

CONF-9608183--1

TITLE: Unitary Symmetry, Combinatorics, and Special Functions

AUTHOR(S): James D. Louck Los Alamos National Laboratory
Theoretical DivisionRECEIVED
JAN 21 1997
OSTI

SUBMITTED TO: Proceedings: Group Theoretical Methods, Goslar, 1996

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

MASTER



Los Alamos National Laboratory, an affirmative action/equal opportunity employer, is operated by the University of California for the U.S. Department of Energy under contract W-7405-ENG-36. By acceptance of this article, the publisher recognizes that the U.S. Government retains a nonexclusive, royalty-free license to publish or reproduce the published form of this contribution, or to allow others to do so, for U.S. Government purposes. The Los Alamos National Laboratory requests that the publisher identify this article as work performed under the auspices of the U.S. Department of Energy.

UNITARY SYMMETRY, COMBINATORICS, AND SPECIAL FUNCTIONS

(Lawrence C. Biedenharn, Jr. Memorial Lecture)

James D. Louck

Los Alamos National Laboratory, Theoretical Division
Los Alamos, New Mexico 87545, USA

Abstract. From 1967 to 1994, Larry Biedenharn and I collaborated on 35 papers on various aspects of the general unitary group, especially its unitary irreducible representations and Wigner-Clebsch-Gordan coefficients. In our studies to unveil comprehensible structures in this subject, we discovered several nice results in special functions and combinatorics. The more important of these will be presented and their present status reviewed.

1. Personal Remarks

The following remark taken from the Preface of *Angular Momentum in Quantum Physics* [1], Vol. 8 characterizes, in my view, the spirit of Larry Biedenharn's approach to physics. It was part of his "bag of tricks," and I personally saw it in action over and over again, sparked by his great intuition for what was important.

"The art of doing mathematics," Hilbert has said. "consists in finding that *special* case which contains all the germs of *generality*." in our view, angular momentum theory plays the role of that "special case" with symmetry--one of the most fruitful themes of modern mathematics and physics--as the 'generality'." We would only amend Hilbert's phrase to include physics as well as mathematics.

George Mackey's comments in Introduction to *The Racah-Wigner Algebra in Quantum Theory* ([1], Vol.9) capture the essence of Biedenharn's emergence as a prominent figure in theoretical physics.

"The year 1949 is a significant one in the history of the development of angular momentum theory. First, it is the year in which Racah completed his celebrated series of four papers on angular momentum theory in atomic spectroscopy. Second, it is the year in which that same Racah, a chief advocate and developer or "purely algebraic" methods, reintroduced group theory and did it with a vengeance. Third, it is the year in which Racah's methods and concepts began to find applications to other parts of physics. Finally, it is the year in which L. C. Biedenharn, the senior author of the present volume, completed his doctorate and formally began his scientific career."

"As mentioned above, Biedenharn, the senior author of this book, was just beginning his scientific career when Racah's paper IV appeared. Its publication coincided with and partly inspired a surge of interest in the theory and applications of Racah's methods. Biedenharn soon became deeply involved, and he is now one of the leading experts on all phases of angular momentum theory, ... "

"In searching for recursion relations for the Racah coefficients, Professor Biedenharn discovered a remarkable new identity between such coefficients. It immediately implies a useful recursion relation and was later found to have an elegant conceptual interpretation. This result was published in the *Journal of Mathematics and Physics* in 1953.

DISCLAIMER

**Portions of this document may be illegible
in electronic image products. Images are
produced from the best available original
document.**

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency thereof, nor any of their employees, make any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

For the next seven years or so, Professor Biedenharn concerned himself almost exclusively with other parts of theoretical physics, but he returned to angular momentum theory in 1961. He has been a prolific and steady contributor ever since."

To George Mackey's observations, I would add: Larry Biedenharn's search for and his success in elucidating comprehensible structure in complex physical and mathematical objects was a commitment that continued throughout his extraordinarily productive career.

The versatility of Biedenharn's talents is well-illustrated by the following:

"A fact is discovered, a theory is invented; ..." J. Bronowski, *The Creative Process*, 1958. (Cited from *Scientific Genius and Creativity*, Owen Gingerich, Readings from *Scientific American*, Freeman, New York, 1986.)

Larry Biedenharn discovered:

$$W(a\alpha b\beta; c\gamma)W(\bar{a}\bar{\alpha}\bar{b}\bar{\beta}; \bar{c}\bar{\gamma}) = \sum_{\lambda} (2\lambda + 1) W(\bar{a}\lambda\alpha c; a\bar{c}) W(b\lambda\bar{c}; \bar{b}c) W(\bar{a}\lambda\gamma b; a\bar{b})$$

This is Eq. (25) from [2], the famous identity mentioned by Mackey (Elliott [3] discovered this relation at about the same time; it is generally known as the Biedenharn-Elliott identity). Who would have guessed that at the very time that the books cited in [1] were being written Askey and Wilson ([4], [5]) were using this relation as a guide to their comprehensive treatment of orthogonal polynomials? Who would have guessed that this relation would have an important role in knot theory and 3-manifolds (Tureav [6]).

Larry Biedenharn invented:

The concept of a **unit tensor operator** in the general unitary group $U(n)$ --a far-reaching generalization of the Racah and Wigner $SU(2)$ (angular momentum) theory.

1. Introduction

I wish to review Larry Biedenharn's contributions to the theory of the general unitary group $U(n)$, which includes its representation functions, its Wigner-Clebsch-Gordan coefficients, its unit tensor operators, and the special functions that arise. This is because it is this part of his many-faceted career with which I am most familiar. Combinatorial aspects of the subject will be noted as well as some of the beautiful mathematics that has emerged and that is still developing. I believe this characterizes aptly the depth and impact of the scientific creativity of Lawrence C. Biedenharn, Jr., although his interests in and contributions to basic physics problems goes well beyond this limited arena and my ability to review it.

In physical applications of symmetry, there are two basic problems to consider. Given a Lie groups G , which is a symmetry group of a physical system:

- I. Determine the unitary irreducible representations of G .
- II. Determine the Clebsch-Gordan coefficients of G .

The first problem is important because the labels of these representations provide the labels of the quantum states of the physical observables associated with the symmetry. The second problem is important because it is the first step in building composite systems

possessing the symmetry group G from elementary systems possessing the same symmetry group G . We will restrict our attention here to the unitary group defined by

$$U(n) = \{U \mid U \text{ is } n \times n \text{ unitary}; UU^\dagger = I\}, \dagger \text{ denotes Hermitian conjugation.} \quad (1)$$

As we shall see, however, many of the results extend to the general linear group $GL(n, \mathbb{C})$, indeed, to arbitrary matrix algebras.

Much of the work done by physicists in the development of group theory, with a view to applications to quantal systems, is carried out in the context of the boson calculus. This is because the creation-annihilation operator approach to the quantal states of a complex system has a wide range of applicability across many areas of physics and chemistry. It is instructive to review briefly this algebra.

