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STABILITY OF HYDROMAGNETIC EQUILIBRIA WITH HELICALLY INVARIANT FIELDS

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Project Matterhorn
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Abstract

An equilibrium situation is obtained for the case of a uniform axial magnetic field with the addition of a superposition of weak helically invariant fields, a field due to a small axial current and a field due to a low pressure infinitely conducting plasma. The energy principle is used to determine conditions for the stability of this equilibrium. Two basic results are obtained: (a) The conditions for the stability of a system which consists of a superposition of helical fields ($\underline{B}(r, \theta, z) = \underline{B}(r, u)$ where $u = \ell\theta - hz$) and "bulge" fields ($\ell = 0$) are determined. In particular, the condition for the inherent stability of a helical field with $\ell = 3$ is $\beta_{critical} = (\rho^{\delta}/R)^2$, where ρ^{δ} is the maximum distortion of the plasma surface from the radius R. (b) It is shown that the addition of a helical field with $\ell = 3$ can produce complete hydromagnetic stability when an axial current is flowing, and can also increase the Kruskal limit on the current for the $m = 1$ mode.

Section I - Introduction

In 1954, Teller[1] raised the question of the stability of the various Sherwood devices and gave strong intuitive reasons for suspecting them to be unstable. By the summer of 1955 the energy principle had been worked out and a treatment for general axisymmetric systems had been carried through. This treatment indicated that the stellarator would be unstable to instabilities in which the lines of magnetic force are essentially interchanged so that matter is carried out toward the walls of the system without the magnetic field energy being changed.

The nature of these interchanges led Spitzer[2] to suggest that these interchange instabilities could be stabilized by changing the pitch of the magnetic lines of force so that in such an interchange, the lines would have to be twisted and the magnetic energy in the system would be increased. A preliminary calculation, in which the twisting of the field lines was produced by an axial current on the surface of the plasma, was encouraging, and was reported[3] at the October 1955 Princeton conference on controlled thermonuclear reactors (CTR).

This method of stabilization is not applicable for a steady state machine, since an emf is necessary to drive the stabilization current. For this reason an investigation of systems in which the stabilizing field is produced by pole pieces placed at right angles to the plasma was begun. The pole pieces were rotated in space in order to make the magnetic field depend on z as well as θ, where r, θ, and z are cylindrical coordinates with axis parallel to the main field. A dependence on z is necessary in order to obtain an equilibrium situation. These fields were helically invariant and to lowest order (in themselves) were proportional to $\sin(\theta - hz)$

or $\cos(\theta - hz)$. It had been pointed out earlier (in February 1955) by H. Koenig[4] that a helical magnetic field of this type would produce a rotational transform, but it was not realized until later that this configuration would have a stabilizing effect because of the variation of transform angle with radial distance. The nature of equilibria in these helical fields was examined and necessary and sufficient conditions were obtained for the system to be stable to all perturbations $\underline{\xi}$ which are periodic over the helical period, $2\pi/h$. In the calculations the equilibria and the perturbations were expanded in the parameters β and δ, where β characterizes the magnitude of the material pressure and δ characterizes the magnitude of the crossed fields. If the order of β is the same as that of δ, $(\beta \sim \delta)$ the system is always unstable. If $\beta \sim \delta^2$, a critical condition for stability is obtained, while if $\beta \sim \delta^3$ or smaller, the system is always stable. These results obtained by Bernstein, Frieman, Johnson, Kruskal, Kulsrud, and Oberman were presented[5] at the June 1956 Gatlinburg CTR conference.

It was realized by Spitzer[6] at this meeting that the rate of change of the pitch of the magnetic lines, with increasing distance from the axis, could be increased by considering more general helically invariant fields which vary as $\cos(\ell\theta - hz)$. The minimization of δW in respect to all $\underline{\xi}$'s which are periodic over the helical period, $2\pi/h$, was carried through, and the stability conditions obtained for more general equilibria in which there is a superposition of these helically invariant fields with arbitrary values of ℓ and bulge fields $(\ell = 0)$. Higher critical values of β (for stability) were obtained.

The effect of these helical fields on the long-wave-length kink instability was then considered in the case where a small axial current is present. To achieve stabilization in any system by means of helical fields, it is necessary that the rotational transform angle, ι, produced by these fields by finite. Since in the present analysis both the helical fields and the axial current are taken to be infinitesimally small, it was necessary to make the length of the system (i.e., the length $2\pi/k$ over which the $\underline{\xi}$'s are periodic) infinitely long and to include an infinite number of helices in order to get stabilization. It was found that the kink instability is stabilized to some extent by these helical fields. Further, it was realized that in the absence of an axial current the system is less stable to these long wave length $\underline{\xi}$'s than to $\underline{\xi}$'s which are periodic over the helical period. These results obtained by Oberman, Kulsrud, Johnson, and Frieman were presented[7] at the February 1957 Berkeley CTR conference.

The methods of calculating equilibrium situations are discussed in Section II and equilibria are obtained for situations in which the pressure distribution is a parabolic function of r. It is assumed that the distortion of the field lines due to the helical fields is infinitesimally small. The twisting of the field lines is described by a function, ι, which is discussed in Appendix IIA. In Appendix IIB other quantities of interest for stability (e.g. V'') are calculated for these helical fields and related to ι.

In Section III the integral, δW, which arises in the treatment of stability by the energy principle[8], is expanded for the general situation. Its minimization is carried through for all components of the perturbation $\underline{\xi}$, except for ξ_r^o, and δW is expressed in terms of ξ_r^o.

In Section IV the final minimization over ξ_r^o is carried through for several special cases: First to be treated is the case of an axially symmetric system in which the ends are identified with a twist. Next we consider the case of a superposition of helically invariant fields with various values of ℓ including $\ell = 0$ (bulge fields), an arbitrary pressure distribution but no axial current. Then situations are discussed in which various distributions of axial current are present, but no helical fields. Finally, the case is treated in which a uniform axial current and a helical field with $\ell = 3$ are present.

Rationalized Gaussian units with $c = 1$ are used throughout this paper.

Section II - Equilibrium

We wish to examine the equilibrium values of the magnetic field $\underline{B}$, material pressure p, and electric current $\underline{j}$, for the case of an ideal plasma contained by a helically invariant magnetic field.[9]

In this first part we limit ourselves to the simple situation of a plasma filling a perfectly conducting tube of a simple helically invariant form, and we permit no net longitudinal currents in the plasma.

In part B we generalize the equilibrium to include the possibility of more complicated bounding surfaces, to permit the presence of longitudinal currents, and to allow a vacuum region to exist between the plasma and the perfectly conducting bounding surface.

Two appendices are included in which certain equilibrium quantities relevant to stability are discussed. In Appendix II A the rotational transform angle[10] for the equilibrium of part A is calculated. In Appendix II B the relation of the function Ψ to the fluxes around the tube is given, and the quantity $\gamma p\,(V''/V' + p'/\gamma p)\,(V''-p'L')/(V' + \gamma p L')$, which was found to be of importance in the axisymmetric situation,[8] is calculated for the equilibrium of part A, and shown to be related to the rotational transform angle.

Part A

The condition of helical invariance requires that in cylindrical coordinates

$$\underline{B} = \underline{B}\,(r, u)\,, \tag{1}$$

where $u = \ell\,\theta - h\,z$, ℓ integer. We thus note that

$$\frac{\partial}{\partial \theta} = \ell \frac{\partial}{\partial u} \; ; \; \frac{\partial}{\partial z} = - h \frac{\partial}{\partial u} \; . \tag{2}$$

The fields may be obtained from a scalar function $\Psi(r, u)$ satisfying

$$\underline{B} \cdot \nabla \Psi = B_r \Psi_r + \frac{1}{r} (\ell B_\theta - h r B_z) \Psi_u = 0 \, , \tag{3}$$

so that $\underline{B}$ and Ψ are related by

$$\Psi_u = \alpha r B_r \; ; \; \Psi_r = - \alpha (\ell B_\theta - h r B_z) \, , \tag{4}$$

where α is an arbitrary function.

The Maxwell equation

$$\nabla \cdot \underline{B} = 0 \tag{5}$$

is satisfied if

$$\Psi_u \alpha_r - \Psi_r \alpha_u = 0 \; , \tag{6}$$

and this relation is in turn satisfied if

$$\alpha = \alpha (\Psi) \text{ only } . \tag{7}$$

For simplicity we shall choose

$$\alpha (\Psi) = 1 \tag{8}$$

any other choice merely constituting a relabeling of the constant Ψ surfaces.

The condition for static equilibrium

$$\nabla p = \underline{j} \times \underline{B} \quad , \tag{9}$$

reads

$$\frac{\partial p}{\partial r} = j_\theta \; B_z - j_z \, B_\theta \quad , \tag{9a}$$

$$\frac{\ell \partial p}{r \partial u} = j_z \; B_r - j_r \; B_z \quad , \tag{9b}$$

$$-\frac{h \partial p}{\partial u} = j_r \; B_\theta \; - j_\theta \; B_r \quad . \tag{9c}$$

The additional Maxwell equation

$$\nabla \times \underline{B} = \underline{j} \quad , \tag{10}$$

reads

$$j_r = \frac{1}{r} \frac{\partial}{\partial u} \left(\ell B_z + h r \; B_\theta \right) \quad , \tag{10a}$$

$$j_\theta = -h \; \frac{\partial B_r}{\partial u} - \frac{\partial B_z}{\partial r} \quad , \tag{10b}$$

$$j_z = \frac{1}{r} \frac{\partial}{\partial r} \; r B_\theta - \frac{\ell}{r} \; \frac{\partial B_r}{\partial u} \quad . \tag{10c}$$

Let us define a scalar function f such that

$$f = \left(\ell B_z + h r \; B_\theta \right) \quad . \tag{11}$$

Then one readily obtains from (10a), (10b), and (10c),

$$j_r = \frac{f_u}{r} \quad , \tag{12a}$$

and

$$h r j_z - \ell j_\theta = f_r \quad . \tag{12b}$$

From (9b) and (9c) one finds

$$B_r (h r j_z - \ell j_\theta) + j_r (\ell B_\theta - hr B_z) = 0 \quad . \tag{13}$$

Using (4), (12a), and (12b), (13) becomes

$$f_r \Psi_u - f_u \Psi_r = 0 \, , \tag{14}$$

or

$$f = f(\Psi) \text{ only} \quad . \tag{15}$$

From (9) ,

$$\underline{B} \cdot \nabla p = 0 \, , \tag{16a}$$

and hence

$$\Psi_u p_r - \Psi_r p_u = 0 \quad , \tag{16b}$$

and

$$p = p(\Psi) \text{ only} \quad . \tag{17}$$

We may now summarize our results obtained so far in the following equations:

$$B_r = \frac{\Psi_u}{r} \quad , \tag{18a}$$

$$B_\theta = \frac{h r f - \ell \Psi_r}{\ell^2 + h^2 r^2} \quad , \tag{18b}$$

$$B_z = \frac{h r \Psi_r + \ell f}{\ell^2 + h^2 r^2} \quad , \tag{18c}$$

$$\ell B_\theta - h r B_z = - \Psi_r \quad , \tag{18d}$$

$$h r B_\theta + \ell B_z = f \quad ; \tag{18e}$$

$$j_r = \frac{f_u}{r} = \frac{f' \Psi_u}{r} \quad , \tag{18f}$$

$$j_\theta = - h r \overline{L} - \frac{\ell f' \Psi_r}{\ell^2 + h^2 r^2} + \frac{2 h^2 r \ell f}{(\ell^2 + h^2 r^2)^2} \quad , \tag{18g}$$

$$j_z = - \ell \overline{L} + \frac{h r f' \Psi_r}{\ell^2 + h^2 r^2} + \frac{2 h \ell^2 f}{(\ell^2 + h^2 r^2)^2} \quad , \tag{18h}$$

$$\ell j_\theta - h r j_z = - f_r = - f' \Psi_r \quad , \tag{18i}$$

$$h r j_\theta + \ell j_z = - (\ell^2 + h^2 r^2) \overline{L} + \frac{2 \ell h f}{(\ell^2 + h^2 r^2)} \quad , \tag{18j}$$

and

$$\overline{L} + \frac{ff'}{\ell^2 + h^2 r^2} - \frac{2 h \ell f}{(\ell^2 + h^2 r^2)^2} = - p' , \tag{18k}$$

where

$$L \equiv \frac{1}{r} \frac{\partial}{\partial r} \frac{r \Psi_r}{\ell^2 + h^2 r^2} + \frac{\Psi_{uu}}{r^2} . \tag{19}$$

It is readily seen that when Ψ & f satisfying (18k) are given, fields $\underline{j}$ and $\underline{B}$ satisfying Maxwell's equation are in hand. If p (Ψ) is given, (18k) gives one relation between Ψ and f. We need yet another to fix them uniquely. We shall take as the additional relation[11, 12]

$$\int_0^{2\pi} \int_0^{r(\Psi, \theta, z)} j_z \, r \, d r \, d \theta = 0 , \tag{20}$$

that is, there shall be no net longitudinal current in the volume contained by <u>any</u> surface of constant Ψ.

