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THEORETICAL STUDIES OF THE ALPHA DECAY OF DEFORMED NUCLEI

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ABSTRACT

The alpha decay of a deformed odd-mass nucleus, U^{233}, is treated by the use of numerical integration on an IBM-650 computer. The results of this treatment are compared with the theory of Bohr, Fröman, and Mottelson.

Approximate analytic methods are developed for calculating the amplitudes of alpha partial waves at the surface of deformed even-even nuclei. A two-term expansion modifying the ordinary Coulomb function to account for nuclear quadrupole coupling is applied. The amplitudes of alpha partial waves at the nuclear surface are tabulated for eight choices of phase and three values of the intrinsic nuclear quadrupole moment. The analytic method is developed to predict the intensities of the higher members of the ground rotational band.

A comparison is made between the numerical integration and the experiments of Roberts, Dabbs, and Parker, in which they examine the angular distribution of alpha particles from aligned U^{233} nuclei.

A detailed comparison is made between the analytic treatment developed here, that of Fröman, and the numerical integration of Rasmussen and Hansen for Cm^{242}. The results of the numerical integration of U^{233} are presented in matrices analogous to those of Fröman.

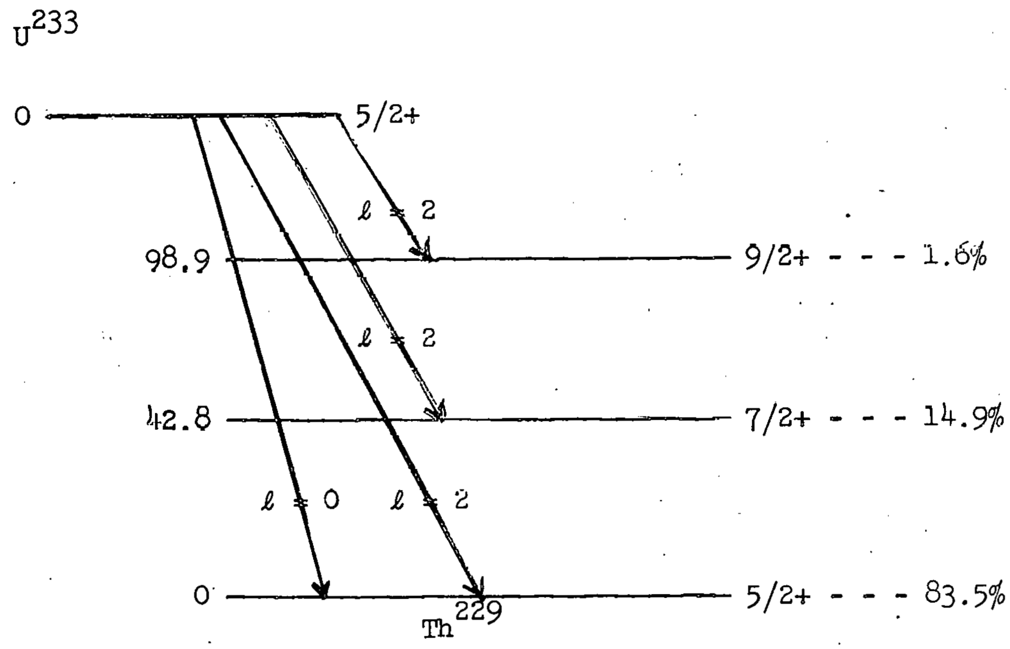
THEORETICAL STUDIES OF THE ALPHA DECAY OF DEFORMED NUCLEI

I. INTRODUCTION

When we consider the interaction of an emitted alpha particle with the daughter nucleus in the heavy element region, the nonsphericity of the nucleus plays an important role. Because the emitted alpha particle may interact with the quadrupole field of the nculeus, the experimental intensities which are observed for decay to various states of the daughter are not the same as the intensities at the nuclear surface, after making a simple coulomb-barrier penetration correction. This quadrupole interaction complicates the solution of the differential equations describing the alpha-decay process because it couples the alpha partial waves that differ from each other in angular momentum by two units. From a numerical integration of these equations, based on experimental alpha-group intensities, one may obtain the amplitudes of the alpha partial waves at the nuclear surface, as well as the amounts of the phase shifts due to the quadrupole terms.

The theory of Bohr-Fröman-Mottelson (B.F.M.) makes definite predictions concerning the amplitudes of alpha partial waves at the nuclear surface, in the case of deformed nuclei.[1] It was decided to test the validity of their predictions in the case of the alpha decay of U^{233} by carrying out extensive numerical integrations of the alpha wave equation including the nuclear quadrupole interaction. The relative intensities of the alpha particles to the low-lying states of Th^{229} have been measured;[2] there is a good deal of confidence in the spin assignments of these levels, and there are estimates of the nuclear quadrupole moment.[3,4] We use the estimates for the quadrupole moment of U^{233}, as there are none available for Th^{229}. We expect the quadrupole moment of Th^{229} to be roughly the same as that of U^{233}. This information is summarized in Fig. 1.

The study of U^{233} is of interest for reasons other than the comparison with B.F.M. At the time the problem was undertaken, alpha intensities had been reported for higher members of the ground rotational



$Q_0 = 13.5 \times 10^{-24}\ cm^2$.

Figure 1. Alpha decay of U^{233}.

band,[5] intensities which differed considerably from B.F.M. predictions. Since that time, however, the gamma rays following the alpha decay have been examined, and the large intensity of high-energy gamma rays indicates that the higher levels populated by the alpha decay are not all members of the ground rotational band.[6]

Roberts, Dabbs, and Parker[7] have aligned U^{233} nuclei at low temperatures in single crystals of $Rb(UO_2)(NO_3)_3$ and examined the anisotropy of emitted alpha particles. They interpret their results as indicating that the $\ell = 2$ wave is out of phase with the $\ell = 0$ alpha partial wave in the alpha group to ground. Their experiment also puts limits on the amount of the $\ell = 2$ wave which populates the ground state of Th^{229}.

We might hope that either the intensity limits of Robsert, Dabbs, and Parker or the boundary conditions at the nuclear surface of B.F.M. will eliminate one of the phase choices for the $\ell = 2$ partial wave relative to the $\ell = 0$.

II. FORMULATION OF THE ALPHA-DECAY PROBLEM

The problem of alpha decay in the region of uranium is complicated, as contrasted to the region of lead, by the existence of large quadrupole moments which interact with the escaping alpha particle. Favored alpha decay, i.e. decay between parent and daughter states having the same nucleonic wave functions and hence the same K (K is the projection of the nuclear spin on the nuclear symmetry axis), has been treated in this region of large quadrupole moments by several authors, both by numerical integrations[8,9,10] and by analytic approximations[11] for the case of even-even nuclei. The quantum mechanical treatment of an odd-even nucleus is quite similar to that of the even-even nucleus.[12] We start from Schroedinger's equation

$$H \Psi = E \Psi \tag{1}$$

Here we have $H = T + V + H_{nuc}$ where T is the kinetic energy of the system, V its potential energy, and H_{nuc} the Hamiltonian for the internal energy of the recoil nucleus. We expand the potential V in spherical coordinates. Making the usual multipole expansion, we obtain

$$V = 2e \sum_{p=1}^{Z} \sum_{\lambda=0}^{\infty} \frac{e_p \, r_p^{\lambda}}{r^{\lambda+1}} P_\lambda (\cos \gamma), \quad (2)$$

where r gives the position of the alpha particle; r_p gives the position of the pth proton in the daughter nucleus, and γ is the angle between r and r_p in the system of the recoil nucleus. In our treatment of the problem, we include the central and quadrupole terms of the potential. We next construct a solution of Schroedinger's equation of the form

$$\Psi = \sum_{\ell', I_f', T'} \left(\frac{1}{r}\right) U_{\ell', I_f'} (r) \; Y^{I,M}_{\ell', I_f', T'} (\theta, \phi, \chi_i) \quad (3)$$

as the first step in the solution. Here

$$Y^{I,M}_{\ell', I_f', T'} (\theta, \phi, \chi_i) = \sum_{m'} \Bigg(\langle \ell', I_f', m', M-m' | I, M \rangle \, Y^{m'}_{\ell'} (\theta, \phi) \; \Phi^{M-m'}_{I_f', T'} (\chi_i) \Bigg) \quad (4)$$

ℓ' is the angular momentum of the alpha particle, and m' is the component of its angular momentum on a space-fixed axis, I_f' and I are the final and initial nuclear angular momenta, and $M - m'$ and M are projections of I_f' and I on the space-fixed axis. Here

$$\Phi^{M-m'}_{I_f', T'} (\chi_i)$$

describes the intrinsic state of the daughter nucleus and the bracketed symbol is a Clebsch-Gordan coefficient. The orthogonality condition on the $Y^{I,M}_{\ell, I_f', T}$ function is

$$\int Y^{I,M*}_{\ell', I_f', T'} \; Y^{I,M}_{\ell, I_f, T} \; d\theta \, d\phi \, d\chi_i = \delta_{\ell, \ell'} \, \delta_{I_f, I_f'} \, \delta_{T, T'} . \quad (5)$$

We next substitute Ψ into Schroedinger's equation, multiply by $Y^{I,M}_{\ell,I_f,T}$ and then integrate over all variables but r. We do this for each value of ℓ, I_f of interest in the daughter nucleus and we are left with a set of coupled ordinary differential equations of the form

$$-\frac{\hbar^2}{2M}\frac{d^2 u_{\ell,I_f}}{dr^2} + \left(\frac{\hbar^2}{2Mr^2}\ell(\ell+1) + E_{I_f,T} - E\right) u_{\ell,I_f} + 2\frac{Ze^2}{r} u_{\ell,I_f} + \sum_{\ell',I_f'} u_{\ell',I_f'} \left\langle Y^{I,M*}_{\ell,I_f,T} \left| \frac{Q_0 e^2}{r^3} P_2(\cos\gamma) \right| Y^{I,M}_{\ell',I_f',T'} \right\rangle = 0 \qquad (6)$$

All terms but the last summation describe the interaction of two charged particles. The last term gives a mixing of states, due to the perturbation induced by the nuclear quadrupole moment.

