LA-UR 96-3704

Alamos National Laboratory is operated by the University of California for the United States Department of Energy under contract W-7405-ENG-36

TITLE: **MULTIGRID FOR REFINED TRIANGLE MESHES**

AUTHOR(S): Yair Shapira

RECEIVED
JAN 21 1997
OSTI

SUBMITTED TO: External Distribution

Los Alamos

Los Alamos National Laboratory
Los Alamos New Mexico 87545

Multigrid for Refined Triangle Meshes

Yair Shapira*

Abstract

A two-level preconditioning method for the solution of (locally) refined finite element schemes using triangle meshes is introduced. In the isotropic SPD case, it is shown that the condition number of the preconditioned stiffness matrix is bounded uniformly for all sufficiently regular triangulations. This is also verified numerically for an isotropic diffusion problem with highly discontinuous coefficients.

1 Introduction

The Black Box Multigrid method of [4] is considered robust for diffusion problems with possibly discontinuous coefficients on structured grids. More specifically, the application of this method requires that the coefficient matrix has a 3^d-coefficient stencil, where d is the dimension of the problem. In the context of finite element schemes for elliptic second order problems, this limits the use of black Box Multigrid to multilinear finite element schemes on a logically cubic meshes. Thus, Black Box Multigrid is not applicable for more complicated (e.g., unstructured) finite element schemes resulting from realistic engineering and applied science problems. Furthermore, it is pointed out in [7] [8] that Black Box Multigrid stagnates for certain diffusion problems with high diffusion areas separated by a thin strip. Surprisingly, this stagnation occurs when the discontinuity curves are aligned with all the coarse grids, case which can be handled easily by either standard multigrid or the method of [3]. The AutoMUG method introduced there avoids this stagnation but diverges for other examples. In [10] this stagnation is explained and a modified version of Black Box Multigrid which avoids it is introduced. This version is related to the method of [5] and based on 'throwing' certain matrix elements to the main diagonal when constructing the prolongation operator from coarse to fine grids. It is shown in [10] that this version is robust both theoretically (for a certain class of problems) and numerically (for the above example and others).

In this work, a method in the spirit of the above version is applied to (locally refined) triangular finite element schemes. The method can thus be considered as a generalization

*Computer Science Department, Technion, Haifa 32000, Israel. Now at Los Alamos National Laboratory, MS B-256, Los Alamos, NM 87545, e-mail: yairs@lanl.gov.

of Black Box Multigrid for certain unstructured grids. Unlike in [3], it is not necessary to assume neither for the implementation nor for the analysis that the curves of discontinuity in the coefficients of the PDE are aligned with the coarse level elements. Also, it is not assumed here that the domain is polygonal; for simplicity, however, we use polygonal domains in the presentation.

The contents of the paper are as follows. In Section 2 the framework and the method are presented. In Section 3 an upper bound for the condition number of the preconditioned coefficient matrix is derived. In Section 4 this result is used to show that the convergence is independent of the mesh size for a certain class of problems. In Section 6 numerical results confirming the analysis are presented. In Section 7 concluding remarks are made.

to the

2 Preliminaries

Consider the problem

$$\text{find } u \in H \text{ such that } a(u,v) = (f,v) \quad \forall v \in H, \tag{1}$$

where H is a Hilbert space, $(f,\cdot)$ is a bounded linear functional on H and $a(\cdot,\cdot)$ is a bilinear bounded form on $H \times H$. Consider also the problem

$$\text{find } u \in V \text{ such that } a(u,v) = (f,v) \quad \forall v \in V, \tag{2}$$

where V is a subspace of H.

Let d be a fixed positive integer, $\Omega \subset R^d$ a bounded domain and L an elliptic differential operator of order 2 in Ω, that is,

$$Lu = -\nabla(D\nabla u) + \vec{\kappa} \cdot \nabla u + \beta u, \tag{3}$$

where D, $\vec{\kappa}$ and β are given functions (D is a $d \times d$ symmetric and uniformly positive definite matrix and $\vec{\kappa}$ is a d-dimensional vector). Let Dirichlet boundary conditions be imposed on a finite set of curves (or domains, if $d > 2$) $\Gamma \subset \partial\Omega$ and other types of boundary conditions on $\partial\Omega \backslash \Gamma$. In our application, $H = H^1_\Gamma(\Omega)$ (the Sobolev space of order 1 of functions defined on Ω and vanishing on Γ), $(\cdot,\cdot) = \int_\Omega \cdot \cdot d\Omega$ is the L_2 inner product, $f \in L_2(\Omega)$ and $a(u,v) = (Lu, v)$.