The main features of the boson operator calculus may be summarized as follows:

• **basic operator algebra (Heisenberg) algebra:**

$$n \text{ commuting creation operators: } (a_1, a_2, \dots, a_n) \quad (2)$$

$$n \text{ commuting annihilation operators: } (\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) \quad (3)$$

commutation relations:

$$[\bar{a}_i, a_j] = \bar{a}_i a_j - a_j \bar{a}_i = \delta_{ij}. \quad (4)$$

• **unitary group action:**

$$U \text{col}(a_1, a_2, \dots, a_n) = \text{col}(a'_1, a'_2, \dots, a'_n), \text{ each } U \in U(n), \quad (5)$$

$$U^\dagger \text{col}(\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n) = \text{col}(\bar{a}'_1, \bar{a}'_2, \dots, \bar{a}'_n), \text{ each } U \in U(n).$$

invariance of commutation relations :

$$[\bar{a}'_i, a'_j] = \bar{a}'_i a'_j - a'_j \bar{a}'_i = \delta_{ij}. \quad (6)$$

$$\text{invariance of the operator: } N = \sum_{i=1}^n a_i \bar{a}_i = \sum_{i=1}^n a'_i \bar{a}'_i. \quad (7)$$

• **group action on the set of polynomials over the creation operators:**

$$(T_U P)(a_1, a_2, \dots, a_n) = P'(a_1, a_2, \dots, a_n) = P(a'_1, a'_2, \dots, a'_n), \quad (8)$$

$$\text{col}(a'_1, a'_2, \dots, a'_n) = U^T \text{col}(a_1, a_2, \dots, a_n), \text{ (}T \text{ denotes matrix transposition).} \quad (9)$$

In general physical applications, N copies of the boson operator structure are used. For the general representation theory of the unitary group $U(n)$, it is sufficient to take n such copies. Each of the n copies then undergoes the same transformation under the action of the unitary group $U(n)$, but the problem of determining those polynomials over the n^2 bosons that transform irreducibly under this action is far from trivial.

In the lecture today, I will outline this theory in an isomorphic form by making the replacements

$$a_i \rightarrow z_i, \bar{a}_i \rightarrow \partial / \partial z_i, \quad (10)$$

where the z_i are taken to be indeterminates. The reasons for this are a commitment

to carrying forward the ideas of L.C. Biedenharn, the personal belief that combinatorics has much to offer in this endeavor, and that what is already developed by physicists can be best communicated to mathematicians in that field by the use of indeterminates.

2. Irreducible Unitary Representations of $U(n)$

2.1. Basic definitions

- **Hilbert space H :** Ring of polynomials over the complex numbers in any number of variables (indeterminates) $z = (z_1, z_2, \dots)$ with inner product:

$$(P, P') = P * (\partial / \partial z) P'(z) \Big|_{z=0}, \quad P, P' \in H. \quad (11)$$

- **basis polynomials:**

$$B_A(Z) = Z^A / A! \stackrel{\text{def}}{=} \prod_{i,j=1}^n (z_{ij})^{a_{ij}} / (a_{ij})!, \quad (12)$$

where Z is an $n \times n$ matrix of (commuting) indeterminates, $Z = (z_{ij})$, $i, j = 1, 2, \dots, n$, and A is an $n \times n$ matrix of nonnegative integer exponents, $A = (a_{ij})$, $i, j = 1, 2, \dots, n$. These polynomials are then orthogonal with respect to the inner product (11):

$$(B_A, B_{A'}) = \delta_{A,A'} (A!)^{-1}. \quad (13)$$

- **group action in H :** The following actions of the unitary group $U(n)$ in the Hilbert space H are important:

$$\text{Left action: } (L_U P)(Z) = P(U^T Z), \text{ each } U \in U(n), \text{ each } P \in H. \quad (14)$$

$$\text{Right action: } (R_V P)(Z) = P(ZV), \text{ each } V \in U(n), \text{ each } P \in H. \quad (15)$$

These two actions commute: $L_U R_V = R_V L_U$.

- **irreducible polynomials basis:** The polynomial basis of H that transforms irreducibly under the action of $U(n)$ are given in terms of the polynomial basis (12) by

$$P_{m m'}^{\mu}(Z) = \sum_{(\alpha:A:\alpha')} C_{m m'}^{\mu}(A) Z^A / A!, \quad (16)$$

where the notations in this relation have the following definitions:

- (1) partition: $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n \geq 0$,

each μ_i a nonnegative integer;

(2) Gel'fand-Zetlin pattern:

$$\begin{pmatrix} \mu \\ m \end{pmatrix} = \begin{array}{ccccccccc} m_{1,n} & m_{2,n} & & \cdots & m_{n,n} \\ & \ddots & \ddots & & \ddots & & \\ & & m_{1,3} & m_{2,3} & m_{3,3} & & & & (\mu_i = m_{i,n}). \\ & & m_{1,2} & & m_{2,2} & & & & \\ & & & m_{1,1} & & & & & \end{array} \quad (18)$$

Each row in this triangular array of integers is itself a partition, one that fits "between" the one above:

$$m_{1,j+1} \leq m_{1,j} \leq m_{2,j+1} \leq \cdots \leq m_{j,j} \leq m_{j+1,j+1} \quad (1 \leq j \leq n-1). \quad (19)$$

This rule of "betweenness" expresses the well-known Weyl group-subgroup rule that the irreducible representation of $U(j+1)$ labeled by partition

$(m_{1,j+1}, m_{2,j+1}, \dots, m_{j+1,j+1})$ reduces on restriction to $U(j)$ to a direct sum of those irreducible representations of $U(j)$ labeled by the partitions $(m_{1,j}, m_{2,j}, \dots, m_{j,j})$, each occurring once.

(3) Double Gel'fand-Zetlin pattern:

$$\begin{pmatrix} \mu \\ m \end{pmatrix} \text{ and } \begin{pmatrix} \mu \\ m' \end{pmatrix}, \text{ written as } \begin{pmatrix} m' \\ \lambda \\ m \end{pmatrix} \text{ or } \begin{array}{c} \lambda \\ m \quad m' \end{array}. \quad (20)$$

In the three-rowed notation, the one pattern is inverted over the other with the shared partition label in the middle, as illustrated by

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & & \end{pmatrix} \leftrightarrow \begin{pmatrix} 2 & & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & & 1 & 1 \end{pmatrix}. \quad (21)$$

(4) The weight (also called a content) of the Gel'fand pattern (18) is the n -tuple defined by

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n),$$

$$\alpha_j = (\text{sum of entries in row } j) - (\text{sum of entries in row } j-1)$$

$$= \sum_{i=1}^j m_{i,j} - \sum_{i=1}^{j-1} m_{i,j-1}, \quad (\alpha_1 = m_{1,1}). \quad (22)$$

(5) The coefficients

$$C_{m m'}^{\mu}(A) \quad (23)$$

may be considered as functions $C_{m m'}^{\mu}$ labeled by the same double Gel'fand patterns that label the functions on the left-hand side in (16), which are defined over the elements of the $n \times n$ array A of exponents that occurs in (12), where the row and column sums of this array are the weights α and α' of the Gel'fand patterns (μ_m) and $(\mu_{m'})$:

$$\text{row } i: \sum_j a_{ij} = \alpha_i; \text{ column } j: \sum_i a_{ij} = \alpha'_j. \quad (24)$$

The "discretized" functions are the elements of an orthogonal matrix that transforms the orthogonal basis (12) of H to the new orthogonal (16) of H . This is discussed further below.