III. A TEST OF THE B.F.M. HYPOTHESES

The hypotheses of B.F.M. may be states as follows: any alpha partial wave has a projection of its angular momentum on the nuclear-symmetry axis which is equal to $K_f \pm K_i$ at the nuclear surface. K_f and K_i are the projections of the spins of the final and initial nuclear states on the nuclear-symmetry axis. For $(K_f + K_i) > \ell$ there is but one permissible value of m_ℓ, which is zero in the case of favored alpha decay. Looking at the B.F.M. hypothesis in an $|I_f, \ell\rangle$ representation, one sees that a given ℓ wave will be apportioned among the states that it populates in proportion to the Clebsch-Gordan coefficient $\langle I_i, \ell, K, 0 | I_f, K\rangle$ at the nuclear surface. This hypothesis will be referred to as B.F.M. - 1. Furthermore, B.F.M. makes the approximation that the relative intensities of alpha decay to the various levels from population by a given ℓ wave will be given by the square of the Clebsch-Gordan coefficient $\langle I_i, \ell, K, 0 | I_f, K\rangle$ times the barrier penetration factor for the particular alpha energy. This approximation, which we shall refer to as B.F.M. - 2, would be exact only in the limit of infinite moment of inertia. In the case of favored alpha decay of even-even nuclei, a given alpha partial wave populates just one level of the daughter nucleus and so affords us no test of the B.F.M. hypotheses.

In the numerical work to be described, B.F.M. - 1 and B.F.M. - 2 are tested separately for the first time. In the case of odd-even nuclei, we may utilize B.F.M. - 1 to set boundary conditions at the nuclear surface for solutions of the alpha wave equations and then compare the alpha intensities from numerical integration with the experimentally observed intensities. This gives us a test of B.F.M. - 1, though our calculations did not include $\ell = 4$ contributions. In the region of U^{233}, the $\ell = 4$ contribution should not be very significant.

IV. NUMERICAL INTEGRATION AND BOUNDARY CONDITIONS

To return to the specific problem of U^{233}, the equations describing the alpha particles as they leave the nucleus are:

$$\frac{d^2u_0}{dr^2} + \left[.929 - \frac{48.78}{r}\right] u_0 - \frac{1}{r^3}\left[101\ u_1 - 117\ u_2 + 69.3\ u_3\right] = 0, \qquad (7a)$$

$$\frac{d^2u_1}{dr^2} + \left[.929 - \frac{48.78}{r} - \frac{6}{r^2}\right] u_1 - \frac{1}{r^3}\left[101\ u_0 + 38.71\ u_1 - 76.0\ u_2 - 26.4\ u_3\right] = 0, \qquad (7b)$$

$$\frac{d^2u_2}{dr^2} + \left[.921 - \frac{48.78}{r} - \frac{6}{r^2}\right] u_2 - \frac{1}{r^3}\left[117\ u_0 - 76.0\ u_1 - 2.58\ u_2 - 76.3\ u_3\right] = 0, \qquad (7c)$$

$$\frac{d^2u_3}{dr^2} + \left[.910 - \frac{48.78}{r} - \frac{6}{r^2}\right] u_3 - \frac{1}{r^3}\left[69.3\ u_0 - 26.4\ u_1 - 76.4\ u_2 - 18.1\ u_3\right] = 0, \qquad (7d)$$

where r is in units of 10^{-13} cm; $\frac{u_0}{r}$ is the radial function of the $\ell = 0$

wave populating the 5/2 state of Th^{229}; and $\frac{u_1}{r}$, $\frac{u_2}{r}$, and $\frac{u_3}{r}$ are the radial functions of the $\ell = 2$ wave populating the 5/2, 7/2, and 9/2 levels of Th^{229}, respectively. This set of equations includes only the $\ell = 0$ and $\ell = 2$ alpha partial waves. In this treatment we use a value of 13.2 barns for Q_0, and the alpha-decay energies are 4.900, 4.857, and 4.801 Mev.

We note that as r approaches ∞ the coupling term, which has a $1/r^3$ dependence, becomes negligible. When we can ignore this term, the equations are decoupled, and their solutions are given by linear combinations of regular (F_ℓ) and irregular (G_ℓ) Coulomb functions. Because the alpha particles are outgoing waves, the solution must be of the form $Ce^{ikx'}$ and not have any component of the form $e^{-ikx'}$, where

$$x' = \rho - \eta \ln 2\rho - \frac{\ell\pi}{2} + \sigma_\ell . \quad (8)$$

The notation is that of the general usage in Coulomb functions.[13] We note that as r approaches ∞, F_ℓ approaches sin x', and G_ℓ approaches cos x'. Therefore, at large distances, our solutions must be of the form $(A + iB)(G_\ell + iF_\ell)$, where A and B are real constants.

The determination of the radial wave functions was accomplished by numerical integration on an IBM-650 computer in the region where the coupling could not be neglected. As we have four second-order differential equations, there are eight boundary conditions which must be applied. We begin by looking at the imaginary part of the solution.

The procedure adopted was to give one of the u's an amplitude of one at the nuclear surface and the other three were given amplitudes of zero. The nonzero function was started off as a regular Coulomb function, and the other three were kept at zero by conditions used on the derivatives. Carrying out the numerical integration for this set of boundary conditions, we obtain one set of solutions. When $u_i = 1$ at the nuclear surface, at infinity we have $u_j = A_{ij} F_j + B_{ij} G_j$. The coefficients A_{ij} and B_{ij} are obtained by fitting the numerical values from the computer program to linear combinations of Coulomb functions at large distances in this case 8.5×10^{-12} cm. By separately setting each of the four u's equal to one, we obtain four independent solutions of the differential equations. The imaginary part of the solution of the physical problem will be some linear combination of these four solutions, i.e., our solutions of physical interest have the form

$$\mathcal{J}(u_i) = \sum_{j=0}^{3} \alpha_j \left(A_{ji} F_i + B_{ji} G_i\right), \tag{9}$$

where α_j is some real number. We make use of the experimental intensities of alpha particles populating the various levels by noting that the intensity is

$$I_i = \left(\sum_{j=0}^{3} \alpha_j A_{ji}\right)^2 + \left(\sum_{j=0}^{3} \alpha_j B_{ji}\right)^2 . \tag{10}$$

The experimental intensities give us three boundary conditions, a trivial condition of over-all normalization and two relative intensities. For the final boundary condition, we make use of B.F.M. - 1 to obtain the ratio of amplitudes of the $\ell = 2$ wave at the nuclear surface populating the 5/2 and 7/2 states of Th^{229} for the real part of the solution. Finally, as a test of B.F.M. - 1, we may examine the relative amplitudes at the nuclear surface of the real part of the $\ell = 2$ wave populating the 5/2 and 9/2 levels of Th^{229}. If the B.F.M. - 1 hypothesis is valid, all five conditions will be satisfied. We have calculated the real part of the wave function by using our knowledge of the asymptotic form of real and imaginary components of the wave function to obtain

$$\mathcal{R}(u_i) = \sum_{j=0}^{3} \alpha_j \left(A_{ji} G_i - B_{ji} F_i\right) \tag{11}$$

at $r = 8.5 \times 10^{-12}$ cm. We integrate inward numerically. A_{ji} and B_{ji} will be the same constants as were obtained from the imaginary part of the solutions, and the set of 4 α_j values that comes closest to satisfying the three intensity conditions and the two constraints put on the real part of the solution by B.F.M. - 1 is used. Only the real part need be considered as obeying B.F.M. - 1, as the imaginary part is negligibly small at the nuclear surface.

V. RESULTS OF NUMERICAL INTEGRATION

We found a set of α_j values which approximately satisfies the conditions mentioned previously for the $\ell = 2$ wave both in phase and out of phase with the $\ell = 0$ wave at the nuclear surface. The conditions were not satisfied exactly in either case, but were satisfied approximately in both cases. The results for the amplitudes of the real parts of the wave functions at the nuclear surface are given in Table I.