Assume further that $\partial\Omega$ is piecewise linear. Let S be a triangulation of Ω, that is, a set of triangles whose interiors are disjoint to each other and whose union is equal to Ω (for $d = 3$, the term *triangle* is replaced by *tetrahedron*, but the formulation is basically the same). Let T be a refinement of S which is defined as follows. According to a certain rule (such as that of [6]), some of the triangles in S are refined. If $s \in S$ is refined, then connect the midpoints

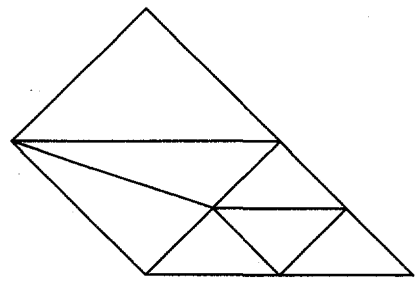

Figure 1: A 2-d example for the refined triangulation T resulting from the coarse triangulation S.

of its edges to each other and include all the resulting triangles in T. If $s \in S$ is not refined but has a neighbor (edge sharing) $s_1 \in S$ which is refined, then connect the midpoint of the edge shared by s and s_1 to the node of s which is in front of it and include the resulting triangles in T. If $s \in S$ is not refined and has no refined neighbors in S, then include it in T. This refinement is illustrated for $d = 2$ in Figure 1 (other refinement strategies are also possible, see [2] [6]).

Let $V \subset H$ be the space of functions which are continuous in Ω and linear in each triangle $t \in T$. For each node n of a triangle in T, let ϕ_n denote the corresponding nodal basis function in V. Let A be the stiffness matrix $(a(\phi_i, \phi_j))$. Then (2) is equivalent to the linear system

$$Ax = b, \tag{4}$$

where b is the vector with components (f, ϕ_i) and the unknowns in the vector x are the coefficients of nodal basis functions in the representation of the unknown function u in (2). It is assumed in the sequel that A is nonsingular.

Let c denote the set of nodes of triangles in S and f the set of nodes of triangles in T which do not belong to c. Hereafter we use this partitioning of nodes also to denote the corresponding partitioning of nodal basis functions. This induces a block partitioning for A:

$$A = \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix} \tag{5}$$

and similarly for other matrices of the same order.

For any set $g \subset c \cup f$, let $J_g : l_2(c \cup f) \to l_2(g)$ denote the injection

$$(J_g w)_j = w_j, \; w \in l_2(c \cup f), \; j \in g.$$

For any matrix M, $M = (m_{i,j})_{1 \le i \le K, \, 1 \le j \le L}$, let $|M| = (|m_{i,j}|)_{1 \le i \le K, \, 1 \le j \le L}$ and

$$rs(M) = diag(\sum_{j=1}^{L} m_{i,j})_{1 \le i \le K}.$$

Define the diagonal matrix G by

$$G = \frac{rs(|A_{fc}|) + rs(|A^*_{cf}|)}{2}.$$

Define the matrices P (prolongation), R (restriction) and Q (coarse-grid coefficient matrix) by

$$P^{-1} = \begin{pmatrix} G & A_{fc} \\ 0 & I \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} G & 0 \\ A_{cf} & I \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} W & 0 \\ 0 & J_cRAPJ_c^t \end{pmatrix},$$

where W is some nonsingular diagonal matrix. Note that R, P, Q and A are of the same order, but the complexity of inverting Q is much smaller than that of inverting A. For the analysis in Section 3 to be valid, one may set $W = I$ or, in the spirit of [4],

$$W = G^{-1} diag(A_{ff}) G^{-1}. \tag{6}$$

It is assumed here that $J_cRAPJ_c^t$ is nonsingular; this is guaranteed when A is SPD and holds in most cases. It is also assumed that G is nonsingular. If G is singular, one may replace its vanishing main diagonal elements by, say, 1. With (6), this choice is immaterial to the two-level algorithm (7), since G is cancelled in the formal triple product $PQ^{-1}R$.