(6) The summation

$$\sum_{(\alpha:A:\alpha')} \quad (25)$$

in (16) is over all $n \times n$ arrays A of nonnegative integers such that the row and column sum restrictions (24) are satisfied.

(7) For fixed partition μ , the Gel'fand patterns $\binom{\mu}{m}$ and $\binom{\mu}{m'}$ enumerate rows and columns, respectively, as m and m' run over all values allowed by the betweenness conditions, of the matrix, denoted

$$P^\mu(Z), \quad (26)$$

of polynomials (16). In order to write out this matrix, we order the Gel'fand patterns by the following rule: Associate with each Gel'fand pattern (18) the sequence W defined by "stringing" together the rows:

$$W = (m_{1,n-1}, \dots, m_{n-1,n-1}; \dots; m_{1,2}, m_{2,2}; m_{1,1}). \quad (27)$$

We order lexicographically the sequences W corresponding to the various Gel'fand patterns having the same partition μ , and then order the Gel'fand patterns by the rule:

$$\binom{\mu}{m} > \binom{\mu}{m'}, \text{ if } W > W'. \quad (28)$$

It is customary in physics to order the columns of the matrix (26) as read from left to right by the greatest to the least pattern, and the rows as read from top to bottom by the same rule. The dimension of the matrix (26) is given by the Weyl dimension formula:

$$\dim P^\mu(Z) = \dim \mu = \prod_{i < j} (\mu_i - \mu_j + j - i) / 1! 2! \dots (n-1)! . \quad (29)$$

2.2. Basic properties of the polynomials $P_{m m'}^\mu$

We give here a partial list of important properties of the polynomials defined by (16). A more complete list may be found in [7]. It is, of course, the discretized functions (23) that must be given unique definition. We list first properties of the polynomials:

- **homogeneity:** The polynomials $P_{m m'}^\mu(Z)$ are homogeneous of degree α_i in $(z_{i1}, z_{i2}, \dots, z_{in})$, of degree α'_j in $(z_{1j}, z_{2j}, \dots, z_{nj})$, and of degree $\lambda_1 + \lambda_2 + \dots + \lambda_n$ in all n^2 variables.

- **transposition and multiplication:**

$$P^\mu(Z^T) = (P^\mu(Z))^T, \quad (30)$$

$$P^\mu(X) P^\mu(Y) = P^\mu(XY), \text{ X and Y arbitrary.}$$

• inner product orthogonality:

$$(P_l^{\lambda} P_{l'}^{\mu}) = \delta_{\lambda, \mu} \delta_{l, m} \delta_{l', m'} M(\lambda), \quad (31)$$

$$M(\lambda) = \prod_i (\lambda_i + n - i)! / \prod_{i < j} (\lambda_i - \lambda_j + j - i)! . \quad (32)$$

• generating-like functions:

$$\prod_{k=1}^n [\det(X^T Z Y)_k]^{\mu_k - \mu_{k+1}} = \sum_{m, m'} P_{m, \max}^{\mu}(X) P_{m' m'}^{\mu}(Z) P_{m', \max}^{\lambda}(Y) \quad (33)$$

$$\prod_{k=1}^n [\det(X^T Z)_k]^{\mu_k - \mu_{k+1}} = \sum_m P_{m, \max}^{\mu}(X) P_{m, \max}^{\mu}(Z).$$

• reduction property:

$$P_{m m'}^{\mu} \begin{pmatrix} Z_{n-1} & \mathbf{0} \\ \mathbf{0} & z_{nn} \end{pmatrix} = z_{nn}^{\mu_n} \delta_{\nu, \nu''} P_{q' q'}^{\nu} (Z_{n-1}),$$

$$(m) = \begin{pmatrix} \nu \\ q \end{pmatrix}, \quad (m') = \begin{pmatrix} \nu' \\ q' \end{pmatrix}. \quad (34)$$

• maximal polynomial:

$$P_{\max, \max}^{\mu}(Z) = \prod_{k=1}^n (\det Z_k)^{\mu_k - \mu_{k+1}}, \mu_{n+1} = 0, \quad (35)$$

$\det Z_k = k \times k$ principal minor of Z ,

where $\begin{pmatrix} \mu \\ \max \end{pmatrix}$ denotes the Gel'fand pattern of weight μ .

2.3 Basic properties of the discretized functions $C_{m m'}^{\mu}$

The coefficients $C_{m m'}^{\mu}(A)$ entering into (16) are the primary objects. By definition, they are taken to be zero unless the weights of the double Gel'fand pattern and the row and column sums satisfy the restrictions (24). Some of their properties follow:

• explicit orthogonality relations:

$$\sum_{(\alpha: A: \alpha')} (A!)^{-1} C_{\ell \ell'}^{\lambda}(A) C_{m m'}^{\mu}(A) = \delta_{\lambda, \mu} \delta_{\ell, m} \delta_{\ell', m'} M(\mu), \quad (36)$$

$$\sum_{\mu \mapsto N} (M(\mu))^{-1} \sum_{m, m'} C_{m m'}^{\mu}(A) C_{m m'}^{\mu}(A) = \delta_{A, A'} A!, \quad (37)$$

$$N = \alpha_1 + \dots + \alpha_n = \alpha'_1 + \dots + \alpha'_n.$$

•reduction property:

$$C_{\nu}^{\mu} \begin{pmatrix} A_{n-1} & \mathbf{0} \\ \mathbf{0} & \mu_n \end{pmatrix} = \mu_n! \delta_{\nu, \nu'} C_{\nu'}^{\mu} (A_{n-1}). \quad (38)$$

This structure reflects the group-subgroup property of the Weyl $U(n) \downarrow U(n-1)$ rule.

Formulas for these "discretized functions" version of the representation functions are known, but are quite difficult [8],[9]. We also have given [7] an explicit recurrence formula for them in which the coefficients at level n are constructed in terms of those at level $n-1$, using property (38). This formula, in turn, is derived from a similar formula in [8] giving the construction of the polynomials (16).

The nontrivial nature of these coefficients is already indicated by relation (35), involving a product of powers of determinants (see [10] for the expansion of the power of a determinant) and by the explicit result at level $n=2$, which already is the WCG-coefficient of $SU(2)$:

$$\begin{aligned} C_{2j_1, j_1+j_2+m}^{(j_1+j_2+j, j_1+j_2-j)} & \begin{pmatrix} j_1+m_1 & j_1-m_1 \\ j_2+m_2 & j_2-m_2 \end{pmatrix} = \\ & = [(2j+1)(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!]^{-1/2} C_{m_1 m_2 m}^{j_1 j_2 j}, \end{aligned} \quad (39)$$

where the C-notation on the right for an $SU(2)$ WCG-coefficients is used in [1]. Formulas for these coefficients for $n=3$ can be found in [9].

