve the equations including gas formation, but assuming infinite residence time for bubbles; this allows discussion of the case in which the reactor period is small compared to the residence time. The major result of this computation is that the product of peak power times reactor period is proportional to the reactivity step; the constant of proportionality is one-half the reciprocal of the energy coefficient of reactivity.

Using $-0.024\%/°C$ for the temperature coefficient of reactivity and 11.1 kcal/°C for the heat capacity of the fuel solution yields -0.5%/Mw-sec for the energy coefficient of reactivity associated with thermal expansion alone. Hence, peak power times period, in kilowatt-seconds, is 1000 times the reactivity step in per cent, as long as the time scale for the excursion is large enough that the formation of radiolytic gas does not contribute appreciably to reactor shutdown.

On the other hand, if the reactor period is sufficiently short that radiolytic-gas bubbles do not have time to escape from the solution and thus contribute significantly to the shutdown, a plot of peak power times period should be expected to yield a straight line, having a smaller slope determined by the total energy coefficient of reactivity.

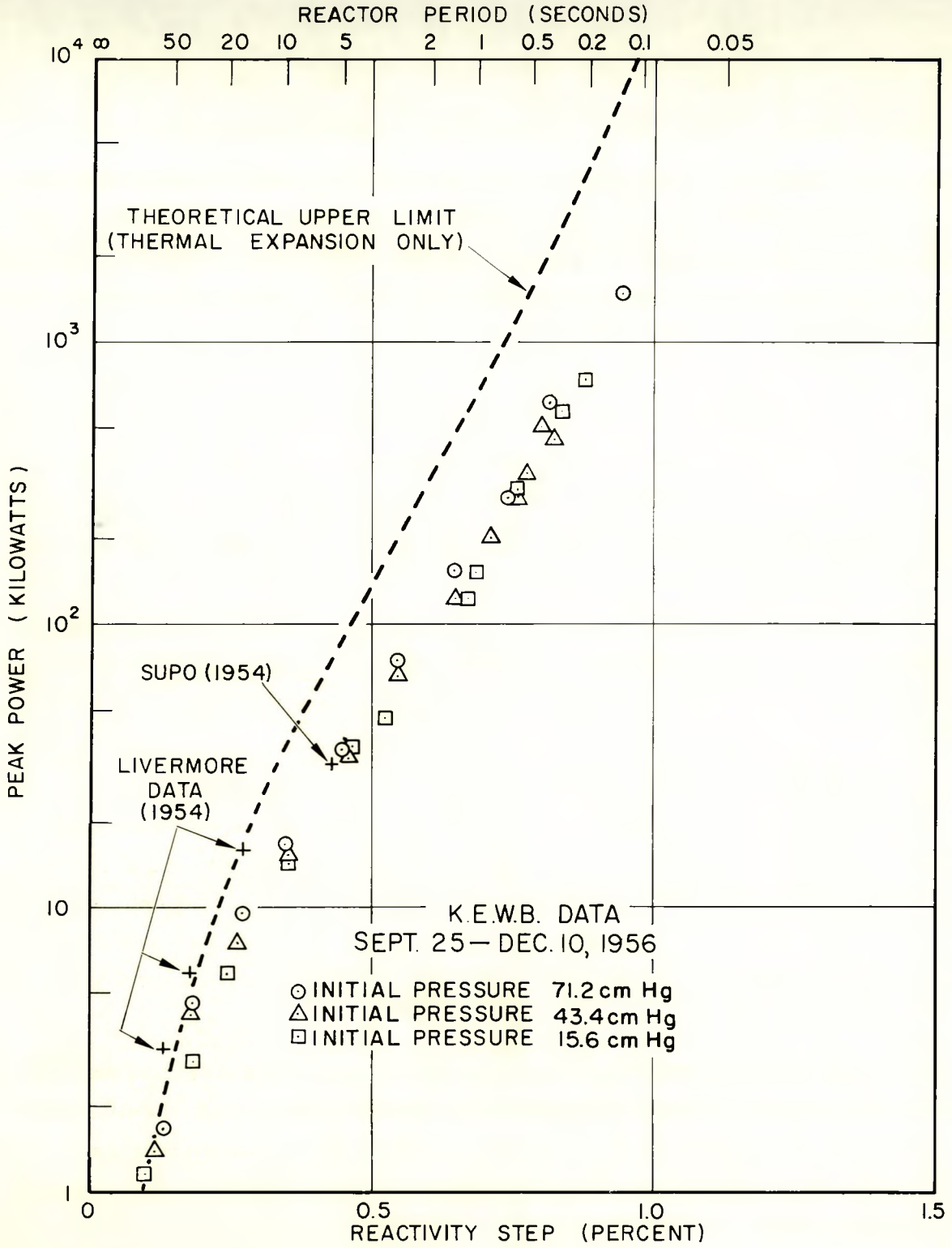

Fig. 2. Peak Power vs Reactivity Step for Various Pressures

Such is apparently the case, as seen in Fig. 3, 4, and 5, where the importance of radiolytic gas is evident for reactor periods smaller than 20 seconds. Since the theoretical model mentioned above is capable of dealing only with limiting cases, and not with a finite bubble residence time, the model cannot predict the intercepts for the straight lines that have been drawn among the data points.

These intercepts are certainly dependent upon the bubble residence time, and probably also dependent upon the heat capacity as well as the volume of the core. Note that in Fig. 3 the larger heat capacity of the Livermore reactor core yields a computed straight line for thermal expansion only, whose slope is 1260 instead of 1000. However, because of the larger core volume, the limited data available do not indicate the region at which the radiolytic-gas production would cause the curve for that reactor to deviate from the calculated curve for thermal expansion.

The ratios of slopes in Fig. 3, 4, and 5 have been used in preparing Table I, which shows the relative effectiveness of shutdown mechanisms for three different starting pressures.

TABLE I

RELATIVE EFFECTIVENESS OF SHUTDOWN MECHANISMS

Initial Pressure (cm Hg)	Ratio of Slopes in Fig. 3, 4, and 5	Effectiveness of Gas Production Relative to Thermal Expansion
15.6	11.8	10.8
43.4	9.5	8.5
71.2	8.0	7.0

Since most of the temperature coefficient of reactivity arises from the effect of core density on the nuclear parameters of the reactor, it is reasonable to estimate a density coefficient of reactivity by dividing the volume coefficient of thermal expansion for the solution into the temperature coefficient. This yields the result that at 30° C a one per cent volume change corresponds to 0.8 per cent reactivity. For the KEWB, this yields a distributed void volume coefficient of