The two-level method is defined by

$$\text{TL}(x_{in}, A, b, x_{out}): \quad x_{out} = x_{in} + PQ^{-1}R(b - Ax_{in}). \tag{7}$$

Since P^{-1} and R^{-1} are triangular with respect to the variable ordering (5), the application of P and R is performed easily by back substitution and forward elimination, respectively. Naturally, one would like to use the present algorithm recursively for the solution of the coarse level problem $Qe = R(b - Ax_{in})$. Fortunately, $J_cRAPJ_c^t$ has the same pattern as that of a linear finite element discretization of (1) using S. Hence, the algorithm is suitable for recursion and multi level implementation.

The method TL may be supplemented with relaxations before and after it in the spirit of multigrid methods. Alternatively, a Lanczos type acceleration may be applied to the basic TL iteration (7). For both approaches, the condition number of the preconditioned system $PQ^{-1}RA$ is an important measure for the rate of convergence. In the following, this condition number is estimated for symmetric positive definite (SPD) problems.

3 Analysis in the SPD Case

Here $(\cdot, \cdot)$ denotes the usual inner product in $l_2(c \cup f)$ and $\|\cdot\|$ denotes the corresponding vector and matrix norms. The following lemma is used in the proof of Theorem 1.

Lemma 1 *Let M be a symmetric and positive semi-definite matrix of the same order as A. Then, for any vector $x \in l_2(c \cup f)$,*

$$(x, Mx) \leq 2(x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x).$$

Proof: Let $\tilde{x} = J_f^t J_f x - J_c^t J_c x$. Then we have

$$0 \leq (\tilde{x}, M\tilde{x}) = (x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x) - (x, (J_f^t J_f M J_c^t J_c + J_c^t J_c M J_f^t J_f)x).$$

The lemma follows from

$$\begin{aligned}(x, Mx) &= (x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x) + (x, (J_f^t J_f M J_c^t J_c + J_c^t J_c M J_f^t J_f)x) \\ &\leq 2(x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x). \quad \Box\end{aligned}$$

Theorem 1 *Assume that A is symmetric and (possibly weakly) diagonally dominant, G is nonsingular and W is SPD. Then the condition number of the preconditioned coefficient matrix $PQ^{-1}RA$ is bounded by*

$$2\max(\|W^{-1}J_f RAPJ_f^t\|, 1)(1 + 2\|P\|\sqrt{\eta\|A\|} + \eta\|RAP\| + \eta\|W\|),$$

where $\eta = (\sqrt{\|A_{ff} - G\|} + \sqrt{\|A\|})^2 \leq 4\|A\|$.

Proof: The proof is in the spirit of those of [10] [11]. Since A is symmetric, $R = P^t$. Since A is symmetric and diagonally dominant, it follows from Gershgorin's theorem that it is positive semi definite. Let $x \in l_2(c \cup f)$ satisfy $\|x\| = 1$ and denote $\varepsilon = (x, Ax)$. Since A is symmetric and positive semi-definite, x may be written as a linear combination of the orthogonal eigenvectors of A. Consequently, $\|Ax\|^2 \leq \|A\|\varepsilon$.

Note that $A_{ff} - G$ and $A - J_f^t(A_{ff} - G)J_f$ are symmetric and diagonally dominant, hence positive semi definite. Using the above argument, we obtain

$$\|J_f^t(A_{ff} - G)J_f x\|^2 \leq \|A_{ff} - G\|(J_f^t(A_{ff} - G)J_f x, x) \leq \|A_{ff} - G\|\varepsilon.$$

Consequently,

$$|\ \|J_f Ax\| - \|J_f P^{-1}x\|\ |^2 \leq \|J_f(A - P^{-1})x\|^2 = \|(A_{ff} - G)J_f x\|^2 \leq \|A_{ff} - G\|\varepsilon,$$

which implies that

$$\|J_f P^{-1}x\| \leq \sqrt{\eta\varepsilon}, \quad \text{where} \quad \eta = (\sqrt{\|A_{ff} - G\|} + \sqrt{\|A\|})^2.$$

As a result, we have

$$\begin{aligned}(x, R^{-1}QP^{-1}x) &= (P^{-1}x, QP^{-1}x) \\ &= (J_c^t J_c x + J_f^t J_f P^{-1}x, Q(J_c^t J_c x + J_f^t J_f P^{-1}x)) \\ &\leq (J_c^t J_c x, RAP(J_c^t J_c x)) + \eta\|W\|\varepsilon \\ &= (P^{-1}x - J_f^t J_f P^{-1}x, RAP(P^{-1}x - J_f^t J_f P^{-1}x)) + \eta\|W\|\varepsilon \\ &\leq (1 + 2\|P\|\sqrt{\eta\|A\|} + \eta\|RAP\| + \eta\|W\|)\varepsilon,\end{aligned}$$