2.4. The irreducible representations of $U(n)$

The irreducible representations functions of $U(n)$ are just the functions (16) obtained by setting $Z = U \in U(n)$:

$$D_m^{\mu} (U) = P_m^{\mu} (U) = \sum_{(\alpha:A:\alpha')} C_m^{\mu} (A) U^A / A!. \quad (40)$$

Thus, we obtain the unitary irreducible representations of $U(n)$ given by

$$\{D^{\mu} (U) \mid U \in U(n)\}, \mu \text{ an arbitrary partition of } n. \quad (41)$$

•representation properties:

$$\begin{aligned} D^{\mu} (U) D^{\mu} (U') &= D^{\mu} (UU'), \\ D^{\mu} (U) (D^{\mu} (U))^{\dagger} &= D^{\mu} (I_n) = I_{\dim \mu}. \end{aligned} \quad (42)$$

This gives all unitary irreducible representations, when multiplied by appropriate powers of $\det U$.

Under the right and left action of $U(n)$ given by (14) and (15) the polynomials (16) undergo the transformations:

$$\begin{aligned} (L_U P_m^{\mu} (m')) (Z) &= P_m^{\mu} (m') (U^T Z) = \sum_{m''} D_{m''}^{\mu} (U) P_{m''}^{\mu} (m') (Z), \\ (R_U P_m^{\mu} (m')) (Z) &= P_m^{\mu} (m') (ZU) = \sum_{m''} D_{m''}^{\mu} (m') (U) P_m^{\mu} (m'') (Z). \end{aligned} \quad (43)$$

2.5. Combinatorial and special function aspects of representation functions

The result presented in Sections 2.1-2.3 abound with combinatorics. We list some of these connections:

(1) The set of Gel'fand patterns corresponding to a given partition μ are one-to-one with the set of standard Young tableaux corresponding to a Young frame of shape μ (see [1] and [11]). Objects having a definition in terms of Gel'fand patterns accordingly have a definition in terms of standard tableaux and conversely. Two examples of this are the classical Schur functions and the Yamanouchi orthogonal representations of the symmetric group (see [8],[12],[13]):

$$\sum_m P_m^\mu \begin{pmatrix} z_1 & 0 \\ 0 & z_n \end{pmatrix} = s_\mu(z_1, z_2, \dots, z_n); \quad (44)$$

$D_{m m'}^\lambda(\Pi)$, $Z = \Pi = n \times n$ permutation matrix,

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots, 0), \lambda_1 + \dots + \lambda_r = n, \lambda_r > 0; \quad (45)$$

0 repeated $n-r$ times, $n \geq r$;

weight α of $m = (1, 1, \dots, 1)$, weight α' of $m = (1, 1, \dots, 1)$

(2) There is a close connection between the polynomials (16) and Rota's double tableau polynomials (see [14] and [15]).

(3) The special case of the polynomials (16) corresponding to the totally symmetric case with $\mu = (k, 0, \dots, 0)$ occur in MacMahon's master theorem (see [15]). The basic result here is

$$1 / \det(I - tXY) = \sum_{k=0}^{\infty} t^k \sum_{\alpha, \beta} \alpha! \beta! P_{\beta, \alpha}^k(X) P_{\alpha, \beta}^k(Y), \quad (46)$$

$$P_{\alpha \beta}^k(Z) = \sum_{(\alpha: A: \beta)} Z^A / A!, \quad (47)$$

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \beta = (\beta_1, \beta_2, \dots, \beta_n), \sum_i \alpha_i = \sum_i \beta_i = k.$$

(4) The normalization factor $M(\lambda)$ in (32) is a "hook" function read off a standard tableau (see [1] and [13]).

(5) One obtains the Gel'fand-Graev [16] (see also [8]) generalized beta function by the specialization

$$P_m^\mu \begin{pmatrix} I_{n-1} & 0 \\ z_1 z_2 \dots z_n & 1 \end{pmatrix} = C_m^\mu \begin{pmatrix} \alpha_1 & 0 & \dots & 0 & 0 \\ \vdots & \alpha_2 & \ddots & \vdots & \vdots \\ 0 & \dots & \alpha_{n-1} & 0 & 0 \\ \alpha'_1 - \alpha_1 & \dots & \alpha'_{n-1} - \alpha_{n-1} & \alpha'_n & \alpha'_n \end{pmatrix} \times \frac{1}{\alpha'_n!} \prod_{i=1}^{n-1} z_i^{\alpha'_i - \alpha_i} / \alpha_i! (\alpha'_i - \alpha_i)! \quad (48)$$

(6) The power of a determinant is an interesting combinatorial object giving rise to mappings between $n \times n$ arrays A with fixed row and column sums and partitions [10].

It is also well-known ([1], [17], [18]) that the power of a 3×3 determinant is the generating function for the $SU(2)$ WCG-coefficients.

(7) The number of $n \times n$ arrays A with fixed row and column sums α and α' is denoted $M_n(\alpha, \alpha')$ and the number of Gel'fand patterns with partition μ having weight α by $K(\mu, \alpha)$ (Kostka number). The fact that the coefficients in (16) constitute an orthogonal transformation, hence square, between bases of the space H implies:

$$M_n(\alpha, \alpha') = \sum_{\lambda \vdash N} K(\lambda, \alpha) K(\lambda, \alpha'). \quad (49)$$

This basic relation in combinatorics was proved by Knuth [19] in the context of double standard tableaux.

(8) The coefficients $C_{m m'}^{\mu}(A)$ are "combinatorial" in every aspect of their labeling: double standard tableaux and square arrays of nonnegative integers having fixed row and column sume. A *combinatorial interpretation and derivation would be major accomplishments for mathematics and physics*.

(9) Special function aspects of the $SU(2)$ representation functions are well-known from the work of Wigner [20] and others [21]- [22], involving Jacobi polynomials, Gegenbauer polynomials, Laguerre polynomials, etc. Very little has been done along these lines for general $U(n)$.

3. Kronecker Products

Given two partitions $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ labeling two irreducible representations of $U(n)$, the abstract Clebsch-Gordan series expressing the decomposition of the Kronecker product into irreducibles and the explicit form of this relation for the irreps (40) are given by

$$\mu \times \nu = \sum_{\lambda} g_{\mu\nu\lambda} \lambda, \quad (50)$$

$$C^T (D^{\mu}(U) \times D^{\nu}(U)) C = \sum_{\lambda} \oplus g_{\mu\nu\lambda} D^{\lambda}(U). \quad (51)$$

In (51), \times denotes the Kronecker of direct product of matrices; \oplus the direct sum, and C is a real orthogonal matrix whose elements are the WCG-coefficients of $U(n)$. The matrix C is of dimension given by:

$$\dim C = \dim \mu \dim \nu = \sum_{\lambda} g_{\mu\nu\lambda} \dim \lambda. \quad (52)$$

The number $g_{\mu\nu\lambda}$ is the Littlewood-Richardson number and gives the number of occurrences of irreducible representation λ in the Kronecker product $\mu \times \nu$.