Since in general we cannot easily solve (18k) and (20), we shall resort to perturbation theory techniques, and write all field quantities as power series in two independent small parameters δ and β which give a measure to the magnitude of the helical fields and the amount of plasma present, respectively. We shall introduce δ and β as follows:

Let us imagine a perfectly conducting right circular cylinder of radius R in which there is a uniform longitudinal magnetic field B^o and in which no plasma is present. This we call the zero-order situation.

Now imagine the boundary of the cylinder deformed to

$$r(\theta, z) = R + \rho^{\delta} \cos u , \tag{21}$$

and define

$$\delta \equiv \frac{\rho^{\delta}}{R} \quad . \tag{22}$$

We shall introduce matter in such a way that

$$p(\Psi) = \beta\,(P_o^{\beta} - \alpha^{\beta}\; g\,(\Psi)\,) \quad , \tag{23}$$

where P_o^{β} and α^{β} are constants of order β . We shall see how (23) defines β in (67). (It is clear that p must vanish on the boundary if there is to be confinement.)

We now write all field quantities as expansions (at a fixed point) in δ and β :

$$\Psi = \Psi^{o} + \delta\,\Psi^{\delta} + \beta\,\Psi^{\beta} + \delta^{2}\,\Psi^{\delta\,\delta} + \delta\beta\,\Psi^{\beta\,\delta} + \beta^{2}\,\Psi^{\beta\beta} + \ldots \,, \tag{24}$$

$$f\,(\Psi) = f^{o}\,(\Psi) + \delta\, f^{\delta}\,(\Psi) + \beta f^{\beta}\,(\Psi) + \delta^{2} f^{\delta\,\delta}\,(\Psi) + \ldots \,, \tag{25a}$$

$$\begin{aligned} &= f^{o}\,(\Psi^{o}) + \delta\,[f^{\delta}\,(\Psi^{o}) + f^{o\,\prime}(\Psi^{0})\Psi^{\delta}] + \beta\,[f^{\beta}\,(\Psi^{o}) + f^{o\,\prime}\,(\Psi^{o})\Psi^{\beta}\,] \\ &\quad + \delta^{2}\,[f^{\delta\,\delta}\,(\Psi^{o}) + \Psi^{\delta} f^{\delta\,\prime}(\Psi^{o}) + \Psi^{\delta\,\delta}\, f^{o\,\prime}(\Psi^{o}) + \frac{{\Psi^{\delta}}^{2}}{2}\; f^{o\,\prime\prime}(\Psi^{o})\,] \\ &\quad + \ldots\ldots \,, \end{aligned} \tag{25b}$$

$$p\,(\Psi) = \beta\,(\underline{P}_o^{\beta} - \alpha^{\beta}\, g^{o}\,(\Psi^{o})) - \beta\delta\,\alpha^{\beta}(g^{\delta}\,(\Psi^{o}) + g^{o\,\prime}(\Psi^{o})\;\Psi^{\delta}\,) - \ldots \,, \tag{26}$$

and so on for all field quantities. (In the future we shall suppress the writing of the powers of the parameters δ and β as coefficients of the terms in the above expansions, a common practice in perturbation theory, and regard these coefficients as adsorbed into the terms themselves.)

We now proceed to solve (18) order by order.

Zero-order Fields

Here we require

$$B_r^{o} = B_\theta^{o} = 0\ ;\ B_z^{o} = B^{o} = \text{const.} \tag{27}$$

From (18abc)

$$\Psi_u^{o} = 0 \quad , \tag{27a}$$

$$\Psi_r^{o} = \frac{hrf^{o}}{\ell} \quad , \tag{27b}$$

and

$$\frac{hr\Psi_r^{o} + \ell f^{o}}{\ell^2 + h^2 r^2} = B^{o} \quad . \tag{27c}$$

Hence

$$f^{o} = \ell B^{o} \tag{28a}$$

and

$$\Psi^{o} = \frac{hr^2 B^{o}}{2} + C^{o} . \tag{28b}$$

The constant C^{o} is made zero by choosing

$$\Psi^{o}(0) = 0 \quad . \tag{29}$$

Vacuum Fields

We now show how to find the vacuum fields (no matter present) to every order in δ .

Here we set $p(\Psi) = 0$ and $\underline{j} = 0$ everywhere interior to our boundary surface. From (18f) and 18i)

$$f' = 0 \tag{30a}$$

so

$$f = \text{const.} \tag{30b}$$

From (18k) the equation satisfied by Ψ then reads

$$\bar{L} \equiv \frac{1}{r} \frac{\partial}{\partial r} \frac{r\Psi_r}{\ell^2+h^2r^2} + \frac{\Psi_{uu}}{r^2} = \frac{2h\ell f}{(\ell^2+h^2r^2)^2} \quad . \tag{31}$$

It is readily verified that the solution of the homogeneous equation

$$\bar{L} = 0 \quad , \tag{32a}$$

regular at the origin, is

$$\Psi = C - \sum_{n=1}^{\infty} A_n\, h\, r\, I'_{n\ell}(nhr) \cos(nu + \alpha_n) \,, \tag{32b}$$

where C, A_n, and α_n are constants and $I_m(x)$ is the modified Bessel function of the first kind.[13] A particular solution of (31) is easily found:

$$\Psi = \frac{hfr^2}{2\ell} \quad . \tag{33}$$

Therefore, the complete solution of (28) which is regular at the origin is

$$\Psi = C + \frac{hr^2f}{2\ell} - \sum_{n=1}^{\infty} A_n\, h\, r\, I'_{n\ell}(nhr) \cos(nu + \alpha_n) \,. \tag{34}$$

The numbers C and A_n are determined from the requirement that the normal component of $\underline{B}$ vanish on the boundary or, equivalently, from (3), that the boundary form a constant Ψ surface. This value of Ψ we shall fix by assigning the same value that was assigned to the cylindrical boundary in the absence of the δ perturbation. Accordingly, we write

$$\Psi^{o}(R) = \Psi(R + \rho^{\delta} \cos u) \quad , \tag{35}$$

which reads after expansion in powers of δ.

$$\Psi^{o}(R) = \Psi^{o}(R) + [\Psi^{\delta}(R) + \rho^{\delta} \cos u \, \Psi_r{}^{o}(R)]$$
$$+ [\Psi^{\delta\delta}(R) + \rho^{\delta} \cos u \, \Psi_r{}^{\delta}(R) + \frac{\rho^2}{2} \cos^2 u \, \Psi_{rr}{}^{o}] + \ldots \tag{36}$$

Equation (34) represents the general solution of the vacuum field, hence we may clearly assume that if we write

$$\Psi_{vac} = \Psi^{o} + \Psi^{\delta} + \Psi^{\delta\delta} + \ldots \quad , \tag{37}$$

then every term of the right hand side of (37), which we denote generically, for the moment, by $\Psi^{(\nu)}$, may be assumed to be of the form,

$$\Psi^{(\nu)} = C^{(\nu)} + \frac{h r^2 f^{(\nu)}}{2\ell} - \sum_{n=1}^{\infty} A_n{}^{(\nu)} h r I'_{n\ell}(nhr) \cos nu \, . \tag{38}$$

Equation (36) is a power series in δ and hence every bracketed term on the right hand side of (36) must separately be zero. We shall choose

$$C^{o} = A_n^{o} = 0 \qquad n = 1, 2, 3, \ldots , \tag{39}$$

in order that we arrive at the zero order situation when δ goes to zero. Since, from (30), f is constant to every order in δ, we may choose, without any essential loss of generality,

$$f^{(\nu)} = 0 \qquad \nu = 1, 2, 3, \ldots , \tag{40}$$

any other choice of $f^{(\nu)}$ leading to a relabeling of the Ψ surfaces (from (38)). (Having chosen $f^{(\nu)} = 0$, we may not in general choose $C^{(\nu)} = 0$ and at the same time satisfy (36). Equation (38) then shows that, in general, this choice for $f^{(\nu)}$ leads to $\Psi\,(r = 0, u) \neq 0$. In some respects the choice $C^{(\nu)} = 0$ (and thus $\Psi^{(\nu)}(r = 0, u) = 0$) would be better, but the fact that now we would have to choose $f^{(\nu)} \neq 0$ in general, that (36) be satisfied, leads to more complicated expressions for the fields.)

From (28b), (36), (38), and (40),

$$C^{\delta} - \sum_{n=1}^{\infty} A_n\, hR\, I'_{n\ell}(nhR) \cos nu = - \rho^{\delta}\, hR\, B^{o} \cos u . \tag{41}$$

Therefore,

$$C^{\delta} = 0 , \tag{42a}$$

$$A_n^{\delta} = 0 , \qquad n \neq 1 \tag{42b}$$

$$A_1^{\delta} = \frac{\rho^{\delta} B^{o}}{I'_{\ell}(hR)} , \tag{42c}$$

and hence

$$\Psi^{\delta} = - B^{o} \rho^{\delta} h r \frac{I_\ell' (hr)}{I_\ell' (hR)} \cos u \quad . \tag{43}$$

From (18abc)

$$B_r^{\delta} = \delta B^{o} X \frac{I_\ell' (x)}{I_\ell' (X)} \sin u \quad , \tag{44a}$$

$$B_\theta^{\delta} = \delta \ell B^{o} X I^{\ell}(X) \frac{R}{r} \frac{I_\ell (x)}{I_\ell (X)} \cos u \, , \tag{44b}$$

and

$$B_z^{\delta} = - \delta B^{o} X^2 I^{\ell} (X) \frac{I_\ell (x)}{I_\ell (X)} \cos u \, , \tag{44c}$$

where

$$\delta = \frac{\rho^{\delta}}{R} \quad , \tag{45a}$$

$$x = h r \quad , \tag{45b}$$

$$X = h R \quad , \tag{45c}$$

and

$$I^{\ell}(y) = \frac{I_\ell (y)}{y I_\ell' (y)} \quad . \tag{45d}$$

In the limit $x \leq X << 1$, (44abc) yield to lowest order in X,

$$B_r^{\delta} = \delta X B^{o} (\frac{r}{R})^{\ell - 1} \sin u \, , \tag{46a}$$

$$B_\theta^{\delta} = \delta X B^{o} (\frac{r}{R})^{\ell - 1} \cos u \, , \tag{46b}$$

$$B_z^{\delta} = -\frac{\delta X^2}{\ell} B^o \left(\frac{r}{R}\right)^{\ell} \cos u \; . \tag{46c}$$

In like fashion we find

$$\Psi^{\delta\delta}(r,u) = -\frac{\delta^2 X^2}{4h} B^o [1-2(\ell^2+X^2)I^{\ell}(X)]\left[1+\frac{r}{R}\frac{I'_{2\ell}(2x)}{I'_{2\ell}(2X)}\cos 2u\right], \tag{47}$$

$$B_r^{\delta\delta}(r,u) = \frac{\delta^2}{2} X B^o [1-2(\ell^2+X^2)I^{\ell}(X)]\frac{I'_{2\ell}(2x)}{I'_{2\ell}(2X)} \sin 2u \quad , \tag{48a}$$

$$B_\theta^{\delta\delta} = \ell\delta^2 X B^o I^{2\ell}(2X)[1-2(\ell^2+X^2)I^{\ell}(X)]\frac{R}{r}\frac{I_{2\ell}(2x)}{I_{2\ell}(2X)} \cos 2u, \tag{48b}$$

$$B_z^{\delta\delta} = -\delta^2 X^2 B^o I^{2\ell}(2X)[1-2(\ell^2+X^2)I^{\ell}(X)]\frac{I_{2\ell}(2x)}{I_{2\ell}(2X)} \cos 2u \; . \tag{48c}$$

In the small X limit these expressions reduce to

$$B_r^{\delta\delta} = -\frac{(2\ell-1)}{2}\delta^2 X B^o \left(\frac{r}{R}\right)^{2\ell-1} \sin 2u \quad , \tag{49a}$$

$$B_\theta^{\delta\delta} = -\frac{(2\ell-1}{2}\delta^2 X B^o \left(\frac{r}{R}\right)^{2\ell-1} \cos 2u \quad , \tag{49b}$$

$$B_z^{\delta\delta} = \frac{(2\ell-1)}{2\ell}\delta^2 X^2 B^o \left(\frac{r}{R}\right)^{2\ell} \cos 2u \; , \tag{49c}$$

In precisely similar fashion, we may use (36) and (38) to write down the fields to any given order in δ .

