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RESEARCH MEMORANDUM

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RESEARCH MEMORANDUM

FREE-FREE GAUNT FACTORS

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SUMMARY

The hydrogenic (unscreened coulomb) free-free gaunt factors are computed for a wide range of initial energies and photon frequencies. In addition, an average over initial energies with the Maxwell-Boltzmann distribution is performed to give the temperature-averaged gaunt factors for use in opacity calculations. These are presented as functions of Z^2/kT and $h\nu/kT$.

The relation between these gaunt factors and the rate of bremsstrahlung energy production is given, as is the total energy emitted as a function of Z^2/kT.

CONTENTS

DISCUSSION

1. Free-Free Absorption

The free-free absorption coefficients in a pure coulomb field are readily computed from the formula for bremsstrahlung, since these are just inverse events. The cross section for absorption of a photon of energy $h\nu$ by an electron of energy $E_i = \frac{\hbar^2 k_i^2}{2m}$, which makes a transition to energy $E_f = \frac{\hbar^2 k_f^2}{2m}$, is related to the emission of the photon $h\nu$ by the electron E_f by the formula:

$$\sigma_{i,\nu \longrightarrow f} = \frac{v_f}{c} \frac{\rho_f}{\rho_{i,\nu}} \sigma_{f \longrightarrow i,\nu} \qquad (1)$$

where v_f is the velocity of the final electron and ρ_f, $\rho_{i,\nu}$ are the density of states for the two final states.

$$\rho_f = 2 \cdot 4\pi \frac{mk_f}{(2\pi)^3 \hbar^2}$$

$$\rho_{i,\nu} = 2 \cdot 4\pi \frac{mk_i}{(2\pi)^3 \hbar^2} \; 2 \cdot 4\pi \frac{\nu^2}{2\pi\hbar c^3} \; , \qquad (2)$$

where the factors of 2 are for electron spin and photon polarizations. The density of states formulae as written imply that the free electrons are to be normalized to plane waves of unit density at infinity.

For bremsstrahlung, we have[1,2]

$$d\sigma_{f \xrightarrow{em} i,\nu} = \frac{32}{3} Z^2 \left(\frac{e^2}{\hbar c}\right)^3 \frac{k_i k_f}{k_f^2} \frac{d(h\nu)}{h\nu} \sum_{l=0}^{\infty} \left\{ (l+1)\tau_{l+1,l}^2 + l\tau_{l-1,l}^2 \right\} , \qquad (3)$$

where
$$\tau_{\ell',\ell} = \int_0^\infty r^2 dr \; \Psi_{\ell'}(r, E_i) \frac{1}{r^2} \Psi_\ell(r, E_f), \tag{4}$$

where asymptotically $\Psi_\ell \sim \dfrac{\sin(kr + \delta_\ell)}{kr}$.

This is the total cross section to emit an unpolarized photon in the energy range $h\nu$ to $h\nu + d(h\nu)$ for electrons of unit density. Thus, for absorption, with the same electron normalization,

$$d\sigma^{ab}_{i,\nu \longrightarrow f} = \frac{\hbar^2 k_f^2 c^2}{4mk_i \nu^2} \, d\sigma^{em}_{f \longrightarrow i,\nu}$$

$$= \frac{8}{3} Z^2 \left(\frac{e^2}{\hbar c}\right)^3 \frac{\hbar^2 c^2 k_f}{m\nu^2} \frac{d(h\nu)}{h\nu} \sum_{\ell=0}^{\infty} \left\{ (\ell+1)\tau^2_{\ell+1,\ell} + \ell\tau^2_{\ell-1,\ell} \right\}$$

$$= \frac{2\pi^2\hbar^2}{mk_i} a_0^2 \left[\frac{64}{3} Z^2 \left(\frac{e^2}{\hbar c}\right)\left(\frac{Ry}{h\nu}\right)^3 \frac{d(h\nu)}{Ry} k_i k_f \sum \left\{ (\ell+1)\tau^2_{\ell+1,\ell} + \ell\tau^2_{\ell-1,\ell} \right\} \right] \tag{5}$$

The sum can be explicitly evaluated[1,2] and we obtain an answer in terms of hypergeometric functions, which can be expressed in several ways.

One way which is convenient is:[2]

$$\sum = \left\{ (k_i^2 + k_f^2 + 2k_f^2 \eta_f^2)(0,1;0) - 2k_i k_f (1 + \eta_i^2)^{1/2} (1 + \eta_f^2)^{1/2} (0,1;1) \right\} (0,1;0), \tag{6}$$

where $$(0,1;\ell) = \int_0^\infty r^2 dr \, \frac{1}{r} \, \Psi_e(r, E_i) \, \Psi_e(r, E_f), \tag{7}$$

$$\eta = \frac{Ze^2}{\hbar v} = \frac{Z}{ka_o}, \tag{8}$$

and again $$\Psi_\ell \sim \frac{\sin(kr + \delta)}{kr}.$$

For coulomb wave functions,

$$(0,1;\ell) = \frac{1}{k_i k_f} I_\ell$$

with

$$I_\ell = \frac{1}{4} \left[\frac{4k_i k_f}{(k_i - k_f)^2} \right]^{\ell+1} e^{\pi|\eta_i - \eta_f|/2} \, \frac{\left| \Gamma(\ell+1+i\eta_i) \Gamma(\ell+1+i\eta_f) \right|}{\Gamma(2\ell+2)} \, G_\ell \tag{9}$$

and G_ℓ (a real function)

$$= \left| \frac{k_f - k_i}{k_f + k_i} \right|^{i\eta_i + i\eta_f} {}_2F_1 \left(\ell+1-i\eta_f, \ \ell+1-i\eta_i; \ 2\ell+2; \ -\frac{4k_i k_f}{(k_i - k_f)^2} \right). \tag{10}$$