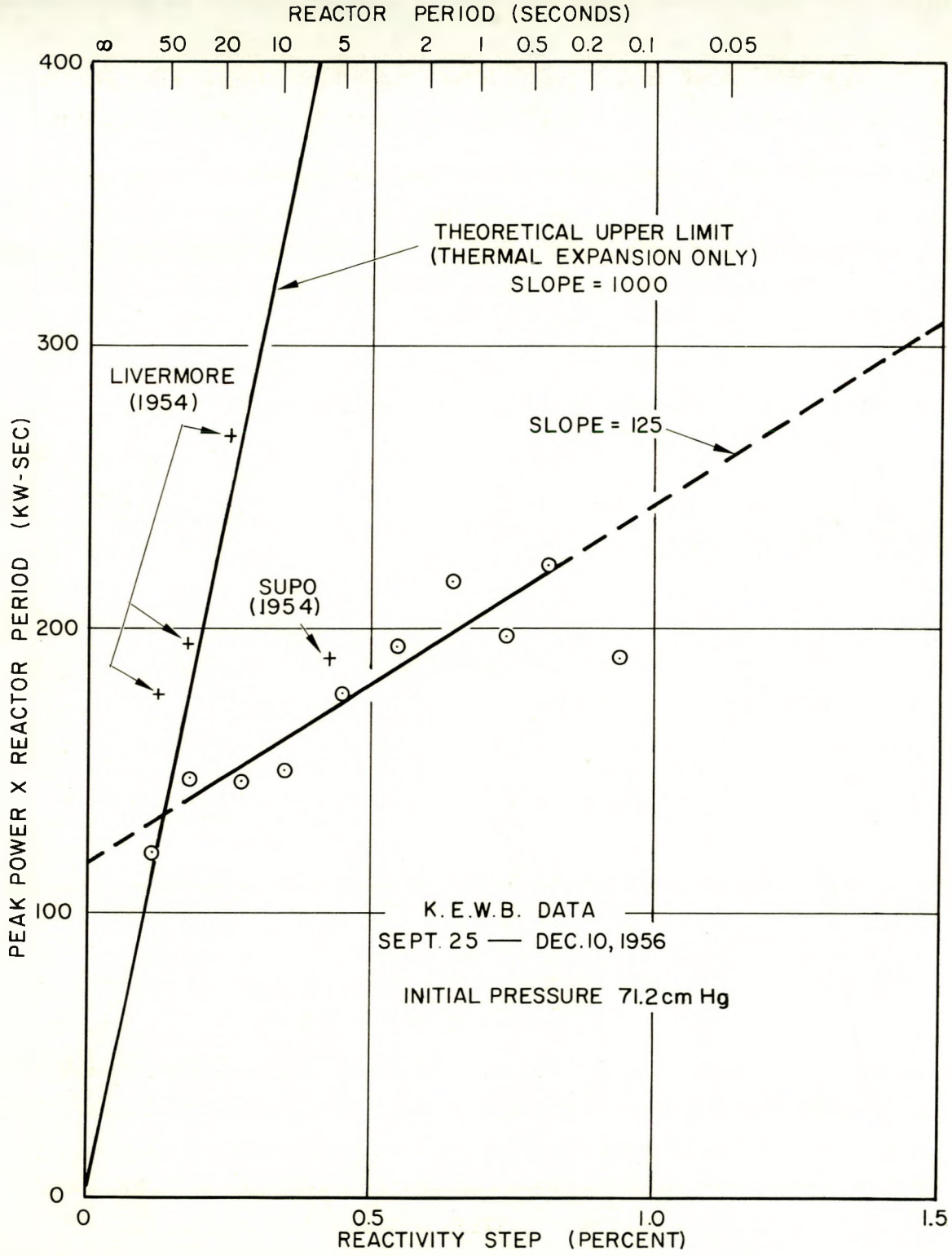

Fig. 3. Peak Power Times Reactor Period vs Reactivity Step at 71.2 cm Pressure

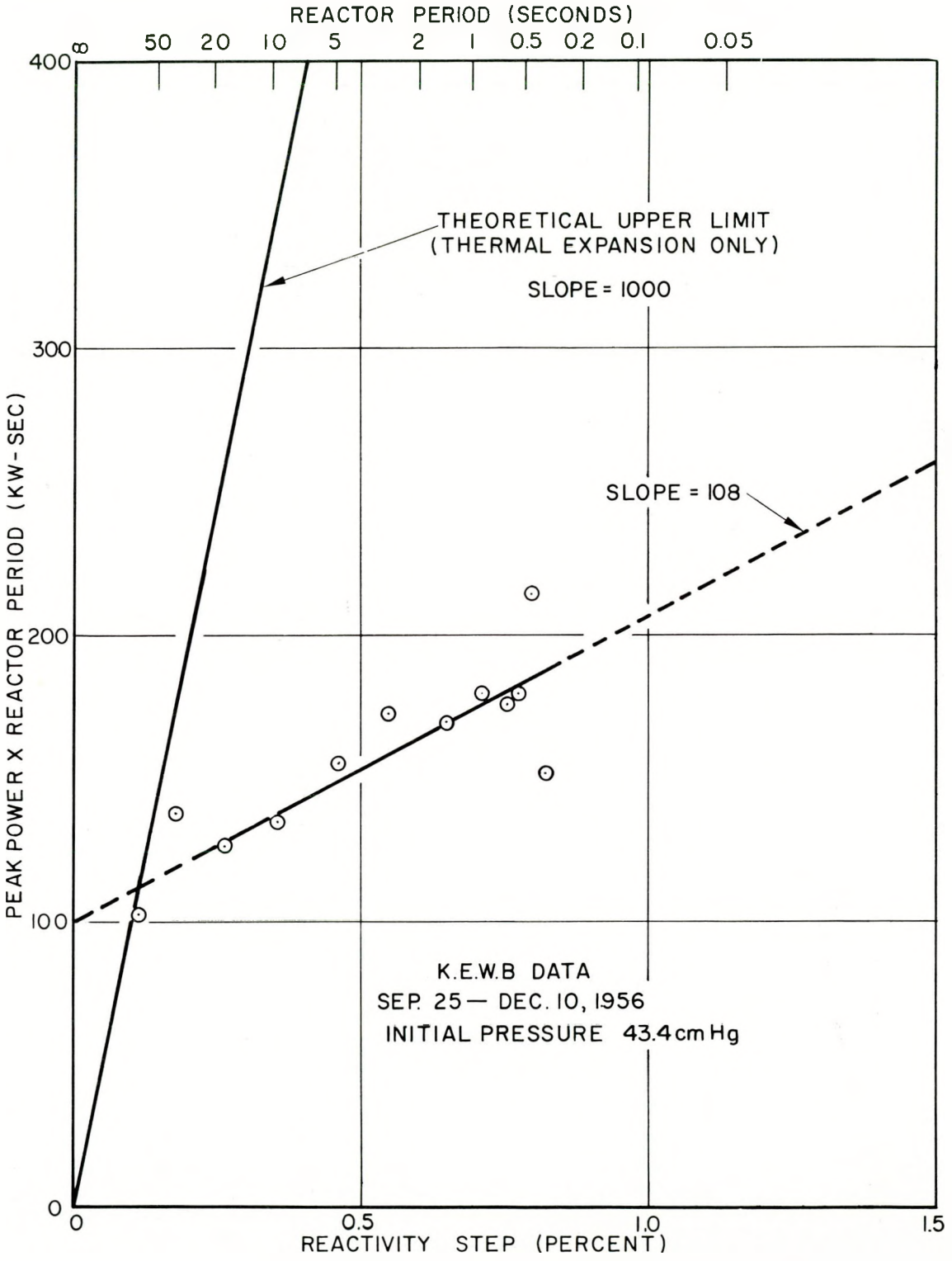

Fig. 4. Peak Power Times Reactor Period vs Reactivity Step at 43.4 cm Pressure

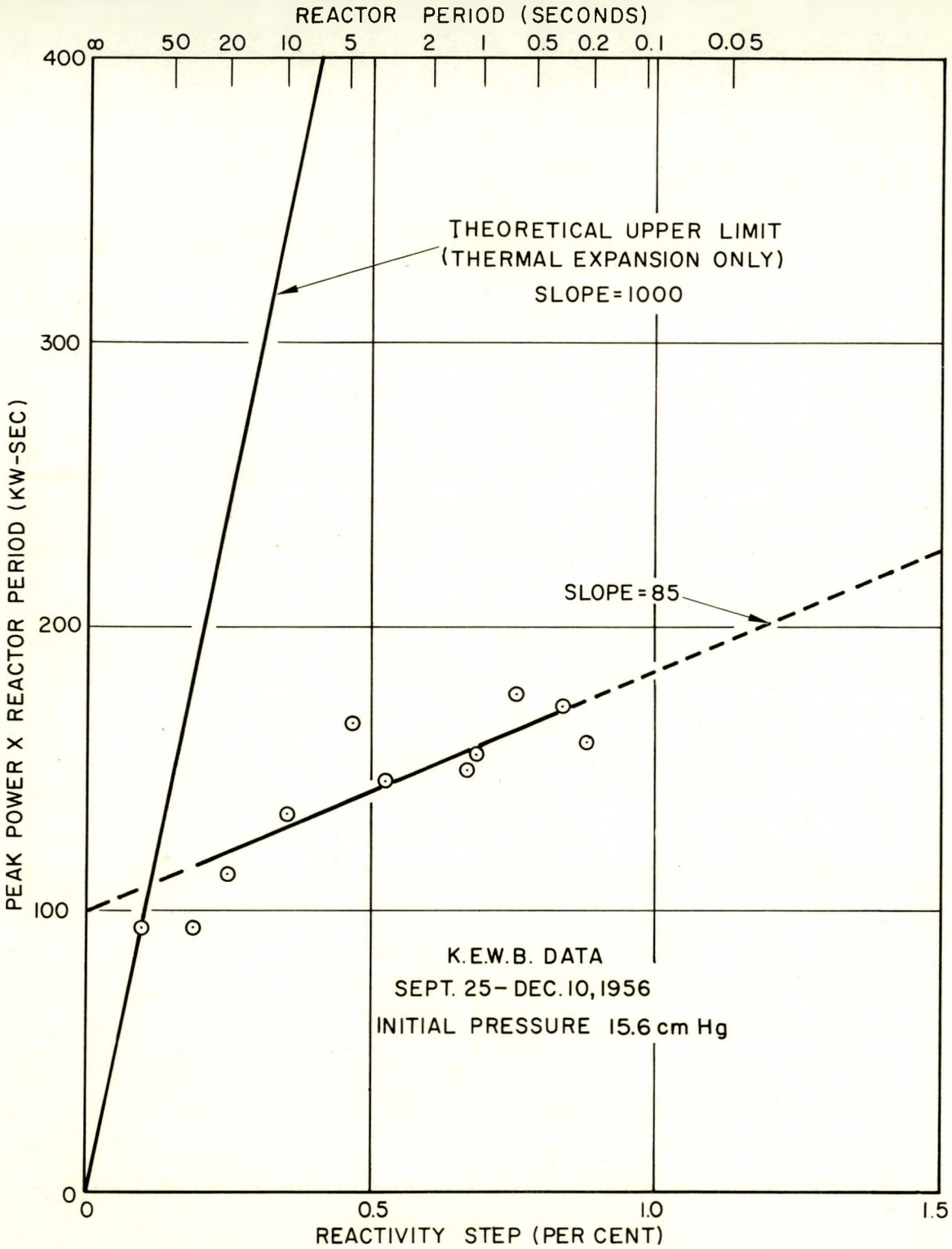

Fig. 5. Peak Power Times Reactor Period vs Reactivity Step at 15.6 cm Pressure

reactivity of -0.007% reactivity/cm^3 of void. This agrees well with a computation using two-group, two-region perturbation theory, which yielded the result -0.008% reactivity/cm^3 for a uniformly distributed density change.

Using $-0.007\%/cm^3$ as an approximate value, together with the observed rate of gas production in steady state (~ 4 cm^3/sec at one kilowatt), yields a rather naively computed "energy coefficient of reactivity" for gas production of -28% reactivity/Mw-sec! This number is 56 times as large as the energy coefficient of reactivity obtained from thermal expansion alone.

The fact that the effectiveness of gas production relative to thermal expansion, as shown in Table I, is much less than the above estimate of 56 indicates (1) that the effectiveness of radiolytic gas cannot be estimated by assuming that gas bubbles are instantaneously available for introducing negative reactivity, and (2) that the density change arising from gas production is not distributed in the same manner as that arising from thermal expansion. Another factor which must be included in improved calculations is provision for time-dependent coefficients in the reactor equations.

The original computations of transient behavior for KEWB were performed with a reduced void volume coefficient of reactivity,[1] somewhat arbitrarily estimated such that the ratio of energy coefficients was seven instead of 56. As the transient data recently obtained in KEWB indicate, the factor seven was a fortuitous choice. However, the model used in these calculations had no provision for time delay in radiolytic-gas bubble formation, and was incapable of predicting the damped oscillations that have been observed.[2]

G. C-SERIES TRANSIENTS

Fourteen transients were examined in this series with the objective of determining the dependence of peak power upon initial temperature. A standard reactivity input of 0.40 per cent was chosen as a convenient value, representing an average transient for the range which has been studied. Since it was desired to run this series out to the boiling point of water, the lowest initial pressure of the B-Series transients, 15.6 cm Hg, was selected as the standard initial pressure for this series. At this pressure, water boils at 60.9° C, a conveniently attainable temperature.

As it was not possible to obtain an *exact* reactivity input of 0.40 per cent for each of these transients, (average deviation during the tests was 0.015 per cent), it was necessary to normalize the observed peak power to that which would have occurred for a reactivity of exactly 0.40 per cent. The normalization was accomplished graphically using data from the B-Series transients. A line parallel to the curve in the B-Series data at a pressure of 15.6 cm Hg was constructed for each data point requiring normalization. This line passed through the point of peak power vs reactivity which had been observed in the C-Series, and its intercept at 0.40 per cent on the B-Series plot yielded the normalized peak power. A plot of normalized peak power vs initial core temperature is shown in Fig. 6.

This technique of using one set of data to normalize a second set, which is used in turn to normalize the first set, implies an infinite series of iterated operations. In this case however, the corrections are initially small and rapidly converging; no significant variations result by carrying the process further than indicated.

In all cases, these transients were run at progressively increasing temperatures, with the pressure adjusted to 15.6 cm Hg before each run. This procedure eliminated the history-dependent effect of gas saturation in the core solution (which later was found to be of importance), since increasing temperature leads to lower gas solubility, thus encouraging saturation.

The wide departure from the curve in Fig. 6 of the two points denoted by the squares is to be noted. These two experiments were the first runs of the two days during which the data were gathered, and no particular effort was taken to insure gas saturation of the core solution. Under these circumstances, the observed transient peak powers would be expected to be abnormally high. Since radiolytic gases would be taken up in saturating the core solution before any bubble formation and density change could be effected, the gas-shutdown mechanism would be delayed in time. The points are included only to illustrate the effect of sub-saturation of the fuel solution; they were not used in arriving at the curve placement.

As illustrated in Fig. 6, peak power decreases gradually with increasing initial core temperature, until the boiling point of the solution is approached. Near the boiling point, peak power falls off rapidly with increasing temperature. The peak power at an initial temperature of 60° C shows a reduction of 75 per cent from the room temperature value (25° C).

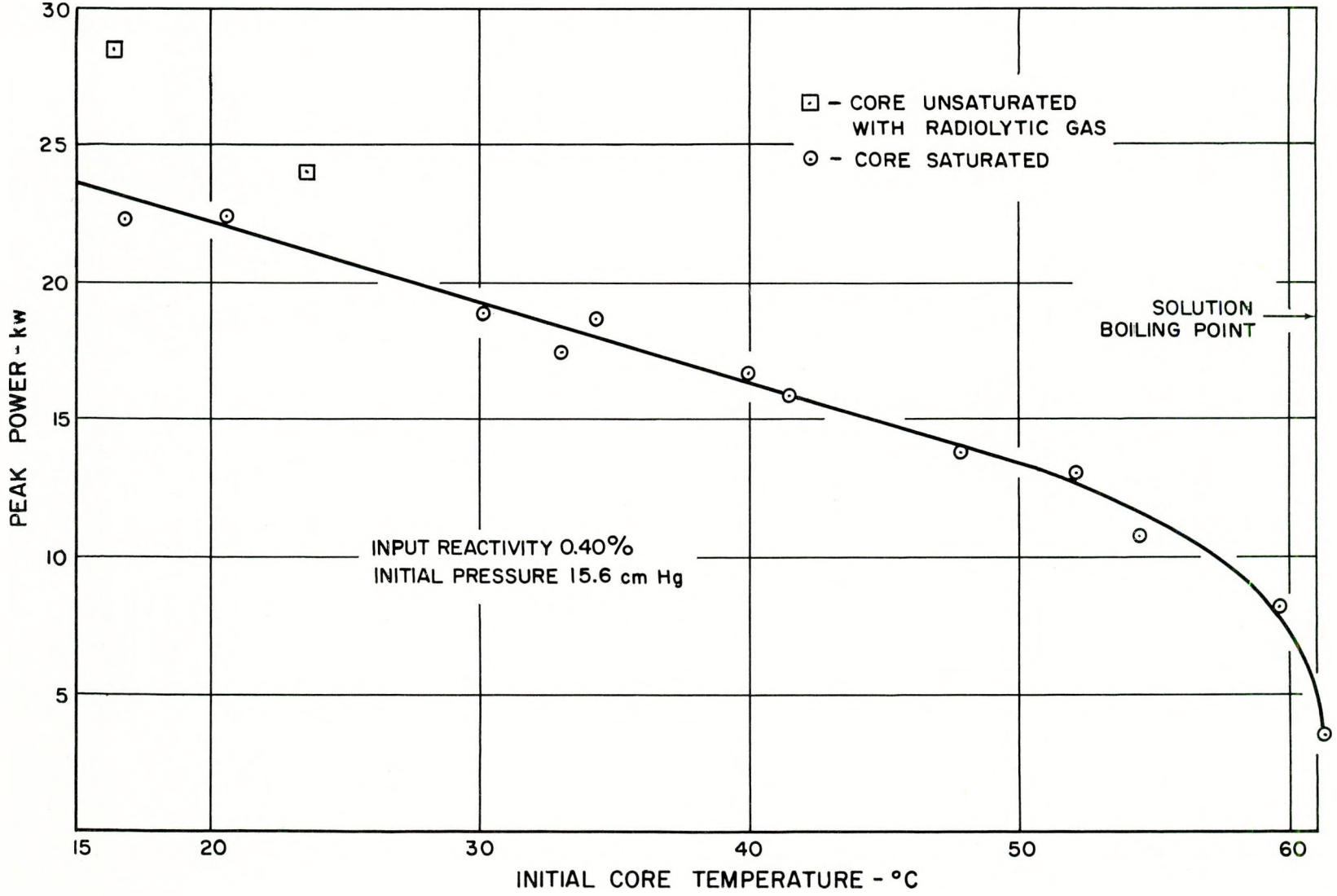

Fig. 6. Peak Power vs Initial Temperature for C-Series Transients

The sharp dependence of peak power upon initial temperature near the boiling point indicates the existence of an additional shutdown mechanism, namely the formation of water vapor in the core solution due to boiling during the transient. Sound-monitoring of the core has permitted detection of violent bubbling sounds during the high temperature transient peaks, which is interpreted to be actual boiling of the core solution.