It is quite significant that (51) is valid when U is replaced by the arbitrary $n \times n$ matrix Z of indeterminates:

$$C^T (D^{\mu}(Z) \times D^{\nu}(Z)) C = \sum_{\lambda} \oplus g_{\mu\nu\lambda} D^{\lambda}(Z). \quad (53)$$

3.1. Biedenharn's operator pattern

It was Biedenharn's great insight that led him to the discovery of a universal labeling of the WCG-coefficients of $U(n)$. He was lead to this discovery by considering the extension of the Wigner-Racah definition of a tensor operator in $SU(2)$ to a unit tensor operator in $U(n)$. This is discussed in the next section. Here we introduce at the outset this labeling directly into the symbol for a WCG-coefficient. The meaning of the notation requires some discussion.

Notation for a WCG – coefficient:
$$\begin{bmatrix} \gamma \\ \lambda & \mu & \nu \\ l & m & q \end{bmatrix}; \quad (54)$$

λ , ν are single Gel' fand patterns; μ is a double Gel' fand pattern.

The occurrence of these patterns is clear in that they enumerate the rows of C :

rows: all $\frac{\mu}{m}$ and $\frac{\nu}{q}$ patterns, $\dim \mu \dim \nu$ labels. (55)

It is the patterns λ , $\dim \lambda$ in number, and the patterns γ , $\dim \mu$ in number, that must provide the column labels of C . This is effected in the following manner, which takes into account two facts: The first fact is that for every partition λ that occurs in $\mu \times \nu$, written $\lambda \in \mu \times \nu$, in the Clebsch-Gordan series (50), there exists a unique weight Δ of the partition μ such that $\lambda = \nu + \Delta$; the second fact is that the maximum value of $g_{\mu, \nu, \nu + \Delta}$ is the Kostka number $K(\mu, \Delta)$. These results are proved in [23] (see also [11] and Kostant [24]). Thus, tentatively, the columns of C are labeled by

columns: all λ patterns, $\dim \lambda$ in number;
all γ patterns such that $\lambda = \nu + \Delta, K(\mu, \Delta)$ in number. (56)

While the maximum value $K(\mu, \Delta)$ of $g_{\mu, \nu, \nu + \Delta}$ is achieved for a denumerable number of partitions ν , it is also true that $g_{\mu, \nu, \nu + \Delta} < K(\mu, \Delta)$ for a denumerable number of partitions ν . Thus, we have

$$\dim \mu \dim \nu = \sum_{\Delta} g_{\mu, \nu, \nu + \Delta} \dim(\nu + \Delta) \leq \sum_{\Delta} K(\mu, \Delta) \dim(\nu + \Delta). \quad (57)$$

In general, we have too many patterns γ in the symbol (54). This is not a fatal flaw. It may be resolved as follows: Let us order the patterns γ by the rule given in (28):

$$\left(\begin{array}{c} \mu \\ \gamma_1 \end{array} \right) > \left(\begin{array}{c} \mu \\ \gamma_2 \end{array} \right) > \dots > \left(\begin{array}{c} \mu \\ \gamma_{\dim \mu} \end{array} \right). \quad (58)$$

We now define the symbol (54) to be zero under the following conditions:

$$\begin{bmatrix} \gamma \\ \lambda & \mu & \nu \\ p & q & r \end{bmatrix} = 0, \text{ unless: } \begin{array}{l} \lambda = \nu + \Delta, \text{ where } \Delta \text{ is a weight of } \mu; \\ \Delta \left(\begin{array}{c} \mu \\ \gamma \end{array} \right) = \Delta, \gamma = \gamma_1, \gamma_2, \dots, \gamma_{\dim \mu} \end{array} \quad (59)$$

By this definition, we now obtain the required number of WCG-coefficients. In the way of nomenclature, the patterns γ are referred to as *operator patterns* because of their role in enumerating unit tensor operators, where all $\dim \mu$ patterns are essential, and the

weight Δ is called a *shift-weight* because of its role in shifting from the partition ν to the partition λ .

The rule (59) for general n is ad hoc. Its merit rests on two points: For $n=2,3$, the concept of the characteristic null space of a unit tensor operator ([1], [23], [25]-[28]) provides a natural ordering in agreement with (58); even if there is no natural ordering, the patterns are *universal* in the sense that it is always a subset of the full set, the full set containing $\dim \mu$ patterns, that effects the labeling, and in denumerably many cases all are required.

It was always Biedenharn's belief that there exists a natural definition of unit tensor operators such that the zeros would fall into place automatically. It is my position that even should this fail, the concept of operator pattern is the most important structural notion introduced into the delineation of the WCG-coefficients of $U(n)$. When it comes to the definition of the Racah coefficients of $U(n)$, operator patterns are indispensable.

The following structural result, known as the factorization lemma [1], [25], [29], is among the most important general relations for $U(n)$. The orthogonal matrix C is moved to the right-hand side in (53) to obtain

$$P_m^\mu P_{m'}^\nu (Z) P_q^\lambda P_{q'}^\nu (Z) = \sum_{\substack{\lambda \\ (\lambda \\ l' \\ l)}} \begin{bmatrix} l' & m' & q' \\ \lambda & \mu & \nu \\ l & m & q \end{bmatrix} P_l^\lambda P_{l'}^\nu (Z), \quad (60)$$

where the square-bracket coefficient is a sum of *double* WCG-coefficients given by

$$\begin{bmatrix} l' & m' & q' \\ \lambda & \mu & \nu \\ l & m & q \end{bmatrix} = \sum \begin{bmatrix} \gamma & & \gamma \\ \lambda & \mu & \nu \\ l & m & q \end{bmatrix} \begin{bmatrix} \gamma & & \gamma \\ \lambda & \mu & \nu \\ l' & m' & q' \end{bmatrix}. \quad (61)$$

Using the inner product (11) gives:

$$(P_l^\lambda, P_m^\mu P_{m'}^\nu P_q^\lambda P_{q'}^\nu) = M(\lambda) \begin{bmatrix} l' & m' & q' \\ \lambda & \mu & \nu \\ l & m & q \end{bmatrix}. \quad (62)$$

The left-hand side of this expression may be given in terms of the coefficients in the expansion (16), hence, the right-hand side of (62) may be considered known. The basic question is: Does there exist a *natural structure*, as discussed above, that allows one to take "apart" the summation over operator patterns in (61) to obtain the WCG-coefficients themselves? The answer is yes for $n=2,3$.

3.1. Combinatorial and special function aspects

(1) Relation (60) is referred to in the mathematical literature as linearization. Use of (44) gives the classical Schur function identity:

$$s_\mu(x) s_\nu(x) = \sum_{\lambda} g_{\mu\nu\lambda} s_\lambda(x). \quad (63)$$

(2) The Littlewood-Richardson numbers may be expressed in terms of the Kostka numbers in (57) by

$$g_{\mu, \nu, \nu + \Delta} = \sum_{\pi \in S_n} \varepsilon_\pi K(\mu, (\nu + \Delta) \circ \pi - \nu), \quad (64)$$

where Δ is a shift-weight of μ such that $\nu + \Delta$ is a partition, and the action of $\pi \in S_n$ on an n -tuple $a = (a_1, a_2, \dots, a_n)$, denoted $a \circ \pi$, is defined by

$$a \circ \pi = (a_{\pi_1} - \pi_1 + 1, a_{\pi_2} - \pi_2 + 2, \dots, a_{\pi_n} - \pi_n + n), \quad (65)$$

$$\pi: (1, 2, \dots, n) \rightarrow (\pi_1, \pi_2, \dots, \pi_n).$$

Relation (64) allows one to investigate the properties of the Littlewood-Richardson numbers. It is known that

$$g_{\mu, \nu, \nu + \Delta} \in \{0, 1, \dots, K(\mu, \Delta)\}, \text{ each partition } \nu. \quad (66)$$

The determination of the set of all ν such that $g_{\mu, \nu, \nu + \Delta} = k, 0 \leq k \leq K(\mu, \Delta)\}$, is very important for the labeling problem of the WCG-coefficients of $U(n)$, as discussed in (56)-(59) above. Only partial progress has been made so far in [23] and in the important work by Baclawski [30]. This is a fascinating problem in pure combinatorics.