If the argument of the hypergeometric function is greater than one, this can be rewritten as

$$G_\ell = \left| \frac{k_f - k_i}{k_f + k_i} \right|^{i\eta_i + i\eta_f} \Gamma(2\ell+2)$$

$$\left\{ \frac{\Gamma(i\eta_f - i\eta_i)}{\Gamma(\ell+1-i\eta_i)\Gamma(\ell+1+i\eta_f)} \left(\frac{4k_f k_i}{(k_f-k_i)^2} \right)^{-\ell-1+i\eta_f} {}_2F_1\left(\ell+1-i\eta_f, -\ell-i\eta_f; 1+i\eta_i-i\eta_f; -\frac{(k_f-k_i)^2}{4k_f k_i} \right) \right.$$

$$\left. + \frac{\Gamma(i\eta_i - i\eta_f)}{\Gamma(\ell+1+i\eta_i)\Gamma(\ell+1-i\eta_f)} \left(\frac{4k_f k_i}{(k_f-k_i)^2} \right)^{-\ell-1+i\eta_i} {}_2F_1\left(\ell+1-i\eta_i, -\ell-i\eta_i; 1+i\eta_f-i\eta_i; -\frac{(k_f-k_i)^2}{4k_f k_i} \right) \right\} \quad (11)$$

In each of these cases it is possible to express the quantity G_ℓ as a real series in the argument. (See Appendix.)

Thus we have

$$d\sigma_{i,\nu \longrightarrow f} = \frac{64}{3} a_o^2 Z^2 \frac{e^2}{\hbar c} \left(\frac{Ry}{h\nu}\right)^3 d\left(\frac{h\nu}{Ry}\right) \frac{1}{k_f k_i} \frac{2\pi^2\hbar^2}{mk_i}$$

$$\left\{ (k_f^2 + k_i^2 + 2k_f^2\eta_f^2) I_o - 2k_f k_i (1+\eta_i^2)^{1/2} (1+\eta_f^2)^{1/2} I_1 \right\} I_o. \quad (12)$$

This is for electrons normalized to unit density. More conveniently, we can normalize the initial electron wave functions on the energy scale (i.e.--in terms of the number per unit volume per unit energy range). The relation pertinent here is

$$\sigma_{\text{energy scale}} = \sigma_{\text{density scale}} \cdot \frac{mk_i}{2\pi^2\hbar^2} \quad (13)$$

Thus,

$$d\sigma_{ff,\ \text{energy scale}} = a_o^2 Z^2 \frac{64}{3} \frac{e^2}{\hbar c} \left(\frac{Ry}{h\nu}\right) \frac{1}{\eta_i \eta_f}$$

$$(\eta_i^2+\eta_f^2+2\eta_i^2\eta_f^2) I_o - 2\eta_i\eta_f(1+\eta_i^2)^{1/2}(1+\eta_f^2)^{1/2} \quad I_1 \qquad I_o \quad . \tag{14}$$

The classical cross section for this process is given by[3]

$$d\sigma_{\text{class; energy scale}} = a_o^2 Z^2 \frac{32\pi}{3\sqrt{3}} \frac{e^2}{\hbar c} \left(\frac{Ry}{h\nu}\right)^3 d\left(\frac{h\nu}{Ry}\right) \quad . \tag{15}$$

Thus the gaunt factor,

$$g_{ff} = \frac{d\sigma_{ff}}{d\sigma_{\text{class}}} = \frac{2\sqrt{3}}{\pi\eta_i\eta_f} \left\{ (\eta_i^2+\eta_f^2+2\eta_i^2\eta_f^2) I_o - 2\eta_i\eta_f(1+\eta_i^2)^{1/2}(1+\eta_f^2)^{1/2} \quad I_1 \right\} \quad I_o \tag{16}$$

where

$$\eta_i^2 = \frac{Z^2 Ry}{E_i} \quad ; \quad \eta_f^2 = \frac{Z^2 Ry}{E_f} \quad . \tag{17}$$

Values of g_{ff} are plotted in Fig. 1 and Fig. 2 as functions of E_i/Z^2 , and h $/Z^2$.

To obtain the total absorption coefficient for a distribution of initial electron energies, we take the cross section on the energy scale and multiply by the number of electrons/unit energy. For the case of the Boltzmann distribution, this is

$$\frac{dn}{dE_i} = \frac{1}{kT} e^{-E_i/kT} \quad . \text{(number/unit volume)}. \tag{18}$$

For many cases of interest in opacity computations the temperatures and densities are such that it is permissible to neglect the degeneracy of the electrons. This is not true, however, of such stellar matter as the white dwarfs, for instance. In general, one ought to use the Fermi-Dirac distribution function, rather than the Boltzmann; it is a straightforward matter to do the integration for any particular case of interest. Because of the difficulty in presenting the results of such calculations (which add a third parameter--the free energy--to the frequency and temperature dependence of the gaunt factor) we have limited ourselves to the Boltzmann distribution.