H. GAS PRODUCTION COEFFICIENTS

The energy coefficient of gas production has been re-examined and the measurements expanded since the final core loading and power calibrations were completed. The reactor was operated at a variety of power levels, temperatures, and pressures with a calibrated void volume, including a ballast tank, of 38.1 liters above the core solution. Considerable difficulty was initially experienced in obtaining reproducible results. It was decided that the difficulty lay with the state of the dissolved gases in the core solution. The state of the dissolved gas could range from sub-saturated to super-saturated in the course of the experiment, introducing appreciable errors in the apparent effective value of the energy coefficient of gas production. A procedure was subsequently devised to control the core history, and in so doing eliminate this source of inconsistency by taking data only under the condition of complete saturation. The procedure consisted of making a preliminary run immediately preceding each experimental run which was expected to yield reliable data. The preliminary run had, in each case, parameters duplicating those of the data run which followed, and was terminated in a manner insuring saturation of the core solution with hydrogen and oxygen. The pressure was then adjusted by evacuating the system to the desired pressure, which further assured saturation. This procedure also excluded air dilution of the system. At no time was air admitted to the system to adjust pressure, the required pressure increases being accomplished by reactor operation. A time schedule was also developed, covering time from the beginning of the preliminary run to the last operation of the data run. The schedule prescribed the times of startup and shutdown, and of each measurement and valve procedure. Using this technique the results obtained are reproducible.

As noted in the discussion of the C-Series transients it appears that the state of the dissolved gases has a measurable effect on peak power resulting from a

given reactivity input. Future transient tests will be made under saturation conditions in order to eliminate this variable. Effects of sub-saturation will, of course, be examined periodically.

The experimental method used for determining the gas production coefficients consisted of observing the pressure change in a known volume at a known temperature. If it is assumed that the gases are ideal and solely occupy the volume, then the S. T. P. volume of gas produced is:

$$\frac{273\,°K \times V \times \Delta_p}{76\ cm\ Hg \times T}$$

where

V = system void volume, liters
Δ_p = pressure change, cm Hg absolute
T = system temperature, °K.

This figure is then divided by the integrated power of the run, in units of kilowatt hours, giving the number of S. T. P. liters of H_2 and O_2 produced per kilowatt-hour of reactor operation.

Due to the complex nature of the chemical reactions which are possible when the products of radiolytically-decomposed water are considered, no immediate effort is contemplated to define effects such as acidity and uranium concentration. Instead, the influence of pressure, temperature, and power density – all of which must be considered in a transient analysis of this particular core – are being examined.

The effect of pressure upon gas production has been completed. In a series of 10 experiments, the production rate decreased from 15.4 to 13.7 liters/kwh, as the initial core pressure was increased from 15.5 cm Hg absolute to 58 cm Hg. Figure 7 shows this decrease to appear essentially linear. The error limits indicate 95 per cent confidence limits for the mean point at each pressure. In each run, the pressure change was small (10 cm), and gas production is plotted vs final pressure. The system temperature was maintained at 30° C.

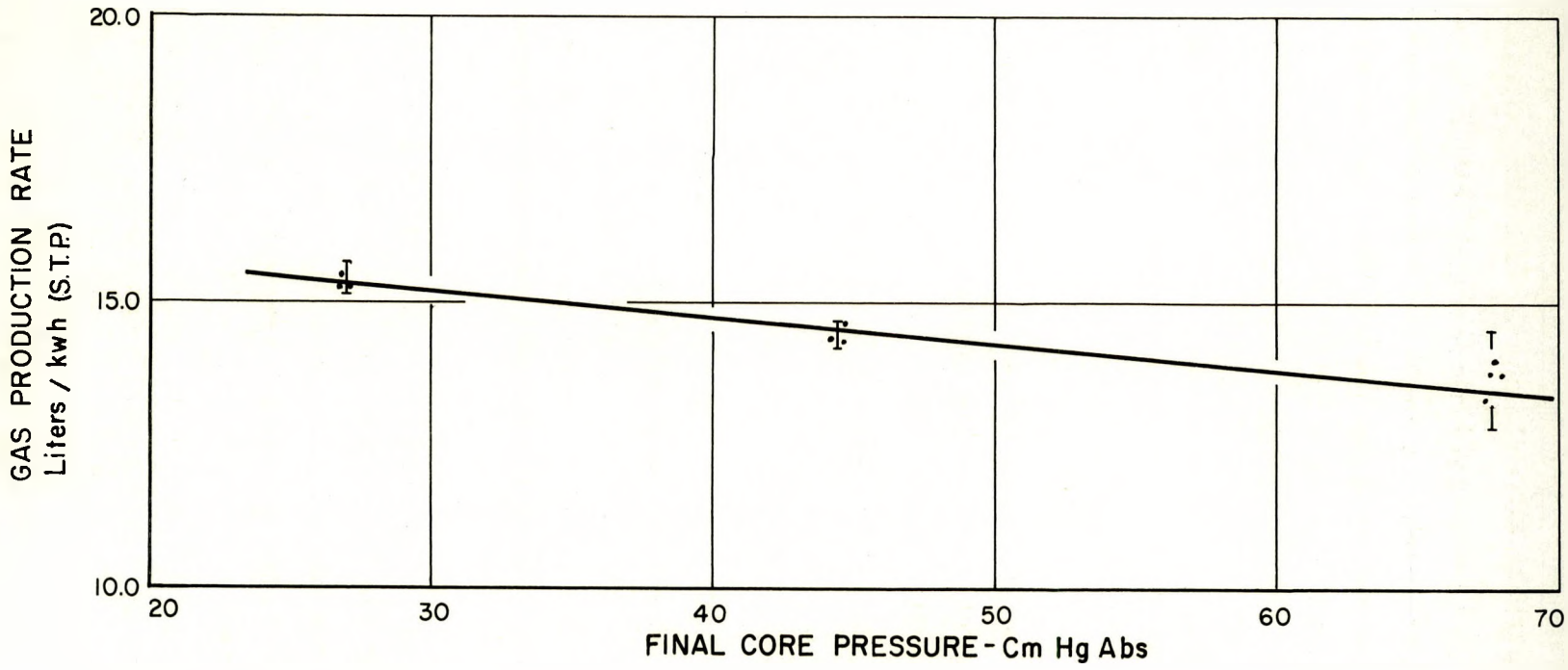

Fig. 7. Energy Coefficient of Gas Production vs Final Pressure

Gas production as a function of power density has also been investigated. Power levels between 500 watts and 20 kw were examined. Initial pressure was 15.5 cm in each case. There does not appear to be any significant variation of gas production within this range of powers. The maximum observed variation in the value was only 1.4 per cent. Because of limitations of the core coolant system, it was necessary to perform these experiments at a higher temperature (~36° C) than those preceding. Even so, the gas production value derived in these experiments was in good agreement with the values determined in the earlier tests which involved comparable final pressures.

APPENDIX I

DESCRIPTION OF KEWB I FUEL SOLUTION

The following describes the uranyl sulphate solution as loaded in the KEWB I reactor October 19, 1956, to give an excess reactivity of four per cent. These data result from experimental analysis of a sample of the fuel solution after completion of the reactor loading.

1. Volume of fuel solution in core, excluding 165 ml of solution in standpipe and drain line	11.5 liters
2. Concentration of fuel solution as determined by volumetric chemical analysis:	
Uranium concentration	176.5 ± 0.2 gm U/liter 0.7504 moles U/liter
Uranium-235 concentration	164.5 gm U^{235}/liter 0.7000 moles U^{235}/liter
Uranyl suphate concentration	272.5 gm UO_2SO_4/liter 21.59 wt per cent
3. Mass of uranium in core	2030 gm
Mass of uranium-235 in core	1892 gm
Gram-atoms uranium in core	8.631 gm-atom
Gram-atoms uranium-235 in core	8.051 gm-atom
4. Fuel density - ~25° C	1.262 gm/ml
5. Hydrogen-uranium ratio	H/U = 141.8
Hydrogen-uranium-235 ratio	H/U^{235} = 152.0
6. Sulphuric acid concentration	39.12 gm H_2SO_4/liter 0.399 moles/liter
7. Hydrogen ion concentration as determined with glass electrode	ph = 0.4
8. Viscosity	1.830 ± 0.060 centipoises @ 20° C 0.902 ± 0.016 centipoises @ 55° C
9. Specific heat between 20 and 24° C	0.797 cal/gm-°C

II. THEORETICAL REACTOR KINETICS STUDIES

(R. E. Skinner and D. L. Hetrick)

A. INTRODUCTION

Reactor kinetics theory provides the foundation for study and evaluation of reactor safety. Experience has shown that present theory is not adequate for such studies. This is due to a lack of knowledge, theoretical and experimental, of time dependent neutron behavior and of special physical and chemical phenomena related to reactor kinetic behavior.

Work was initiated in October 1956 on a project in basic reactor kinetics with the broad objectives of:

1) Fundamental theoretical studies to improve basic reactor kinetics theory. These are to include studies of the pile oscillator technique, development of new techniques of measuring the reactor parameters, and studies of special physical and chemical phenomena related to reactor kinetic theory, e.g., inertial stresses and bubble formation.

2) Application of results to safety and hazard studies of homogeneous reactor systems.

In selecting those problems to be studied during the first phase of the program, an attempt has been made to select problems which will aid in the design, analysis, and interpretation of reactor kinetics experiments with particular emphasis upon problems related to current experimental kinetic and safety programs. The problems which were selected for investigation during the first phase of the program are:

1) The theory of reactor transfer functions, the experimental determination of the functions, the use of the experimental transfer function to determine the reactor kinetic parameters, and the relationship between the transfer function and non-linear kinetic behavior.

2) The theory of a modified boron-bubble experiment and an analysis of the original experiment to make possible the independent measurement of the effective fraction of delayed neutrons.

3) Refinement of the theory of inertial stresses resulting from fast power transients with application to inertial pressure effects in homogeneous reactors and inertial thermal stresses in fast reactor fuel rods.

4) Theory of gas bubble formation and application to the kinetic behavior of homogeneous reactors.

5) The writing of a general reactor kinetics code using the space independent kinetic equations and application of the code in a general parameter survey of startup accidents.

B. CURRENT STATUS

During the report period theoretical investigations were started on three problems:

1) Refinement of the theory of the water boiler reactor transfer function and of the derivation of the space independent kinetic equations as given in Ref. 1.

2) The development of the theory of a modified boron-bubble experiment based on the original experiments conducted at Los Alamos[6] using neutron transport theory.

3) Development and evaluation of the theory of various means of determining the reactor kinetic parameters from the measured reactor transfer function. Particular emphasis was placed on the use of the generalized least squares process as the most promising method.

Details of the results obtained during the report period are covered in this report.

C. THE WATER BOILER TRANSFER FUNCTION AND KINETIC EQUATIONS

1. Introduction

The theory of the water boiler transfer function for the case of one delayed neutron group is given in Ref. 1 along with preliminary results for the case of six delayed neutron groups. Further work on the extension to six delayed groups has been carried out.[7]

It is the aim of the current investigation to refine the results of the above investigations and to examine methods, other than Bode diagrams, of exhibiting the transfer function. The results that have been obtained should form a good basis and reference point for understanding the implications, with regard to kinetic behavior, of the results of the intended study of radiolytic-gas bubble formation. Also, the results on the water boiler transfer function provide an excellent example for illustrating various aspects of the theory of measuring a reactor transfer function and the kinetic parameters.

2. The Non-Linear and Linear Space Independent Kinetic Equations

The space independent kinetic equations for a water boiler reactor for the case of m delayed neutron groups are taken as†

$$\frac{dP(t)}{dt} + \frac{\beta_{eff} - \rho(t)}{l} P(t) - \sum_{j=1}^{m} \lambda_j C_j(t) = 0, \qquad \ldots(1.1)$$

$$\frac{dC_j(t)}{dt} + \lambda_j C_j(t) = \frac{\beta_{eff} a_j}{l} P(t), (j = 1,2,\ldots,m). \qquad \ldots(1.2)$$

$$\frac{dT(t)}{dt} + \gamma T(t) = K[P(t) - P_0]. \qquad \ldots(1.3)$$

$$\frac{dV(t)}{dt} + \sigma V(t) = G[P(t) - P_0], \qquad \ldots(1.4)$$

and

$$\rho(t) = \rho_1(t) + \alpha T(t) + \phi V(t). \qquad \ldots(1.5)$$

This system of non-linear space independent kinetic equations can be linearized for the case

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† See Appendix II for the definition of the symbols appearing in these equations. The derivation and limitations of these equations are discussed in the later part of this section.

$$\frac{\delta P(t)}{P_0} << 1,$$

where $\delta P(t) = [P(t) - P_0]$, by neglecting the term $[\rho(t)\,\delta P(t)/l]$ in Eq. (1.1). Making this approximation, combining Eq. (1.1) and (1.5), and rewriting all equations in terms of deviations from equilibrium, it is found that the linerarized space independent kinetic equations are given by

$$\frac{d\delta P(t)}{dt} + \frac{\delta P(t)}{l^*} - \frac{\alpha^*}{l^*} P_0 T(t) - \frac{\phi^*}{l^*} P_0 V(t) - \sum_{j=1}^{m} \lambda_j \delta C_j(t) = \frac{P_0}{l^*} \rho_1^*(t), \qquad \ldots(2.1)$$

$$\frac{d\delta C_j(t)}{dt} + \lambda_j \delta C_j(t) = \frac{a_j}{l^*} \delta P(t), (j=1,2,\ldots,m), \qquad \ldots(2.2)$$

$$\frac{dT(t)}{dt} + \gamma T(t) = K\,\delta P(t), \qquad \ldots(2.3)$$

and

$$\frac{dV(t)}{dt} + \sigma \mathrm{V}(t) = G\,\delta P(t), \qquad \ldots(2.4)$$

where $\delta C_j(t) = C_j(t) - C_{j0}; C_{j0}$ the value of $C_j(t)$ for equilibrium operation at the fission rate P_0 .