Matter Fields with $\delta = 0$

Here we set $\delta = 0$, and hence all field quantities shall be independent of u. We shall assume for the present that $g(\Psi)$ in (23) has the simple form

$$g(\Psi) = \Psi . \tag{50}$$

From (18a) and (18f)

$$B_r = j_r = 0 . \tag{51}$$

From (18h) and (18k)

$$j_z = - \ell\alpha^\beta + \frac{\ell f f'}{\ell^2 + h^2 r^2} + \frac{h r f' \Psi_r}{\ell^2 + h^2 r^2} . \tag{52}$$

To order β

$$j_z^\beta = - \ell\alpha^\beta + \frac{\ell f^o f^{\beta\prime}(\Psi^o)}{\ell^2 + h^2 r^2} + \frac{h r f^{\beta\prime}(\Psi_o)\Psi_r{}^o}{\ell^2 + h^2 r^2} . \tag{53}$$

By means of (27c), (53) reduces to

$$j_z^\beta = - \ell\alpha^\beta + \frac{f^{\beta\prime}(\Psi^o) f^o}{\ell} . \tag{54}$$

If we write the equation of a constant Ψ surface as

$$r(\Psi) = \eta^o(\Psi) + \eta^\beta(\Psi) + \eta^{\beta\beta}(\Psi) + \ldots \tag{55}$$

$$= \eta^o(\Psi^o) + [\eta^\beta(\Psi^o) + \Psi^\beta \eta^{o\prime}(\Psi^o)] + \ldots, \tag{56}$$

then condition (20) yields, to order β,

$$2\pi \int_0^{\eta_o} j_z^{\beta} \, r \, dr = 0 \quad , \tag{57}$$

or

$$\int_0^{\eta_o} [-\ell\alpha^{\beta} + \frac{f^o f^{\beta\prime}(\Psi^o)}{\ell}] \, r \, dr = 0 \; . \tag{58}$$

Since Ψ^o is a function of r only and since we require (58) to be true for all values of η^o, the integrand itself must vanish and hence

$$f^{\beta\prime}(\Psi^o) = \frac{\ell^2 \alpha^{\beta}}{f^o} \quad , \tag{59}$$

and thus

$$f^{\beta}(r) = \frac{h\ell\alpha^{\beta} r^2}{2} \quad . \tag{60}$$

We have found f^{β} and may now solve (18k),

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{r\Psi_r^{\beta}}{\ell^2+h^2r^2} + \frac{\ell^2\alpha^{\beta}}{\ell^2+h^2r^2} - \frac{h^2\ell^2\alpha^{\beta}r^2}{(\ell^2+h^2r^2)^2} = \alpha^{\beta} \, , \tag{61}$$

for Ψ^{β}.

The solution of (61) which is regular at the origin and satisfies $\Psi^{\beta}(R) = 0$ (see discussion following equation (34)) is easily found:

$$\Psi^{\beta} = \frac{h^2\alpha^{\beta}}{8}(r^4 - R^4) \quad . \tag{62}$$

We may now write down the fields to order β using (18abc):

$$B_r^{\beta} = B_{\theta}^{\beta} = 0 \quad , \tag{63}$$

$$B_z^{\beta} = \frac{h r^2 \alpha^{\beta}}{2} = \frac{\beta B^o}{2} \frac{r^2}{R^2} , \tag{64}$$

$$j_r^{\beta} = j_z^{\beta} = 0 , \tag{65a}$$

$$j_{\theta}^{\beta} = - \alpha^{\beta} h r = - \frac{\beta B^o r}{R^2} , \tag{65b}$$

$$p^{\beta} = \frac{\alpha^{\beta} h B^o}{2} (R^2 - r^2) = \frac{\beta B^{o\,2}}{2} \frac{(R^2 - r^2)}{R^2} , \tag{66}$$

where we have now defined β such that

$$\beta = \frac{2 p^{\beta}(r=0)}{B^{o\,2}} = \frac{\alpha^{\beta} h R^2}{B^o} . \tag{67}$$

We could proceed to find in a similar fashion the higher order terms in β .

Interaction between Matter and Vacuum Fields

We shall write down the interaction terms of lowest order, those of order $\beta\delta$.

From (18h), (18k), and (27)

$$\frac{1}{r} \frac{\partial}{\partial r} \frac{r \Psi_r^{\beta\delta}}{\ell^2 + h^2 r^2} + \frac{\Psi_{uu}^{\beta\delta}}{r^2} - \frac{2h\ell (f^{\beta\delta} + \Psi^{\delta} f^{\beta\prime}(\Psi^o))}{(\ell^2 + h^2 r^2)^2} + \frac{f^o f^{\beta\delta\prime}(\Psi^o)}{\ell^2 + h^2 r^2} = 0 , \tag{68}$$

and

$$j_z^{\beta\delta} = \frac{f^{\beta\delta\prime}(\Psi^o) f^o}{\ell} + \frac{h r f^{\beta\prime}(\Psi^o)}{\ell^2 + h^2 r^2} \Psi_r^{\delta} . \tag{69}$$

The condition (20) yields to order $\beta\,\delta$,

$$\int_0^{2\pi}\int_0^{\eta_0} j_z^{\beta\delta}\, r\,dr\,d\theta = 0 \quad . \tag{70}$$

When (69) is inserted into (70), we arrive at

$$f^{\beta\delta} = \text{constant} \quad , \tag{71}$$

after noting the second term on the right hand side of (69) vanishes upon integration over θ and using arguments on the first term entirely analogous to those used in arriving at (59).

Since Ψ^{δ} and f^{β} are known we may now solve (68), obtaining

$$\Psi^{\beta\delta} = C^{\beta\delta} + \frac{h r^2 f^{\beta\delta}}{2\ell} + \frac{\ell^2 \alpha^{\beta}}{h}\, \rho^{\delta}\, \frac{I_{\ell}(h r)}{I_{\ell}'(h R)} \cos u$$

$$- \sum_{n=1}^{\infty} A_n^{\beta\delta}\, h r\, I_{n\ell}'(n h r) \cos n u \quad . \tag{72}$$

The condition (35) yields to order $\beta\delta$

$$\Psi^{\beta\delta}(R) + \rho^{\delta} \cos u\, \Psi_r^{\beta}(R) = 0 \ , \tag{73}$$

and now the coefficients in (61) are specified. Again, we shall choose $f^{\beta\delta}$ equal to zero, without essential loss of generality, and thus write

$$\Psi^{\beta\delta} = \frac{\delta\beta B^{o}}{h}\left[\ell^2 I^{\ell}(X)\frac{I_\ell(x)}{I_\ell(X)} - \frac{1}{2}\left(2\ell^2 I^{\ell}(X) + X^2\right)\frac{r}{R}\frac{I_\ell'(x)}{I_\ell'(X)}\right]\cos u . \qquad (74)$$

Thus,

$$B_r^{\beta\delta} = \frac{\Psi_u^{\beta\delta}}{r} = \delta\beta\, B^{o}\left\{-\frac{\ell^2}{x} I^{\ell}(X)\frac{I_\ell(x)}{I_\ell(X)} + \frac{1}{2X}\left(2\ell^2 I^{\ell}(X) + X^2\right)\frac{I_\ell'(x)}{I_\ell'(X)}\right\}\sin u, \qquad (75a)$$

$$B_\theta^{\beta\delta} = \frac{h\, r\left(f^{\beta\delta}(\Psi^{o}) + f^{\beta'}(\Psi^{o})\Psi^{\delta}\right) - \ell\Psi_r^{\delta}}{\ell^2 + h^2 r^2}$$

$$= \ell\delta\beta\, B^{o}\left\{-\frac{I_\ell'(x)}{X I_\ell'(X)} + \frac{I_\ell(x)}{2x I_\ell(X)} I^{\ell}(X)\left(2\ell^2 I^{\ell}(X) + X^2\right)\right\}\cos u, \qquad (75b)$$

and

$$B_z^{\beta\delta} = -\frac{1}{2}\delta\beta B^{o}\left(2\ell^2 I^{\ell}(X) + X^2\right) I^{\ell}(X)\frac{I_\ell(x)}{I_\ell(X)}\cos u . \qquad (75c)$$

From (69), (18f), and (18i)

$$j_r^{\beta\delta} = \ell\beta\delta\, B^{o}\frac{h}{X}\frac{I_\ell'(x)}{I_\ell'(X)}\sin u , \qquad (76a)$$

$$j_\theta^{\beta\delta} = \ell^2\beta\delta\, B^{o} h I^{\ell}(X)\frac{I_\ell(x)}{x I_\ell(X)}\cos u , \qquad (76b)$$

and

$$j_z^{\beta\delta} = -\ell\beta\delta\, B^{o} h I^{\ell}(X)\frac{I_\ell(x)}{I_\ell(X)}\cos u . \qquad (76c)$$

From (23) and (50) we have

$$p^{\beta\delta} = -\alpha^{\beta}\Psi^{\delta} = \beta\delta\, B^{o}\, \frac{x}{X}\, \frac{I_{\ell}'(x)}{I_{\ell}'(X)}\, \cos u . \qquad (77)$$

We have now found the field quantities to sufficiently high order for subsequent work. It is clear how to proceed to obtain field quantities of higher order.

Part B

At this point it is desirable to generalize the nature of the equilibrium in several respects.

We wish to include the effect of "bulges" in the field lines which, for instance, may be due to the finite spacing between coils producing the main B^{o} field. We wish also to consider the case where the plasma aperature is smaller than that of the containing tube, that is, we shall assume there exists a vacuum region surrounding the plasma. In addition, it is of interest to consider the case when longitudinal currents exist in the plasma (e.g. during the ohmic heating process). This means we abandon (20).

The use of a Ψ function, as in the previous case, is now complicated by the presence of fields which are not helically invariant. Since it is the equilibrium values of the field quantities $\underline{B}$, $\underline{j}$, and p we are primarily interested in, it is perhaps more straightforward to solve for them directly using (5), (9) and (10) together, of course, with the appropriate jump conditions on the field quantities at the plasma-vacuum interface, and the outer boundary.

The zero-order situation we shall take to be again

$$B_r^o = B_\theta^o = 0, \ B_z = B^o = \text{const.}, \ r \leq S \,. \tag{78}$$

The helical (δ) field and bulge (ϵ) field shall be introduced by imagining the perfectly conducting tube deformed from a circular one of radius S , to

$$r(\theta, z) = S + \sigma^\delta \cos u + \sigma^\epsilon \cos h_\epsilon z \,, \tag{79}$$

where

$$u = \ell\theta - h_\delta z \quad . \tag{80}$$

We now shall define $\delta = \frac{\sigma^\delta}{S}$ and $\epsilon = \frac{\sigma^\epsilon}{S}$, and assume for simplicity $h_\epsilon \neq h_\delta$. The parameters β and η , which shall give measure to the amount of matter present and the magnitude of the heating current, will be defined in a moment. (It is clear that when β, ϵ, and η all equal zero the δ-fields which we arrived at earlier in this section shall prevail (with S replacing R)).