Thus the average cross section at temperature T for absorption of photons of frequency $h\nu$ by free electrons is:[(4)]

$$\overline{d\sigma_{ff}} = \frac{1}{kT} \int e^{-E_i/kT} \, dE_i \, d\sigma_{ff}(E_i, E_i + h\nu) \ . \tag{19}$$

In terms of the parameters η, we have

$$\frac{1}{\eta_f^2} = \frac{1}{\eta_i^2} + \frac{h\nu}{Z^2} \ , \tag{20}$$

so that we obtain for the temperature-averaged free-free gaunt factor

$$\overline{g_{ff}}\,(u, \gamma^2) = \int_0^\infty dx \, e^{-x} \, g_{ff}\left(\eta_i = \gamma x^{-1/2}, \frac{h\nu}{Z^2} = \frac{u}{\gamma^2}\right) , \tag{21}$$

where

$$u = h\nu/kT$$

$$\gamma^2 = \frac{Z^2 Ry}{kT} \ .$$

This gaunt factor is plotted in Figs. 3, 4 and 5.

2. Bremsstrahlung

For those more interested in the emission of radiation than its absorption, we indicate how these calculations apply.

By comparing Eqs. (3), (6), (9) and (16), we see that we can write the cross section for an electron of energy E_o, normalized to unit density at infinity, to emit a photon in the energy range $h\nu$ to $h\nu + d(h\nu)$ in terms of the free-free gaunt factor:

$$d\sigma^{em} = \frac{16\pi}{3\sqrt{3}} \left(\frac{e^2}{\hbar c}\right)^3 Z^2 \frac{\hbar^2}{2mE_o} \frac{d(h\nu)}{h\nu} g_{ff} (E_i = E_o - h\nu, E_f = E_o)$$

The energy emitted per unit time is found by multiplying by $h\nu$ and by the electron flux, $n_e v_o$:

$$W(\nu)\, d\nu = \frac{8\pi}{3} \left(\frac{e^2}{\hbar c}\right)^3 Z^2 n_e \frac{\hbar^2}{m} \left(\frac{2}{3m}\right)^{1/2} E_o^{-1/2} d(h\nu)\, g_{ff}$$

$$= \frac{128\pi^2}{3\sqrt{3}} \left(\frac{e^2}{\hbar c}\right)^3 Z^2 (n_e a_o^3) (E_o/Ry)^{-1/2} d\nu\, g_{ff}\, Ry.$$

To find the energy emitted per unit time by a Maxwellian distribution of electron energies we average this spectrum:

$$\overline{W(\nu)}\, d\nu = \frac{\int_{h\nu}^{\infty} dE_o\, E_o^{1/2}\, e^{-E_o/kT}\, W(\nu)\, d\nu}{\int_0^{\infty} dE_o\, E_o^{1/2}\, e^{-E_o/kT}}$$

If we change to integration variable $E_o - h\nu$ in the numerator, we are evaluating exactly the same integral that gives the temperature-averaged gaunt factor. Thus, after simplifying, and rearranging the constants, we find

$$\overline{W(\nu)}\, d\nu = \frac{2^5 \pi e^6}{3hmc^3}\; Z^2 n_e \left(\frac{2\pi kT}{3m}\right)^{1/2} du\, e^{-u}\; \overline{g_{ff}}\,(u,\ \gamma^2)$$

$$= 1.42 \times 10^{-27}\, Z^2 n_e\; T^{1/2}\, du\, e^{-u}\, \overline{g_{ff}}\,(u,\ \gamma^2) \text{ ergs/sec}$$

where $u = h\nu/kT$; $\gamma^2 = Z^2 Ry/kT$, and T is in $^{\circ}K$ in the second expression.

The energy emitted per unit volume per unit time is found by multiplying by the density of positive ions which serve as targets for the bremsstrahlung production. To find the total energy emitted, we integrate over the photon spectrum

$$\epsilon = \left(\frac{2\pi kT}{3m}\right)^{1/2} \frac{2^5 \pi e^6}{3hmc^3}\; Z^2 n_e n_i \int_0^{\infty} du\, e^{-u}\; \overline{g_{ff}}\,(u,\ \gamma^2)$$

$$= 1.42 \times 10^{-27}\, Z^2 n_e n_i\, T^{1/2} \int_0^{\infty} du\, e^{-u}\; \overline{g_{ff}}\,(u,\ \gamma^2)\, \frac{\text{ergs}}{\text{cm}^3\ \text{sec}}\ .$$

$$= \epsilon_0 < \overline{g_{ff}}\,(\gamma^2) > \ .$$

This corresponds to formula (5-49) in Spitzer's book, which is equivalent to the assumption $\overline{g_{ff}} \equiv 1$.

The integral, $< \overline{g_{ff}}\,(\gamma^2) > = \epsilon/\epsilon_0$, is plotted as a function of γ^2 in Fig. 6.