Table I

	$\ell = 2$ partial wave		
I_f	B.F.M. - 1 Prediction	$\ell = 2$ in phase	$\ell = 2$ out of phase
5/2	-0.86	-0.86	-0.88
7/2	1	1	1
9/2	-0.59	-0.55	-0.51

Another way of stating B.F.M. - 1 is as follows: in an $|\ell, m_\ell\rangle$ representation only the component having $m_\ell = 0$ will be present in favored alpha decay. We transformed the wave functions to an $|\ell, m_\ell\rangle$ representation and found the $m_\ell = 0$ component of the $\ell = 2$ wave to be some two orders of magnitude larger than the other m_ℓ components. The deviations from perfect agreement with B.F.M. - 1 may be attributed to slight inaccuracies in the reported alpha intensities, and incorrect choice of quadrupole moment in this calculation, or, finally, to the neglect of alpha particles with angular momenta greater than $\ell = 2$. Our calculations support the validity of B.F.M. - 1.

Finally we give in Tables II and III numerical values of the radial wave functions $u_{I_f,\ell}$, for several values of r.

Table II

Real part of alpha wave functions for U^{233} decay with ℓ=0 and ℓ=2 in phase

r x 10^{-13} cm	$\vert 5/2,0\rangle$	$\vert 5/2,2\rangle$	$\vert 7/2,2\rangle$	$\vert 9/2,2\rangle$
9.0	1.52(17)[a]	1.26(17)[a]	-1.46(17)[a]	8.08(16)[a]
9.2	9.89(16)	8.23(16)	-9.47(16)	5.22(16)
9.6	4.28(16)	3.54(16)	-4.07(16)	2.24(16)
10.0	1.89(16)	1.55(16)	-1.79(16)	9.81(15)
10.4	8.55(15)	6.99(15)	-8.02(15)	4.39(15)
11.0	2.70(15)	2.18(15)	-2.51(15)	1.37(15)
12.0	4.32(14)	3.45(14)	-3.95(14)	2.14(14)
14.0	1.49(13)	1.16(13)	-1.32(13)	7.05(12)
16.0	7.08(11)	5.39(11)	-6.11(11)	3.23(11)
18.0	4.42(10)	3.30(10)	-3.72(10)	1.94(10)
20.0	3.48(9)	2.55(9)	-2.86(9)	1.47(9)
25.0	1.40(7)	9.87(6)	-1.09(7)	5.42(6)
30.0	1.51(5)	1.03(5)	-1.11(5)	5.35(5)
35.0	3.74(3)	2.49(3)	-2.60(3)	1.20(3)
40.0	1.98(2)	1.29(2)	-1.29(2)	5.68(1)
45.0	2.29(1)	1.45(1)	-1.38(1)	5.61(0)
50.0	6.51(0)	3.99(0)	-3.54(0)	1.30(0)
55.0	2.17(0)	1.38(0)	-1.36(0)	5.39(-1)
60.0	-2.65(0)	-1.51(0)	1.03(0)	-2.25(-1)
65.0	-8.95(-1)	-6.58(-1)	8.61(-1)	-4.01(-1)
70.0	2.62(0)	1.61(0)	-1.40(0)	4.39(-1)
75.0	-2.66(0)	-1.55(0)	1.09(0)	-2.16(-1)
80.0	2.14(0)	1.19(0)	-6.18(-1)	2.31(-3)

[a]Number in parentheses indicates power of ten to which function is raised.

Table III

Real part of alpha functions for U^{233} decay with ℓ=0 and ℓ=2 out of phase

$r \times 10^{-13}$ cm	$\vert 5/2,0 \rangle$	$\vert 5/2,2 \rangle$	$\vert 7/2,2 \rangle$	$\vert 9/2,2 \rangle$
9.0	5.65(17)[a]	-6.76(17)[a]	7.68(17)[a]	-3.92(17)[a]
9.2	3.81(17)	-4.45(17)	5.05(17)	-2.58(17)
9.6	1.75(17)	-1.95(17)	2.22(17)	-1.13(17)
10.0	8.16(16)	-8.78(16)	9.98(16)	-5.08(16)
10.4	3.87(16)	-4.02(16)	4.57(16)	-2.32(16)
11.0	1.30(16)	-1.30(16)	1.47(16)	-7.45(15)
12.0	2.26(15)	-2.12(15)	2.41(15)	-1.22(15)
14.0	8.79(13)	-7.58(13)	8.59(13)	-4.31(13)
16.0	4.55(12)	-3.71(12)	4.18(12)	-2.08(12)
18.0	3.02(11)	-2.36(11)	2.64(11)	-1.31(11)
20.0	2.48(10)	-1.88(10)	2.10(10)	-1.03(10)
25.0	1.08(8)	-7.67(7)	8.42(7)	-4.04(7)
30.0	1.22(6)	-8.33(5)	8.97(5)	-4.18(5)
35.0	3.13(4)	-2.07(4)	2.17(4)	-9.79(3)
40.0	1.70(3)	-1.09(3)	1.11(3)	-4.79(2)
45.0	2.01(2)	-1.26(2)	1.21(2)	-4.92(1)
50.0	5.83(1)	-3.56(1)	3.22(1)	-1.19(1)
55.0	1.70(1)	-1.10(1)	1.14(1)	-4.34(0)
60.0	-2.50(1)	1.42(1)	-1.01(1)	3.29(0)
65.0	-6.96(0)	5.22(0)	-7.30(0)	3.25(0)
70.0	2.35(1)	-1.44(1)	1.29(1)	-4.27(0)
75.0	-2.47(1)	1.43(1)	-1.05(1)	2.62(0)
80.0	2.05(1)	-1.14(1)	6.45(0)	-7.69(-1)

[a]Number in parentheses indicates power of ten to which function is raised.

From our wave functions, we may calculate the phase shifting of the alpha partial waves caused by the nuclear quadrupole moment. This information will be of interest in the case of an odd-even nucleus, because it emters into calculations of angular distributions from angular correlation experiments and nuclear alignment and nuclear polarization experiments. The phase shifting of the ith alpha partial wave by the nuclear quadrupole moment is given by the relation

$$\theta_i = \tan^{-1}\left(\sum_{j=0}^{3} \alpha_j \; B_{ji} \Big/ \sum_{j=0}^{3} \alpha_j \; A_{ji} \right) \qquad (12)$$

where θ_i is given in radians.

The calculated phase shifts are given in Table IV in degrees.

Table IV

Phase shifts caused by nuclear quadrupole moment

I_f, ℓ	ℓ=0 and ℓ=2 in phase	ℓ=0 and ℓ=2 out of phase
5/2,0	-1.1°	$+1.94^\circ$
5/2,2	-4°	$+0.87^\circ$
7/2,2	-5.3°	-0.179°
9/2,2	-2.8°	$+7.16^\circ$

These phase shifts may be compared with those calculated for even-even alpha emitters in this region, $\simeq -3^\circ$ for the $\ell = 2$ wave when it is in phase with the $\ell = 0$ wave. In nuclear-alignment experiments, only the difference in phase shift of the $|5/2,2\rangle$ and $|5/2,0\rangle$ is of direct interest. The interference term contributing to anisotropy has a factor $\cos(\theta_0 - \theta_2)$. The phase shifting due to angular momentum is $\simeq -7^\circ$ and, correcting for the quadrupole interaction, we have $\simeq -10^\circ$ for the $\ell = 0$ partial wave in phase with the $\ell = 2$ wave, and $\simeq -6^\circ$ partial wave out of phase with the $\ell = 2$ wave.

Next let us examine the bearing of this calculation on the validity of B.F.M. - 2 approximation. From B.F.M. - 2, one may predict the amount of $\ell = 2$ wave and the amount of $\ell = 0$ wave populating the 5/2 state of Th^{229} independent of nuclear-alignment experiments. Using data from neighboring even-even nuclei, B.F.M. conclude that 81% of the ground-state intensity is due to the $\ell = 0$ wave, and 19% comes from the $\ell = 2$ alpha wave.[1] In the numerical calculation, our modified prediction is that only

75% of the ground-state intensity is due to the $\ell = 0$ wave, and 25% is due to the $\ell = 2$ wave. This conclusion holds for both choices of phase of the $\ell = 2$ wave, under the constraint that B.F.M. - 1 and the intensity conditions hold. This discrepancy suggested that one might find some other approximation that is superior to B.F.M. - 2 but simpler than numerical integration to predict unseen intensities of odd-even nuclei and partial wave amplitudes at the nuclear surface for even-even nuclei.

VI. AN APPROXIMATE TREATMENT OF ALPHA DECAY

An approximate method that was developed and was used to treat even-even nuclei will be described. The equations for favored alpha decay of even-even nuclei, which have been derived previously,[2] are

$$u_0''(r) - \left(\frac{4mZe^2}{\hbar^2 r} - \frac{2mE_0}{\hbar^2}\right) u_0(r) = \frac{2mQ_0 e^2}{\hbar^2 r^3} \frac{u_2(r)}{\sqrt{5}} , \tag{13a}$$

$$u_2''(r) - \left(\frac{4mZe^2}{\hbar^2 r} - \frac{2mE_2}{\hbar^2} + \frac{6}{r^2}\right) u_2(r) = \frac{2mQ_0 e^2}{\hbar^2 r^3} \left(\frac{u_0(r)}{\sqrt{5}} + \frac{2\, u_2(r)}{\sqrt{7}} + \frac{6\, u_4(r)}{7\sqrt{5}}\right) \tag{13b}$$

$$u_4''(r) - \left(\frac{4mZe^2}{\hbar^2 r} - \frac{2mE_4}{\hbar^2} + \frac{20}{r^2}\right) u_4(r) = \frac{2mQ_0 e^2}{\hbar^2 r^3} \left(\frac{6u_2(r)}{7\sqrt{5}} + \frac{20u_4(r)}{77} + \frac{15u_6(r)}{11\sqrt{13}}\right) \tag{13c}$$

$$u_6''(r) - \left(\frac{4mZe^2}{\hbar^2 r} - \frac{2mE_6}{\hbar^2} + \frac{42}{r^2}\right) u_6(r) = \frac{2mQ_0 e^2}{\hbar^2 r^3} \left(\frac{15u_4(r)}{11\sqrt{13}} + \frac{14u_6(r)}{55}\right) . \tag{13d}$$

Here only the alpha partial waves having angular momenta 0, 2, 4, and 6 are included, Z is the charge of the daughter nucleus, m is the reduced mass of the system, E_ℓ is the total decay energy of the ℓth alpha wave, and Q_0 is the intrinsic nuclear quadrupole.