3. The Transfer Function

The generalized definition of the transfer function $G(iw)$ in terms of Fourier transforms is given by

$$G(iw) = \frac{\mathcal{F}[r(t); iw]}{\mathcal{F}[f(t); iw]}, \qquad \ldots(3)$$

where $r(t)$ is the output function corresponding to the input function $f(t)$; $\mathcal{F}[f(t); iw]$ is the Fourier transform of $f(t)$ and is given by

$$F(iw) = \mathcal{F}[f(t); iw] = \int_{-\infty}^{\infty} f(t)\, e^{-iwt} dt \ ; \qquad \ldots(4.1)$$

and the inverse transform $f(t) = \frac{1}{2\pi}\mathcal{F}[F(iw); -it]$. is given by

$$f(t) = \frac{1}{2\pi}\mathcal{F}[F(iw); -it] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(iw)\, e^{iwt} dw. \qquad \ldots(4.2)$$

Further it is seen that for all functions $f(t)$ and $r(t)$ whose Laplace transforms exist[8] the definition of the transfer function $G(iw)$, given by Eq. (3), reduces to the usual definition in terms of Laplace transforms upon setting $s = iw$. Thus,

$$G(iw) = G(s) = \frac{\mathcal{L}[r(t); s]}{\mathcal{L}[f(t); s]}, \qquad \ldots(5)$$

where $\mathcal{L}[f(t); s]$ is the Laplace transform of $f(t)$.

It can be shown that the transfer function $G(iw)$ as defined by Eq. (3), for $w = \omega$. , ω real, is given by the response function $A(\omega)$ of the system to pure oscillatory inputs, $f(t) = f_0 e^{i\omega t}$, divided by the input amplitude f_0, i.e.,

$$G(i\omega) = \frac{A(\omega)}{f_0} \ . \qquad \ldots(6)$$

Thus, it is seen that the important aspects of the generalized definition, Eq. (3), of the transfer function are: (1) it provides in explicit form, in terms of the transforms of the corresponding input and output functions, the analytic continuation of the transfer function as defined by Laplace transforms, and

(2) it provides the mathematical foundation and justification for combining data from pile oscillator experiments and other types of experiments when measuring the transfer function for the case $w = \omega, \omega$ real.

4. The Water Boiler Transfer Function

For a physical system that is governed by a system of linear differential equations there exist a number of variables. Consequently, we may define a transfer function for each variable or certain combinations of variables. For present purposes, it is sufficient to define

1. The power transfer function, $G_{\bar{P}}(iw; P_0)$.

$$G_{\bar{P}}(iw; P_0) = \frac{\mathcal{F}[\delta P_1(t); iw]}{\mathcal{F}[\rho_1(t); iw]}, \qquad \ldots(7.1)$$

2. The normalized power transfer function (reactivity in dollars), $G_P(iw; P_0)$.

$$G_P(iw; P_0) = \frac{\beta_{eff}}{P_0} G_{\bar{P}}(iw; P_0). \qquad \ldots(7.2)$$

3. The temperature loop transfer function, $G_T(iw; P_0)$,

$$G_T(iw; P_0) = \frac{\mathcal{F}[T_1(t); iw]}{\mathcal{F}[\delta P_1(t); iw]}. \qquad \ldots(7.3)$$

with $\delta P_1(t)$ the power response and $T_1(t)$ the variation in the space average temperature as a function of time for the reactivity input, $\rho_1(t)$, at the equilibrium fission rate, P_0.

Using the above definitions and taking the Fourier transforms of the linear system, Eq. (2.1) through (2.4), it is found that the normalized water boiler reactor power transfer function is given by

$$G_P(iw; P_0) = \left[iwl^* + iw \sum_{j=1}^{m} \frac{a_j}{iw + \lambda_j} - \frac{\alpha^* K P_0}{iw + \gamma} - \frac{\phi^* G P_0}{iw + \sigma} \right]^{-1} . \qquad \ldots (8.1)$$

and the temperature loop transfer function by

$$G_T(iw; P_0) = \frac{K}{iw + \gamma} \quad . \qquad \ldots (8.2)$$

Letting $iw = u + i\omega$, **Eq. (8.1) and (8.2) can be written in the useful form**

$$G_P(u + i\omega; P_0) = \{ [E_1(u + i\omega; P_0)]^2 + [E_2(u + i\omega; P_0)]^2 \}^{-1/2} \exp [i\delta_1(u + i\omega; P_0)] \quad , \qquad \ldots (9.1)$$

and

$$G_T(u + i\omega; P_0) = K [\omega^2 + (\gamma + u)^2]^{-1/2} \exp [i\delta_2(u + i\omega; P_0)] \quad , \qquad \ldots (9.2)$$

where

$$\delta_1(u + i\omega; P_0) = \tan^{-1} [-E_2/E_1]. \qquad \ldots (9.3)$$

$$\delta_2(u + i\omega; P_0) = \tan^{-1} [-\omega/(\gamma + u)]. \qquad \ldots (9.4)$$

$$E_1(u + i\omega; P_0) = \left[l^* u + \sum_{j=1}^{m} \frac{a_j [(\lambda_j + u) u + \omega^2]}{\omega^2 + (\lambda_j + u)^2} - \frac{(\gamma + u) \overline{K}}{\omega^2 + (\gamma + u)^2} - \frac{(\sigma + u) \overline{G}}{\omega^2 + (\sigma + u)^2} \right], \qquad \ldots (9.5)$$

$$E_2(u + i\omega; P_0) = \omega \left[l^* + \sum_{j=1}^{m} \frac{a_j \lambda_j}{\omega^2 + (\lambda_j + u)^2} + \frac{\overline{K}}{\omega^2 + (\lambda + u)^2} + \frac{\overline{G}}{\omega^2 + (\sigma + u)^2} \right], \qquad \ldots (9.6)$$

and

$$K = K\alpha^* P_0, \bar{G} = G\phi^* P_0. \qquad \ldots(9.7)$$

From Eq. (9.1) through (9.7) it follows immediately that the water boiler transfer function depends only on the kinetic parameters $l^*, K\alpha^*, G\phi^*, \lambda_j, a_j,$ and the equilibrium power P_0. Further, by means of any measurements based on the space independent kinetic equations, it is possible to measure the reactivity only in units of "dollars"; that is, to measure ρ^*. Consequently, it is possible to measure within the framework of the space independent kinetic equations only the quantities $l^*, K\alpha^*, G\phi^*, \lambda_j, a_j,$ and ρ^*. The parameter G is easily measured, and the parameter K can be determined from the measurement of G_T. As a result, the parameters α^* and ϕ^* can be determined by the addition of two more measurements, one of which is kinetic.

The following asymptotic expressions have been obtained; for the case $u = 0$; as $\omega \to 0$

$$|G_P(i\omega; P_0)| \sim \begin{cases} [(\bar{K}/\gamma) + (\bar{G}/\sigma)]^{-1}; & P_0 \neq 0 \qquad \ldots(10.1) \\ \left\{ \omega \left[l^* + \sum_{j=1}^{m} (a_j/\lambda_j) \right] \right\}^{-1}; & P_0 = 0 \qquad \ldots(10.2) \end{cases}$$

$$\tan \delta_1 (i\omega; P_0) \sim \begin{cases} \dfrac{\omega \left\{ l^* + \sum_{j=1}^{m} (a_j/\lambda_j) + \left[(\bar{K}/\gamma^2) + (\bar{G}/\sigma^2) \right] \right\}}{[\bar{K}/\gamma) + (\bar{G}/\sigma)]} \to 0; P_0 \neq 0 & \ldots(10.3) \\ -\dfrac{\left[l^* + \sum_{j=1}^{m} (a_j/\lambda_j) \right]}{\omega \sum_{j=1}^{m} (a_j/\lambda_j^2)} \to \infty ; & P_0 = 0 \qquad \ldots(10.4) \end{cases}$$

as $\omega \to \infty$

$$|G_P(i\omega; P_0)| \sim \frac{1}{\omega l^*} \qquad \ldots(10.5)$$

$$\tan \delta_1 (i\omega; P_0) \sim (-\omega l^*). \qquad \ldots(10.6)$$

The quantity $[(K\alpha^*/\gamma) + (G\phi^*/\sigma)]$ appearing in Eq. (10.1) and (10.3) is called the zero frequency power coefficient of reactivity.

Equation (10.3) and (10.4) are special cases of the general result that, if $10^n \omega$ is equal to or less than the smallest of $\lambda_1, \ldots, \lambda_m, \gamma$, and σ, (n = 1, 2, ...) ; then

$$\tan \delta_1 (i\omega; P_0) \cong \frac{-\omega \left\{ l^* + \sum_{j=1}^{m} (a_j/\lambda_j) + [(\bar{K}/\gamma^2 + (\bar{G}/\sigma^2)] \right\}}{\left\{ \omega^2 \sum_{j=1}^{m} (a_j/\lambda_j^2) - [(\bar{K}/\gamma) + (\bar{G}/\sigma)] \right\}} . \qquad \ldots(11)$$

to better than (n + 1) significant figures. Further, for small enough P_0 and angular frequencies larger than those for which Eq. (10.1) through (10.4) are valid but less than the smallestofthe λ_j, it is found that

$$|G_P (i\omega; P_0)| \sim \left\{ \omega \left[l^* + \sum_{j=1}^{m} (a_j/\lambda_j) \right] \right\}^{-1} \qquad \ldots(11.1)$$

and

$$\tan \delta_1 (i\omega; P_0) \sim \left\{ -\omega \left[l^* + \sum_{j=1}^{m} (a_j/\lambda_j) \right] \right\} \left\{ \omega^2 \sum_{j=1}^{m} (a_j/\lambda_j^2) - [(\bar{K}/\gamma) + (\bar{G}/\sigma)] \right\}^{-1} . \qquad \ldots(11.2)$$

If P_0 be small enough that $[(\bar{K}/\gamma) + (\bar{G}/\sigma)] << \omega^2$, Eq. (11.2) reduces to

$$\tan \delta_1 (i\omega; P_0) \sim -\left[l^* + \sum_{j=1}^{m} (a_j/\lambda_j) \right] \left[\omega \sum_{j=1}^{m} (a_j/\lambda_j^2) \right]^{-1} . \qquad \ldots(11.3)$$

If $\ell^* << 1/\lambda$,Eq. (11.1) reduces to

$$|G_P(i\omega; P_0)| \sim \lambda/\omega, \qquad \ldots(11.4)$$

and Eq. (11.3) becomes

$$\tan \delta_1 (i\omega; P_0) \sim -\left[\omega\lambda \sum_{j=1}^{m} (a_j/\lambda_j^2)\right]^{-1}, \qquad \ldots(11.5)$$

where

$$1/\lambda = \sum_{j=1}^{m} (a_j/\lambda_j) \quad ; \qquad \ldots(11.6)$$

λ the "equivalent" one group decay constant. The major difference between the transfer function for m delayed neutron groups and the one delayed neutron group transfer function is displayed in Eq. (11.2), (11.3), or (11.5). This difference results from the fact that $(1/\lambda^2) \neq \sum_{j=1}^{m} (a_j/\lambda_j^2)$. Numerically it is found using the data given in Table II that $(1/\lambda^2) = 145$ while $\sum_{j=1}^{6} (a_j/\lambda_j^2) = 417$.

In Fig. 8 and 9 are plotted the Bode diagrams (amplitude and phase shift) of $G_P(i\omega; P_0)$ for $P_0 = 10^{-3}, 10^{-2}, 10^{-1}, 1, 10,$ and 50 kw. The values of the delayed neutron constants and of the kinetic parameters used in these and all other calculations for the case of six delayed neutron groups are given in Tables II and III respectively. It has been assumed here for purposes of calculation that the kinetic parameters are independent of power. The intercepts of the various asymptotes given by Eq. (10.1) through (10.6), (11.1) and (11.2) are indicated.