APPENDIX

We have the real expression

$$G_{\ell} = \left| \frac{\eta_i - \eta_f}{\eta_i + \eta_f} \right|^{i\eta_i + i\eta_f} {}_2F_1\left(\ell+1-i\eta_f,\ \ell+1-i\eta_i;\ 2\ell+2;\ -\frac{4\eta_i\eta_f}{(\eta_i-\eta_f)^2}\right), \quad \text{(A.1)}$$

where

$${}_2F_1(a,b;c;x) = \sum_{0}^{\infty} \frac{\Gamma(a+m)\,\Gamma(b+m)\,\Gamma(c)}{\Gamma(c+m)\,\Gamma(a)\,\Gamma(b)} \frac{x^m}{m!} .$$

Let $x = -\dfrac{4\eta_i\eta_f}{(\eta_i-\eta_f)^2}$; then $1-x = \left(\dfrac{\eta_i+\eta_f}{\eta_i-\eta_f}\right)^2$. (A.2)

Let

$$a = \ell+1-i\eta_f$$

$$b = \ell+1-i\eta_i$$

$$c = 2\ell+2$$

$$d = \frac{1}{2}(a+b-c) .$$

Then $G = (1-x)^d F(a,b;c;\ x)$, (A.3)

or $F = (1-x)^{-d} G$. (A.4)

The hypergeometric function, F, satisfies the differential equation

$$x(1-x)F'' + \left[c-(a+b+1)x \right] F' - abF = 0 \;, \tag{A.5}$$

so G satisfies

$$x(1-x)^2 G'' + (1-x) \left[c + x(2d-a-b-1) \right] G'$$
$$+ \left[x \left(d^2+ab-d(a+b) \right) - ab+dc \right] G = 0 \;. \tag{A.6}$$

Substituting for a,b,c,d, we have

$$x(1-x)^2 G'' + (1-x) \left[2\ell+2 - x(2\ell+3) \right] G'$$
$$+ \left\{ x \left[\left(\frac{\eta_i - \eta_f}{2} \right)^2 + (\ell+1)^2 \right] + \eta_1 \eta_2 - (\ell+1)^2 \right\} G = 0 \;. \tag{A.7}$$

The regular solution to this differential equation can be expressed in the usual manner as a power series. The result is

$$G = \sum_{n=0}^{\infty} a_n x^n \tag{A.8}$$

with $$a_o = 1$$

$$a_1 = \frac{(\ell+1)^2 - \eta_i \eta_f}{2\ell+2}$$

$$a_n = \frac{1}{n(2\ell+1+n)} \left\{ a_{n-1} \left[(n-1)(2n+4\ell+1) + (\ell+1)^2 - \eta_i \eta_f \right] \right.$$

$$\left. - a_{n-2} \left[(n-2)(n+2\ell) + (\ell+1)^2 + \frac{\eta_i - \eta_f}{2} \right]^2 \right\}$$

and

$$x = - \frac{4\eta_i \eta_f}{(\eta_i - \eta_f)^2} . \qquad (A.9)$$

for $|x| > 1$ we find the series solution of the differential equation in terms of a series in inverse powers. With $y = -1/x$, we can write

$$G = y^{\ell+1} \left[\cos(\lambda \; \ell n \; y) \; A(y) + \sin(\lambda \; \ell n \; y) \; B(y) \right] \qquad (A.10)$$

where

$$\lambda = \frac{1}{2} (\eta_i - \eta_f)$$

and A and B are regular series in y. Substituting into the differential equation we obtain two coupled differential equations:

$$y(1+y)^2 A'' + (1+y)(1+2y)A' + 2\lambda(1+y)^2 B'$$

$$- (1+y)(\ell^2+\ell+\lambda^2)A - (\lambda^2+\eta_i\eta_f)A + \lambda(1+y)B = 0$$

and

$$y(1+y)^2 B'' + (1+y)(1+2y)B' - 2\lambda(1+y)^2 A'$$

$$- (1+y)(\ell^2+\ell+\lambda^2)B - (\lambda^2+\eta_i\eta_f)B - \lambda(1+y)A = 0 . \qquad (A.11)$$

As before the solutions are of the form

$$A = \sum_{n=0}^{\infty} a_n y^n$$

$$B = \sum_{n=0}^{\infty} b_n y^n \qquad \text{(A.12)}$$

with

$$a_o = 2 \,\mathrm{Re} \left\{ \frac{\Gamma(2\ell+2)\,\Gamma(i\eta_f - i\eta_i)}{\Gamma(\ell+1-i\eta_i)\,\Gamma(\ell+1+i\eta_f)} \right\}$$

$$b_o = -\, 2 \,\mathrm{Im} \left\{ \frac{\Gamma(2\ell+2)\,\Gamma(i\eta_f - i\eta_i)}{\Gamma(\ell+1-i\eta_i)\,\Gamma(\ell+1+i\eta_f)} \right\}$$

and the recursion relations

$$n(4\lambda^2+n^2)\, a_n = -\,\lambda(3n-2+2\alpha)b_{n-1} - \lambda(3n-4+2\beta)b_{n-2}$$

$$+ \left[n\alpha - n(n-1)(2n-1) - 2\lambda^2(4n-3) \right] a_{n-1}$$

$$+ \left[n\beta - n(n-1)(n-2) - 2\lambda^2(2n-3) \right] a_{n-2}$$

and

$$n(4\lambda^2+n)b_n = \lambda(3n-2+2\alpha)a_{n-1} + \lambda(3n-4+2\beta)\, a_{n-2}$$

$$+ \left[n\alpha - n(n-1)(2n-1) - 2\lambda^2(4n-3) \right] b_{n-1}$$

$$+ \left[n\beta - n(n-1)(n-2) - 2\lambda^2(2n-3) \right] b_{n-2}\, , \qquad \text{(A.13)}$$