We can see that the solutions of the set of equations (13) would be regular or irregular Coulomb functions were it not for the quadrupole moment, that is, the right hand side of the equations would vanish if Q_0 were zero. From an examination of the series expansion of the W.K.B. integrand, we surmised that the radial wave functions of the alpha partial waves might be well represented by functions of the form

$$\left(\alpha_\ell + \frac{\beta_\ell}{r^{3/2}}\right) G_\ell (r)$$

in the region of the nuclear surface. Here α_ℓ and β_ℓ are parameters fixed over all r values, and G_ℓ is the irregular Coulomb function. It is clear that $\beta_\ell / r^{3/2}$ approaches zero as r approaches ∞, so one may identify α_ℓ as the square root of the quotient of the alpha partial-wave intensity intensity and its velocity. We determine β_ℓ by substituting the approximate solution into the differential equations and demanding that they be exactly satisfied at some arbitrary intermediate distance (2.0×10^{-12} cm seemed to be optimum). The equations which one obtains are algebraic; their right-hand sides are the same as those of equations (13), with

$$\left(\alpha_\ell + \frac{\beta_\ell}{r^{3/2}}\right) G_\ell (r)$$

simply substituted for $u_\ell (r)$.

$$\beta_0 \left(\frac{-3\,G_0'(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_0(r)}{r^{7/2}}\right) = \text{right hand side of Eq. 13a,} \qquad (14a)$$

$$\beta_2 \left(\frac{-3\,G_2'(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_2(r)}{r^{7/2}}\right) = \text{right hand side of Eq. 13b,} \qquad (14b)$$

$$\beta_4 \left(\frac{-3\,G_4'(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_4(r)}{r^{7/2}}\right) = \text{right hand side of Eq. 13c,} \qquad (14c)$$

$$\beta_6 \left(\frac{-3\,G_6'(r)}{r^{5/2}} + \frac{15}{4}\,\frac{G_6(r)}{r^{7/2}}\right) = \text{right hand side of Eq. 13d.} \qquad (14d)$$

When we obtain the values of α_ℓ and β_ℓ, we may readily compute the alpha partial wave amplitudes at the nuclear surface by simply setting $r = r_0$ in the approximate solution.

The distance at which we demand that the approximate solutions satisfy the differential equations is somewhat arbitrary; the reasons for choosing 2.0×10^{-12} cm are completely pragmatic — using this value, we found that we could get best agreement with results of detailed numerical integrations for both Cm^{242} [9] and U^{233}. If we choose 1.5 or 2.5×10^{-12} cm, the variations in surface amplitudes over this range is about 10%. It seems quite reasonable that if we were to add another term to the approximate solutions, i e., $(\alpha_\ell + \beta_\ell/r^{3/2} + \gamma_\ell/r^3)\, G_\ell\,(r)$, these variations could be minimized.

VII. AMPLITUDES OF ALPHA PARTIAL WAVES AT THE NUCLEAR SURFACE

As quadrupole moments are not known very well in the heavy-element region, we have calculated the surface amplitudes for several values of the nuclear quadrupole moment. Because α_ℓ is the square root of the intensity of the ℓ wave, divided by its velocity, there is a sign ambiguity. We have included all eight phase choices here. Angular correlation work on Am^{241} [14] seems to indicate that the D wave is in phase with the S wave within the barrier, but the interpretation of the U^{233} alignment experiment is somewhat ambiguous.[7]

The data used in this analysis (Table V) come from two summary compilations.[2,15] Results of the analysis are shown in Table VI.

Table V

Data used in analysis of alpha-decay of even-even heavy elements

Isotope	Energy (Mev)	Abundance (%)	$\sqrt{I}$ [a]
	5.421	71	8.426
	5.338	28	5.292
${}_{90}Th^{228}$	5.173	0.2	0.448
	4.883	0	0 [b]

Table V (continued)

Isotope	Energy (Mev)	Abundance (%)	$\sqrt{I}$ [a]
$_{90}Th^{230}$	4.682	76	8.718
	4.615	24	4.899
	4.467	0.2	0.4472
	4.206	0	0 [b]
$_{92}U^{230}$	5.884	67.2	8.197
	5.813	32	5.656
	5.685	0.3	0.550
	5.380	0	0 [b]
$_{92}U^{232}$	5.318	68	8.246
	5.261	32	5.656
	5.134	0.32	0.566
	4.919	0	0 [b]
$_{92}U^{234}$	4.768	72	8.485
	4.717	28	5.292
	4.594	0.3	0.550
	4.397	0	0 [b]
$_{92}U^{236}$	4.499	73	8.544
	4.449	27	5.196
	4.339	0.5	0.71
$_{94}Pu^{236}$	5.763	69	8.306
	5.716	31	5.567
	5.610	0.2	0.4472
	5.442	0.002	0.0447
$_{94}Pu^{238}$	5.495	72	8.485
	5.452	28	5.292
	5.352	0.09	0.300
	5.204	0.004	0.0632

Table V (continued)

Isotope	Energy (Mev)	Abundance (%)	$\sqrt{I}$ [a]
$_{94}Pu^{240}$	5.162	76	8.718
	5.118	24	4.899
	5.014	0.1	0.316
	4.851	0.003	0.055
$_{94}Pu^{242}$	4.898	76	8.718
	4.858	24	4.899
$_{96}Cm^{242}$	6.110	73.7	8.585
	6.066	26.3	5.130
	5.965	0.035	0.187
	5.806	0.006	0.0775
$_{96}Cm^{244}$	5.802	76.7	8.758
	5.760	23.3	4.83
	5.662	0.017	0.13
	5.510	0.004	0.0632
$_{98}Cf^{246}$	6.753	78	8.832
	6.711	22	4.690
	6.615	0.16	0.4
	6.469	0.015	0.123
$_{98}Cf^{250}$	6.024	83	9.11
	5.980	17	4.120
$_{98}Cf^{252}$	6.112	82	9.055
	6.069	15	3.872
	5.969	0.2	0.447
	5.811	0	0 [b]
$_{100}Fm^{254}$	7.20	85	9.22
	7.16	15	3.872
	7.064	0.4	0.632
	6.96	0	0 [b]

[a] The difference in velocity of the various waves was ignored in calculating these boundary conditions.

[b] No α intensity given.

Table VI

Results

The first two columns of the data compilation give the charge Z and mass A of the parent nucleus. The next four columns give the relative amplitudes of the alpha partial waves on the spherical surface given by $R = R_0 A^{1/3}$ (where R_0 is 1.45×10^{-13} cm). The amplitudes are given for all eight choices of phase and three values of the intrinsic quadrupole moment, Q_0, are used in the calculation for each phase choice.

Z	A	Relative phase Plus $\ell = 0$	Plus $\ell = 2$	Plus $\ell = 4$	Plus $\ell = 6$
$Q_0 = 8$					
90	228	1.0000	1.5541	0.8884	0.2156
90	230	1.0000	1.4577	0.9354	0.2215
92	230	1.0000	1.5048	0.7491	0.1779
92	232	1.0000	1.4742	0.7948	0.1773
92	234	1.0000	1.4113	0.8359	0.1838
92	236	1.0000	1.4378	1.0522	0.2368
94	236	1.0000	1.3350	0.5678	0.3233
94	238	1.0000	1.2568	0.4598	0.3757
94	240	1.0000	1.2045	0.4874	0.4456
94	242	1.0000	1.1671	0.2095	0.0183
96	242	1.0000	1.1400	0.3409	0.3160
96	244	1.0000	1.1241	0.3053	0.2867
98	246	1.0000	1.0819	0.4245	0.3763
98	250	1.0000	0.9750	0.1693	0.0151
98	252	1.0000	0.9640	0.4155	0.0820
100	254	1.0000	0.9282	0.4403	0.0913
$Q_0 = 11$					
90	228	1.0000	1.6016	0.9661	0.2932
90	230	1.0000	1.5162	1.0028	0.2994
92	230	1.0000	1.5527	0.8291	0.2451
92	232	1.0000	1.5242	0.8676	0.2423
92	234	1.0000	1.4688	0.9021	0.2498
92	236	1.0000	1.5002	1.1090	0.3175
94	236	1.0000	1.3923	0.6512	0.3605
94	238	1.0000	1.3169	0.5443	0.4014
94	240	1.0000	1.2702	0.5694	0.4723
94	242	1.0000	1.2227	0.2833	0.0341
96	242	1.0000	1.2078	0.4234	0.3360
96	244	1.0000	1.1910	0.3851	0.3041
98	246	1.0000	1.1622	0.5057	0.4048
98	250	1.0000	1.0489	0.2338	0.0288
98	252	1.0000	1.0504	0.4770	0.1175
100	254	1.0000	1.0236	0.5037	0.1303
$Q_0 = 14$					
90	228	1.0000	1.6446	1.0410	0.3716
90	230	1.0000	1.5691	1.0685	0.3770
92	230	1.0000	1.5965	0.9064	0.3146
92	232	1.0000	1.5696	0.9382	0.3084
92	234	1.0000	1.5210	0.9668	0.3162
92	236	1.0000	1.5563	1.1647	0.3964
94	236	1.0000	1.4454	0.7323	0.4017
94	238	1.0000	1.3729	0.6268	0.4314
94	240	1.0000	1.3315	0.6499	0.5028
94	242	1.0000	1.2749	0.3564	0.0551
96	242	1.0000	1.2712	0.5052	0.3610
96	244	1.0000	1.2539	0.4644	0.3263
98	246	1.0000	1.2374	0.5871	0.4374
98	250	1.0000	1.1188	0.2999	0.0475
98	252	1.0000	1.1318	0.5405	0.1564
100	254	1.0000	1.1134	0.5698	0.1731