The resonance in the transfer function that occurs for large enough P_o (P_o = 10 *kw and* P_o = 50 *kw in Fig. 8*) indicates that for such P_0 the reactor is less than critically damped *in the linear approximation.* Hence, for such P_0 the transient solution contains a damped oscillatory term with an angular frequency equal to the angular frequency at which the maximum of the resonance occurs. If both α^* and φ^* are negative, as they are for the water boiler, then the

TABLE II

PROPERTIES OF DELAYED NEUTRONS IN THERMAL FISSION OF U^{235} †

Group j	Half Life T_j Sec	Mean Lifetime τ_j Sec	Decay Constant λ_j Sec^{-1}	Fraction a_j
1	54.31	78.338	0.01276	0.035
2	19.78	28.538	0.03504	0.230
3	5.50	7.935	0.126	0.180
4	2.11	3.044	0.3285	0.406
5	0.434	0.6334	1.579	0.124
6	0.114	0.1645	6.079	0.025

† These values are as taken from Ref. 9. The values given in the post-conference copy of Ref. 9 and more recent values[10] are slightly different.

TABLE III

VALUES OF THE KINETIC PARAMETERS USED IN THE CALCULATIONS FOR THE WATER BOILER REACTOR TRANSFER FUNCTION††

$\beta_{eff} = 0.0085$	$\alpha = -2.6 \times 10^{-4}$ per °C
$l = 1.2 \times 10^{-4}$ sec	$G = 4.16\,cm^3$/kw-sec at STP
$K = 0.01428$° C/kw-sec	$\sigma = 0.5\ sec^{-1}$ at STP
$\gamma = 10^{-2}\ sec^{-1}$	$\varphi = -7 \times 10^{-6}\ cm^{-3}$ at STP

†† Except for β_{eff} and K the values of the parameters are as given in Ref. 1. The value of β_{eff} is that measured for Lopo.[6]

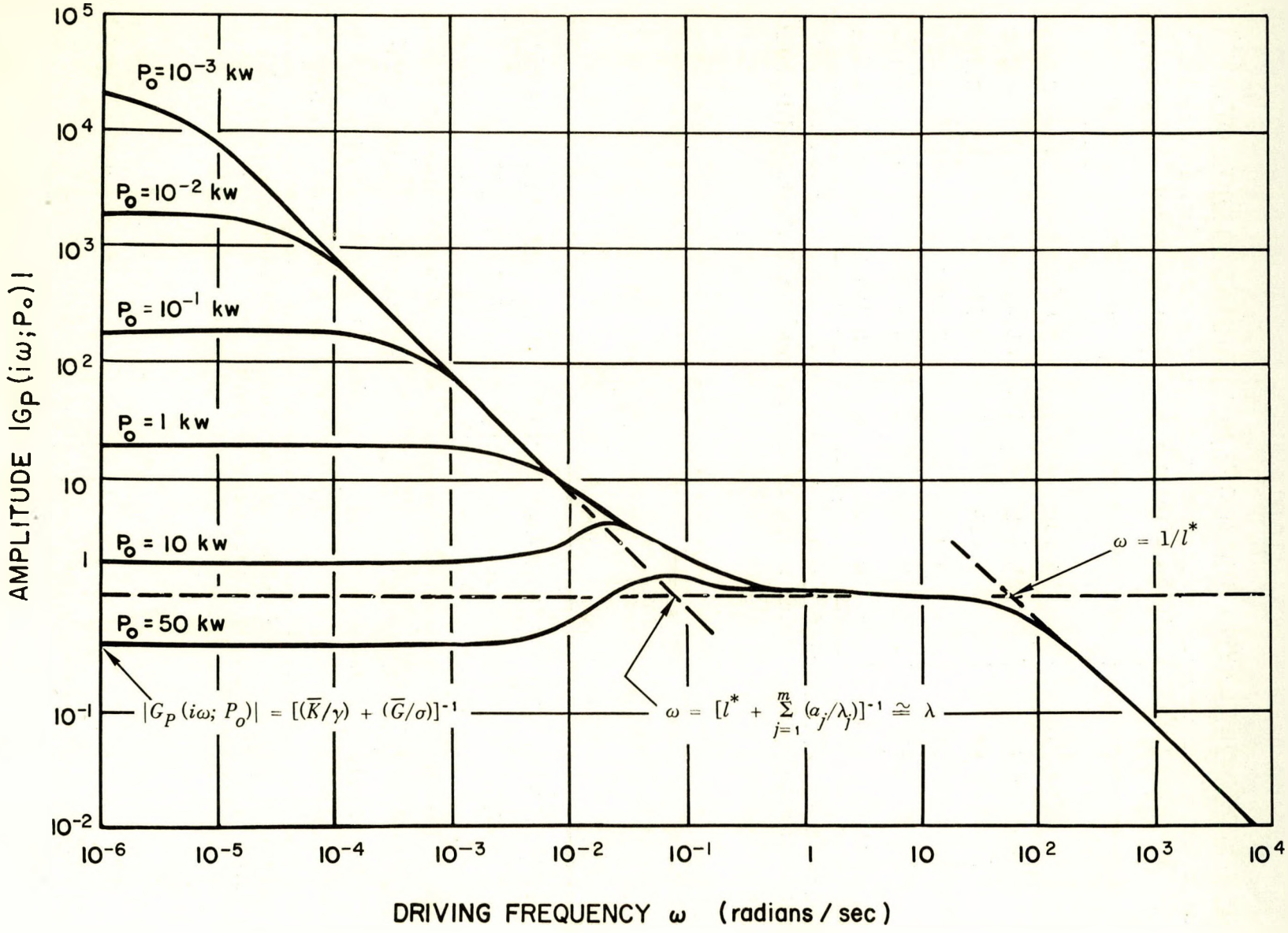

Fig. 8. Normalized Water Boiler Reactor Power Transfer Function – Amplitude $G_P(iw; P_o)$ – (6 groups of delayed neutrons)

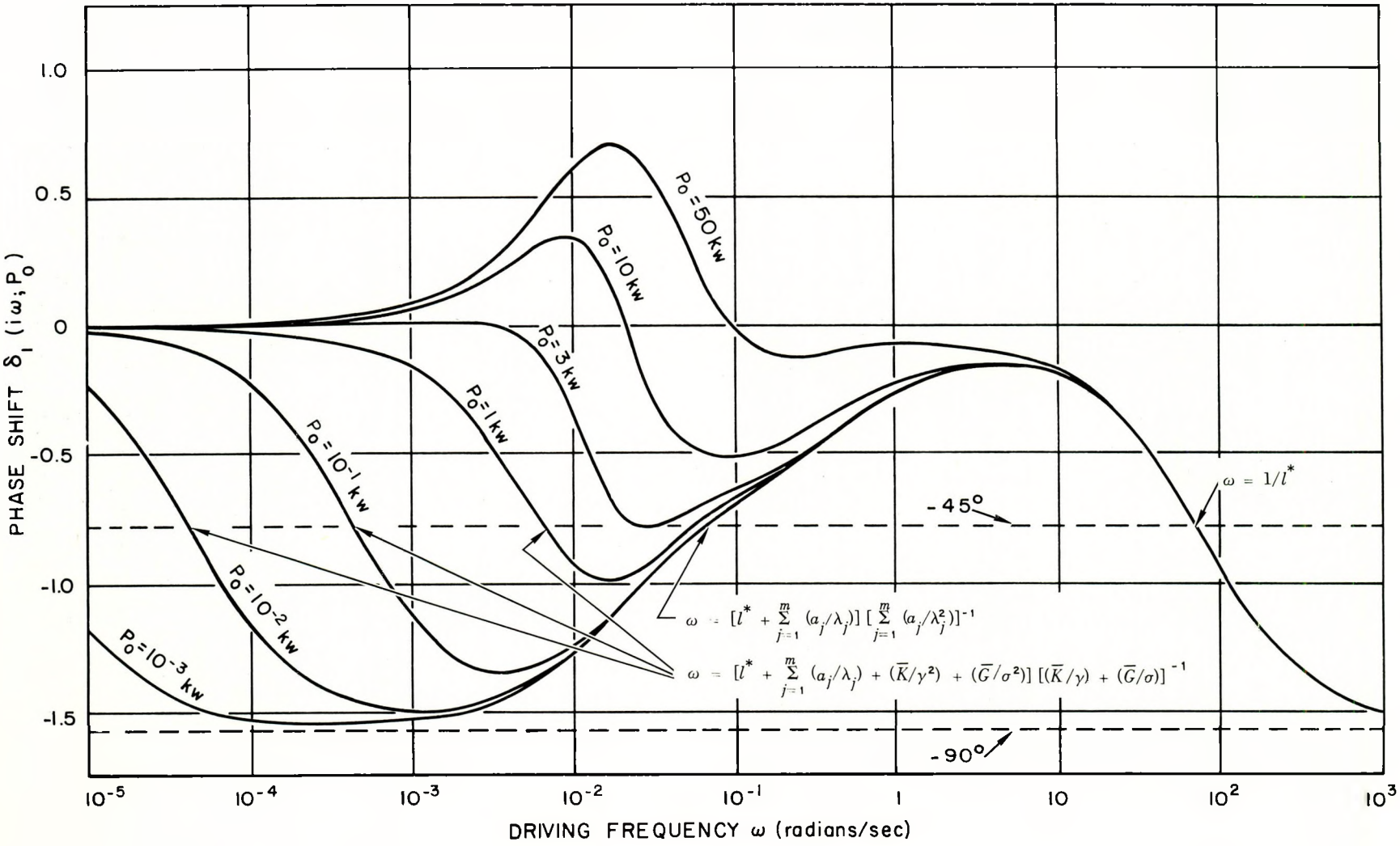

Fig. 9. Phase Shift $\delta_1(i\omega;P_o)$, for Water Boiler Reactor Power Transfer Function – (6 groups of delayed neutrons)

non-linear transient solution will also contain a damped oscillatory term. No infinite resonance as discussed by Bethe[11] is possible, since both α^* and φ^* are negative and there is no transport delay time in the model under consideration. It is interesting to note that there appears to be a correlation between the maximum of the resonance and the frequency at which $\delta_1 = 0$. However, no equation has so far been found to show that this relationship is a general one. Also, for the case $P_0 = 50\,\text{kw}$, a second very wide resonance is beginning to appear with a maximum at $\omega \approx 5 \times 10^{-1}\ rad/sec$. Hence, for large enough P_0 the transient solution in the linear approximation will contain two damped oscillatory terms with angular frequencies equal to the angular frequencies at which the maxima of the resonances occur. Also, since α^* and φ^* are negative, the non-linear transient solution will contain at least two damped oscillatory terms.

In Fig. 10 is plotted a log plot of $|\tan \delta_1|$ as a function of the angular frequency ω. It is to be noted that the result for the intercept of the high frequency asymptote of the log log plot of $|\tan \delta_1|$ as a function of ω is completely general, while the result $\delta_1 = 45^c$ at $\omega = (1/l^*)$ in the high frequency region is not valid in general for sufficiently large P_0. The same remark holds for the low frequency asymptotes of $|\tan \delta_1|$. In Fig. 11 are shown the log log plots of $|G_T(i\omega; P_0)|$ and $|\tan \delta_2|$ with the intercepts of the asymptotes indicated.

5. Other Representations

In Fig. 12 is shown a polar plot of $G_P(i\omega; P_0)^{-1}$, , the inverse of the normalized water boiler reactor power transfer function.† For the one delayed neutron group model it can be shown that at zero power, the inverse polar plot for small angular frequencies is a semi-circle passing through the origin with its diameter along the polar axis. At higher frequencies this semi-circle becomes distorted and finally for high enough frequencies becomes the straight vertical line seen in Fig. 12.

Another useful way to treat a reactor is as two complex impedances in parallel; one is the complex impedance presented to the reactivity input function $\rho_1^*(t)$ due to the effects of prompt and delayed neutrons alone, $Z_0(iw)$; the other is the impedance due to power reactivity feedback effects, $Z_P(iw; P_0)$. For a water boiler the power reactivity feedback effects are those due to temperature

† In Fig. 12, 14, and 16 the solid line represents the function for the frequency range $0 \leq \omega \leq \infty$ and the dashed line represents the frequency range $-\infty < \omega \leq 0$.