where
$$\alpha = \ell(\ell+1) + 2\lambda^2 + \eta_i\eta_f$$

$$\beta = \ell(\ell+1) + \lambda^2$$

$$\lambda = \frac{1}{2}(\eta_i - \eta_f) \ .$$

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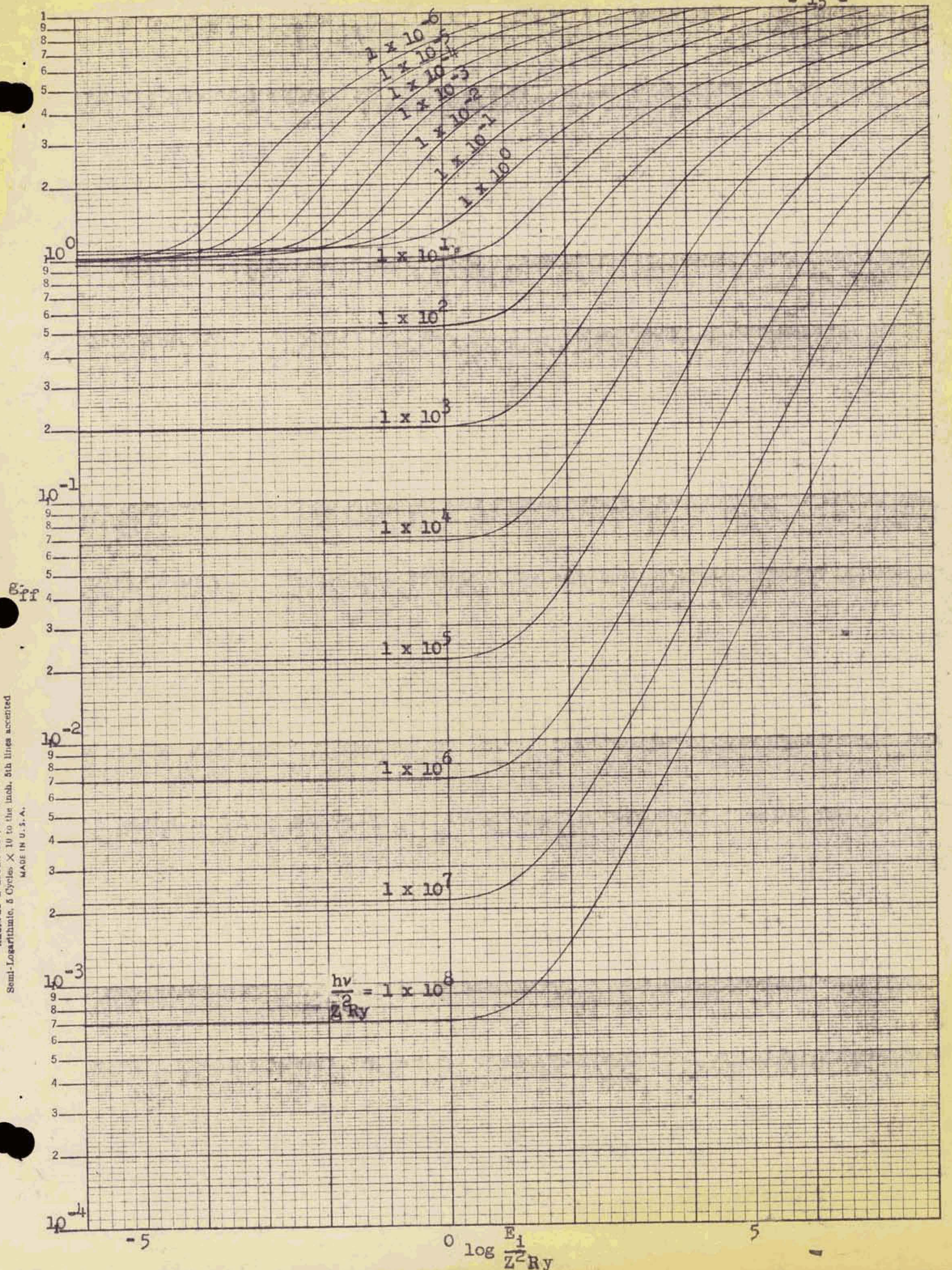
3. Menzel and Pekeris, Monthly Notices of the Royal Astronomical Society 96, 77 (1935).

4. H. Mayer, Methods of Opacity Calculations, LA-647, Los Alamos Scientific Laboratory, (1948). Note that formula (2.31) p. 15, is in error.

Figure 1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1 x 10^-5
1 x 10^-4
1 x 10^-3
1 x 10^-2
1 x 10^-1
1 x 10^0
1 x 10^1
1 x 10^2
1 x 10^3
1 x 10^4
1 x 10^5
1 x 10^6
1 x 10^7
hν/Z²Ry = 1 x 10^8
g_ff
10^0
10^-1
10^-2
10^-3
10^-4
-5
0
5
log E_i/Z²Ry
KEUFFEL & ESSER CO., N. Y. NO. 359-91
Semi-Logarithmic, 5 Cycles X 10 to the inch, 5th lines accented
MADE IN U.S.A.