		Relative phase					
		Plus	Plus	Minus		Plus	
Z	A	$l = 0$	$l = 2$	$l = 4$		$l = 6$	
$Q_0 = 8$							
90	228	1.0000	1.4181	0.2691	-	0.1530	-
90	230	1.0000	1.3030	0.4027	-	0.1721	-
92	230	1.0000	1.3947	0.1543	-	0.1158	-
92	232	1.0000	1.3511	0.2420	-	0.1256	-
92	234	1.0000	1.2748	0.3363	-	0.1398	-
92	236	1.0000	1.2510	0.5682	-	0.1962	-
94	236	1.0000	1.2630	0.0232	-	0.1562	
94	238	1.0000	1.2088	0.0623		0.2674	
94	240	1.0000	1.1480	0.0120		0.3194	
94	242	1.0000	1.1671	0.2095		0.0183	
96	243	1.0000	1.1147	0.1351		0.2590	
96	244	1.0000	1.1052	0.1496		0.2450	
98	246	1.0000	1.0327	0.0335		0.2671	
98	250	1.0000	0.9750	0.1693		0.0151	
98	252	1.0000	0.9020	0.0954	-	0.0536	-
100	254	1.0000	0.8565	0.1241	-	0.0622	-
$Q_0 = 11$							
90	228	1.0000	1.4321	0.1511	-	0.1796	-
90	230	1.0000	1.3215	0.2971	-	0.2095	-
92	230	1.0000	1.4154	0.0442	-	0.1320	-
92	232	1.0000	1.3699	0.1367	-	0.1481	-
92	334	1.0000	1.2965	0.2386	-	0.1695	-
92	236	1.0000	1.2641	0.4679	-	0.2440	-
94	236	1.0000	1.3015	0.0749		0.1442	
94	238	1.0000	1.2558	0.1551		0.2606	
94	240	1.0000	1.1979	0.1022		0.3076	
94	242	1.0000	1.3227	0.2833		0.0341	
96	243	1.0000	1.1754	0.2206		0.2614	
96	244	1.0000	1.1668	0.2315		0.2493	
98	246	1.0000	1.0989	0.1184		0.2613	
98	250	1.0000	1.0489	0.2338		0.0288	
98	252	1.0000	0.9695	0.033[illegible]	-	0.0630	-
100	254	1.0000	0.9299	0.0620	-	0.0743	-
$Q_0 = 14$							
90	228	1.0000	1.4481	0.0405	-	0.1929	-
90	230	1.0000	1.3412	0.1973	-	0.2352	-
92	230	1.0000	1.4372	0.0599		0.1361	-
92	232	1.0000	1.3899	0.0373	-	0.1597	-
92	234	1.0000	1.3188	0.1456	-	0.1892	-
92	236	1.0000	1.2790	0.3733	-	0.2809	-
94	236	1.0000	1.3388	0.1694		0.1412	
94	238	1.0000	1.3009	0.2453		0.2617	
94	240	1.0000	1.2458	0.1903		0.3030	
94	242	1.0000	1.2749	0.3564		0.0551	
96	242	1.0000	1.2328	0.3052		0.2703	
96	244	1.0000	1.2251	0.3127		0.2596	
98	246	1.0000	1.1619	0.2034		0.2625	
98	250	1.0000	1.1188	0.2999		0.0475	
98	252	1.0000	1.0341	0.0308		0.0664	-
100	254	1.0000	1.0003	0.0029		0.0799	-

		Relative phase							
		Plus	Minus		Plus		Minus		
Z	A	$\ell = 0$	$\ell = 2$		$\ell = 4$		$\ell = 6$		
$Q_0 = 8$									
90	228	1.0000	1.7168	-	0.5373		0.2771		
90	230	1.0000	1.4269	-	0.7083		0.2861		
92	230	1.0000	1.6578	-	0.3310		0.2095		
92	232	1.0000	1.5605	-	0.4628		0.2179		
92	234	1.0000	1.3735	-	0.5858		0.2284		
92	236	1.0000	1.3348	-	0.9460		0.3154		
94	236	1.0000	1.3202	-	0.0876		0.2452	-	
94	238	1.0000	1.1906	-	0.0490	-	0.4062	-	
94	240	1.0000	1.0631	-	0.0256		0.4675		
94	242	1.0000	1.1017	-	0.2661	-	0.0232		
96	242	1.0000	0.9800	-	0.1509	-	0.3747		
96	244	1.0000	0.9612	-	0.1714	-	0.3491		
98	246	1.0000	0.8024	-	0.0009		0.3720		
98	250	1.0000	0.7132	-	0.1859	-	0.0166		
98	252	1.0000	0.5943	-	0.1662		0.0739		
100	254	1.0000	0.4914	-	0.2069		0.0849		
$Q_0 = 11$									
90	228	1.0000	1.9144	-	0.4715		0.4187		
90	230	1.0000	1.5204	-	0.6958		0.4286		
92	230	1.0000	1.8500	-	0.2304		0.3098		
92	232	1.0000	1.7097	-	0.4007		0.3243		
92	234	1.0000	1.4591	-	0.5589		0.3388		
92	236	1.0000	1.3917	-	0.9798		0.4734		
94	236	1.0000	1.4205	-	0.0357	-	0.2636	-	
94	238	1.0000	1.2637	-	0.1826	-	0.4607	-	
94	240	1.0000	1.1025	-	0.0847	-	0.5167	-	
94	242	1.0000	1.1633	-	0.3968	-	0.0478	-	
96	242	1.0000	1.0134	-	0.2765	-	0.4313	-	
96	244	1.0000	0.9906	-	0.2925	-	0.4028	-	
98	246	1.0000	0.7935	-	0.0907	-	0.4073	-	
98	250	1.0000	0.7008	-	0.2671	-	0.0329	-	
98	252	1.0000	0.5479	-	0.1315		0.1017		
100	254	1.0000	0.4267	-	0.1817		0.1180		
$Q_0 = 14$									
90	228	1.0000	2.1911	-	0.3815		0.5985		
90	230	1.0000	1.6419	-	0.6813		0.6030		
92	230	1.0000	2.1195	-	0.0930		0.4314		
92	232	1.0000	1.9120	-	0.3193		0.4542		
92	234	1.0000	1.5697	-	0.5262		0.4714		
92	236	1.0000	1.4623	-	1.0236		0.6660		
94	236	1.0000	1.5545	-	0.1925	-	0.2970	-	
94	238	1.0000	1.3604	-	0.3482	-	0.5369	-	
94	240	1.0000	1.1548	-	0.2172	-	0.5827	-	
94	242	1.0000	1.2433	-	0.5547	-	0.0858	-	
96	242	1.0000	1.0589	-	0.4268	-	0.5070	-	
96	244	1.0000	1.0306	-	0.4361	-	0.4740	-	
98	246	1.0000	0.7867	-	0.1957	-	0.4539	-	
98	250	1.0000	0.6894	-	0.3576	-	0.0566	-	
98	252	1.0000	0.4965	-	0.0964		0.1301		
100	254	1.0000	0.3547	-	0.1583		0.1527		