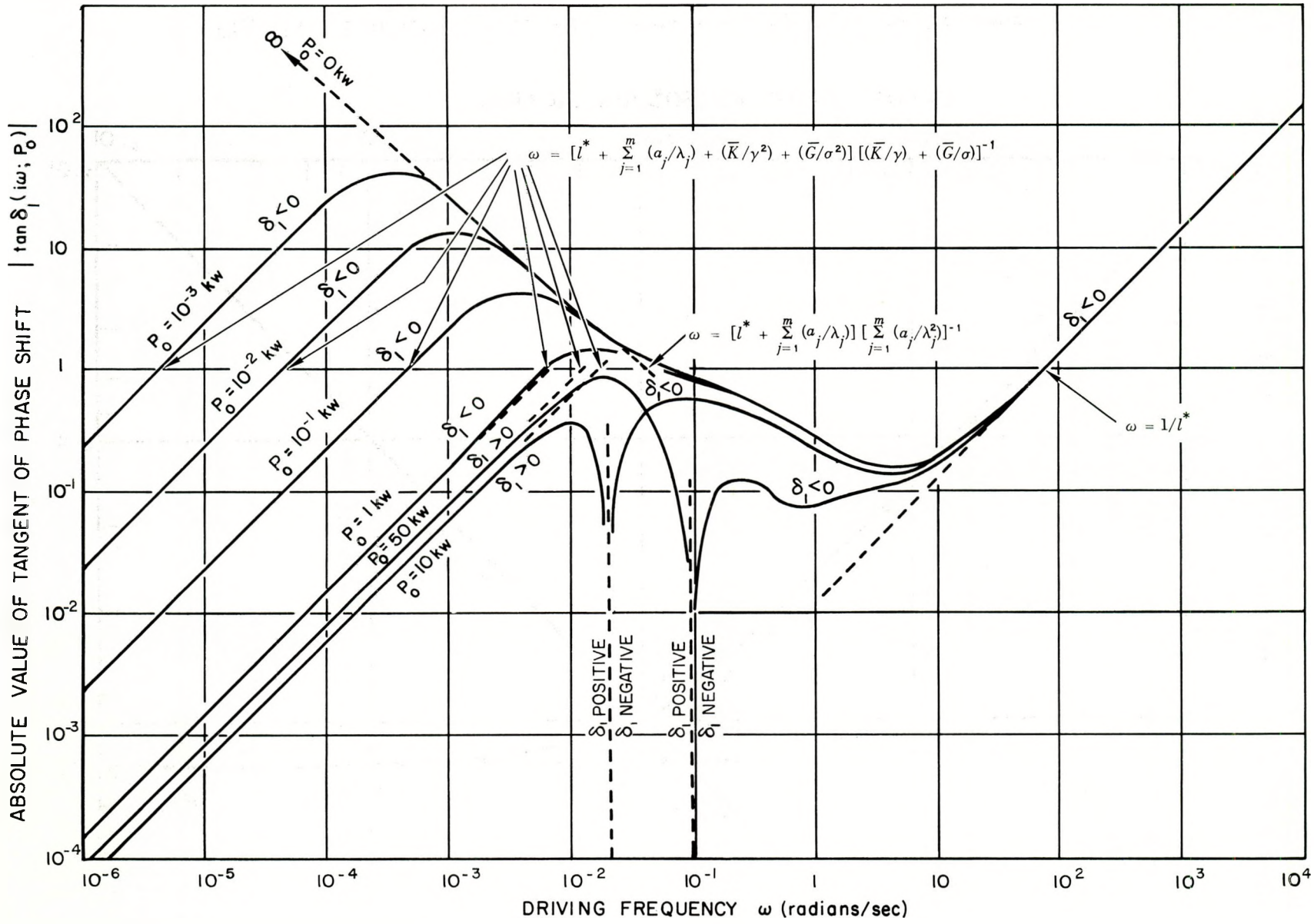

Fig. 10. Tangent of Phase Shift for Water Boiler Reactor Power Transfer Function – (6 groups of delayed neutrons)

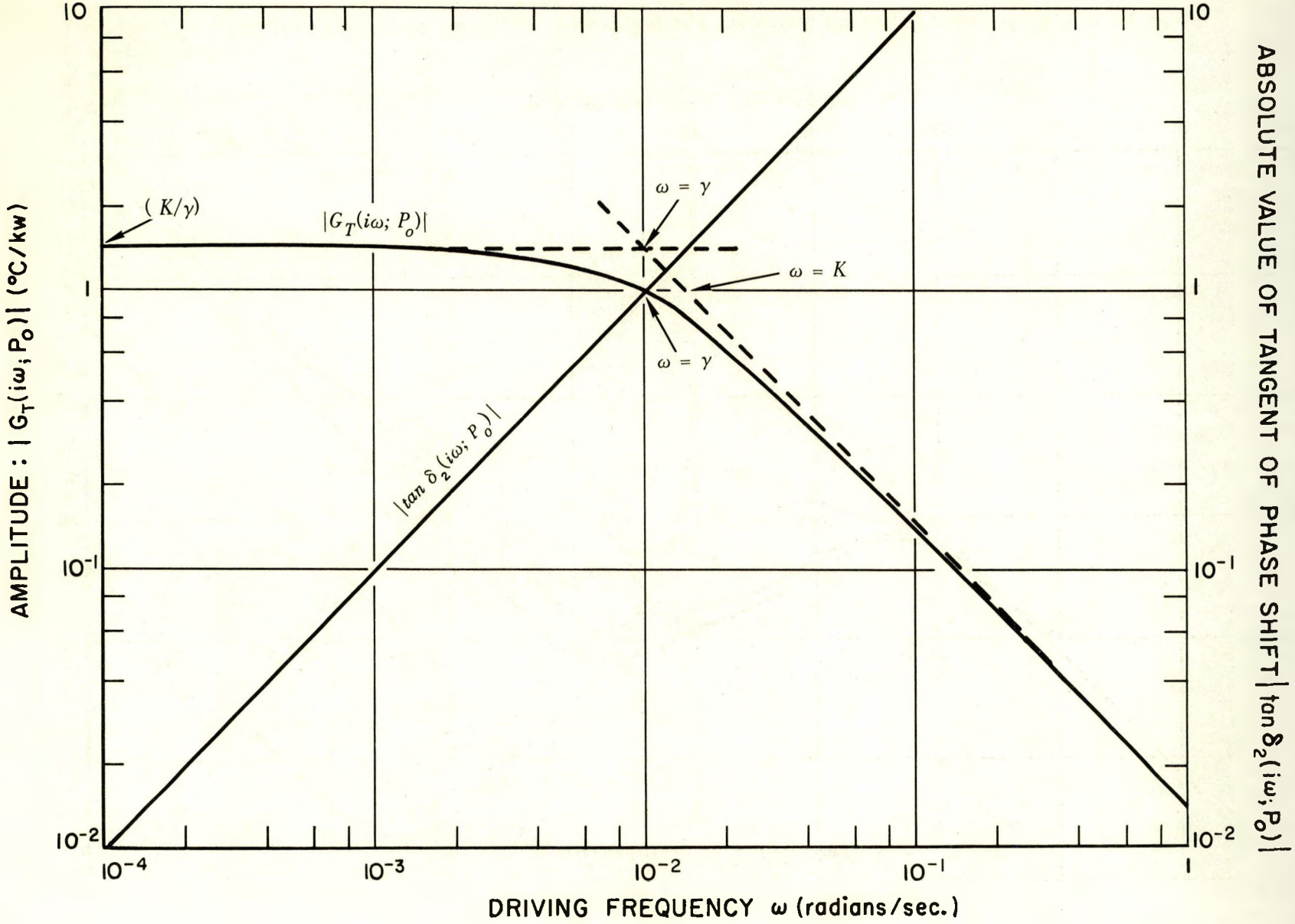

Fig. 11. Temperature Transfer Function Amplitude and Tangent of Phase Shift for Water Boiler Reactor

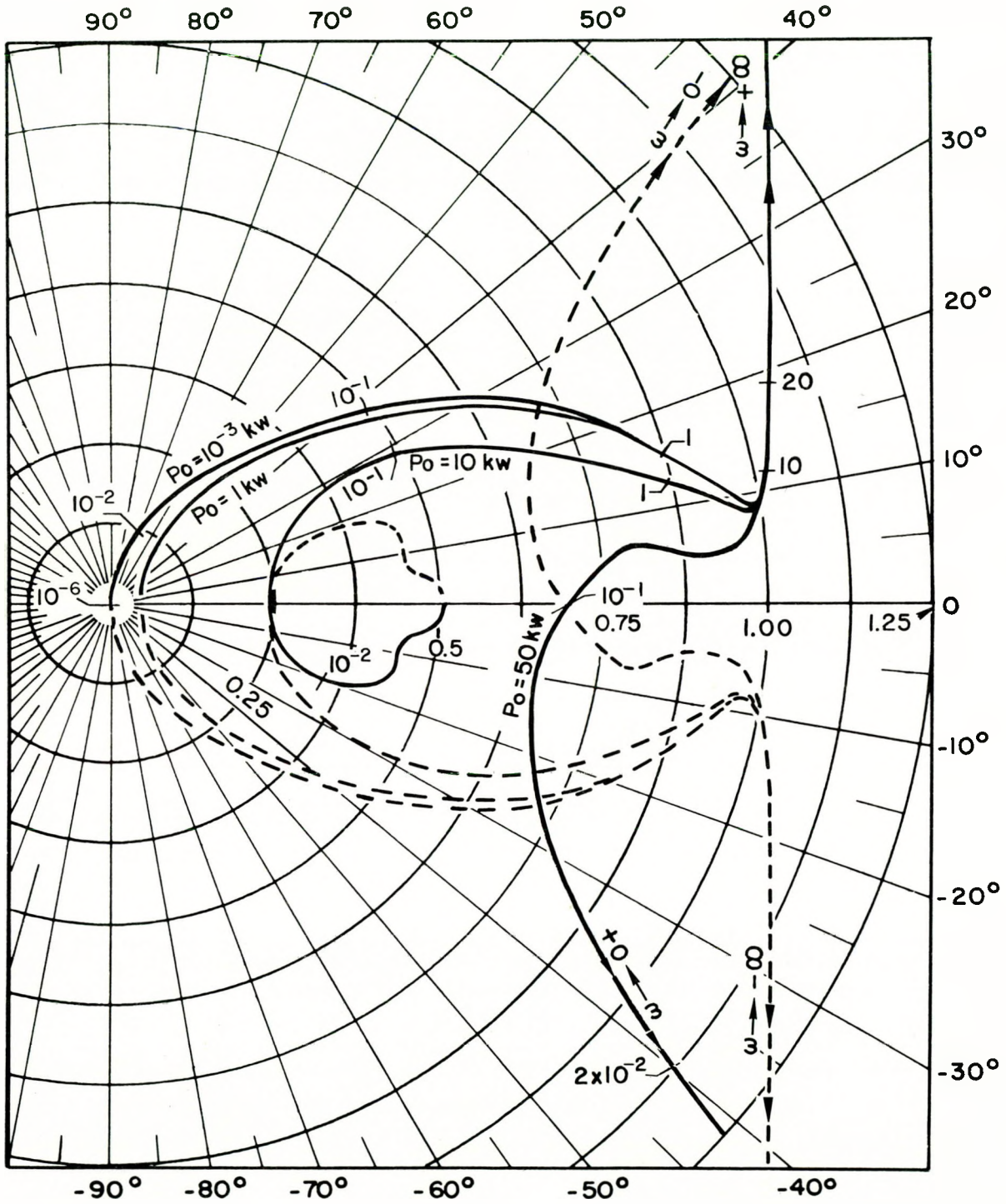

Fig. 12. Inverse of Normalized Water Boiler Reactor Power Transfer Function, $Gp(i\omega; P_o)^{-1}$ – (6 groups of delayed neutrons)

and radiolytic-gas generation. This situation is illustrated in Fig. 13.

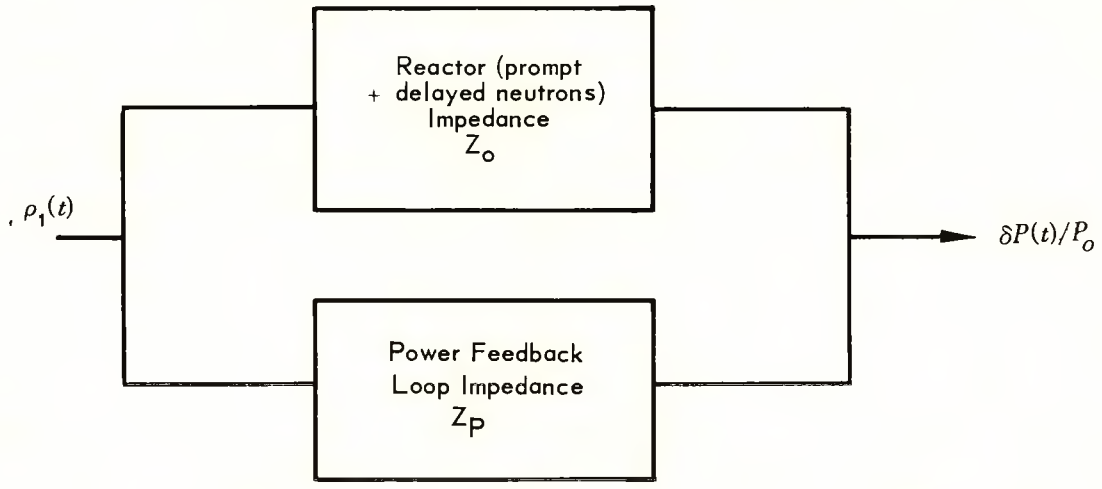

Fig. 13. Parallel Impedance Model Block Diagram

The overall complex impedance presented to the input function of reactivity $\rho_1^*(t)$ is then $Z(iw; P_0)$, where

$$Z(iw; P_0)^{-1} = Z_0(iw)^{-1} + Z_P(iw; P_0)^{-1}. \qquad \ldots(12)$$

Thus, $Z(iw; P_0)$ is simply the normalized reactor power transfer function (reactivity expressed in dollars) $G_P(iw; P_0)$ and $Z_0(iw) = G_P(iw; 0)$. The advantage of this approach lies in the fact that it enables one to separate the effects of the reactivity power feedback on the transfer function. The function $[-P_0 Z_P(iw; P_0)]^{-1}$ is called the complex power coefficient of reactivity.