Figure 2

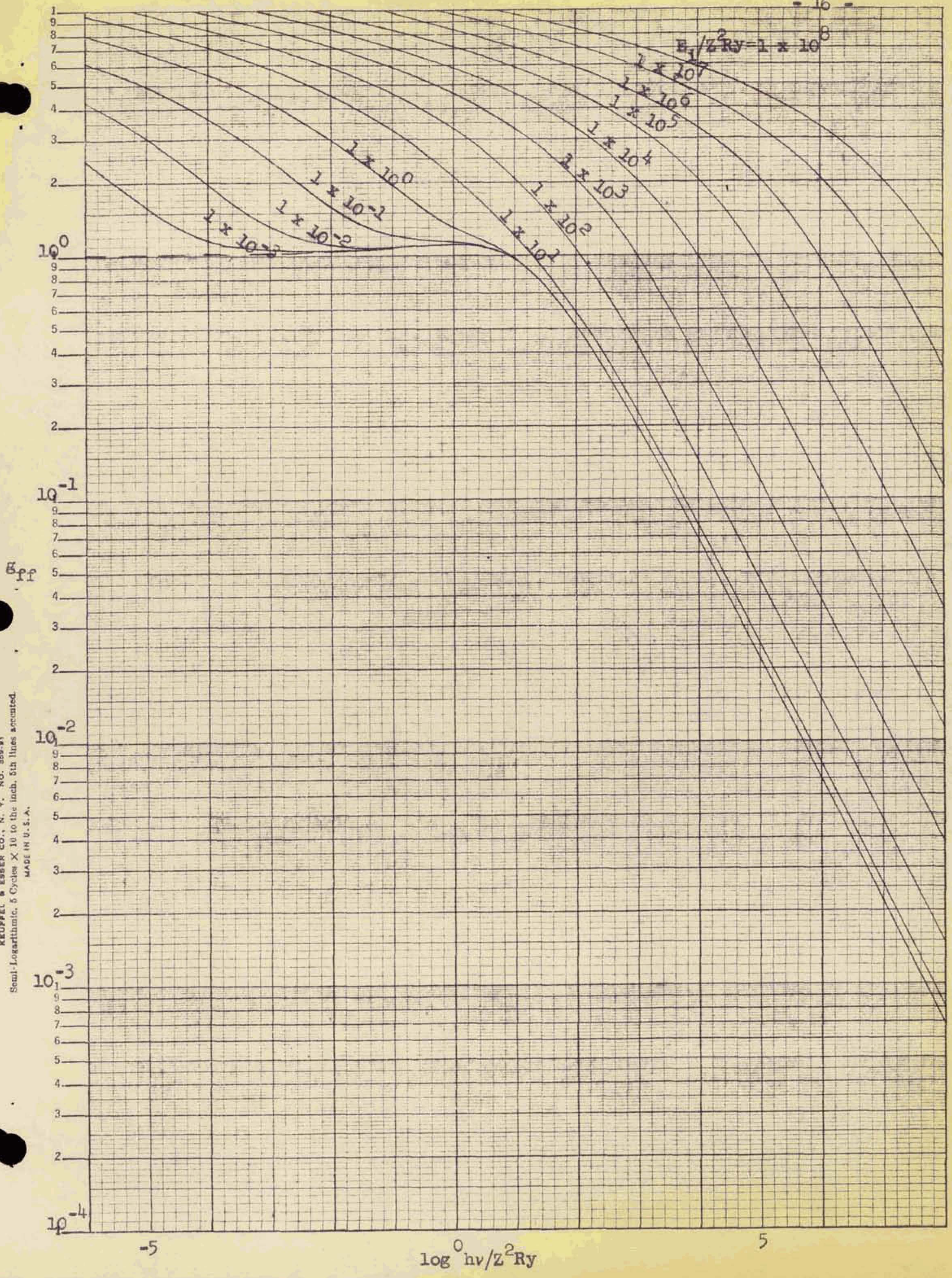
$E_1/Z^2Ry = 1 \times 10^8$
1×10^7
1×10^6
1×10^5
1×10^4
1×10^3
1×10^2
1×10^1
1×10^0
1×10^{-1}
1×10^{-2}
1×10^{-3}
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
g_{ff}
-5
0
5
$\log_{10} h\nu/Z^2Ry$
KEUFFEL & ESSER CO., N. Y. NO. 359-91
Semi-Logarithmic, 5 Cycles X 10 to the inch, 5th lines accented.
MADE IN U.S.A.