		Relative Phase						
		Plus	Minus		Minus		Minus	
Z	A	$l=0$	$l=2$		$l=4$		$l=6$	
$Q_0=8$								
90	228	1,0000	2,0118	-	1,5490	-	0,3846	-
90	230	1,0000	1,7284	-	1,5163	-	0,3663	-
92	230	1,0000	1,8933	-	1,2802	-	0,3120	-
92	232	1,0000	1,8119	-	1,3218	-	0,3010	-
92	234	1,0000	1,6326	-	1,3229	-	0,2969	-
92	236	1,0000	1,6865	-	1,6749	-	0,3830	-
94	236	1,0000	1,4557	-	0,8685	-	0,5186	-
94	238	1,0000	1,2760	-	0,6641	-	0,5768	-
94	240	1,0000	1,1584	-	0,6819	-	0,6594	-
94	242	1.0000	1,1017	-	0,2661	-	0,0232	-
96	242	1.0000	1,0221	-	0,4540	-	0,4600	-
96	244	1.0000	0,9919	-	0,3970	-	0,4104	-
98	246	1.0000	0,8806	-	0,5546	-	0,5300	-
98	250	1.0000	0,7132	-	0,1860	-	0,0166	-
98	252	1.0000	0,6839	-	0,5084	-	0,1048	-
100	254	1.0000	0.5936	-	0.5308	-	0.1153	-
$Q_0=11$								
90	228	1.0000	2,4274	-	2,1179	-	0,6653	-
90	230	1,0000	2,0221	-	1,9731	-	0,6078	-
92	230	1.0000	2,2571	-	1,7621	-	0,5414	-
92	232	1.0000	3,1360	-	1,7709	-	0.5121	-
92	234	1.0000	1,8859	-	1,7122	-	0,4898	-
92	236	1.0000	1,9695	-	2,1290	-	0,6256	-
94	236	1.0000	1,6429	-	1,1736	-	0,6966	-
94	238	1.0000	1,4009	-	0,9030	-	0,7272	-
94	240	1.0000	1,2532	-	0,9019	-	0,0130	-
94	242	1.0000	1,1633	-	0,3968	-	0.0478	-
96	242	1.0000	1,0797	-	0,6261	-	0,5624	-
96	244	1.0000	1,0386	-	0,5510		0,4966	-
98	246	1,0000	0,9146	-	0,7251	-	0,6478	-
98	250	1,0000	0,7008	-	0,2671	-	0,0329	-
98	252	1.0000	0,6829	-	0,6204	-	0,1631	-
100	254	1.0000	0,5800	-	0.6401	-	0,1779	-
$Q_0=14$								
90	228	1.0000	3,0635	-	2,9673	-	1,1087	-
90	230	1,0000	2,4425	-	2.6080	-	0,9602	-
92	230	1,0000	2,8055	-	2,4689	-	0,9011	-
92	232	1,0000	2,6105	-	2,4103	-	0,8300	-
92	234	1,0000	2,2416	-	2,2417	-	0,7663	-
92	236	1,0000	2,3687	-	2,7510	-	0,9711	-
94	236	1,0000	1,9029	-	1,5796	-	0,9471	-
94	238	1,0000	1,5694	-	1,2080	-	0,9300	-
94	240	1,0000	1,3791	-	1,1754	-	1,0132	-
94	242	1.0000	1,2433	-	0,5547	-	0,0858	-
96	242	1.0000	1,1570	-	0,8357	-	0,6963	-
96	244	1.0000	1,1012	-	0,7362	-	0,6085	-
98	246	1.0000	0.9625	-	0,9285	-	0,7971	-
98	250	1.0000	0,6894	-	0,3576	-	0,0566	-
98	252	1,0000	0,6861	-	0,7471	-	0,2356	-
100	254	1,0000	0.5690	-	0,7629	-	0,2550	-

		Relative phase			
		Plus	Plus	Plus	Minus
Z	A	$l = 0$	$l = 2$	$l = 4$	$l = 6$
$Q_0 = 8$					
94	236	1.0000	1.3331	0.5217	0.1090 -
94	238	1.0000	1.2543	0.3958	0.2254 -
94	240	1.0000	1.2015	0.4117	0.2810 -
96	242	1.0000	1.1377	0.2834	0.2198 -
96	244	1.0000	1.1219	0.2527	0.2086 -
98	246	1.0000	1.0791	0.3579	0.2294 -
$Q_0 = 11$					
94	236	1.0000	1.3891	0.5915	0.0572 -
94	238	1.0000	1.3125	0.4610	0.1825 -
94	240	1.0000	1.2650	0.4705	0.2361 -
96	242	1.0000	1.2036	0.3480	0.1880 -
96	244	1.0000	1.1873	0.3159	0.1810 -
98	246	1.0000	1.1572	0.4180	0.1900 -
$Q_0 = 14$					
94	236	1.0000	1.4405	0.6604	0.0024 -
94	238	1.0000	1.3662	0.5260	0.1360 -
94	240	1.0000	1.3236	0.5298	0.1879 -
96	242	1.0000	1.2649	0.4134	0.1513 -
96	244	1.0000	1.2482	0.3800	0.1487 -
98	246	1.0000	1.2298	0.4801	0.1460 -

		Relative phase			
		Plus	Plus	Minus	Minus
Z	A	$l = 0$	$l = 2$	$l = 4$	$l = 6$
$Q_0 = 8$					
94	236	1.0000	1.2612	0.0696 -	0.2781 -
94	238	1.0000	1.2062	0.0019 -	0.3355 -
94	240	1.0000	1.1450	0.0640 -	0.4097 -
96	242	1.0000	1.1123	0.0774	0.2777 -
96	244	1.0000	1.1031	0.0969	0.2510 -
98	246	1.0000	1.0299	0.0334 -	0.3405 -
$Q_0 = 11$					
94	236	1.0000	1.2983	0.0145	0.2770 -
94	238	1.0000	1.2513	0.0712	0.3265 -
94	240	1.0000	1.1927	0.0025	0.4053 -
96	242	1.0000	1.1712	0.1449	0.2641 -
96	244	1.0000	1.1630	0.1621	0.2368 -
98	246	1.0000	1.0939	0.0300	0.3369 -
$Q_0 = 14$					
94	236	1.0000	1.3338	0.0963	0.2681 -
94	238	1.0000	1.2940	0.1434	0.3110 -
94	240	1.0000	1.2377	0.0687	0.3945 -
96	242	1.0000	1.2264	0.2128	0.2443 -
96	244	1.0000	1.2193	0.2280	0.2170 -
98	246	1.0000	1.1541	0.0950	0.3264 -

		Relative phase					
		Plus	Minus		Plus		Plus
Z	A	$\ell = 0$	$\ell = 2$		$\ell = 4$		$\ell = 6$
$Q_0 = 8$							
94	236	1.0000	1.3168	-	0.1618		0.4495
94	238	1.0000	1.1861	-	0.0496		0.5189
94	240	1.0000	1.0581	-	0.1377		0.6082
96	242	1.0000	0.9761	-	0.0664	-	0.4107
96	244	1.0000	0.9577	-	0.0954	-	0.3655
98	246	1.0000	0.7980	-	0.0952		0.4844
$Q_0 = 11$							
94	236	1.0000	1.4127	-	0.0806		0.5469
94	238	1.0000	1.2537	-	0.0302	-	0.6043
94	240	1.0000	1.0917	-	0.0863		0.7055
96	242	1.0000	1.0049	-	0.1477	-	0.4606
96	244	1.0000	0.9831	-	0.1771	-	0.4045
98	246	1.0000	0.7841	-	0.0514		0.5543
$Q_0 = 14$							
94	236	1.0000	1.5391	-	0.0211	-	0.6609
94	238	1.0000	1.3411	-	0.1269	-	0.7016
94	240	1.0000	1.1343	-	0.0271		0.8160
96	242	1.0000	1.0429	-	0.2428	-	0.5130
96	244	1.0000	1.0166	-	0.2721	-	0.4440
98	246	1.0000	0.7691	-	0.0042		0.6302

		Relative Phase					
		Plus	Minus		Minus		Plus
Z	A	L=0	L=2		L=4		L=6
$Q_0 = 8$							
94	236	1.0000	1.4522	-	0.7935	-	0.1812
94	238	1.0000	1.2714	-	0.5649	-	0.3526
94	240	1.0000	1.1534	-	0.5690	-	0.4218
96	242	1.0000	1.0182	-	0.3693	-	0.3273
96	244	1.0000	0.9885	-	0.3208	-	0.3054
98	246	1.0000	0.8761	-	0.4598	-	0.3303
$Q_0 = 11$							
94	236	1.0000	1.6349	-	1.0549	-	0.1270
94	238	1.0000	1.3908	-	0.7486	-	0.3484
94	240	1.0000	1.2421	-	0.7285	-	0.4225
96	242	1.0000	1.0711	-	0.4964	-	0.3340
96	244	1.0000	1.0311	-	0.4351	-	0.3136
98	246	1.0000	0.9050	-	0.5812	-	0.3230
$Q_0 = 14$							
94	236	1.0000	1.8865	-	1.4011	-	0.0407
94	238	1.0000	1.5494	-	0.9812	-	0.3318
94	240	1.0000	1.3579	-	0.9245	-	0.4137
96	242	1.0000	1.1407	-	0.6494	-	0.3332
96	244	1.0000	1.0870	-	0.5707	-	0.3156
98	246	1.0000	0.9444	-	0.7240	-	0.3060

VIII. APPROXIMATE METHOD FOR OBTAINING PHASE SHIFTS

Although the phase shifting due to the quadrupole interaction does not affect any experimental observables in alpha decay of even-even nuclei, it will enter into such things as alpha-gamma angular correlations and alpha angular distributions for aligned nuclei in odd-mass nuclei.