For a water boiler reactor Z_P is found to be given by

$$Z_P(iw; P_0) = -P_0^{-1}\left[\frac{K\alpha^*}{iw+\gamma} + \frac{G\varphi^*}{iw+\sigma}\right]^{-1} . \qquad \ldots(13)$$

The function $\{\beta_{eff}/[P_0 Z_P(i\omega; P_0)]\}$ for a water boiler reactor is plotted in Fig. 14.

In order to apply the generalized Nyquist criterion[12] in its usual form to the linear behavior of a water boiler reactor, it is convenient to formulate the derivation of the transfer function in a different manner. The reactor in the linear approximation can be regarded as a system which includes only the effects of prompt neutrons on the power output plus a series of parallel feedback loops. These feedback loops consist of one for each group of delayed neutrons, one for the effect of temperature on reactivity, and one for the effects of radiolytic-gas bubbles on reactivity. The block diagram for this model is shown in Fig. 15. From this approach it is found that $G_P(iw; P_0)$ can be written in the form

$$G_P(iw; P_0) = G(iw)/[1 + G(iw) H(iw; P_0)] , \qquad \ldots(14.1)$$

or

$$[1 + G(iw) H(iw; P_0)] = [G(iw)/G_P(iw; P_0)] , \qquad \ldots(14.2)$$

where

$$H(iw; P_0) = -\left[\sum_{j=1}^{m} H_j(iw) + H_T(iw; P_0) + H_V(iw; P_0)\right]. \qquad \ldots(14.3)$$

The expressions for $G(iw)$, $H_j(iw)$, $H_T(iw; P_0)$, and $H_V(iw; P_0)$ are given in Fig. 15. The generalized Nyquist stability criterion for the water boiler reactor power transfer function reduces to the statement that the system is stable in the linear approximation if the polar plot of $[1 + G(i\omega) H(i\omega; P_0)]$ as ω goes from $-\infty$ to $+\infty$ does not encircle the point $[1 + G(iw) H(iw; P_0)] = 0$. †

The modified Nyquist plots of the water boiler reactor power transfer function for $P_0 = 10^{-3}$, $P_0 = 1$, $P_0 = 10$, and $P_0 = 50$ kw are shown in Fig. 16. If the model used is valid, these plots show that the water boiler is stable *in the linear approximation.* † Even for the case where $P_0 = 0$, , which is equivalent to the behavior with no power reactivity feedback terms, it can be shown that the modified Nyquist plot

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† The sense of the word "stable" as used here is that of impossibility of divergent oscillations.

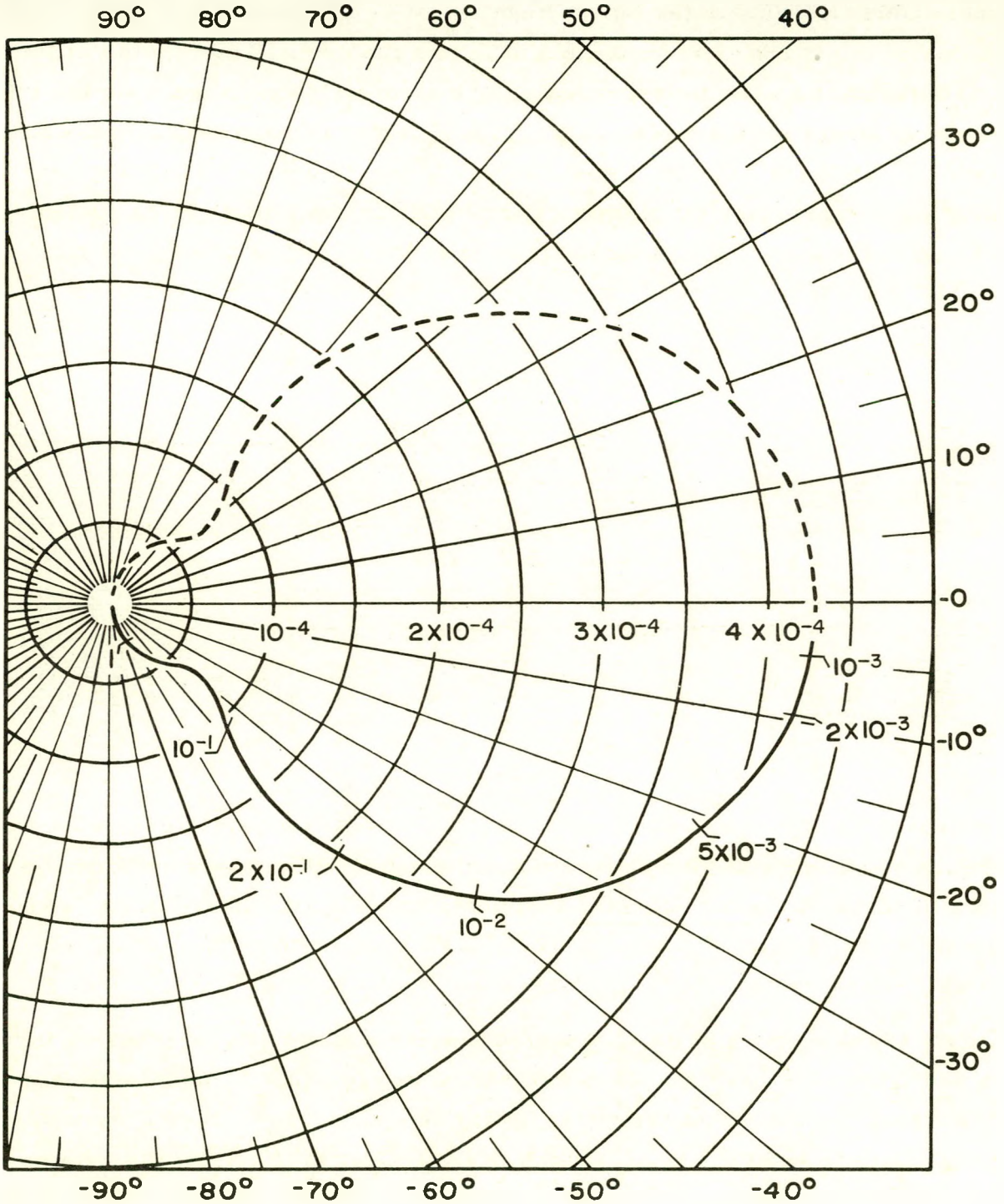

Fig. 14. $\{\beta_{eff}/[P_0 Z_P(i\omega; P_0)]\}$ for Water Boiler Reactor Power Transfer Function

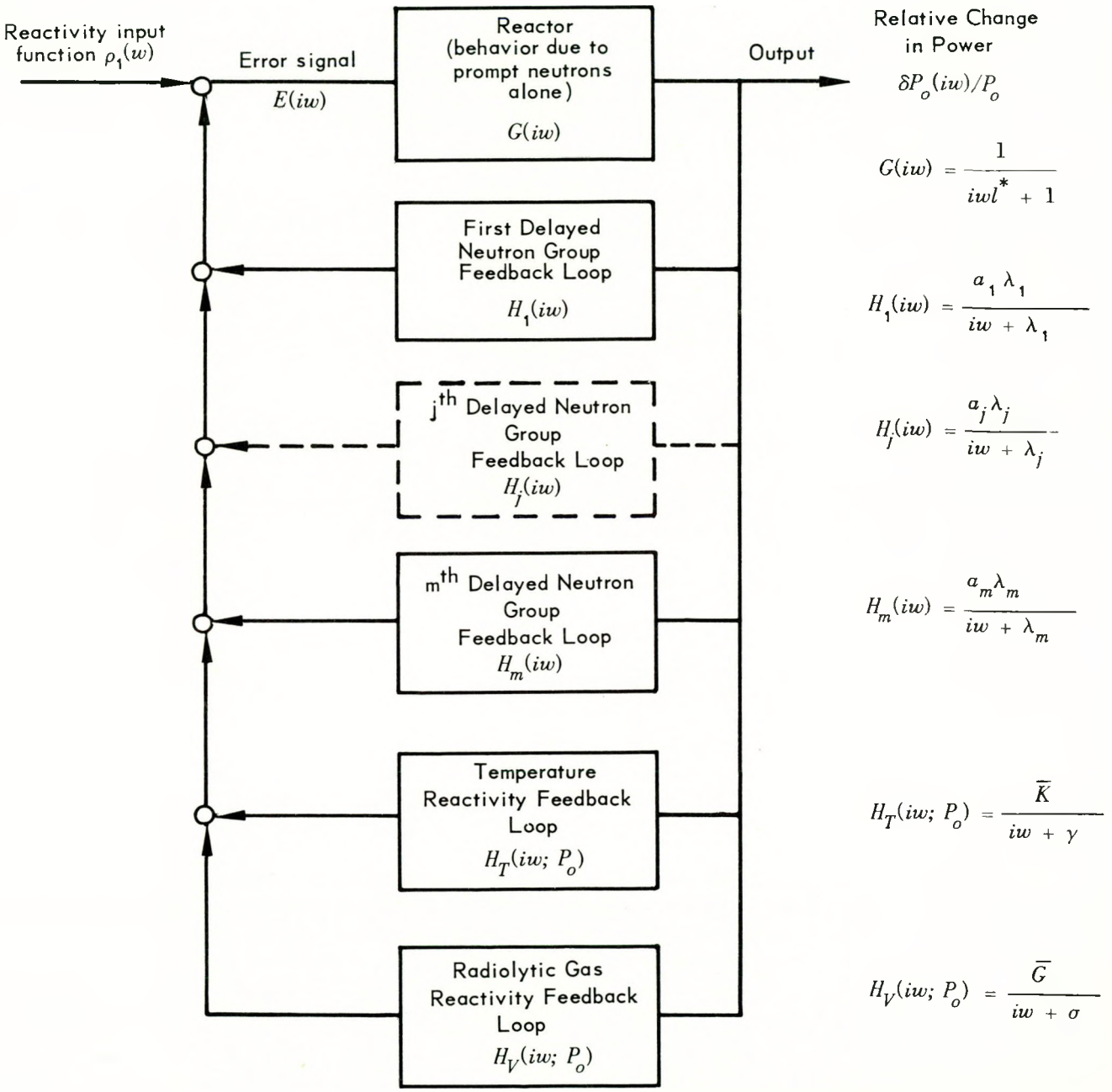

Fig. 15. Block Diagram Representation for Application of Nyquist Criterion

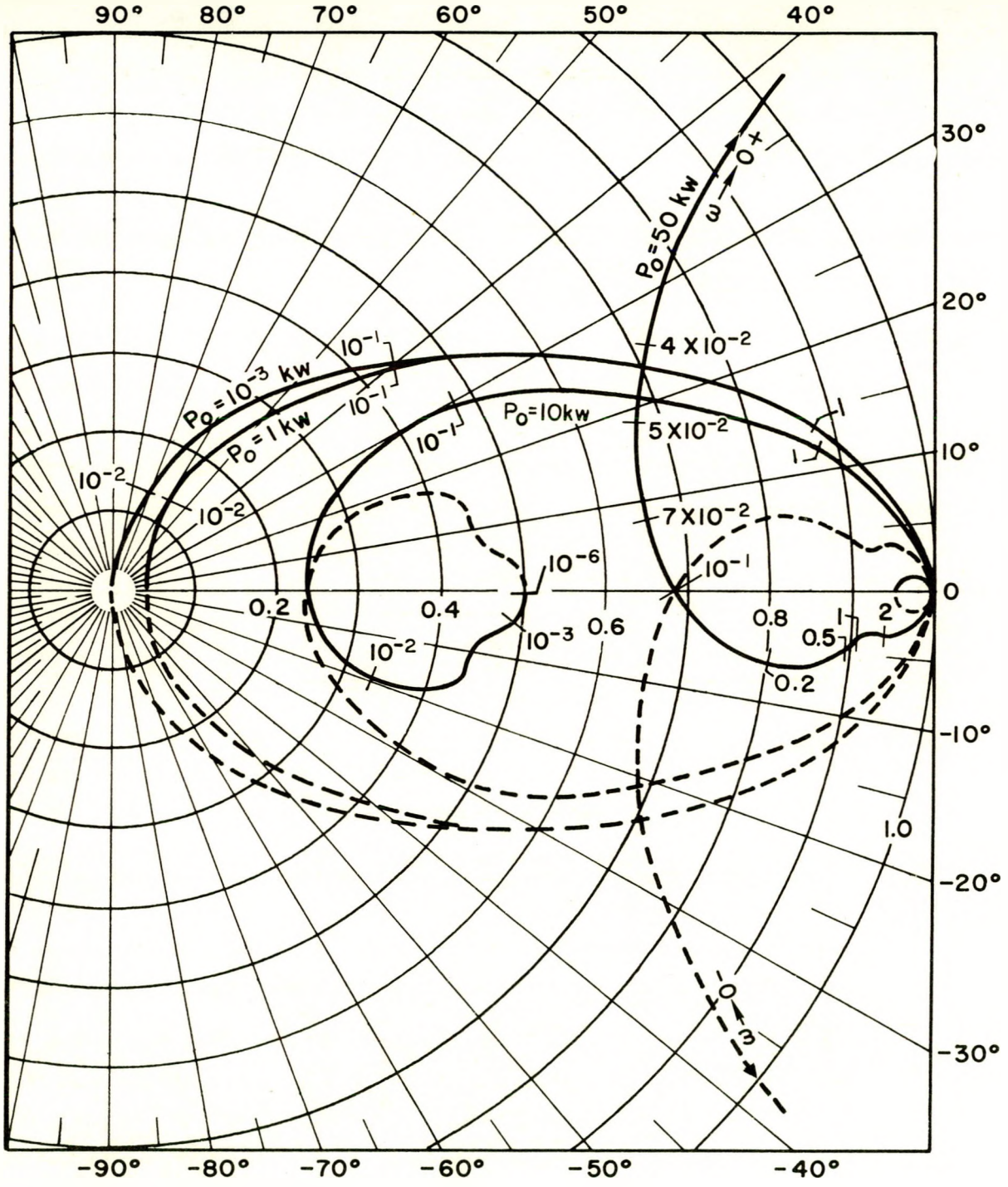

Fig. 16. Modified Nyquist Plot for Water Boiler Reactor Power Transfer Function. $1 + G(iw)H(iw; P_0)$ – (6 groups of delayed neutrons)

does not encircle the point zero. However, the modified Nyquist plot in this case passes through the point zero which indicates that sustained, but *non–divergent* oscillations are possible *in the linear approximation,* Now it is a well-known fact that, in the actual non-linear theory, a reactor is anything but stable for this case. This emphasizes the extremely important and not always appreciated fact that, *even though a system is stable in the linear approximation, it can be disastrously unstable in the non–linear theory.*