Figure 3

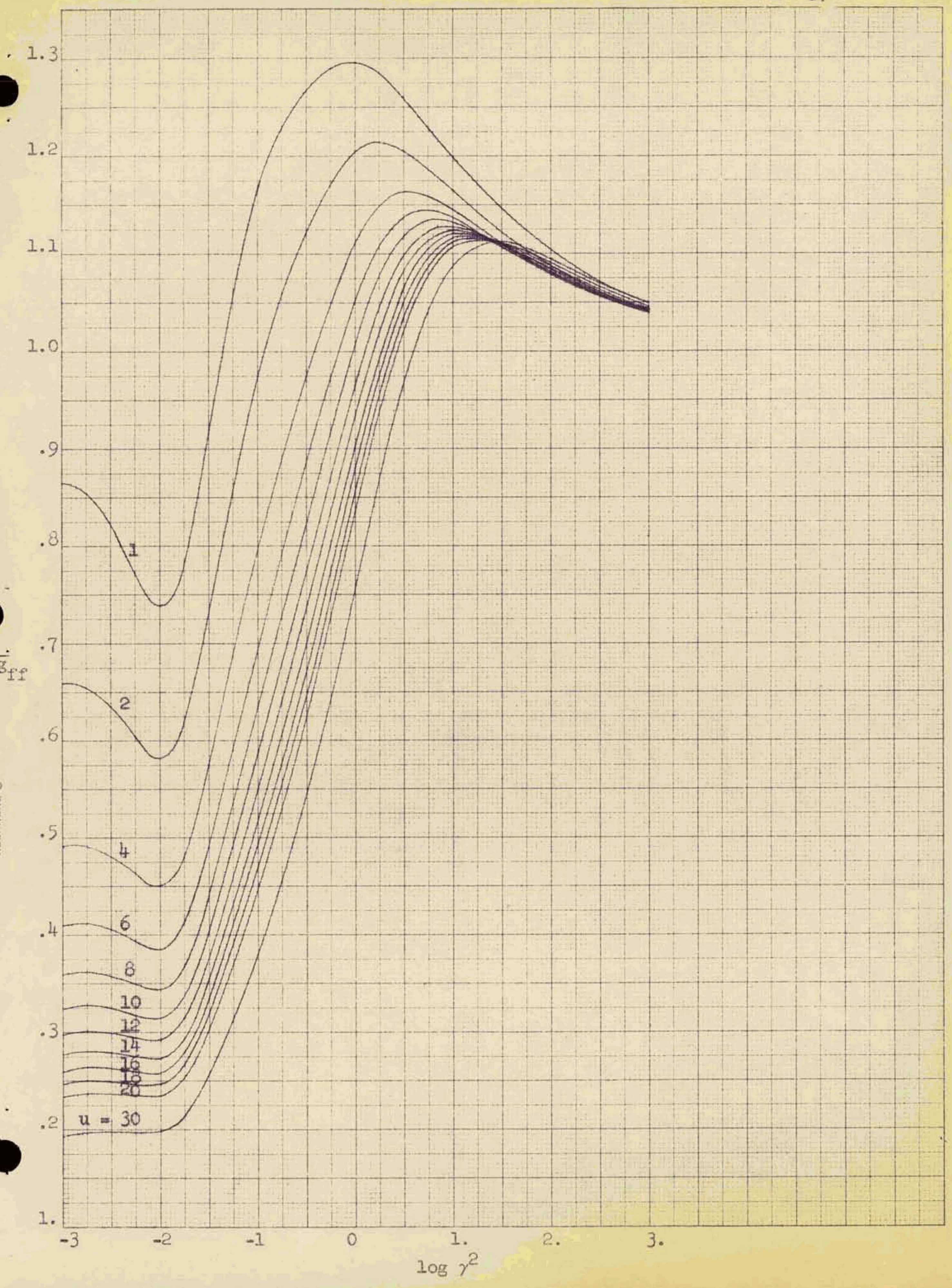
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
1.
$\bar{g}_{ff}$
1
2
4
6
8
10
12
14
16
18
20
u = 30
-3
-2
-1
0
1.
2.
3.
$\log \gamma^2$

Figure 4

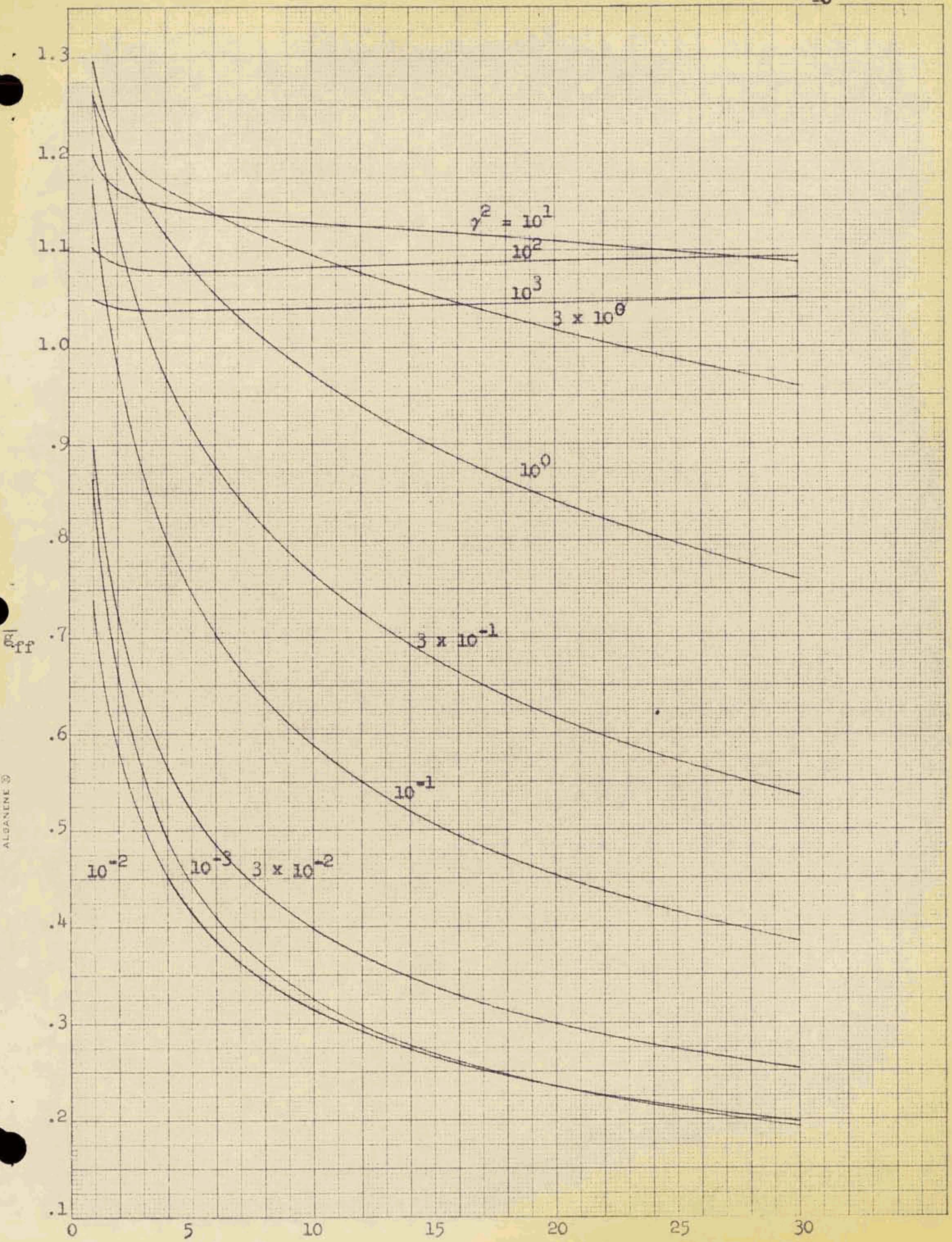
1.3
1.2
1.1
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
$\bar{g}_{ff}$
$\gamma^2 = 10^1$
10^2
10^3
3×10^0
10^0
3×10^{-1}
10^{-1}
10^{-2}
10^{-3}
3×10^{-2}
0
5
10
15
20
25
30
u