We may obtain approximate relations for the phase shifting due to the quadrupole interaction by noting a few things. The main contribution to the quadrupole phase shifting comes when $u''_i = 0$, (i.e., near the classical turning point). When this is the case, the functions u_i may be represented by straight lines. On the other hand, the u_i's are also representable as Coulomb functions in this region. Using this information, we may estimate the quadrupole phase shifting. We set $u_i = \alpha_i (r - r_i)$, where α_i is the square root of the intensity and r_i is the point at which the Coulomb function of this E and ℓ would have its node. As the formulae are well known for the phase shifting due to angular momentum, we treat the quadrupole interaction as a pseudo-angular momentum. We then set

$$\ell_i' (\ell_i' + 1) = \ell_i (\ell_i + 1) + \frac{2mQ_0e^2}{\hbar^2 r} \sum_j a_{ij} \frac{u_j}{u_i} . \qquad (15)$$

Evaluating the second term at the point where $u''_i = 0$, that is, r_i^o, we obtain

$$\ell_i' (\ell_i' + 1) = \ell_i (\ell_i + 1) + \frac{2mQ_0e^2}{\hbar^2 r_i^o} \sum_j a_{ij} \frac{\alpha_j (r_i^o - r_j)}{\alpha_i (r_i^o - r_i)} . \qquad (16)$$

The numerical coefficients a_{ij} are given on the right-hand side of equations (13). To determine the points r_i and r_i^o, we make the relations given for irregular Coulomb functions at the turning point,[16]

$$\frac{G_i'}{G_i} = - \frac{\Gamma(2/3)}{\Gamma(1/3)} \left(\frac{\rho_i^o}{3}\right)^{-1/3} \left[1 + \frac{\ell_i (\ell_i + 1)}{\rho_i^{o2}}\right]^{1/3} . \qquad (17)$$

Because the functions u_i are also straight lines in this region, we have $G_i' = \alpha_i/k_i$; $\rho_i = k_i r$. We calculate ρ_i^o (the point at which $u_i'' = 0$) by the relation[16]

$$\rho_i^o = \eta + \left[\eta^2 + \ell_i' (\ell_i' + 1)\right]^{1/2} \qquad (18)$$

and by the use of an iterative procedure. First we calculate ρ_i^o neglecting the quadrupole interaction; we then put in the quadrupole interaction to calculate $\ell_i' (\ell_i' + 1)$ and then calculate ρ_i^o. We then calculate another value for $\ell_i' (\ell_i' + 1)$ and continue until ρ_i^o does not change. $\rho_i^o - \rho_i^{o'}$ is the quadrupole phase shift.

We may compare phase shifts calculated in this manner with those obtained through the detailed calculations of Rasmussen and Hansen[9] for Cm^{242} assuming all alpha partial waves in phase, (Table VII).

Table VII

Quadrupole phase shifts (radians)					
	R and H	Approximation		R and H	Approximation
θ_0	-0.009	-0.03	$\theta_0 - \theta_2$	0.051	0.07
θ_2	-0.060	-0.10	$\theta_0 - \theta_4$	0.67	0.67
θ_4	-0.68	-0.70	$\theta_0 - \theta_6$	0.066	0.06
θ_6	-0.075	-0.09			

We note that the agreement is fairly good for the differences in phase shift. It is the cosine of the difference in phase shift that enters into angular-correlation experiments, and the approximation seems useful for calculations of correlations for odd-mass nuclei.

IX. APPLICATION OF APPROXIMATE METHOD TO U^{233}

This method was also applied in treating the alpha partial-wave intensities in the decay of U^{233} to form Th^{229}. To determine the values of the coefficients α_ℓ, I_f, and β_ℓ, I_f, we apply B.F.M. - 1 to obtain two conditions, i.e., the partition of the $\ell = 2$ alpha wave between the 5/2, 7/2, and 9/2 states at the nuclear surface. We then substitute the analytic approximation into the differential equations and demand that the equations be satisfied exactly at some arbitrary intermediate distance (2.0×10^{-12} cm) to obtain four more conditions. The over-all normalization gives us a seventh condition. To obtain the final condition we may do one of two things: (a) we can use the ratio of the $\ell = 2$ wave to the $\ell = 0$ wave at the nuclear surface obtained from a neighboring even-even nucleus, U^{232} or U^{234}, or (b) we can use the ratio of any two experimental intensities, bearing in mind that the observed alpha intensity to any level I is equal to $\sum_{\ell} |\alpha_{\ell,I_f}|^2$. Using (b) to obtain the final condition, we are then able to check the approximation with the third experimental intensity and with the amount of $\ell = 2$ wave calculated to populate the 5/2 state in the numerical integration. We compare the intensity predictions of this treatment with the predictions of B.F.M. We may compare several things in the following manner. We may use B.F.M. - 1 as a boundary condition at the nuclear surface and then use B.F.M. - 2 and the analytic method described here to calculate intensities at the nuclear surface. We may also use the results of the numerical integration to provide boundary conditions at the nuclear surface. We shall adjust the B.F.M. intensity predictions by the use of the relative intensities of the 5/2 and 7/2 states. The comparisons are found in Table VIII.

Table VIII

Test of B.F.M. - 2

I_f	Experimental intensity	Boundary conditions at nuclear surfaces: B.F.M. - 1		Boundary conditions at nuclear surfaces: Numerical integration	
		B.F.M. - 2	Analytic approx.	B.F.M. - 2	Analytic approx.
5/2	100%	100%	100%	100%	100%
7/2	17.9%	17.9%	17.9%	17.9%	17.9%
9/2	1.9%	2.54%	2.35%	2.2%	1.85%

The agreement with experiment is fairly good for $\ell = 2$ alpha partial waves, using the B.F.M. - 2 approximation; however, this method does not take into account different phase choices for the alpha partial waves. If we consider an $\ell = 4$ wave, the terms in the radial equations due to the nonvanishing nuclear quadrupole moment may be important in an intensity prediction of alpha decay, and the predictions may vary considerably, depending on the choice of partial wave phases. It is for this application that we feel that the approximate method described here has a considerable advantage over the B.F.M. - 2 approximation.

A calculation was made by the use of the analytic approximation including the $\ell = 4$ partial wave in the alpha decay of U^{233} to form Th^{229}. We can then predict the alpha intensities populating the 11/2 and 13/2 states of Th^{229} that are members of the ground-state rotational band. If we neglect the $\ell = 6$ contributions (and including them would be a Herculean task) and apply the data on relative amplitudes of alpha partial waves from the neighboring even-even nuclides, we obtain the intensity predictions for four phase choices. We compare these with B.F.M. - 2 and experimental observation in Table IX.

Table IX

Intensity predictions							
I_f	Relative Phase					B.F.M.	Experimental
	$\ell = 0$	+	+	+	+		
	$\ell = 2$	+	+	-	-		
	$\ell = 4$	+	-	+	-		
5/2		100	100	100	100	100	100
7/2		16	18.3	16.8	15.5	13	17.9
9/2		2.46	3.01	2.70	2.37	1.8	1.9
11/2		0.036	0.286	0.180	0.016	0.2	? [a]
13/2		0.007	0.020	0.012	0.0035		?

[a]The large limits quoted on this experimental intensity make it useless for distinguishing between possible phase choices.

X. COMPARISON WITH NUCLEAR-ALIGNMENT EXPERIMENT

Some experimental data are available on the relative phases of the $\ell = 0$ and $\ell = 2$ alpha partial waves. Roberts, Dabbs, and Parker have aligned U^{233} nuclei in a single crystal of $Rb(UO_2)(NO_3)_3$ and have obtained an angular distribution of alpha particles.[7] They have interpreted their results as indicating that the $\ell = 2$ partial wave populating the ground state of Th^{229} is out of phase with the $\ell = 0$ wave. To arrive at this conclusion, they make the assumption that the quadrupole coupling constant,

$$q = \frac{3\, e\, Q_{spec.}}{4I\,(2I-1)} \left\langle \frac{\partial^2 V(0)}{\partial z^2} \right\rangle$$

is negative. Here e is the electronic unit of charge, $Q_{spec.}$ is the spectroscopic value of the nuclear quadrupole moment and

$$\left\langle \frac{\partial^2 V(0)}{\partial z^2} \right\rangle$$

is the gradient of the electronic field evaluated at the surface of U^{233}. The calculations of Eisinstein and Pryce[17] are interpreted by Roberts et al. as indicating that

$$\left\langle \frac{\partial^2 V(0)}{\partial z^2} \right\rangle$$

is positive in the UO_2^{++} ion. This conclusion does not seem to be completely warranted, as we have found that for both unscreened nonrelativistic and screened relativistic[18] wave functions $\left\langle \frac{1}{r^3} \right\rangle_{6d,7s}$ is negative, so there is a possibility that q will be positive. If we define the percent of $\ell = 2$ admixture in the population of the 5/2 state as $100\,\delta^2/1+\delta^2$; Roberts et al. show from the measurements that they have made that

$$\frac{0.795\,\delta^2 + 4.145\,\delta + 0.226}{1 + 0.835\,\delta^2} \cdot \frac{q}{k} = 0.0625 \pm .0025 \qquad (19)$$

Using the values of δ which we obtained in the numerical integration of U^{233}, we may then calculate a value for q. If the $\ell = 2$ wave is in phase with the $\ell = 0$ wave, we have $\delta = 0.577$; if the $\ell = 2$ wave is out of phase with the $\ell = 0$ wave, we have $\delta = -0.577$. For the $\ell = 2$ wave in phase we calculate $\frac{q}{k} = 0.0277^{\circ}K$; for the $\ell = 2$ wave out of phase, we calculate

$\frac{q}{k} = -0.0418^{o}K$. Roberts et al. give a value for $|\frac{q}{k}|$ of $0.0388 \pm .0086^{o}K$ from specific heat measurements, but the sign is not determined in these measurements. Roberts et al. argue that the sign of q is negative in a manner analogous to Bleaney et al. for Np^{237}.[19] From paramagnetic-resonance measurements, Bleaney shows that the magnetic moment of Np^{237} and the quadrupole coupling constant of $Rb(NpO_2)(NO_3)_3$ must have opposite signs. Bleaney suggests that the magnetic moment, μ, is positive and q is negative on theoretical grounds. Our value calculated for the $\ell = 2$ wave out of phase with the $\ell = 0$ is well within the limits of error of their measurement, and the value for the $\ell = 2$ wave in phase with the $\ell = 0$ seems to be outside the limits of error.