(3) For $SU(2)$, the combinatorial problems and special function aspects of WCG-coefficients and the associated $3n - j$ coefficients are almost boundless when one considers the reduction into irreducibles of multiple Kronecker products. This leads to the Biedenharn-Elliott identity, which inspired Askey and Wilson toward their synthesis of orthogonal polynomials. One encounters here the fascinating relationship between $3n - j$ coefficients, labeled binary trees, Cayley trivalent trees, cubic graphs, triangle polynomials, and generating functions. There is no space here to discuss this, and we refer to [7] and [9] for a recent accounting. For $U(n)$, there is almost nothing known about multiple Kronecker products.

5. Biedenharn's Abstract Theory of Unit Tensor Operators

Larry Biedenharn discovered many of the above *facts*, and used them to *invent* a comprehensive theory of tensor operators. We take a quite pedestrian approach here and regard the WCG-coefficients of $U(n)$ as known, although this is not necessary. Biedenharn recognized that while the Gel'fand patterns encode a Weyl group-subgroup property, the patterns γ encode an entirely different kind of information: these patterns encode a *shift action* associated with a weight of partition μ in going from partition ν to the partition $\lambda = \nu + \Delta \in \mu \times \nu$. In order to give an operator formulation of this, we assume as given a vector space H_ν of $\dim H_\nu = \dim \nu$, given by the Weyl dimension formula (29), on which there is defined an action of $U(n)$ such that

$$T_U \left| \begin{smallmatrix} \nu \\ q' \end{smallmatrix} \right\rangle = \sum_q D_q \left| \begin{smallmatrix} \nu \\ q' \end{smallmatrix} \right\rangle (U) \left| \begin{smallmatrix} \nu \\ q \end{smallmatrix} \right\rangle, \quad (67)$$

where an orthonormal basis of H_ν is given by the set of ket vectors

$$B_\nu = \left\{ \left| \begin{smallmatrix} \nu \\ q \end{smallmatrix} \right\rangle \mid q \text{ runs over all Gel'fand patterns} \right\} \quad (68)$$

These are, in fact, not assumptions, since one already has at hand many such vector spaces as given by the normalized polynomials

$$\left\langle Z \left| \begin{smallmatrix} \nu \\ q \quad q' \end{smallmatrix} \right\rangle \right\rangle = [M(\nu)]^{-1/2} P_q \left| \begin{smallmatrix} \nu \\ q \quad q' \end{smallmatrix} \right\rangle (Z), \quad (69)$$

on which one fixes q' and chooses T_U to be the left action L_U (see (14), (40), and (43)). One can interpret the right action in (43) similarly.

In order that the shift action intrinsic to the shift-weight Δ of a pattern γ map a vector space into a vector space, we introduce the model Hilbert space as a direct sum of perpendicular spaces H_γ , each taken exactly once:

$$H = \sum_{\gamma} \bigoplus H_\gamma. \quad (70)$$

We now define shift operators mappings of H into H as follows:

$$\left\langle \begin{array}{c} \gamma \\ \mu \\ m \end{array} \right| \left| \begin{array}{c} \nu \\ q \end{array} \right\rangle = \sum_{\gamma, l} \left[\begin{array}{ccc} \nu + \Delta(\gamma) & \gamma & \nu \\ l & \mu & m \\ & m & q \end{array} \right] \left| \begin{array}{c} \nu + \Delta(\gamma) \\ l \\ m \\ q \end{array} \right\rangle. \quad (71)$$

It is then a consequence of the definition of the WCG-coefficients that:

$$T_U \left\langle \begin{array}{c} \gamma \\ \mu \\ m' \end{array} \right| T_U^{-1} = \sum_m D_m \left\langle \begin{array}{c} \gamma \\ \mu \\ m \end{array} \right|, \text{ each pattern } \gamma. \quad (72)$$

A set of operators

$$\left\{ \left\langle \begin{array}{c} \gamma \\ \mu \\ m \end{array} \right| \mid m \text{ runs over all Gel' fand patterns} \right\} \quad (73)$$

with the transformation property (72) constitute what physicists call an *irreducible tensor operator* of $U(n)$. Each pattern γ , $\dim \mu$ in number, gives a tensor operator (73) with $\dim \mu$ components. The term *unit tensor operator* originates from the fact that the WCG-coefficients in the definition (71) are elements of an orthogonal matrix, and an arbitrary tensor operator with the transformation property (72) is a sum of such unit tensor operators with coefficients that are invariants under the action of $U(n)$.

It cannot be emphasized too strongly that all operator patterns γ , $\dim \mu$ in number, enter into definition (71), the seros of the WCG-coefficients being associated with entire vector spaces lying in the null space of such an operator. It was this approach, through tensor operators, that led Biedenharn to the introduction of operator patterns.

It is not my intention here to go further into the properties of $U(n)$ WCG-coefficients, except to note that all their properties flow from the definition (71). Some new ideas are presented in [31]. In the mid-seventies, attention shifted to the detailed construction of the $U(3)$ WCG-coefficients [32]-[36], since the characteristic null space classification was complete. It was at this point that Max Lohe joined the efforts and was instrumental in advancing this subject. Here I will describe one small aspect of these difficult calculations because of the richness of mathematical constructs that emerged from them.

6. Combinatorial Mathematics Originating from the Study of $U(3)$ Unit Tensor Operators

The flavor of the mathematics encountered in constructing the WCG-coefficients for $U(3)$, in both its difficulty and elegance, can be sensed by describing some of the properties of a very special family of polynomials that arises from just one small piece of the WCG problem for $U(3)$. These are the $G_q^t(\Delta_1, \Delta_2, \Delta_3; x_1, x_2, x_3)$ polynomials which are defined for each nonnegative integer q and each $t = 0, 1, \dots, q$. The quantity $\Delta = (\Delta_1, \Delta_2, \Delta_3)$ is the shift-weight of a $U(3)$ unit tensor operator, and the real variables