We now write

$$\underline{B} = \underline{B}^o + \underline{B}^\beta + \underline{B}^\delta + \underline{B}^\epsilon + \underline{B}^\eta + \underline{B}^{\beta\delta} + \ldots \,, \tag{80a}$$

with similar expressions for $\underline{j}$, p , etc. The equation of a normal vector at the outer boundary is

$$\underline{n} = \nabla (r - S - \sigma^\delta \cos u - \sigma^\epsilon \cos h_\epsilon z) \,, \tag{80b}$$

(where we have not normalized $\underline{n}$ since the condition $\underline{B} \cdot \underline{n} = 0$ on the boundary is the only condition imposed, and clearly this is independent of the magnitude of $\underline{n}$). Thus we may write

$$\underline{n} = \underline{n}^{o} + \underline{n}^{\delta} + \underline{n}^{\epsilon} + \underline{n}^{\epsilon\delta} + \ldots , \tag{80c}$$

$$\underline{n} = \underline{e}_r + \delta \ell \sin u \, \underline{e}_\theta - \delta h_\delta S \sin u \, \underline{e}_z$$

$$+ \epsilon h_\epsilon S \sin h_\epsilon z \, \underline{e}_z + \ldots \quad . \tag{80d}$$

The equation of the plasma-vacuum interface we shall take to be

$$r = R + \rho = R + \rho^{\delta}(\theta, z) + \rho^{\epsilon}(z) + \rho^{\beta}(\theta, z) + \rho^{\eta}(\theta, z) + \rho^{\epsilon\delta} + \ldots , \tag{80e}$$

with the unit (outward) normal

$$\underline{n} = \frac{\nabla (r - R - \rho)}{|\nabla (r - R - \rho)|} , \tag{80f}$$

or

$$\underline{n} = \frac{\underline{e}_r - \frac{1}{r}\frac{\partial \rho}{\partial \theta}\underline{e}_\theta - \frac{\partial \rho}{\partial z}\underline{e}_z}{[1 + \frac{1}{r^2}(\frac{\partial \rho}{\partial \theta})^2 + (\frac{\partial \rho}{\partial z})^2]^{1/2}} , \tag{80g}$$

which yields after expansion,

$$\underline{n} = \underline{e}_r - \frac{\underline{e}_\theta}{R}\left(\frac{\partial \rho^{\delta}}{\partial \theta} + \frac{\partial \rho^{\epsilon}}{\partial \theta} + \frac{\partial \rho^{\beta}}{\partial \theta} + \frac{\partial \rho^{\eta}}{\partial \theta}\right)$$

$$- \underline{e}_z\left(\frac{\partial \rho^{\delta}}{\partial z} + \frac{\partial \rho^{\epsilon}}{\partial z} + \frac{\partial \rho^{\beta}}{\partial z} + \frac{\partial \rho^{\eta}}{\partial z}\right) \quad +$$

$$+ \underline{e}_\theta \frac{(\rho^\delta + \rho^\epsilon + \rho^\beta + \rho^\epsilon)}{R^2} \left(\frac{\partial\rho^\delta}{\partial\theta} + \frac{\partial\rho^\epsilon}{\partial\theta} + \frac{\partial\rho^\beta}{\partial\theta} + \frac{\partial\rho^\eta}{\partial\theta}\right)$$

$$- \frac{\underline{e}_\theta}{R} \left(\frac{\partial\rho^{\delta\delta}}{\partial\theta} + \frac{\partial\rho^{\delta\epsilon}}{\partial\theta} + \frac{\partial\rho^{\epsilon\epsilon}}{\partial\theta} + \ldots\right)$$

$$- \underline{e}_z \left(\frac{\partial\rho^{\delta\delta}}{\partial z} + \frac{\partial\rho^{\delta\epsilon}}{\partial z} + \frac{\partial\rho^{\epsilon\epsilon}}{\partial z} + \ldots\right) \tag{80h}$$

$$- \frac{\underline{e}_r}{2} \left[\frac{1}{R^2} \left(\frac{\partial\rho^\delta}{\partial\theta} + \frac{\partial\rho^\epsilon}{\partial\theta} + \ldots\right)^2 + \left(\frac{\partial\rho^\delta}{\partial z} + \frac{\partial\rho^\epsilon}{\partial z} + \ldots\right)^2\right] + \ldots \quad .$$

The quantity ρ is determined from the conditions

$$\underline{B} \cdot \underline{n} = 0 \tag{80i}$$

and $p + \frac{B^2}{2}$ continuous at the plasma-vacuum interface. We shall now compute the fields.

β-order

Here

$$\frac{\partial p^\beta}{\partial r} = j_\theta^{\ \beta} B^o \quad , \tag{81a}$$

$$\frac{\partial p^\beta}{r\partial\theta} = - j_r^{\ \beta} B^o \quad , \tag{81b}$$

$$\frac{\partial p^\beta}{\partial z} = 0 \quad , \tag{81c}$$

and

$$j_r^{\beta} = \frac{1}{r}\frac{\partial}{\partial\theta} B_z^{\beta} - \frac{\partial B_\theta^{\beta}}{\partial z} \quad , \tag{82a}$$

$$j_\theta^{\beta} = \frac{\partial B_r^{\beta}}{\partial z} - \frac{\partial B_z^{\beta}}{\partial r} \quad , \tag{82b}$$

$$j_z^{\beta} = \frac{1}{r}\frac{\partial}{\partial r} r B_\theta^{\beta} - \frac{1}{r}\frac{\partial B_r^{\beta}}{\partial\theta} \quad , \tag{82c}$$

and

$$\nabla \cdot \underline{B} = 0 \quad . \tag{83}$$

The boundary condition that $\underline{B} \cdot \underline{n}$ be zero on the plasma-vacuum interface and at the outer boundary reads, to order β,

$$B_r^{\beta}\Big|_{r=R} = 0 = B_r^{\beta}\Big|_{r=S} \quad . \tag{84}$$

The condition of continuity of the total pressure, magnetic plus material, across the interface reads, to order β,

$$B^o \bar{B}_z^{\beta}\Big|_{r=R} = (p^{\beta} + \bar{B}^o B_z^{\beta})\Big|_{r=R} \quad , \tag{85}$$

where we have designated quantities outside the plasma region by bars, and where we have made use of the fact that the material pressure vanishes outside the plasma region.

For the sake of simplicity we shall assume at once

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial z} = 0 \quad . \tag{86}$$

Therefore, (81abc) read,

$$\frac{\partial p^\beta}{\partial r} = j_\theta^\beta B^o \quad , \tag{87a}$$

$$\frac{\partial p^\beta}{r \partial \theta} = - j_r^\beta B^o = 0 \quad , \tag{87b}$$

$$\frac{\partial p^\beta}{\partial z} = 0 \quad , \tag{87c}$$

and (82 abc) read

$$j_r^\beta = 0 \quad , \tag{88a}$$

$$j_\theta^\beta = - \frac{\partial B_z^\beta}{\partial r} \quad , \tag{88b}$$

$$j_z^\beta = \frac{1}{r} \frac{\partial}{\partial r} r B_\theta^\beta \quad . \tag{88c}$$

Equation (83) reads

$$\frac{1}{r} \frac{\partial}{\partial r} r B_r^\beta = 0 \quad , \tag{89}$$

and hence

$$B_r^{\beta} = 0 \tag{90}$$

inside and outside the plasma, when regularity at the origin is required and (84) is invoked.

From (87a) j_{θ}^{β} is known when $p^{\beta}(r)$ is known, and hence from (88b)

$$B_z^{\beta}(r) = - \int j_{\theta}^{\beta}\, dr + \text{const.} \quad , \tag{91}$$

and

$$\bar{B}_z^{\beta}(r) = \text{const.} \tag{92}$$

We now make the following observation. Because of our choice of the zero-order situation (only $B_z^{o} \neq 0$), we find that j_z^{β} and B_{θ}^{β}, connected by (88c), are not coupled to the material pressure in any way, either by (84) or (85), and hence are really independent of β, which we shall attach to the material pressure in a moment. We shall, therefore, choose

$$j_z^{\beta} = B_{\theta}^{\beta} = 0 \quad , \tag{93}$$

and use the distinct parameter η to admit the existence of a j_z and B_{θ} independent of the pressure.

We shall for the present make a reasonable choice for $p^{\beta}(r)$ and adopt the parabolic profile

$$p^{\beta}(r) = C^{\beta}\left(1 - \frac{r^2}{R^2}\right) \quad , \tag{94}$$

where C^{β} is constant. From (87a)

$$j_{\theta}^{\beta} = \frac{1}{B^{o}} \frac{\partial p^{\beta}}{\partial r} = -\frac{2C^{\beta}}{B^{o}} \frac{r}{R^2} \quad , \tag{95}$$

and from (91) and (92)

$$B_z^{\beta} = \frac{C^{\beta}}{B^{o}} \frac{r^2}{R^2} + \text{const.} \quad , \tag{96a}$$

and

$$\overline{B}_z^{\beta} = \text{const.} \tag{96b}$$

We shall choose the constant of integration in (96a) equal to zero, and then, using (85) obtain

$$\overline{B}_z^{\beta} = \frac{C^{\beta}}{B^{o}} \quad . \tag{97}$$

We now define β by choosing

$$\beta = \frac{2p^{\beta}(0)}{B^{o2}} = \frac{2C^{\beta}}{B^{o2}} \quad , \tag{98a}$$

or

$$C^{\beta} = \beta \frac{B^{o\,2}}{2} \quad . \tag{98b}$$

Hence,

$$B_r^{\beta} = B_{\theta}^{\beta} = 0 = \bar{B}_r^{\beta} = \bar{B}_{\theta}^{\beta} \quad , \tag{99a}$$

$$B_z^{\beta} = \frac{\beta B^{o}}{2} \frac{r^2}{R^2} \quad , \tag{99b}$$

$$\bar{B}_z^{\beta} = \frac{\beta B^{o}}{2} \quad , \tag{99c}$$

$$j_r^{\beta} = j_z^{\beta} = 0 \, ; \quad \bar{\underline{j}}^{\beta} = 0 \quad , \tag{99d}$$

$$j_{\theta}^{\beta} = - \beta B^{o} \frac{r}{R^2} \quad , \tag{99e}$$

and

$$p^{\beta} = \frac{\beta B^{o\,2}}{2} \left(1 - \frac{r^2}{R^2}\right) \quad . \tag{99f}$$

ϵ - order

Here we put

$$p^{\epsilon} = 0 \quad , \tag{100a}$$

$$\underline{j}^{\epsilon} = 0 \quad , \tag{100b}$$

and

$$\frac{\partial}{\partial \theta} = 0 \tag{100c}$$

again because of invariance of boundary. ($\eta = \beta = \delta = 0$) against rotations about z-axis. Maxwell's equations now read

$$\frac{\partial B_\theta^\epsilon}{\partial z} = 0 \quad , \tag{101a}$$

$$\frac{\partial B_r^\epsilon}{\partial z} = \frac{\partial B_z^\epsilon}{\partial r} \quad , \tag{101b}$$

$$\frac{1}{r} \frac{\partial}{\partial r} r B_\theta^\epsilon = 0 \quad , \tag{101c}$$

and

$$\frac{1}{r} \frac{\partial}{\partial r} r B_r^\epsilon + \frac{\partial B_z^\epsilon}{\partial z} = 0 \quad . \tag{102}$$

From (100c), (101a), and (101c)

$$B_\theta^\epsilon = 0 \quad , \tag{103}$$

since we demand regularity at the origin. From (101b) and (102) we find

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial B_z^\epsilon}{\partial r} + \frac{\partial^2 B_z^\epsilon}{\partial z^2} = 0 \quad . \tag{104}$$

The solution of (104) regular everywhere in the region is immediately given by

$$B_z^{\epsilon} = \sum_{n=1}^{\infty} A_n^{\epsilon} I_o(nh_{\epsilon}r) \cos(nh_{\epsilon}z + \phi_n) \quad . \tag{105a}$$

From (101b) and (102) we find

$$B_r^{\epsilon} = \sum_{n=1}^{\infty} A_n^{\epsilon} I_o'(nh_{\epsilon}r) \sin(nh_{\epsilon}z + \phi_n) \tag{105b}$$

By (79) and (80b) the boundary condition $\underline{B} \cdot \underline{n} = 0$ on the outer boundary is, to order ϵ,

$$B_r^{\epsilon}(r=S,z) + B^o h_{\epsilon} \sigma^{\epsilon} \sin h_{\epsilon} z = 0 \ . \tag{106}$$

Therefore,

$$A_n^{\epsilon} = 0 \quad , \qquad n \neq 1 \tag{107a}$$

and

$$A_1^{\epsilon} = - \frac{B^o h_{\epsilon} \sigma^{\epsilon}}{I_o'(h_{\epsilon}S)} \quad . \tag{107b}$$

Accordingly, we write

$$B_r^{\epsilon} = B^o \epsilon h_{\epsilon} S \frac{I_o'(h_{\epsilon}r)}{I_o'(h_{\epsilon}S)} \sin(-h_{\epsilon}z) \ , \tag{108a}$$

$$B_\theta^\epsilon = 0 \quad , \tag{108b}$$

and

$$B_z^\epsilon = - B^o \epsilon h_\epsilon^2 S^2 I^o (h_\epsilon S) \frac{I_o(h_\epsilon r)}{I_o(h_\epsilon S)} \cos(-h_\epsilon z) \ , \tag{108c}$$

where, we remember,

$$\epsilon = \frac{\sigma^\epsilon}{S} \quad , \tag{109}$$

and where we have introduced $(-h_\epsilon z)$ as the argument of the circular functions rather than $h_\epsilon z$ in order to exhibit the parallelism to (44abc). In fact, these results could be obtained most quickly by setting $\ell = 0$ in the δ-order and substituting ϵ for δ .