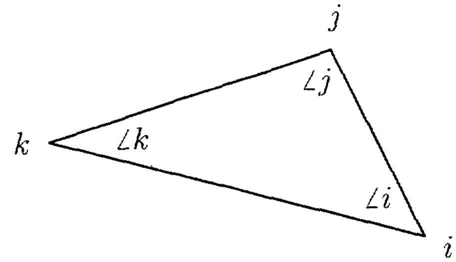

Figure 2: Vertices and angles of a triangle.

which implies that the function $(x, R^{-1}QP^{-1}x)/(x, Ax)$ is bounded. On the other hand, we have from Lemma 1 that, for any $y \in l_2(c \cup f)$,

$$(y, RAPy) \leq 2(y, (J_f^t J_f RAPJ_f^t J_f + J_c^t J_c RAPJ_c^t J_c)y) \leq 2\max(\|W^{-1}J_f RAPJ_f^t\|, 1)(y, Qy),$$

which implies that the function $(x, Ax)/(x, R^{-1}QP^{-1}x)$ is bounded. □

4 Application to Isotropic Problems

Here we assume that $a(\cdot,\cdot)$ is symmetric, which implies that $\vec{\kappa} \equiv 0$ in (3). Also, we assume that L in (3) is isotropic, that is, $D = \delta I$, where δ is a function on Ω satisfying $\delta \geq \alpha$, for some constant $\alpha > 0$. Assume also that $\beta \geq 0$ in (3). This implies that A is SPD.

In the sequel we use $d = 2$; the multi dimensional case is similar. Denote the vertices of a triangle s by i_s, j_s and k_s. Denote the positive angles inside s, vertexed at i_s, j_s and k_s, by $\angle i_s$, $\angle j_s$ and $\angle k_s$, respectively (see Figure 2). Assume that

$$\max_{s\in S} \max_{n\in\{i_s,j_s,k_s\}} \angle n \leq \theta, \tag{8}$$

where θ is a constant satisfying $\pi/3 \leq \theta < \pi/2$.

Let $\triangle(s)$ denote the area of the triangle s (or, if $d > 2$, the volume of the element s). Define

$$\triangle_{min} = \min_{s\in S} \triangle(s) \quad \text{and} \quad \triangle_{max} = \max_{s\in S} \triangle(s).$$

Assume that

$$\triangle_{max}/\triangle_{min} \leq r, \tag{9}$$

where r is a constant. In the sequel, the triangulation S is unspecified apart from the fact that (8) and (9) are satisfied for fixed θ and r. Assume that (4) is multiplied by $\triangle_{min}^{2/d-1}$, so that (using (9)) $\|A\|$ and $diag(A)^{-1}$ are bounded independently of $\triangle_{min}$.

Assume that $\triangle_{min}$ is so small that, for any pair ϕ_i, ϕ_j of nodal basis functions in V, the sign of $a(\phi_i, \phi_j)$ is determined by that of $\int_\Omega \delta\nabla\phi_i \cdot \nabla\phi_j d\Omega$. Let i, j and k be the vertices

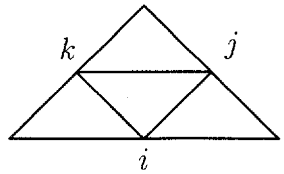

Figure 3: Fine level nodes participating in the two-step competition (11).

of a triangle $t \in T$ such that either i or j are in f (see Figure 2). Note that from (8) and Figure 1 $\angle k < \pi/2$. Let $\xi = (k-i)/||k-i||$ and $\eta = \xi^{\perp}$ be a pair of orthonormal vectors in R^2. Let F be the 2 by 2 matrix whose first column is ξ and second column is η. Denote by $\nabla_F = (\partial/\partial\xi, \partial/\partial\eta)^t$ the vector of derivative operators in directions ξ and η. We have

$$\begin{aligned}\int_t \delta\nabla\phi_i \cdot \nabla\phi_j d\Omega &= \int_t \delta F^*\nabla\phi_i \cdot F^*\nabla\phi_j d\Omega \\ &= \int_t \delta\nabla_F\phi_i \cdot \nabla_F\phi_j d\Omega \\ &= \int_t \delta(\phi_i)'_\eta(\phi_j)'_\eta d\Omega \\ &= -(|k-i|\tan(\angle k))^{-1}(|k-j|\sin(\angle k))^{-1}\int_t \delta d\Omega \\ &= -\frac{\int_t \delta d\Omega}{2\triangle(s)\tan(\angle k)} \le -\frac{\alpha}{2\tan(\angle k)}. \qquad (10)\end{aligned}$$