$x = (x_1, x_2, x_3)$ are arbitrary. The properties of these polynomials are developed in a number of papers [32]-[36]. They have the astonishing property of vanishing on the points of the general weight space associated with irrep $(q-t, 0, -t+1)$ of $U(3)$ with the multiplicity of a given zero being equal to the multiplicity of the weight space point. Such weight space points may be positioned in the Möbius plane (positive axes at 120° with coordinates determined by perpendicular projection onto the three axes) by the coordinates $a = (a_1, a_2, a_3)$,

$$a_1 = \Delta_3 - t + 1 - \alpha_1, a_2 = -\Delta_2 - \Delta_3 + q - 1 - \alpha_2, a_3 = \Delta_2 - t + 1 - \alpha_3, \quad (74)$$

obtained by letting $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ run over all weights of irrep $(q-t, 0, -t+1)$. The multiplicity of the weight space point $a = (a_1, a_2, a_3)$ is given by

$$M_q^t(\Delta; a) = \min(t, q-t+1, 1+d_t(a)), \quad (75)$$

where $d_t(a)$ is the "distance" from the lattice point a to the nearest boundary of the weight space diagram, measured along the appropriate coordinate axis (1 lattice spacing = 1 unit of distance). We then have that

$$G_q^t(\Delta; a) = 0, \text{ each point } a \text{ of the weight space,} \quad (76)$$

where the multiplicity of the zero a is $M_q^t(\Delta; a)$. The polynomial $G_q^t(\Delta; x)$ also vanishes at some lattice points obtained from the weight space diagram by symmetries, but at no other lattice points in the Möbius plane. Moreover, each polynomial $G_q^t(\Delta; x)$ is irreducible in that it cannot be factored over the lattice points of the Möbius plane.

The polynomials $G_q^t(\Delta; x)$ have been given explicitly [32]-[36]. We mention this quite difficult subject here because it was the "source" of some quite nice combinatorial-like mathematics that arose from our studies of the symmetries and zeros of the $G_q^t(\Delta; x)$ polynomials, which we now mention briefly.

• **hypergeometric Schur functions:** These functions combine certain hypergeometric coefficients with the classical Schur functions $s_\lambda(x_1, x_2, \dots, x_n)$:

$${}_p\mathfrak{S}_q(a; b; x) = \sum_{\lambda} \langle {}_p\mathfrak{S}_q(a; b) | \lambda \rangle s_\lambda(x), \quad (77)$$

where the hypergeometric coefficient depends on p numerator parameters $a = (a_1, a_2, \dots, a_p)$ and q denominator parameters $b = (b_1, b_2, \dots, b_q)$ and is defined by

$$\langle {}_p\mathfrak{S}_q(a; b) | \lambda \rangle = \frac{1}{M(\lambda)} \prod_{s=1}^n \frac{\prod_{i=1}^p (a_i - s+1) \lambda_s}{\prod_{j=1}^q (b_j - s+1) \lambda_s}, \quad (78)$$

in which $(z)_k = z(z+1)\dots(z+k-1)$, $k = 0, 1, \dots$ with $(z)_0 = 1$, denotes a rising factorial.

The functions ${}_2\mathfrak{S}_1(a; b; x)$ were introduced in [37] (It turns out they have been discovered in a completely different context by James [38]). These functions and their generalizations (77) have been studied extensively [39]-[41].

• **factorial Schur functions:** These functions arose from our studies of the symmetries of the $G_q^t(\Delta; x)$ polynomials, and some of their properties are presented in [42]-[45]. The factorial Schur functions are polynomials in n indeterminates $z = (z_1, z_2, \dots, z_n)$ and are defined in terms of the set of Gel'fand patterns

$$G_\lambda = \left\{ \binom{\lambda}{m} \middle| \text{all } m \text{ satisfying the betweenness conditions} \right\} \quad (79)$$

by the following formulas:

$$\begin{aligned} t_\lambda(z) &= \sum_{m \in G_\lambda} t_m(z), \\ t_m(z) &= \prod_{j=1}^n \prod_{i=1}^j (z_j - m_{i,j} - j + i + 1)_{m_{i,j} - m_{i,j+1}}. \end{aligned} \quad (80)$$

In this last relation, we have $\lambda_i = m_{i,n}$. These functions may be written in a number of alternative forms paralleling the standard Schur functions and have turned out to be quite interesting for combinatorial mathematics [46]-[49].

7. Concluding Remarks

I have reviewed (rather too quickly, I fear) one small corner of the many-faceted interests of Lawrence C. Biedenharn and tried to illustrate the unique viewpoints that he brought to all of his research, viewpoints deeply rooted in an intuitive geometrical and quantum world-view that guided him to the basic foundations of a subject, perspectives that often eluded others.

I have not touched on his work, mostly in nuclear physics, prior to his moving into the field of symmetry and its applications, nor have I mentioned the papers with Le Blanc, Hecht, and Rowe, exploring the use of the coherent state approach to unitary groups and WCG-coefficients, his work with Gustafson and Milne on a class of $U(n)$ generalizations, nor his work with Johndale Solem in gamma ray lasers, to mention a few omissions.

I need also to remark that Larry, with the help of Max Lohe, and many colleagues present here today, have made enormous advances in recent years in giving the q -versions of much that I have mentioned in this lecture.

Finally, I apologize to the numerous investigators whose contributions to unitary symmetry have been enormous, and whom I have not mentioned, since it was my intention not to review the field, but to try to give a coherent picture of the Biedenharn approach.

One of my own goals is to continue Larry's work by bringing a firm combinatorial foundation to the subject that I have just reviewed.

Acknowledgments

I am deeply honored to be asked to give this lecture by the organizers, and especially thank Prof. Dr. Doebner for his role and hospitality. Work performed under auspices of the U.S. Department of Energy.