Figure 5

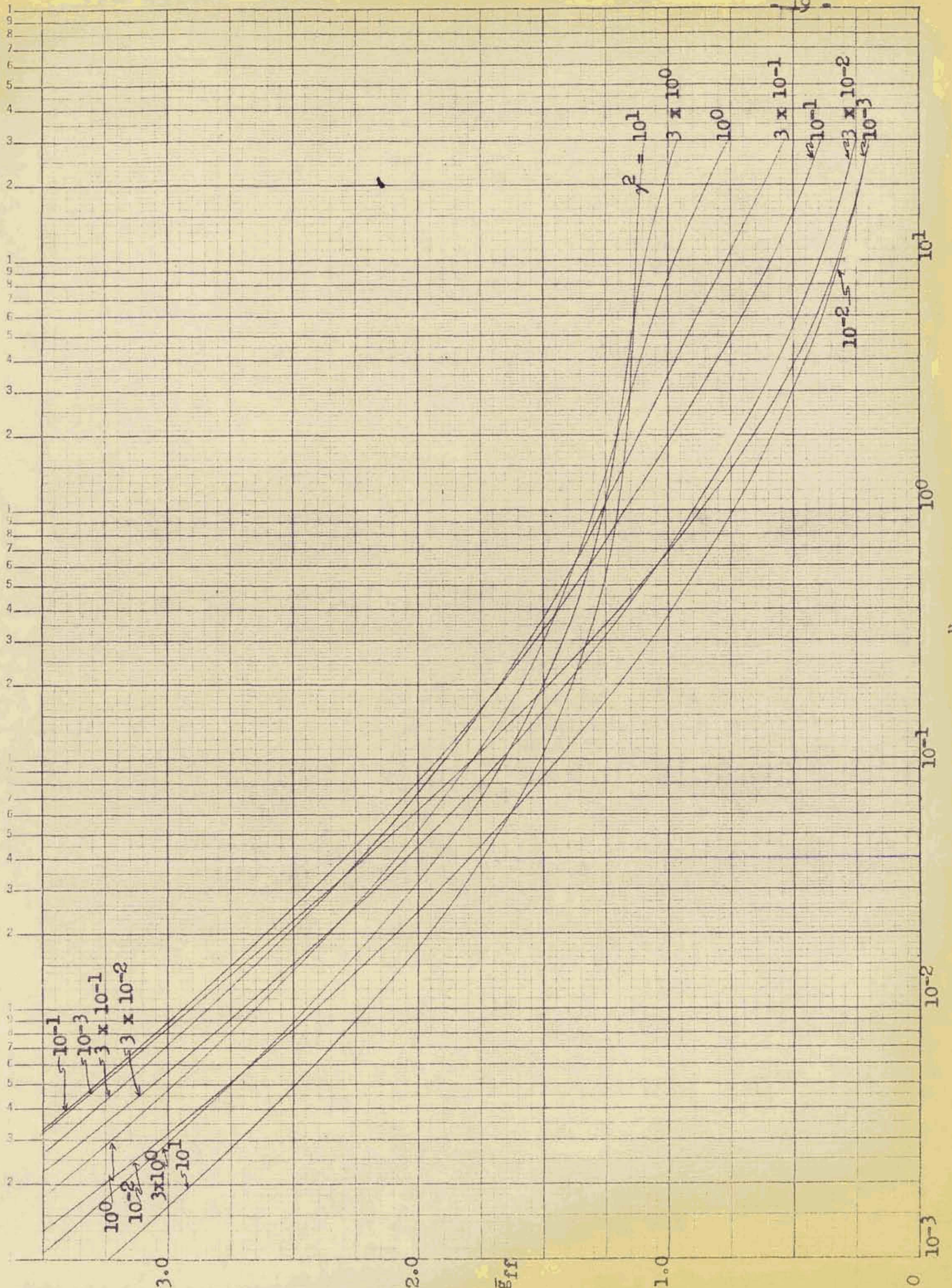
$\gamma^2 = 10^1$
3×10^0
10^0
3×10^{-1}
10^{-1}
3×10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^{-3}
3×10^{-1}
3×10^{-2}
10^0
10^{-2}
3×10^0
10^1
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
u
3.0
2.0
$\bar{\sigma}_{ff}$
1.0
0
SEMI-LOGARITHMIC 359-91
KEUFFEL & ESSER CO.
5 CYCLES X 70 DIVISIONS

Figure 6