This guarantees that the main diagonal elements of G are bounded away from zero, so that $||G^{-1}||$ (and hence $||P||$ and $||W||$) are bounded independently of the specific triangulation S used. Finally, assume that A is (possibly weakly) diagonally dominant.

Corollary 1 *With the assumptions in this section, the condition number of the preconditioned coefficient matrix $PQ^{-1}RA$ is bounded, independently of the specific choice of the triangulation S.*

5 A Two-Step Algorithm

It is seen from Theorem 1 that a necessary condition for the boundedness of the condition number of $PQ^{-1}RA$ is the boundedness of $||G||$. Reducing $||G||$ might thus improve the performance of the two-level algorithm (7). To achieve this, we suggest the following modification of the TL method introduced in Section 2.

Split f into two sets as follows. Let i, j and k be edge midpoints of a refined triangle $s \in S$ (see Figure 3). Let $G_{i,i}$, $G_{j,j}$ and $G_{k,k}$ be the main diagonal elements of G corresponding to i, j and k, respectively. If

$$G_{i,i}/A_{i,i} \ge \max_{l\in\{i,j,k\}} G_{l,l}/(2A_{l,l}), \qquad (11)$$

then include i in f_1; otherwise, include it in f_2. The same procedure is employed for j and k. Clearly, f_1 and f_2 are disjoint and $f = f_1 \cup f_2$. This induces the following block partitioning of A:

$$A = \begin{pmatrix} A_{f_2f_2} & A_{f_2f_1} & A_{f_2c} \\ A_{f_1f_2} & A_{f_1f_1} & A_{f_1c} \\ A_{cf_2} & A_{cf_1} & A_{cc} \end{pmatrix}.$$

Define the diagonal matrices G_1 and G_2 by

$$G_1 = \frac{rs(|A_{f_1c}|) + rs(|A^*_{cf_1}|)}{2} \quad \text{and} \quad G_2 = \frac{rs(|A_{f_2c}|) + rs(|A^*_{cf_2}|)}{2} + \frac{rs(|A_{f_2f_1}|) + rs(|A^*_{f_1f_2}|)}{2}.$$

Define the matrices P, R and Q by

$$P^{-1} = \begin{pmatrix} G_2 & A_{f_2f_1} & A_{f_2c} \\ 0 & G_1 & A_{f_1c} \\ 0 & 0 & I \end{pmatrix}, \quad R^{-1} = \begin{pmatrix} G_2 & 0 & 0 \\ A_{f_1f_2} & G_1 & 0 \\ A_{cf_2} & A_{cf_1} & I \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} W & 0 \\ 0 & J_cRAPJ_c^t \end{pmatrix},$$

where W is some nonsingular matrix. For the analysis in Section 3 to be valid W should be SPD. The reasonable choices are

$$W = I \tag{12}$$

or, in the spirit of [4],

$$W = R_{ff}diag(A_{ff})diag(P_{ff}). \tag{13}$$

Although W in (13) is not SPD, it is spectrally equivalent to the SPD matrix $diag(W)$. Hence, the application of the proof of Theorem 1 (see below) is essentially unchanged.

The TL method (7) is implemented with the P, R and Q defined above. Note that $J_cRAPJ_c^t$ has a slightly wider pattern than that of a linear finite element discretization of (1) using S. Indeed, $(J_cRAPJ_c^t)_{i,j}$ might be nonzero for vertices $i \in s$ and $j \in s_1$ if s and s_1 are in S and share a vertex. Nevertheless, when multi-level implementations are considered this pattern may be resolved recursively by a similar approach, using some splitting of the next fine grid $f_c \subset c$ derived from c the same way f was derived from $c \cup f$. A reasonable choice is to use the above splitting method also for f_c; it is recommended to construct the next prolongation and restriction operators $P_c : c \to c$ and $R_c : c \to c$ not from $J_cRAPJ_c^t$ but from a modification of it, in which elements which are outside the pattern of the corresponding linear finite element scheme are 'thrown' to the main diagonal (see [9] [10]). With this implementation, the pattern of the coarse grid coefficient matrix is preserved also for the next coarse level coefficient matrix, which results in an efficient multi-level implementation.