η - order

We remark at once that the η fields are force free since we have chosen $p^\eta = p^{\eta\eta} = \cdots = 0$. Because of the same invariance as in the β - order and in consequence of the discussion following (92) we write

$$j_r^\eta = j_\theta^\eta = 0 \quad , \tag{110a}$$

$$B_r^\eta = B_z^\eta = 0 \quad , \tag{110b}$$

and

$$j_z^\eta = \frac{1}{r} \frac{\partial}{\partial r} r \ B_\theta^\eta (r) \quad . \tag{110c}$$

When j_z^η is known, B_θ^η is determined. We now consider two simple choices of the form of j_z^η. Case (a), j_z^η uniform, $r \leq R$, $\bar{j}_z^\eta = 0$; case (b), a surface current j_z^η at R.

Case (a), $j_z^\eta = \text{const.}, r \leq R, \bar{j}_z^\eta = 0$

The solution of the homogeneous equation

$$\frac{1}{r} \frac{\partial}{\partial r} r B_\theta^\eta(r) = 0 \tag{111}$$

is

$$B_\theta^\eta = \frac{C^\eta}{r} \quad . \tag{112}$$

A particular solution of (110c) when j_z is constant is

$$B_\theta^\eta = \frac{j_z^\eta}{2} r \quad . \tag{113}$$

Invoking regularity at the origin, we find

$$B_\theta^\eta = \frac{j_z^\eta}{2} r \quad , \tag{114a}$$

and

$$\bar{B}_\theta^\eta = \frac{C^\eta}{r} \quad . \tag{114b}$$

Since the tangential component of $\underline{B}$ is continuous in this case,

$$\bar{B}_\theta^\eta (R) = B_\theta^\eta (R) \quad , \tag{115}$$

or

$$C^{\eta} = \frac{j_z^{\eta} R^2}{2} \quad . \tag{116}$$

Thus

$$B_\theta^{\eta} = \eta \frac{r}{R} B^{o} \quad , \tag{117a}$$

and

$$\overline{B}_\theta^{\eta} = \eta B^{o} \frac{R}{r} \quad , \tag{117b}$$

where we have defined η by

$$\eta = \frac{\overline{B}_\theta^{\eta}(R)}{B^{o}} = \frac{j_z^{\eta} R}{2 B^{o}} \quad . \tag{118}$$

Case (b), sheet current at $r = R$

Clearly in this case

$$B_\theta^{\eta} = 0 \quad , \tag{119a}$$

$$\overline{B}_\theta^{\eta} = \frac{C^{\eta}}{r} = \eta B^{o} \frac{R}{r} \tag{119b}$$

and

$$j_z^{\eta} = \eta B^{o} \quad , \tag{119c}$$

where, again, we have defined

$$\eta = \frac{\overline{B}_\theta^{\eta}(R)}{B^{o}} \quad . \tag{120}$$

$\beta\delta$ - order

We shall piece together this order from our earlier work, since this situation is almost the same as the earlier $\beta\delta$ - order, with the exception that we have two distinct regions to consider, inside and outside the plasma.

Let us first consider the region outside the plasma, a vacuum region, for which we know the general helically invariant solution (34) applies. (However, the modified Bessel functions of the second kind[13] must be admitted since regularity at the origin is no longer required in this region.) Thus, the condition $\underline{B} \cdot \underline{n}$ equal zero on the outer boundary

$$\bar{B}_r^{\beta\delta}(S, u) - \sigma^\delta h_\delta \sin u \, \bar{B}_z^\beta(S) = 0 \quad , \tag{121}$$

enables us to write

$$\bar{B}_r^{\beta\delta} = [\bar{C}^{\beta\delta} h_\delta I_\ell'(h_\delta r) + \bar{D}^{\beta\delta} h_\delta K_\ell'(h_\delta r)] \sin u \quad , \tag{122a}$$

$$\bar{B}_\theta^{\beta\delta} = [\bar{C}^{\beta\delta} \frac{\ell}{r} I_\ell(h_\delta r) + \bar{D}^{\beta\delta} \frac{\ell}{r} K_\ell(h_\delta r)] \cos u \quad , \tag{122b}$$

$$\bar{B}_z^{\beta\delta} = [-\bar{C}^{\beta\delta} h_\delta I_\ell(h_\delta r) - \bar{D}^{\beta\delta} h_\delta K_\ell(h_\delta r)] \cos u \quad , \tag{122c}$$

with

$$\bar{C}^{\beta\delta} h_\delta I_\ell'(h_\delta S) + \bar{D}^{\beta\delta} h_\delta K_\ell'(h_\delta S) = \frac{\delta\beta h_\delta B^0}{2} \quad . \tag{123}$$

In the interior region we have the vacuum solution for the magnetic field plus a particular solution due to the currents $\underline{j}^{\beta}$. From (9c), (44a), and (99e), we find to this order

$$p^{\beta\delta} = \beta\delta B^{o\,2} \frac{S}{R^2} r \frac{I_\ell'(h_\delta r)}{I_\ell'(h_\delta S)} \cos u \quad , \tag{124}$$

and from (9ab)

$$j_r^{\beta\delta} = \beta\delta \ell B^o \frac{S}{R^2} \frac{I_\ell'(h_\delta r)}{I_\ell'(h_\delta S)} \sin u \quad , \tag{125a}$$

$$j_\theta^{\beta\delta} = \beta\delta \ell^2 B^o \frac{S^2}{R^2} \frac{I^\ell(h_\delta S)}{r} \frac{I_\ell(h_\delta r)}{I_\ell(h_\delta S)} \cos u \quad , \tag{125b}$$

and

$$j_z^{\beta\delta} = -\beta\delta \ell B^o h_\delta \frac{S^2}{R^2} I^\ell(h_\delta S) \frac{I_\ell(h_\delta r)}{I_\ell(h_\delta S)} \cos u \quad , \tag{125c}$$

(where we have used the fact $\nabla \cdot \underline{j}^{\beta\delta} = 0$ and permitted no net longitudinal current in this order because η is not involved). Thus,

$$B_r^{\beta\delta} = [-\ell^2 \beta\delta B^o \frac{S^2}{R^2} I^\ell(h_\delta S) \frac{I_\ell(h_\delta r)}{h_\delta r I_\ell(h_\delta S)} + C^{\beta\delta} h_\delta I_\ell'(h_\delta r)] \sin u \, , \tag{126a}$$

$$B_\theta^{\beta\delta} = [-\ell\beta\delta B^o \frac{S^2}{R^2} \frac{I_\ell'(h_\delta r)}{h_\delta S I_\ell'(h_\delta S)} + C^{\beta\delta} \frac{\ell}{r} I_\ell(h_\delta r)] \cos u \; , \tag{126b}$$

and

$$B_z^{\beta\delta} = -C^{\beta\delta}\, h_\delta I_\ell(h_\delta r) \cos u \quad , \tag{126c}$$

with $C^{\beta\delta}$ yet to be determined. The condition (80i) at the plasma-vacuum interface reads to this order

$$B_r^{\beta\delta}(R, u) - \rho^\delta h_\delta \sin u\, B_z^\beta(R) - B^o \frac{\partial\rho^{\beta\delta}}{\partial z} = 0 \,, \tag{127a}$$

and

$$\overline{B}_r^{\beta\delta}(R, u) - \rho^\delta h_\delta \sin u\, \overline{B}_z^\beta(R) - B^o \frac{\partial\rho^{\beta\delta}}{\partial z} = 0 \,, \tag{127b}$$

which yields after elimination of $\frac{\partial\rho^{\beta\delta}}{\partial z}$, $\overline{B}^{\beta\delta}(R)$ equal to $B_r^{\beta\delta}(R)$, or

$$\overline{C}^{\beta\delta} h_\delta I_\ell'(h_\delta R) + \overline{D}^{\beta\delta} h_\delta K_\ell'(h_\delta R)$$

$$= -\ell^2 \beta\delta B^o \frac{S^2}{h_\delta R^3} I^\ell(h_\delta S) \frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)} + C^{\beta\delta} h_\delta I_\ell'(h_\delta R) \,. \tag{128}$$

We now have two relations among the three quantities $\overline{C}^{\beta\delta}$, $\overline{D}^{\beta\delta}$, and $C^{\beta\delta}$, and thus one more is needed. This additional relation comes from imposing continuity of total pressure across the plasma-vacuum interface. To order $\beta\delta$ we have

$$p^{\beta\delta}(R) + \rho^\delta \cos u \frac{\partial p^\beta}{\partial r} + B^o B_z^{\beta\delta}(R) + B_z^\beta(R) B_z^\delta(R) + B^o \rho^\delta \cos u \frac{\partial B_z^\beta}{\partial r}$$

$$= B^o \overline{B}_z^{\beta\delta}(R) + \overline{B}_z^\beta B_z^\delta(R) \,. \tag{129}$$

($\rho^{\delta}(\theta, z)$ is readily determined from (80i) expanded to order δ.) After again noting that $\bar{B}_z^{\beta}(R)$ equals $B_z^{\beta}(R)$ and verifying that the first two terms cancel, (129) reduces to

$$B_z^{\beta\delta}(R) + \beta\delta \ B^o \frac{S}{R} \frac{I_\ell'(h_\delta R)}{I_\ell'(h_\delta S)} \cos u = \bar{B}_z^{\beta\delta}(R), \tag{130}$$

and hence

$$-\bar{C}^{\beta\delta} h_\delta I_\ell(h_\delta R) - \bar{D}^{\beta\delta} h_\delta K_\ell(h_\delta R) + C^{\beta\delta} h_\delta (h_\delta R)$$

$$= \beta\delta \ B^o \frac{S}{R} \frac{I_\ell'(h_\delta R)}{I_\ell'(h_\delta S)} , \tag{131}$$

and this order is now complete. Solving (123), (128), and (131) for $C^{\beta\delta}$ $\bar{C}^{\beta\delta}$, $\bar{D}^{\beta\delta}$, we obtain

$$h_\delta C^{\beta\delta} = \frac{A}{I_\ell(h_\delta R)} + \frac{B}{I_\ell'(h_\delta S)} + \left[\frac{K_\ell(h_\delta R)}{I_\ell(h_\delta R)} - \frac{K_\ell'(h_\delta S)}{I_\ell'(h_\delta S)}\right]\left[\frac{CI_\ell(h_\delta R) + AI_\ell'(h_\delta R)}{K_\ell'(h_\delta R)I_\ell(h_\delta R) - I_\ell'(h_\delta R)K_\ell(h_\delta R)}\right], \tag{132a}$$

$$h_\delta \bar{C}^{\beta\delta} = \frac{B}{I_\ell'(h_\delta S)} - \frac{K_\ell'(h_\delta S)}{I_\ell'(h_\delta S)} \left[\frac{CI_\ell(h_\delta R) + AI_\ell'(h_\delta R)}{K_\ell'(h_\delta R) I_\ell(h_\delta R) - I_\ell'(h_\delta R) K_\ell(h_\delta R)}\right] , \tag{132b}$$

and

$$h_\delta \bar{D}^{\beta} = \frac{CI_\ell(h_\delta R) + AI_\ell'(h_\delta R)}{K_\ell'(h_\delta R) I_\ell(h_\delta R) - I_\ell'(h_\delta R) K_\ell(h_\delta R)} , \tag{132.c}$$

which may be simplified somewhat using the Wronskian relation[13]

$$I_\ell(y)\,K_\ell'(y) - K_\ell(y)\,I_\ell'(y) = -\frac{1}{y}\,, \quad y \neq 0\,, \tag{132d}$$

to

$$h_\delta C^{\beta\delta} = \frac{A}{I_\ell(h_\delta R)} + \frac{B}{I_\ell'(h_\delta S)} + h_\delta R\left[\frac{K_\ell'(h_\delta S)}{I_\ell'(h_\delta S)} - \frac{K_\ell(h_\delta R)}{I_\ell(h_\delta R)}\right]\left[C I_\ell(h_\delta R) + A I_\ell'(h_\delta R)\right]\,, \tag{133a}$$

$$h_\delta \bar{C}^{\beta\delta} = \frac{B}{I_\ell'(h_\delta S)} + h_\delta R\,\frac{K_\ell'(h_\delta S)}{I_\ell'(h_\delta S)}\,\left[C I_\ell(h_\delta R) + A I_\ell'(h_\delta R)\right] \tag{133b}$$

and

$$h_\delta \bar{D}^{\beta\delta} = -\,h_\delta R\left[C I_\ell(h_\delta R) + A I_\ell'(h_\delta R)\right]\,, \tag{133c}$$

where

$$A = \beta\delta\, B^o\,\frac{S}{R}\,\frac{I_\ell'(h_\delta R)}{I_\ell'(h_\delta S)}\,, \tag{134a}$$

$$B = \beta\delta\, B^o\,\frac{h_\delta S}{2}\,, \tag{134b}$$

and

$$C = -\,\beta\delta\, B^o\,\ell^2\,\frac{S^2}{h_\delta R^3}\,I^\ell(h_\delta S)\,\frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)}\,. \tag{134c}$$