D. DERIVATION AND LIMITATIONS OF THE WATER BOILER SPACE INDEPENDENT KINETIC EQUATIONS

1. Introduction

There are limitations to the validity of the model represented by the space independent kinetic equations given by Eq. (1.1) through (1.5). An attempt was made to ascertain more precisely than was stated in Ref. 1 what these limitations are and some estimation of their degree of seriousness. Also, a better understanding of their derivation will help when considering modifications of the present model on the basis of results of intended future studies of special chemical and physical phenomena.

2. Equations for Neutron Behavior

Of the derivations in the literature that were examined, that of Ussachoff,[13] that in the Reactor Handbook,[14] and that of Hurwitz[15] were found to be the best. A modification of the derivation given by Ussachoff has been carried out with the result that Eq. (1.1) and (1.2) were obtained with the quantity $(\beta_{eff} a_j)$ replaced by $(\beta \epsilon_j a_j)$, $\epsilon = \sum_{j=1}^{m} \epsilon_j a_j$, $l = [l_0 / k_{eff}]$, and $C_j(t) = [\epsilon_j c_j(t)/(vl)]$ where

l_0 = the prompt neutron lifetime and is given by the ratio of the integral importance of all neutrons to the rate of destruction of importance,

ϵ_j = the relative importance of delayed neutrons belonging to the j^{th} delayed neutron group as compared to prompt neutrons and is given by the ratio of the rate of creation of importance if all neutrons were born with the energy spectrum of the j^{th} delayed group to the total rate of creation of importance,

ν = the average number of neutrons per fission and is dependent upon the neutron energy spectrum of the reactor, and,

$c_j(t)$ = the concentration of delayed neutron percursor atoms for the j^{th} delayed neutron group at time t.

The usual assumption that l is a constant is equivalent to the first assumption made by Bethe.† This assumption is also one that is made in the modified derivation.

The other assumptions made in the derivation are that: (1) the delayed neutron percursor atoms stay where they are formed; and (2) that

$$\int dVol \int_{\widehat{\Omega}} d\widehat{\Omega} \int_0^\infty \left\{ \frac{1}{v} \frac{\partial}{\partial t} [F^+(\boldsymbol{r}, E, \widehat{\Omega}, t) F(\boldsymbol{r}, E, \widehat{\Omega}, t)] \right\} dE$$

$$>> \int dVol \int_{\widehat{\Omega}} d\widehat{\Omega} \int_0^\infty \left[\frac{F(\boldsymbol{r}, E, \widehat{\Omega}, t)}{v} \frac{\partial F^+(\boldsymbol{r}, E, \widehat{\Omega}, t)}{\partial t} \right] dE,$$

where $F(\boldsymbol{r}, E, \widehat{\Omega}, t)$ is the neutron flux of energy E at point r at time t per unit energy per unit solid angle about $\widehat{\Omega}$, and where $F^+(\boldsymbol{r}, E, \widehat{\Omega}, t)$ is the adjoint flux or neutron importance function at point r at time t for neutrons of energy E with velocities in the direction $\widehat{\Omega}$.

The second assumption is equivalent to the assumption that the change in the fission rate density spatial distribution takes place slowly in time or that it takes place in a very small volume compared to the size of the reactor core. What happens if this approximation is not valid has been discussed in a number of places.[16] It is to be remarked, however, that this approximation is certainly valid over a wider range than it would be if the space independent kinetics equations were formulated in terms of the total importance of all neutrons or the neutron flux level, as is usually the case.

3. Equation for Radiolytic Gas

The equation governing the behavior of radiolytic gas, Eq. (1.4), is easily derived by assuming that all of the radiolytic gas that is formed appears

† The first assumption made by Bethe is that the excess multiplication, $k_{ex} = k_{eff} - 1$ can be neglected with respect to one.

immediately in the form of bubbles, i.e., that bubbles are formed instantaneously, and that the rate of removal of gas is directly proportional to the volume of gas present. Other possibilities have been considered by Wilkens.[17] It is to be expected that this equation will require a number of modifications which cannot properly be made until the process of gas bubble formation is better understood.

4. Equation for Temperature

Equation (1.3) which is taken to describe the temperature behavior can be derived by assuming that the heat generation at r at time t is directly proportional to the fission rate density $P(r,t)$ at r at time t, and that

$$\frac{d\theta(\boldsymbol{r},t)}{dt} = -f[\theta(\boldsymbol{r},t), v(\boldsymbol{r},t), P(\boldsymbol{r},t)] + K(\boldsymbol{r})P(\boldsymbol{r},t), \qquad \ldots(15)$$

where $\theta(r,t)$ is the temperature at r at time t, $v(r,t)$ is the radiolytic gas density at r at time t, $K(r)$ is directly proportional to the reciprocal heat capacity at r at time t, and $f(\theta,v,P)$ is directly proportional to the rate of removal of heat. The function f is then expanded in a Taylor series about the equilibrium temperature $\theta_0(r)$, radiolytic gas density $v_0(r)$, and fission rate density $P_0(r)$ at r for the equilibrium total fission rate P_0. All terms except the one involving the first partial of f with respect to θ are neglected; finally, integrating the resultant equation over the core, one obtains Eq. (1.3) where

$$T(t) = V^{-1}\int_{core} T(\boldsymbol{r},t)\,dVol\ , \qquad \ldots(16.1)$$

$$\gamma = [1/T(t)]\int_{core}\left\{\left[\frac{\partial f[\theta_0(\boldsymbol{r}), v_0(\boldsymbol{r}), P_0(\boldsymbol{r})]}{\partial\theta_0(\boldsymbol{r})}\right]T(\boldsymbol{r},t)\right\}dVol\ , \qquad \ldots(16.2)$$

and

$$K = [1/\delta P(t)]\int_{core}[K(\boldsymbol{r})\,\delta P(\boldsymbol{r},t)]\,dVol \qquad \ldots(16.3)$$

where

$$T(\boldsymbol{r},t) = \theta(\boldsymbol{r},t) - \theta_0(\boldsymbol{r}) \qquad \delta P(\boldsymbol{r},t) = P(\boldsymbol{r},t) - P_0(\boldsymbol{r}).$$

$$\delta P(t) = \int_{core} \delta P(\boldsymbol{r},t)\, dVol. \qquad P_0 = \int_{core} P_0(\boldsymbol{r})\, dVol.$$

and V is the volume of the core.

From the derivation, it is clear the Eq. (1.3) is strictly valid for small deviations from equilibrium or for the first part of fast transients in which case $\theta(\boldsymbol{r},t)$ is directly proportional to $P(\boldsymbol{r},t)$. However, for the proper heat removal mechanisms it can also be valid for larger deviations from equilibrium. Further, the derivation is quite general and applies to all reactors.

In the case of solution type reactors, the neglect of the first order term in the Taylor series expansion of f involving the radiolytic-gas density $v(\boldsymbol{r},t)$ may not be strictly valid.[18] This derivation also has the advantage that if one possesses an expression for $f(\theta, v, P)$ and $K(\boldsymbol{r})$ it enables him to calculate the parameters γ and K.

5. Reactivity Feedback Equation

The reactivity feedback equation, Eq. (1.5), also involves an approximation. It is assumed that all effects of temperature and radiolytic gas on the kinetic behavior of the fission rate enter through the multiplication constant and can be represented by a function g of the space average temperature $\theta_0(t)$, volume of radiolytic gas in the form of bubbles $v(t)$, and the total fission rate $P(t)$ at the time t. The expression for g is obtained from that given by Ussachoff[13] for the reactivity. The term $[\alpha T(t) + \phi V(t)]$ appearing in Eq. (1.5) is then obtained by expanding g in a Taylor series about equilibrium with the result that

$$\alpha = \frac{\partial g(\theta_0, v_0, P_0)}{\partial \theta_0} \text{ and } \phi = \frac{\partial g(\theta_0, v_0, P_0)}{\partial v_0}. \qquad \ldots(17)$$

Clearly the validity of this approximation is also in the strict sense restricted to small deviations from equilibrium. However, as in the case of the temperature equation the range of validity may actually be much larger.

From a pragmatic viewpoint the range of validity of the space independent kinetic equations may be extended by regarding the parameters $K, \alpha, \gamma, G, \phi$, and σ as functions of the equilibrium power given all other external experimental parameters constant and measured as such from the experimental transfer function.

APPENDIX II

Definition of Quantities Appearing in the Space Independent Kinetic Equation for a Water Boiler Reactor.

$P(t)$ = total fission rate (or power) at time t .†

P_0 = the equilibrium or steady state total fission rate (or power).†

$C_j(t)$ = the effective "latent" fission rate of the reactor stored in the form of percursor atoms of the j^{th} group of delayed neutrons at the time t .

$T(t)$ = the space average over the core of the temperature at the time t , measured with respect to the same space average when operating at the steady state total fission rate P_0 .

$V(t)$ = the total volume at STP of radiolytic gas in solution in the form of gas bubbles at the time t , measured with repsect to the total volume of gas in solution in the form of gas bubbles when operating at the steady state total fission rate (or power) P_0 .

$\rho(t)$ = $(k_{eff} - 1)/k_{eff}$, where k_{eff} the effective multiplication factor = the reactivity in "absolute" units at time t .

$\rho_1(t)$ = the external input function of reactivity, in "absolute" units, e.g., by the motion of control rod.

β = the total fraction of all fission neutrons which are delayed.

a_j = the fraction of all delayed neutrons which belong to the j^{th} delayed group, $\sum_{j=1}^{m} a_j = 1$.

βa_j = the fraction of all fission neutrons which belong to the j^{th} group of delayed neutrons.

ε = the relative effectiveness of delayed neutrons compared to prompt neutrons.

β_{eff} = $\varepsilon\beta$ the effective total fraction of all neutrons which are delayed.

l = l_0/k_{eff}, the "effective" neutron generation time in the reactor (assumed to have negligible time dependence).

l_0 = the neutron lifetime.

K = the adiabatic rate of increase of the space average temperature per unit total fission rate (or power) for the reactor.

† The terms "power" and "total fission rate" or simply "fission rate" are used synonymously throughout this report.

G = the abiabatic rate of increase of the total volume at STP of radiolytic gas in solution per unit total fission rate (or power) for the reactor.

γ = the reciprocal of the characteristic mean time for loss of heat from the core, such that $\gamma T(t)$ is the rate of decrease of $T(t)$.

σ = the reciprocal of the characteristic time for loss of radiolytic gas from the core, such that $\sigma V(t)$ gives the total rate of loss of gas from the "soup."

α = the increase in reactivity of the reactor due to a unit increase in T, the temperature coefficient of reactivity in "absolute" units.†

ϕ = the increase in reactivity of the reactor due to a unit increase in radiolytic gas volume (STP), i.e., the void coefficient of reactivity in "absolute" units.†

$\rho^* = \rho/\beta_{eff}$, the reactivity in dollars.††

$l^* = l/\beta_{eff}$.††

$\alpha^* = \alpha/\beta_{eff}$, the temperature coefficient of reactivity in units of dollars.††

$\phi^* = \phi/\beta_{eff}$, the void coefficient of reactivity in units of dollars.††

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† It is to be noted that in the case of all water boiler type reactors built to date α and ϕ are negative

†† The superscript * is used to denote the symbol without the * divided by β_{eff} .

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