Theorem 1 applies also to the two-step algorithm, provided that nonsingularity of G_1 and G_2 is assumed. There is only a slight change in the proof and the definition of η. This is described next.

Let

$$A_1 = \begin{pmatrix} \frac{rs(|A_{f_2f_1}|)+rs(|A^*_{f_1f_2}|)}{2} & A_{f_2f_1} & 0 \\ A_{f_1f_2} & A_{f_1f_1} - G_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} A_{f_2f_2} - G_2 & & \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since A_1, A_2, $A-A_1$ and $A-A_2$ are symmetric and diagonally dominant, it follows from Gershgorin's theorem that they are positive semi definite. Using the argument in the beginning of the proof of Theorem 1, we obtain

$$\|A_n x\|^2 \leq \|A_n\|(x, A_n x) \leq \|A_n\|\varepsilon, \quad n = 1, 2.$$

Note that

$$J_{f_n} A_n = J_{f_n}(A - P^{-1}), \quad n = 1, 2.$$

Consequently,

$$\begin{aligned} | \, \|J_f A x\| - \|J_f P^{-1} x\| \, |^2 &\leq \|J_f(A - P^{-1})x\|^2 \\ &= \sum_{n=1}^{2} \sum_{i \in f_n} |(A_n x)_i|^2 \\ &\leq \sum_{n=1}^{2} \|A_n x\|^2 \leq (\|A_1\| + \|A_2\|)\varepsilon, \end{aligned}$$

which implies that

$$\|J_f P^{-1} x\| \leq \sqrt{\eta\varepsilon}, \quad \text{where} \quad \eta = (\sqrt{\|A_1\| + \|A_2\|} + \sqrt{\|A\|})^2 \leq (\sqrt{2} + 1)^2 \|A\|.$$

The proof of Theorem 1 proceeds as before with the newly defined η. Note that the original proof of Theorem 1 is obtained from this modified proof by setting $f_1 = \emptyset$ or $f_2 = \emptyset$ in the two-step algorithm.

6 Numerical Experiments

Here we apply the TL method to a diffusion problem with discontinuous coefficient. As a test problem we take the 'staircase' problem (Example IV in [1]). This is an isotropic diffusion equation (see Section 4) whose diffusion coefficient is 1000 inside the staircase and 1 outside it. The domain is a square. Mixed (vacuum) boundary conditions are given on two nonparallel edges of the domain and Neumann boundary conditions are given on the other two edges. Although a finite volume discretization is used, it may be thought of as a finite element scheme, using right angled triangles of which two edges are equal in length and parallel to a Cartesian coordinate axis. Full refinement is used, namely, all the elements are refined. This is equivalent to the standard Cartesian grid coarsening used in [1] [4]. The grid size is $N \times N$ ($N = 17$ yields the example tested in [1]).

Since the triangles are right angled, it follows from (10) that the one step algorithm is insufficient. Hence, the two-step algorithm is used. From (11), it follows that fine level nodes which lie between two coarse level nodes in a Cartesian direction belong to f_1, whereas fine level nodes which lie between two coarse level nodes in an oblique direction belong to f_2. This yields the following grid partitioning:

$$\begin{array}{ccccc} f_2 \cdot & f_1 \cdot & f_2 \cdot & f_1 \cdot & f_2 \cdot \\ f_1 \cdot & c* & f_1 \cdot & c* & f_1 \cdot \\ f_2 \cdot & f_1 \cdot & f_2 \cdot & f_1 \cdot & f_2 \cdot \\ f_1 \cdot & c* & f_1 \cdot & c* & f_1 \cdot \\ f_2 \cdot & f_1 \cdot & f_2 \cdot & f_1 \cdot & f_2 \cdot \end{array},$$

where points in c, f_1 and f_2 are denoted by '$c*$', '$f_1 \cdot$' and '$f_2 \cdot$', respectively. Hence, in the interior of the domain the stencil of the coarse grid coefficient matrix is identical to that of the method of [4].