$\beta\beta$ - order

If we choose

$$p^{\beta\beta} = 0\,, \tag{135}$$

then in a fashion entirely similar to that used in arriving at (86) through (99), we find

$$j_r^{\beta\beta} = j_z^{\beta\beta} = 0; \; j_\theta^{\beta\beta} = - j_\theta^{\beta} \frac{B_z^{\beta}}{B_o} = \frac{\beta^2 B^o}{2} \frac{r^3}{R^4} , \tag{136a}$$

$$B_r^{\beta\beta} = B_\theta^{\beta\beta} = 0 \; ; \; B_z^{\beta\beta} = - \frac{\beta^2 B^o}{8} \frac{r^4}{R^4} , \tag{136b}$$

and

$$\overline{B}_r^{\beta\beta} = \overline{B}_\theta^{\beta\beta} = 0 , \; \overline{B}_z^{\beta\beta} = - \frac{\beta^2 B^o}{8} . \tag{136c}$$

$\eta\eta$ - order

Following through with our two cases of the η - order, we readily obtain

Case (a)

$$j_\theta^{\eta\eta} = j_z^{\eta} \frac{B_\theta^{\eta}}{B^o} = 2\eta^2 B^o \frac{r}{R^2} , \tag{137a}$$

$$B_z^{\eta\eta} = - \eta^2 B^o \frac{r^2}{R^2} , \tag{137b}$$

$$\overline{B}_z^{\eta\eta} = - \eta^2 B^o . \tag{137c}$$

If we choose

$$j_z^{\eta\eta} = 0 , \tag{138}$$

then clearly

$$B_\theta^{\eta\eta} = 0 \qquad (139a)$$

and

$$\bar{B}_\theta^{\eta\eta} = 0 \quad . \qquad (139b)$$

Case (b)

Again, choosing $j_z^{\eta\eta} = 0$, we find

$$B_\theta^{\eta\eta} = \bar{B}_\theta^{\eta\eta} = 0 \quad , \qquad (140a)$$

$$B_z^{\eta\eta} - \bar{B}_z^{\eta\eta} = \frac{\eta^2 B^o}{2} = j_\theta^{\eta\eta} \quad , \qquad (140b)$$

where we have again invoked continuity of the total pressure. (Here $j_\theta^{\eta\eta}$ has the same dimensions as $\underline{B}$ because $\underline{j}$ is a sheet current.)

$\beta\eta$ - order

If here we choose

$$p^{\beta\eta} = j_z^{\beta\eta} = 0 \quad , \qquad (141)$$

it is easily verified that all fields vanish in this order.

$\eta\delta$ - order

Here we have

$$p^{\eta\delta} = 0 \quad , \qquad (142)$$

and hence from (9) we have, to order $\eta\delta$,

$$j_r^{\eta\delta} = j_z^{\eta} \frac{B_r^{\delta}}{B^o} \quad , \tag{143a}$$

and

$$j_\theta^{\eta\delta} = j_z^{\eta} \frac{B_\theta^{\delta}}{B^o} \quad , \tag{143b}$$

where for the moment we make no distinction between cases (a) and (b).

If we require no net longitudinal current <u>in this order</u>, we find (using $\nabla \cdot \underline{j} = 0$),

$$j_z^{\eta\delta} = j_z^{\eta} \frac{B_z^{\delta}}{B_o} \tag{144}$$

It is easy to exhibit a particular set of fields $\underline{B}^{\eta\delta}$ with these currents as sources. Namely, since

$$j_r^{\eta\delta} = \frac{\partial B_z^{\eta\delta}}{r\,\partial\theta} - \frac{\partial B_\theta^{\eta\delta}}{\partial z} \quad , \tag{145a}$$

$$j_\theta^{\eta\delta} = \frac{\partial B_r^{\eta\delta}}{\partial z} - \frac{\partial B_z^{\eta\delta}}{\partial r} \quad , \tag{145b}$$

and

$$j_z^{\eta\delta} = \frac{1}{r} \frac{\partial}{\partial r} r\, B_\theta^{\eta\delta} - \frac{\partial B_r^{\eta\delta}}{r\,\partial\theta} \tag{145c}$$

we simply choose

$$B_z^{\eta\delta} = 0 \quad . \tag{146}$$

From (145ab) and (143ab) we then have

$$B_r^{\eta\delta} = - j_z^{\eta}\delta\,\ell\,\frac{S^2}{r}\,I^{\ell}(h_\delta S)\,\frac{I_\ell(h_\delta r)}{I_\ell(h_\delta S)}\,\sin u \quad , \tag{147a}$$

and

$$B_\theta^{\eta\delta} = - j_z^{\eta}\delta\,S\,\frac{I_\ell'(h_\delta r)}{I_\ell'(k_\delta S)}\,\cos u \quad . \tag{147b}$$

It is easily verified that $\nabla \cdot B^{\eta\delta} = 0$. We now write for the general solution in this order

$$B_r^{\eta\delta} = (-j_z^{\eta}\delta\,\ell\,\frac{S^2}{r}\,I^{\ell}(h_\delta S)\,\frac{I_\ell(h_\delta r)}{I_\ell(h_\delta S)} + C^{\eta\delta}\,h_\delta I_\ell'(h_\delta r))\,\sin u, \tag{148a}$$

$$B_\theta^{\eta\delta} = (-j_z^{\eta}\,\delta S\,\frac{I_\ell'(h_\delta r)}{I_\ell'(h_\delta S)} + C^{\eta\delta}\,\frac{\ell}{r}\,I_\ell(h_\delta r)\,)\,\cos u \quad , \tag{148b}$$

$$B_z^{\eta\delta} = - C^{\eta\delta}\,h_\delta\,I_\ell\,(h_\delta r)\,\cos u \quad , \tag{148c}$$

and

$$\bar{B}_r^{\eta\delta} = (\bar{C}^{\eta\delta}\,h_\delta\,I_\ell'(h_\delta r) + \bar{D}^{\eta\delta}\,h_\delta\,K_\ell'(h_\delta r))\,\sin u \quad , \tag{149a}$$

$$\bar{B}_\theta^{\eta\delta} = (\bar{C}^{\eta\delta}\,\frac{\ell}{r}\,I_\ell(h_\delta r) + D^{\eta\delta}\,\frac{\ell}{r}\,K_\ell\,(h_\delta r)\,)\,\cos u \,, \tag{149b}$$

$$\bar{B}_z^{\eta\delta} = -\,(\bar{C}^{\eta\delta}\,h_\delta\,I_\ell(h_\delta r) + \bar{D}^{\eta\delta}\,h_\delta\,K_\ell(h_\delta r))\,\cos u \,. \tag{149c}$$

That $\underline{B} \cdot \underline{n}$ be zero on outer boundary requires

$$\overline{B}_r^{\eta\delta}(S) + \delta \ell \sin u \; \overline{B}^{\eta}(S) = 0 \quad , \tag{150a}$$

or

$$\overline{C}^{\eta\delta} h_\delta I_\ell'(k_\delta S) + \overline{D}^{\eta\delta} h_\delta K_\ell'(h_\delta S) = -\delta \eta \ell B^o \frac{R}{S} \; . \tag{150b}$$

That $\underline{B} \cdot \underline{n}$ be zero at the plasma-vacuum interface yields

$$B_r^{\eta\delta}(R) - \frac{\partial \rho^\delta}{R\partial\theta} B_\theta^\eta(R) - B^o \frac{\partial \rho^{\eta\delta}}{\partial z} = 0 \quad , \tag{151a}$$

and

$$\overline{B}_r^{\eta\delta} - \frac{\partial \rho^\delta}{R\partial\theta} \overline{B}_\theta^\eta(R) - B^o \frac{\partial \rho^{\eta\delta}}{\partial z} = 0 \quad . \tag{151b}$$

After eliminating $\frac{\partial \rho^{\eta\delta}}{\partial z}$, we obtain

$$\overline{C}^{\eta\delta} h_\delta I_\ell'(h_\delta R) + \overline{D}^{\eta\delta} h_\delta K_\ell'(h_\delta R) + j_z^\eta \delta \ell \frac{S^2}{R} I^\ell(h_\delta S) \frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)}$$

$$- C^{\eta\delta} h_\delta I_\ell'(h_\delta R) = - \frac{\ell \rho^\delta}{R} [B_\theta^\eta]_R = -\ell\delta \frac{S}{R} \frac{I_\ell'(h_\delta R)}{I_\ell'(h_\delta S)} [B_\theta^\eta]_R \quad , \tag{152}$$

where

$$[B_\theta^\eta]_R = \overline{B}_\theta^\eta(R) - B_\theta^\eta(R) \quad . \tag{153}$$

Continuity of total pressure across the plasma-vacuum interface yields

$$B^o[B_z^{\eta\delta}]_R = - B_\theta^\delta(R)[B_\theta^\eta]$$

$$= - \delta \ell B^o h_\delta S I^\ell(h_\delta S) \frac{S}{R} \frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)} [B_\theta^\eta]_R \quad . \tag{154}$$

We now consider our two different forms for the current distribution j_z^η

Case (a)

$$[B_\theta^\eta] = 0 \ ; \qquad j_z^\eta = \frac{2\eta B^o}{R} \quad . \tag{155}$$

In this case (150b), (152), and (154) read

$$\bar{C}^{\eta\delta} h_\delta I_\ell'(h_\delta S) + \bar{D}^{\eta\delta} h_\delta K_\ell'(h_\delta S) = - \ell\delta\, \eta B^o \frac{R}{S} \quad , \tag{156a}$$

$$\bar{C}^{\eta\delta} h_\delta I_\ell'(h_\delta R) + \bar{D}^{\eta\delta} h_\delta K_\ell'(h_\delta R) - C^{\eta\delta} h\delta\, I_\ell'(h_\delta R)$$

$$= -2\eta\delta\,\ell \frac{S^2}{R^2} B^o I^\ell(h_\delta S) \frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)} \quad , \tag{156b}$$

and

$$- \bar{C}^{\eta\delta} h_\delta I_\ell(h_\delta R) - \bar{D}^{\eta\delta} h_\delta K_\ell(h_\delta R) + \bar{C}^{\eta\delta} h_\delta I_\ell(h_\delta R) = 0 \quad . \tag{156c}$$

The solution of this system of equations is

$$C^{\eta\delta} = - \frac{\ell\delta\,\eta B^o R}{h_\delta S I_\ell'(h_\delta S)} + [2\eta\delta\ell \frac{S^2}{R^2} B^o I^\ell(h_\delta S) \frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)}] [\frac{1}{h_\delta I_\ell'}(h_\delta R) +$$

$$\frac{R K_\ell'(h_\delta R) I_\ell(h_\delta R)}{I_\ell'(h^\delta R)} - \frac{R K_\ell'(h_\delta S) I_\ell(h_\delta R)}{I_\ell'(h_\delta S)}] \tag{157a}$$

$$\bar{C}^{\eta\delta} = - \frac{\ell\delta\,\eta B^o R}{h_\delta S I_\ell'(h_\delta S)} - 2\eta\delta\,\ell \frac{S^2}{R} B^o I^\ell(h_\delta S) \frac{I_\ell^2(h_\delta R) K_\ell'(h_\delta S)}{I_\ell'(h_\delta S) I_\ell(h_\delta S)} , \tag{157b}$$

$$\bar{D}^{\eta\delta} = 2\eta\,\delta\,\ell\,\frac{S^2}{R}\,B^o I^\ell(h_\delta S)\,\frac{I_\ell^2(h_\delta R)}{I_\ell(h_\delta S)} \quad . \tag{157c}$$