References

1. L.C. Biedenharn and J.D. Louck, *Angular Momentum in Quantum Physics*, Vol. 8; *The Racah-Wigner Algebra in Quantum Theory*, Vol 9., *Encyl. of Mathematics and Its Applications* (ed., G.-C.Rota, Cambridge Univ. Press, Cambridge, 1981.)
2. L.C. Biedenharn, An identity satisfied by the Racah coefficients, *J. Math. Phys.* 31 (1953) 287-293.
3. J.P. Elliott, Theoretical studies in nuclear spectroscopy V. The matrix elements of non-central forces with applications to the 2p-shell, *Proc. Roy. Soc. A* (1953) 345-370.
4. R. Askey and J.A. Wilson, A set of hypergeometric polynomials that generalize the Racah or 6-j symbols, *SIAM J. Math. Anal.* 10 (1979), 1008-1016.
5. R. Askey and J.A. Wilson, A set of hypergeometric orthogonal polynomials, *SIAM J. Math. Anal.* 13 (1982) 651-655.
6. Tureav , Invariant , Knot Theory, and 3-Manifolds.
7. J.D. Louck, Unitary symmetry, combinatorics and generating functions (presented at Conference on Combinatorics and Its Applications, Nankai University, China, June 1996; submitted for publication in *Discrete Mathematics*).
8. J.D. Louck and L.C. Biedenharn, The structure of the canonical tensor operators in the unitary groups. III. Further developments of the boson polynomials and their implications, *J. Math. Phys.* 14 (1973) 1336-1357.
9. L.C. Biedenharn, W.Y.C. Chen, M.A. Lohe, and J.D. Louck, The role of 3n-j coefficients in $SU(3)$, in: *Symmetry and Structural Properties of Condensed Matter* (eds., T. Lulek, W. Florek, and S. Walcerz, *Proc. Third SSCPM*, World Scientific, Singapore, 1995) 150-182.
10. J. D. Louck, Power of a determinant with two applications (submitted for publication, November 1996).
11. G.E.Baird and L.C. Biedenharn, On the representations of the semisimple Lie groups. IV. A canonical classification for tensor operators in SU_3 , *J. Math. Phys.* 5 (1965) 1730-1747.
12. G.D. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Vol. 16, *Encyl. of Mathematics and Its Applications* (ed., G.-C.Rota, Cambridge Univ. Press, Cambridge, 1981.)
13. I.G. Macdonald, *Symmetric Functions and Hall Polynomials* (Oxford Univ. Press, London/NY, 1979).
14. J.P.S. Kung and G.-C.Rota, The invariant theory of binary forms, *Bull. Amer. Math. Soc.* 10 (1984) 27-85.
15. J.D. Louck, MacMahon's master theorem, double tableau polynomials, and representations of groups, *Adv. Appl. Math.* 17 (1996) 143-168.
16. I.M. Gel'fand and M.I. Graev, Finite-irreducible full representations of the unitary and linear group and its related special functions, *Izv. Akad. Nauk. SSSR Mekh. Mashinostr.* 29 (1965) 1329-1356 [Am. Math. Soc. Transl. 64 (1967) 116-146].
17. J. Schwinger, On Angular Momentum. U.S. Atomic Energy Commissin Report NYO-3071, 1952 (unpublished). Published in *Quantum Theory of Angular Momentum* (eds., L. C. Biedenharn and H. van Dam, Academic Press, NY, 1965) 229-279.
18. T. Regge, Symmetry properties of Clebsch-Gordan coefficients, *Nuovo Cimento* 10(1958) 544-545.
19. D.E. Knuth, Permutations, matrices and generalized Young tableaux, *Pac. J. Math.* 34 (1970) 709.
20. E.P. Wigner, Application of Group Theory to Special Functions of Mathematical Physics (Unpublished lecture notes, Princeton Univ. 1955).
21. J. Talman, *Special Function: A Group Theoretical Approach*, Benjamin, NY 1968.

22. N. Vilenkin, Special Functions and the Theory of Group Representations, Transl. Amer. Math. Soc. 22, 1968.
23. J.D. Louck and L.C. Biedenharn, Some properties of the intertwining number of the general linear group, Science and Computers, Adv. Math. Suppl. Studies (2) 10 (1986) 265-311.
24. B. Kostant, A formula for the multiplicity of a weight, Trans. Amer. Math. Soc. 93 (1959) 53-73..
25. J.D. Louck, Recent progress toward a theory of tensor operators in the unitary groups, Am. J. Phys. 38 (1970) 3-42.
26. J.D. Louck and L.C. Biedenharn, Canonical unit adjoint operators in $U(n)$, J. Math. Phys. 11 (1970) 2368-2414.
27. J.D. Louck and L.C. Biedenharn, On the structure of the canonical tensor operators in the unitary groups: II. The tensor operators in $U(3)$ characterized by maximal null space, J. Math. Phys. 13 (1972) 1985-2001.
28. L.C. Biedenharn, M.A. Lohe, and J.D. Louck, The canonical resolution of the multiplicity problem for $U(3)$: An explicit and complete constructive solution, in: Group Theoretical Methods in Physics (eds., A. Janner, T. Janssen, and M. Boon, Springer-Verlag, 1976) 395.
29. L.C. Biedenharn, A. Giovannini, and J.D. Louck, Canonical definition of Wigner coefficients in $U(n)$, J. Math. Phys. 8 (1967) 691-700.
30. K. Baclawski, A new rule for computing Clebsch-Gordan series, Adv. Appl. Math. 5 (1984) 416-432.
31. J.D. Louck, Combinatorial aspects of representations of the unitary group, in: Symmetry and Structural Properties of Condensed Matter (eds., T. Lulek and W. Florek, Fourth SSCP, World Scientific, Singapore, 1997).
32. J. D. Louck, M.A. Lohe, and L.C. Biedenharn, Structural properties of the canonical $U(3)$ Racah functions and the $U(3):U(2)$ projective functions. J. Math. Phys. 16 (1975) 2408-2426.
33. M.A. Lohe, L.C. Biedenharn, and J.D. Louck, Structural properties of the self-conjugate $SU(3)$ tensor operators, J. Math. Phys. 18 (1977) 1883-1891.
34. L.C. Biedenharn, R.S. Gustafson, M.A. Lohe, J.D. Louck, and S.C. Milne, Special functions and group theory in theoretical physics, in: Special Functions: Group Theoretical Aspects and Applications (eds., R.A. Askey, et al., D. Riedel Publ. Co., 1984) 129-162.
35. L.C. Biedenharn, M.A. Lohe, and J.D. Louck, On the denominator function for canonical $SU(3)$ tensor operators, J. Math. Phys. 26 (1985) 1458-1492.
36. L.C. Biedenharn, M.A. Lohe, and J.D. Louck, On the denominator function for canonical $SU(3)$ tensor operators. II. Explicit polynomial form, J. Math. Phys. 29 (1988) 1106-1117.
37. J.D. Louck and L.C. Biedenharn, A generalization of the Gauss hypergeometric series, J. Math. Analysis and Appl. 59 (1977) 423-431.
38. Co-discoverer----find important
39. D.P. Shulka, Certain transformations of generalized hypergeometric series, Indian J. Pure Appl. Math. 12 (1981) 994-1000.
40. L.C. Biedenharn and J.D. Louck, Some properties of generalized hypergeometric coefficients, Adv. Appl. Math. 9 (1988) 447-464.
41. L.C. Biedenharn, A.M. Bincer, M.A. Lohe, and J.D. Louck, New relations and identities for generalized hypergeometric coefficients, Adv. Appl. Math. 13 (1992) 62-121.
42. L.C. Biedenharn and J.D. Louck, A new class of symmetric polynomials defined by tableaux, Adv. Appl. Math. 10 (1989) 396-438.
43. L.C. Biedenharn and J.D. Louck, Inhomogeneous basis set of symmetric polynomials defined by tableaux, Proc. Natl. Acad. USA 87 (1990) 1441-1445.

44. J.D. Louck and L.C. Biedenharn, Special functions associated with SU(3) Wigner-Clebsch-Gordan coefficients, *Symmetry and Structural Properties of Condensed Matter* (eds., T. Lulek, W. Florek, and S. Walcerz, Proc. Third SSCPM, World Scientific, Singapore, 1995) 150-182
45. W.Y.C. Chen and J.D. Louck, The factorial Schur function, *J. Math. Phys.* 34 (1993) 4144-4160.
46. I.G. Macdonald, Schur functions: Themes and variations. *Pulb. IRMA Strasbourg*, 1992, 498/S-27. *Actes 28^e Séminare Lotharingien*, pp. 5-39.
47. I.P. Goulden and A.M. Hamel, Shift operators and factorial symmetric functions, *Univ. Waterloo preprint CORR 92-10*, 1992.
48. I Goulden and C. Greene, A new tableau representation for supersymmetric Schur functions, *Univ. Waterloo preprint CORR 92-35*. (date).
49. *Adv. Inst. paper*. (very important).