We will be able to make a definite phase choice only when more experimental data become available. Either a high-precision determination of the populations of the 11/2 and 13/2 levels of Th^{229} by alpha decay, or a measurement of the sign of

$$\left\langle \frac{\partial^2 V(o)}{\partial z^2} \right\rangle$$

will definitely determine the relative phases of the $\ell = 0$ and the $\ell = 2$ partial waves.

XI. COMPARISON OF EVEN-EVEN APPROXIMATE TREATMENT WITH FRÖMAN'S TREATMENT

If we let a_ℓ' be the expansion coefficients of the Legendre functions on our spherical nuclear surfaces and b_ℓ be the reciprocal of the product of the hindrance factor and the centrifugal-barrier reduction factor, there exists a matrix such that $b_\ell = \sum_{\ell'} k_{\ell,\ell'}(B)\, a_{\ell'}$ for all phase choices of b_ℓ. Fröman has derived the elements of such matrices in his treatment of the alpha decay of spheroidal nuclei.[11] To calculate the value of B, we use Fröman's equation (VI-9), leaving out the unity term in the final factor and using (VI-2) to obtain his q. This matrix has been calculated for Cm^{242} alpha decay as well as an equivalent matrix derived by Rasmussen and Hansen from numerical integration of the wave equations.[9] In Table X we compare these two matrices to the one obtained from our analytic approximation developed here.

Table X

Comparison of Fröman-like matrices

Matrix derived from numerical integration by Rasmussen and Hansen[a]

$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
1.015+0.0116i	-0.1674-0.0176i	0.01166+0.00217i	-0.0005093-0.000130i
-0.2107-0.0456i	0.9542-0.00158i	-0.1195-0.00592i	0.007260+0.000679i
0.02114+0.0135i	-0.1899-0.0595i	0.9191-0.00360i	-0.1008-0.00187i
-0.001089-0.00216i	0.01885+0.0187i	-0.2052-0.0893i	0.9086-0.0241i

Matrix derived from Fröman's treatment[b]

$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
1.019	-0.193	0.014	-0.0005
-0.193	0.908	-0.158	0.014
0.014	-0.158	0.917	-0.155
-0.0005	0.014	-0.155	0.917

Matrix derived from analytic treatment of this paper[c]

$\ell = 0$	$\ell = 2$	$\ell = 4$	$\ell = 6$
1.034	-0.1855	0.0166	-0.00121
-0.2405	0.9181	-0.1213	0.00879
-0.0358	-0.2010	0.9197	-0.0979
-0.00684	0.0383	-0.2579	0.9070

[a] $Q_0 = 9 \times 10^{-24}$ cm^2; $R_0 = 10.1 \times 10^{-13}$ cm.

[b] $B = -0.455$.

[c] $Q_0 = 8 \times 10^{-24}$ cm^2; $R_0 = 8.9 \times 10^{-13}$ cm.

It is interesting to note that the Fröman treatment gives a matrix which corresponds rather closely to the real elements of the numerical integration treatment, although Fröman's treatment necessitates a symmetric matrix, which is not the case in the numerical treatment. The present treatment corresponds more nearly to the sum of the real and imaginary parts of the matric elements of the numerical treatment, and the matrix is not symmetric.

XII. RESULTS OF THE NUMERICAL INTEGRATION OF U^{233}

The results of the numerical integrations of U^{233} may be expressed in several ways. In analogy with Fröman, we give matrices through which one may convert amplitudes of partial waves at the nuclear surface of amplitudes at infinity which are (intensity/velocity)$^{1/2}$.

Let a_t' be a column vector giving the amplitudes of partial waves at the nuclear surface, where t denotes indices ℓ and I_f. We may relate this to a column vector b_t, which gives the amplitudes of the partial waves at infinity, by an equation of the form $b_t = \Sigma_{t'} k_{t,t'} a_{t'}$. We then factor $k_{t,t'}$ into two matrices,

$$k_{t,t'} = \Sigma_{t''} \quad [G^{-1}_{t''} (R) \quad \delta_{t,t''}] \; k'_{t'',t'} \tag{20}$$

in the case of the real (irregular) components, and in the case of the imaginary parts,

$$k_{t,t'} = \Sigma_{t''} \quad [F^{-1}_{t''} (R) \quad \delta_{t,t''}] \; k'_{t'',t'} \; . \tag{21}$$

In both cases the Coulomb functions are evaluated at the nuclear radius, in our work chosen to be 9.0×10^{-13}cm. The matrices $k'_{t,t'}$ are similar to those given by Fröman[9] and by Rasmussen and Hansen[11] and are a convenient way of displaying the detailed effects of the quadrupole interaction. The matrices $k'_{t,t'}$ become simple unit matrices $\delta_{t,t'}$ in the limit of zero nuclear quadrupole moment. It should be pointed out that these matrices apply to a spherical surface at the nucleus whereas Fröman's matrices are given for a spheroidal nuclear surface.

From the imaginary part of the numerical integration we obtain the matrix $k'_{t,t'}$ (Table XI).

Table XI

$k'_{t,t'}$ matrix from imaginary part of numerical integration				
	$\ell' = 0$, $I'_f=5/2$	$\ell' = 2$, $I'_f=5/2$	$\ell' = 2$, $I'_f=7/2$	$\ell' = 2$, $I'_f=9/2$
$\ell = 0$, $I_f=5/2$	1.000 -0.010i	0.150 +0.030i	-0.180 +0.043i	0.118 -0.030i
$\ell = 2$, $I_f=5/2$	0.1541 -0.0195i	0.886 -0.010i	-0.111 +0.022i	-0.054 +0.0049i
$\ell = 2$, $I_f=7/2$	-0.173 +0.027i	-0.106 +0.013i	0.849 -0.0353i	-0.110 +0.022i
$\ell = 2$, $I_f=9/2$	0.096 -0.0067i	-0.021 +0.0024i	-0.099 +0.0076i	0.854 +0.0006i

Table XII shows the matrix $k'_{t,t'}$ obtained from the real part of the numerical integration.

Table XII

$k'_{t,t'}$ matrix from real part of numerical integration				
	$\ell' = 0$, $I'_f=5/2$	$\ell' = 2$, $I'_f=5/2$	$\ell' = 2$, $I'_f=7/2$	$\ell' = 2$, $I'_f=9/2$
$\ell = 0$, $I_f=5/2$	1.000 +0.013i	-0.166 -0.016i	0.1953 +0.013i	-0.1024 -0.0037i
$\ell = 2$, $I_f=5/2$	-0.1948 -0.022i	0.9053 +0.012i	0.116 +0.0068i	0.059 +0.0058i
$\ell = 2$, $I_f=7/2$	0.2421 +0.022i	0.128 +0.0052i	1.007 -0.027i	0.126 +0.0005i
$\ell = 2$, $I_f=9/2$	-0.143 -0.048i	0.057 +0.032i	0.093 + 0.015i	0.911 +0.019i

XIII. SUMMARY

In summary we believe this detailed numerical integration of the alpha-decay wave equation for U^{233} shows the validity of the Bohr-Fröman-Mottelson hypothesis (B.F.M. - 1) that for favored alpha decay there is zero projection of alpha angular momentum on the nuclear-symmetry axis while the alpha is near the surface. The approximation (B.F.M. - 2) that the projection remains zero near the classical turning point is shown to be a fairly good approximation for the relatively abundant $\ell = 2$ wave but a very poor approximation for the weak $\ell = 4$ wave. The analytical approximation based on modified Coulomb functions is shown to give results nearer those of the numerical integration than does the B.F.M. - 2 approximation. The extra phase shifts due to the quadrupole interaction were derived, and the shifts most significant to the interpretation of nuclear-alignment experiments were shown to be negligibly small.

Through approximate methods, we are able to obtain information concerning the alpha decay of deformed nuclei which was heretofore obtainable only through detailed numerical integration. We have developed approximate methods for calculating both alpha partial wave amplitudes at the nuclear surface and phase shifting caused by nuclear-quadrupole deformation, using a small fraction of the computer time involved in a numerical integration.

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