The Conjugate Gradient Squared (CGS) method of [12] is used to accelerate the basic TL iteration (7). This method is chosen because it does not use the transpose of the preconditioner, which is not always available. The method is iterated until the l_2 norm of the residual is reduced by six orders of magnitude.

The results in Table 1 show the insensitivity of the rate of convergence to problem size for both the implementations (12) and (13). This is as predicted by Theorem 1 (with the version used in Section 5).

Table 1: Iteration numbers for the two-step algorithm (accelerated by CGS) for the 'staircase' problem on $N \times N$ grids.

N	(12)	(13)
17	57	19
33	39	17
65	39	17
129	39	15

7 Conclusions

The present two-level algorithm for the solution of linear finite element discretizations of elliptic second order boundary value problems is analyzed in the SPD case under the additional assumption that the stiffness matrix is diagonally dominant. For isotropic problems, the bound derived for the condition number of the preconditioned stiffness matrix is uniform for all sufficiently regular triangulations. It is verified numerically for an isotropic diffusion problem with highly discontinuous coefficients that the rate of convergence is independent of the size of the problem. Unlike in [3], it is not assumed here that the possible discontinuities in the diffusion coefficient coincide with edges of finite elements. This makes the method applicable to problems in which the location of these discontinuities is not known in advance.

The pattern of the coarse level preconditioning matrix is preserved in further coarser levels in multi-level implementations. However, the analysis is not always carried over to the coarse system, which is not necessarily diagonally dominant. Hence, it is desirable that the diagonal dominance assumption be removed. This is left to future research.

References

[1] Alcouffe, R.; Brandt, A.; Dendy, J. E.; and Painter J.: The Multigrid Method for the Diffusion Equation with Strongly Discontinuous Coefficients. *SIAM J. Sci. Stat. Comput.,* vol. 2, 1981, pp. 430-454.

[2] Bank, R. E.; Dupont, T. F.; and Yserentant, Y.: The Hierarchical Basis Multigrid Method. *Numer. Math.* 52 (1988), 427-458.

[3] Bramble J. H.; Pasciak J. E.; and Shatz A. H.: The Constructuring of Preconditioners for Elliptic Problems on Regions Partitioned into Substructures *II. Math. Comp.* 47 (1986) 103-134.

[4] Dendy, J. E.: Black Box Multigrid. *J. Comp. Phys.* 48 (1982), pp. 366-386.

[5] Kettler, R.; and Meijerink, J.A.: A Multigrid Method and a Combined Multigrid-Conjugate Gradient Method for Elliptic Problems with Strongly Discontinuous Coefficients in General Domains. Shell Publ. 604, KSELP, Rijswijk, The Netherlands, 1981.

[6] Mitchell, W. F.: Optimal Multilevel Iterative Methods for Adaptive Grids. *SIAM J. Sci. Stat. Comput.* 13 (1992), pp. 146-167.

[7] Shapira, Y.: Multigrid Methods for 3-D Definite and Indefinite Problems. Technical Report #834 (revised version), Computer Science Department, Technion, Israel, Oct. 1994, submitted to *Appl. Numer. Math.*.

[8] Shapira, Y.: Multigrid Techniques for Highly Indefinite Equations. 7th Copper Mountain Conference on Multigrid Methods, McCormick, S. F. and Manteuffel, T. A. (eds.), NASA, Langley Research Center, Hampton, VA (1995), in press.

[9] Shapira, Y.: Black-Box Multigrid Solver for Definite and Indefinite Problems. In Algebraic Multi-Level Iteration Methods with Applications, O. Axelsson and B. Polman (eds.), Nijmegen, The Netherlands, 1996, pp. 235–250.

[10] Shapira, Y.: Two-Level Analysis and Multigrid Methods for SPD, Non-Normal and Indefinite Problems. submitted to *SIAM J. Sci. Comput.*.

[11] Shapira, Y.: Domain Decomposition Multigrid for Unstructured Grids. Ninth Int'l Conference on Domain Decomposition Methods, Bjorstad, P., Espedal, M. and Keyes, D. (eds.), Willey and sons, Bergen, Norway, 1996 (submitted).

[12] Sonneveld, P.: CGS, a Fast Lanczos-Type Solver for Nonsymmetric Linear Systems. *SIAM J. Sci. Statist. Comput.* 10 (1989), pp. 36-52.