Case (b)

$$[B_\theta{}^\eta]_R = \eta\, B^o \quad ; \qquad j_z{}^\eta = 0 \quad . \tag{158}$$

In this case (150b), (152), and (154) read

$$\bar{C}^{\eta\delta} h_\delta I_\ell'(h_\delta S) + \bar{D}^{\eta\delta} h_\delta K_\ell'(h_\delta S) = -\ell\delta\eta\, B^o\,\frac{R}{S} \quad , \tag{159a}$$

$$\bar{C}^{\eta\delta} h_\delta I_\ell'(h_\delta R) + \bar{D}^{\eta\delta} h_\delta K_\ell'(h_\delta R) - C^{\eta\delta} h_\delta I_\ell'(h_\delta R)$$

$$= -\ell\eta\delta\,\frac{S}{R}\,B^o\,\frac{I_\ell'(h_\delta R)}{I_\ell'(h_\delta S)} \quad , \tag{159b}$$

and

$$-\bar{C}^{\eta\delta} h_\delta I_\ell(h_\delta R) - \bar{D}^{\eta\delta} h_\delta K_\ell(h_\delta R) + C^{\eta\delta} h_\delta I_\ell(h_\delta R)$$

$$= -\delta\ell\eta\, B^o h_\delta \frac{S^2}{R} I^\ell(h_\delta S)\,\frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)} \quad . \tag{159c}$$

The solution of this system of equations is

$$\bar{C}^{\eta\delta} = \frac{1}{h_\delta I_\ell'(h_\delta S)}\,[\,a + h_\delta R K_\ell'(h_\delta S)\,[b I_\ell(h_\delta R) + C I_\ell'(h_\delta R]\,]\,, \tag{160a}$$

$$\bar{D}^{\eta\delta} = -R\,[b I_\ell(h_\delta R) + C I_\ell'(h_\delta R)\,] \quad , \tag{160b}$$

$$C^{\eta\delta} = \frac{a}{h_\delta I_\ell'(h_\delta S)} - \frac{b}{h_\delta I_\ell'(h_\delta R)} - R\left[\frac{K_\ell'(hR)}{I_\ell'(hR)} - \frac{K_\ell'(hS)}{I_\ell'(hS)}\right][b I_\ell(hR) +$$

$$C I_\ell'(hR)\,] \quad , \tag{160c}$$

with

$$a = - \ell \delta \eta B^o \frac{R}{S} , \qquad (161a)$$

$$b = - \ell \delta \eta \frac{S}{R} B^o \frac{I_\ell'(h_\delta R)}{I_\ell'(h_\delta S)} , \qquad (161b)$$

and

$$c = - \delta \ell \eta B^o h_\delta \frac{S^2}{R} I^\ell(h_\delta S) \frac{I_\ell(h_\delta R)}{I_\ell(h_\delta S)} . \qquad (161c)$$

$\beta\epsilon$ - order

This order is most quickly arrived at by setting $\ell = 0$ in the $\beta\delta$ - order.

Thus

$$p^{\beta\epsilon} = \beta\epsilon B^{o\,2} \frac{S}{R^2} r \frac{I_o'(h_\epsilon r)}{I_o'(h_\epsilon S)} , \qquad (162)$$

$$\underline{j}^{\beta\epsilon} = 0 , \qquad (163)$$

$$B_r^{\beta\epsilon} = C^{\beta\epsilon} h_\epsilon I_o'(h_\epsilon r) \sin(-h_\epsilon z) , \qquad (164a)$$

$$B_\theta^{\beta\epsilon} = 0 = \bar{B}_\theta^{\beta\epsilon} \qquad (164b)$$

$$B_z^{\beta\epsilon} = - C^{\beta\epsilon} h_\epsilon I_o(h_\epsilon r) \cos(-h_\epsilon z) , \qquad (164c)$$

$$\bar{B}_r^{\beta\epsilon} = (\bar{C}^{\beta\epsilon} h_\epsilon I_o'(h_\epsilon r) + \bar{D}^{\beta\epsilon} h_\epsilon K_o'(h_\epsilon r) \sin(-h_\epsilon z) \qquad (165a)$$

$$\bar{B}_z^{\beta\epsilon} = - (\bar{C}^{\beta\epsilon} h_\epsilon I_o(h_\epsilon r) + \bar{D}^{\beta\epsilon} h_\epsilon K^o(h_\epsilon r)) \cos(-h_\epsilon z) , \qquad (165b)$$

where

$$h_\epsilon C^{\beta\epsilon} = \frac{A}{I_o(h_\epsilon R)} + \frac{B}{I_o'(h_\epsilon R)} + Ah_\epsilon RI_o'(hR)\left[\frac{K_o'(h_\epsilon S)}{I_o'(h_\epsilon S)} - \frac{K_o(h_\epsilon R)}{I_o(h_\epsilon R)}\right] , \tag{166a}$$

$$h_\epsilon \overline{C}^{\beta\epsilon} = \frac{B}{I_o'(h_\epsilon S)} + h_\epsilon R \frac{K_o'(h_\epsilon S)}{I_o'(h_\epsilon S)} AI_o'(h_\epsilon R) , \tag{166b}$$

$$h_\epsilon \overline{D}^{\beta\epsilon} = - h_\epsilon RAI_o'(h_\epsilon R) , \tag{166c}$$

and

$$A = \beta\epsilon B^o \frac{S}{R} \frac{I_o'(h_\epsilon R)}{I_o'(h_\epsilon S)} ; \quad B = \beta\epsilon B^o \frac{h_\epsilon S}{2} . \tag{167}$$

$\eta\epsilon$ - order

Again we arrive at this order by setting $\ell = 0$ in the $\eta\delta$ - order.

Thus

$$p^{\eta\epsilon} = 0 , \tag{168}$$

$$\underline{j}^{\eta\epsilon} = j_z^\eta \frac{\underline{B}^\epsilon}{B^o} . \tag{169}$$

Case (a)

$$B_r^{\eta\epsilon} = 0 , \tag{170a}$$

$$B_\theta^{\eta\epsilon} = - j_z^\eta \epsilon S \frac{I_o'(h_\epsilon r)}{I_o'(h_\epsilon S)} \cos(-h_\epsilon z) , \tag{170b}$$

$$B_z^{\eta\epsilon} = 0 , \tag{170c}$$

$$\underline{B}^{\eta\epsilon} = 0 . \tag{171}$$

Case (b)

$$p^{\eta\epsilon} = 0 \quad , \tag{172}$$

$$\underline{j}^{\eta\epsilon} = 0 \quad , \tag{173}$$

$$\underline{B}^{\eta\epsilon} = \underline{\bar{B}}^{\eta\epsilon} = 0 \quad . \tag{174}$$

Section III - Stability of Equilibria I ; Minimization of δW

In this section we are interested in the stability of those equilibria considered in Section II. In particular, we are interested in the effect of the helically invariant fields on the interchange instability[8] in the presence of a bulge, and on the kink instability.[14] The equilibrium of this section is one close to a cylinder with a boundary

$$r = S + \delta \sum_i \sigma_i^\delta \cos(\ell_i \theta - p_i hz) + \epsilon \sum_i \sigma_i^\epsilon \cos(-q_i hz) \qquad (1)$$

on which $\underline{B} \cdot \underline{n} = 0$. The results of Section II are readily generalized to a superposition of such perturbations of the boundary. Each perturbation is periodic in z over a length $2\pi/h$ and p_i and q_i denote the number of times the respective perturbations fit into this length. The interger ℓ_i denotes the variation of the perturbation with θ . We explicitly exclude the cases $p_i = q_j$ for any i and j and $p_i = p_j$ for $i \neq j$, since in this case the equilibria do not superpose easily. $\delta\, \sigma_i^\delta$ and $\epsilon \sigma_i^\epsilon$ represent the amplitudes of the perturbations and δ and ϵ are small expansion parameters. We also include an arbitrary pressure distribution $\beta p^\beta(r)$ and an arbitrary longitudinal current $\eta j^\eta(r)$. We temporarily exclude any surface currents or discontinuities in p and j and assume all quantities are finite and continuous.

The quantities needed in the treatment of stability are the zeroth order field $\underline{B}^0$, a constant field in the z direction, the fields of order δ

and ϵ which may be written from equations (44), (105) of Section II,

$$B_{r\,i}^{\delta} = \hat{B}_{r\,i}^{\delta} \sin u = p_i h \sigma_i^{\delta} B^{o} \frac{I_{\ell_i}'(p_i h r)}{I_{\ell_i}'(p_i h S)} \sin u_i \ , \tag{2a}$$

$$B_{\theta\,i}^{\delta} = \hat{B}_{\theta\,i}^{\delta} \cos u = \frac{\ell_i \sigma_i^{\delta} B^{o}}{r} \frac{I_{\ell_i}(p_i h r)}{I_{\ell_i}'(p_i h S)} \cos u_i \ , \tag{2b}$$

$$B_{z\,i}^{\delta} = \hat{B}_{z\,i}^{\delta} \cos u = - p_i h \sigma_i^{\delta} B^{o} \frac{I_{\ell_i}(p_i h r)}{I_{\ell_i}'(p_i h S)} \cos u_i \ , \tag{2c}$$

$$\underline{B}^{\delta} = \sum_i \underline{B}_i^{\delta} \ , \tag{2d}$$

with $u_i = \ell_i \theta - p_i h z$, and

$$B_{r\,i}^{\epsilon} = \hat{B}_{ri}^{\epsilon}(r) \sin(-q_i hz) = \frac{q_i h \sigma_i^{\epsilon} B^{o}}{I_o'(q_i h S)} I_o'(q_i hr) \sin(-q_i h z), \tag{3a}$$

$$B_{\theta i}^{\epsilon} = 0 \ , \tag{3b}$$

$$B_{zi}^{\epsilon} = \hat{B}_{z\,i}^{\epsilon}(r) \cos(-q_i hz) = - \frac{q_i h \sigma_i^{\epsilon} B^{o}}{I_o'(q_i h S)} I_o(q_i hr) \cos(-q_i hz) \ , \tag{3c}$$

$$\underline{B}^{\epsilon} = \sum_i \underline{B}_i^{\epsilon} \ . \tag{3d}$$

Further we need

$$B_z^{\beta}(r) = \frac{p^{\beta}(0) - p^{\beta}(r)}{B^{o}} \tag{4a}$$

$$j_\theta^\beta(r) = -\frac{\partial B_z^\beta(r)}{\partial r} , \qquad (4b)$$

and all other components of $\underline{B}^\beta$ and $\underline{j}^\beta$ are zero. Finally

$$B_\theta^\eta(r) = \frac{1}{r}\int_0^r r\, j_z^\eta(r)\, d r , \qquad (5)$$

$\underline{j}^\eta$ is in the z direction, and all other components of $\underline{B}^\eta$ are zero. We have collected these results here for ready reference in the ensuing stability analysis. We will not need any information about fields of higher order except the fact that all δ^2, $\delta\epsilon$, and ϵ^2 fields are sinusoidal in z. This remark is not true in the excluded cases $p_i = q_i$ etc.

No vacuum regions are allowed in our equilibrium but regions where p = 0, (pressureless plasmas)[8] are considered. Later in this section it is shown how the stability is affected by replacing these regions by vacua.

The stability of these equilibria is treated by means of an energy principle.[8] This principle reduces the question of the stability of a magnetostatic equilibrium to the problem: can the quadratic functional of $\underline{\xi}$

$$2\,\delta W = \int \{ \underline{Q}^2 + \underline{j} \cdot \underline{\xi} \times \underline{Q} + \underline{\xi} \cdot \nabla p\, \nabla \cdot \underline{\xi} + \gamma\, p (\nabla \cdot \underline{\xi})^2 \}\, d\tau , \qquad (6)$$

with

$$\underline{Q} = \nabla \times (\underline{\xi} \times \underline{B}) , \qquad (7)$$

be made negative for any choice of $\underline{\xi}(r)$? $\underline{B}$, p and $\underline{j}$ represent the equilibrium values. $\underline{\xi}$ is imagined to be an arbitrary (virtual) displacement from the equilibrium, subject only to the condition $\underline{\xi} \cdot \underline{n} = 0$ on a rigid boundary , and δW is the resulting second order (in $\underline{\xi}$) change in the potential energy. If δW can be made negative for some $\underline{\xi}$, it can be made negative for a normal mode $\underline{\xi}$, some potential energy is turned into kinetic energy, and the system is unstable. We ask for those values of our parameters $p^{\beta}, \sigma^{\delta}$, etc. for which the system is stable and for those values for which it is unstable. It should be noted that the question of stability is unaffected by the distribution in matter density which only affects the rate of growth of an instability.

Our equilibria have many parameters of expansion β , δ , etc., but these may be expressed in terms of one parameter λ . For example, we may assume δ and ϵ are proportional to λ and β and η are proportional to λ^2. The stability problem is much easier for a one-parameter equilibrium and we express our equilibrium in terms of λ with the above choice. There is no loss in generality in assuming the proportionality factors are one so $\delta = \epsilon = \lambda$ and $\beta = \eta = \lambda^2$.

In carrying out the expansion in λ , it is necessary also to expand the trial functions $\underline{\xi}$. This can be seen from the following argument: The value $\lambda = 0$ makes the equilibrium that of a cylinder with constant field, which is neutral. If we regard the parameters σ^{δ} , σ^{ϵ}, etc. other than λ as fixed , our equilibrium will either be stable for all sufficiently small λ or unstable for all sufficiently small λ . It is clear that in the second case a $\underline{\xi}$ which makes δW negative for one λ need not make it negative for another. Thus one must allow $\underline{\xi}$ to depend arbitrarily on λ as well as $\underline{r}$. Since λ is small we expand $\underline{\xi}$ in it and regard the coefficients in the expansion as arbitrary.

One now expands δW as a power series in λ and examines the lowest order which gives a decisive answer as to stability. A criterion expressed in terms of the parameters σ^{δ}, σ^{ϵ}, p^{β}, $\underline{j}^{\eta}$ is then found.

In applying these results to the stellarator we imagine it straightened out and neglect any effect of the curvature of the tube. However, we wish to keep partially the effect of the closed machine by demanding our equilibrium and our perturbations $\underline{\xi}$ be periodic over a length $L = 2\pi/k$ equal to the length of the machine. We require that h be an integral multiple of k so that each perturbation of the boundary fits into this length.

We shall choose k to be small of order λ^2 and expand in it as well. The reason for this is to keep the rotational angle ι, through which a line rotates over the whole machine, finite as λ goes to zero. The displacement tries to follow the lines and the requirement that $\underline{\xi}$ be periodic in L would make the perturbation very irregular as λ went to zero, unless ι for the machine were finite. That ι is finite for the entire machine can be seen from Appendix II A, for

$$\iota/\text{per helix} \sim \delta^2 , \tag{8a}$$

$$\text{no. of helices} \sim \frac{1}{k} \sim \frac{1}{\delta^2} , \tag{8b}$$

$$\text{total } \iota \sim 1 . \tag{8c}$$

To conclude this introduction it should be observed that any finite situation may be approached by a system expanded in λ in any number of ways. For instance if $k \sim \lambda^2$, k will be finite when λ becomes finite. Alternatively, the finite situation could have been approached also by keeping k fixed, as λ increased from zero, at its final finite value. The best expansion is the one that leads to the most rapidly converging scheme. Since we do not examine the situation beyond the lowest significant order, we cannot apply this criterion to our scheme. However, our choice of orders leads to an expression which remains "uniformly" valid as we arbitrarily shift our choice of orders, while another choice does not. In particular, it is shown in Appendix III A that results obtained by treating hR as finite remain valid when hR is made transcendentally small in respect to λ. Further, our expansion readily yields itself to physical interpretation so in some sense it is the "best" expansion.

To proceed with the stability analysis we expand δW to zeroth order in λ to obtain

$$2\,\delta W^{o} = \int \underline{Q}^{o2}\, d\tau \ , \tag{9}$$

$$\underline{Q}^{o} = \nabla \times (\underline{\xi}^{o} \times \underline{B}^{o}) = B^{o} \frac{\partial \bar{\underline{\xi}}^{o}}{\partial z} - \underline{B}^{o} (\nabla \cdot \bar{\underline{\xi}}^{o}) \ . \tag{10}$$

We have introduced a convenient notation of a bar to distinguish the r and θ components of a vector so that for an arbitrary vector $\underline{A}$,

$$\underline{A} = \bar{\underline{A}} + \underline{e}_z A_z \ . \tag{11}$$

We cannot expand our $\underline{\xi}$'s at a fixed point since they will have wave lengths of order L and the expansion would not be uniform over L. To get around the difficulty we first Fourier analyze $\underline{\xi}$ in z

$$\underline{\xi} = \sum_{s,n} \underline{\xi}(s,n)\, e^{ishz + inkz} \tag{12}$$

with s and n finite integers. We assume that $\underline{\xi}$ has no Fourier components except those with finite wave numbers or wave numbers proportional to λ^2. It will be shown in Appendix III B that this assumption cannot affect the question of stability since the nonvanishing of such components always makes δW positive. It is now expected that $\underline{\xi}(s, n)$ can be expanded uniformly in λ. It should be noted that $\partial/\partial z$ operating on $\underline{\xi}$ can change its order.

It is obvious that δW^o is non-negative. It can be made zero only by the choices

$$\underline{\xi}^o(s, n) = 0, \qquad s \neq 0 \tag{13}$$

and

$$\nabla \cdot (\underline{\xi}^o) = 0 . \tag{14}$$

Any other choices make δW positive. Our $\underline{\xi}$'s are restricted to $\underline{\xi} \cdot \underline{n} = 0$ which to lowest order says $\xi_r^o(S) = 0$. This restriction will be relaxed in higher orders temporarily. If δW is always positive without this restriction, it is certainly positive with it. If δW can be made negative, it will be shown that the $\underline{\xi}$ which makes it negative can be chosen also to satisfy this restriction to all orders.

Since $\underline{Q}^o = 0$, it is clear that $\delta W^1 = 0$. The second order part of δW is

$$2\,\delta W^2 = \int \left\{ \underline{Q}^{\lambda 2} + \gamma p^{\lambda\lambda} \left(\frac{\partial \xi_z^o}{\partial z}\right)^2 + \underline{\xi}^o \cdot \nabla p^{\lambda\lambda} \left(\frac{\partial \xi_z^o}{\partial z}\right) \right\} d\tau . \tag{15}$$

By equations (4) and (13) the last term is to this order a z derivative which vanishes on integration. The region of integration is the zeroth order region i.e. a cylinder of radius S and length L. δW^2 is thus non-negative. It is clear that if δW is to be negative $\partial \xi_z^o/\partial z = 0$

wherever $p \neq 0$, i.e.

$$\xi_z^o(s,n) = 0 \qquad \text{for } s \neq 0 \text{ and } p \neq 0 \quad . \tag{16}$$

Before proceeding to make $\underline{Q}^{\lambda}$ zero, we will effect a considerable simplification by demonstrating that we may consider only $\underline{\xi}$'s with $\xi_z^o = 0$. To do this we momentarily drop our expansion and consider the exact change in δW produced by changing $\underline{\xi}$ to $\underline{\xi}'$, where

$$\underline{\xi}' = \underline{\xi} + f\,\underline{B} \quad . \tag{17}$$

We have $\underline{Q} = \underline{Q}'$ where primes represent quantities containing $\underline{\xi}'$. Thus

$$\Delta 2\delta W = 2\delta W' - 2\delta W = \int \{ \underline{j} \cdot f\,\underline{B} \times \nabla \times (\underline{\xi} \times \underline{B}) + \underline{\xi} \cdot \nabla p \quad \underline{B} \cdot \nabla f$$

$$+ \gamma p (2\,\nabla \cdot \underline{\xi}\;\underline{B} \cdot \nabla f + (\underline{B} \cdot \nabla f)^2 \}\, d\tau \tag{18}$$

where we have used $\underline{B} \cdot \nabla p = 0$.

Let us consider the first term which may be written

$$I_1 = - \int f\,\nabla \cdot [(\underline{j} \times \underline{B}) \times (\underline{\xi} \times \underline{B})]\, d\tau = - \int f\,\nabla \cdot [\underline{j} \cdot \underline{\xi} \times \underline{B}\;\underline{B}]\, d\tau$$

since $\nabla \times (\underline{j} \times \underline{B}) = 0$. Integrating by parts and making use of the fact that $\underline{B} \cdot \underline{n} = 0$ on the boundary, we have

$$I_1 = \int \underline{j} \cdot \underline{\xi} \times \underline{B}\;\underline{B} \cdot \nabla f\, d\tau = - \int \underline{\xi} \cdot \underline{j} \times \underline{B}\;\underline{B} \cdot \nabla f\, d\tau$$

which just cancels the second term of (18). Thus

$$\Delta 2 \delta W = \int \gamma p \left[2 (\nabla \cdot \underline{\xi}) \underline{B} \cdot \nabla f + (\underline{B} \cdot \nabla f)^2 \right] d\tau \quad . \tag{19}$$

In the region where p is zero the contribution is zero. Since elsewhere p is second order and $(\nabla \cdot \underline{\xi})$ is first order $\Delta 2 \delta W$ is higher than fourth order if $\underline{B} \cdot \nabla f$ is higher than first order. It is now clear that if any $\underline{\xi}$ makes δW negative with ξ_z^o unequal to zero, one can change this $\underline{\xi}$ by $f^o \underline{B}^o$ + $(f^\lambda \underline{B}^o + f^o \underline{B}^\lambda)$ with $f^o = - \xi_z^o$. Since $\underline{B} \cdot \nabla f = \underline{B}^o \cdot \nabla f^o + [\underline{B}^\lambda \cdot \nabla f^o + \underline{B}^o \cdot \nabla f^\lambda]$, f^λ may be chosen to make the bracket zero, and $\underline{B}^o \cdot \nabla f^o$ is of second order in the region where $p = 0$, $\underline{B} \cdot \nabla f$ is second order for this choice. $\Delta 2 \delta W$ is thus zero to fourth order and ξ_z^o can be chosen zero. In the region where $p \neq 0$, it is obvious that $\Delta 2 \delta W$ is zero and ξ_z^o can also be chosen zero in this region.

To make $\underline{Q}^\lambda$ zero, we must first have

$$B^o \frac{\partial \xi_r^\lambda}{\partial z} = - \underline{e}_r \cdot \nabla \times (\underline{\xi}^o \times \underline{B}^\lambda) \tag{20a}$$

$$B^o \frac{\partial \xi_\theta^\lambda}{\partial z} = - \underline{e}_\theta \cdot \nabla \times (\underline{\xi}^o \times \underline{B}^\lambda) \tag{20b}$$

which makes $\underline{Q}^\lambda = 0$. It is clear that $\underline{\xi}^\lambda$ can be chosen to satisfy (20a) and (20b) since the right hand sides must integrate to zero over the length L. $\underline{\xi}^\lambda (0, n)$ is still arbitrary. The vanishing of Q_z^λ requires

$$B^o \nabla \cdot \underline{\xi}^\lambda = \underline{e}_z \cdot \nabla \times (\underline{\xi}^o \times \underline{B}^\lambda) \quad , \tag{21}$$

and from (20a) and (20b) it is easily shown that $\partial/\partial z\ Q_z^\lambda = 0$. By choosing $\underline{\xi}^\lambda(0,n)$ to make $\nabla\cdot\underline{\xi}^\lambda(0,n)$ zero one sees that $Q_z^\lambda(0,n)$ can be made zero.

To third order we have

$$2\delta W^3 = \int \underline{\xi}^o \cdot \nabla p^{\lambda\lambda}\left[(\nabla\cdot\underline{\xi}^\lambda) + \frac{\partial \xi_z^\lambda}{\partial z}\right] d\tau \qquad (22)$$

The second term goes out upon integration by parts while the first term in the bracket integrates out since $\nabla\cdot\underline{\xi}^\lambda \sim e^{ishz}$. The term in (15) which previously went out on integration over the zeroth order region is now zero since it is proportional to ξ_z^o . Thus its contribution to δW in the third order in the region between S and the boundary given by (1) is zero.

Finally, in fourth order

$$2\delta W^4 = \int_V \{(\underline{Q}^{\lambda\lambda})^2 - j_\theta^{\lambda\lambda}\,\xi_r^o\,Q_z^{\lambda\lambda} + j_z^{\lambda\lambda}[\xi_r^o\,Q_\theta^{\lambda\lambda} - \xi_\theta^o\,Q_r^{\lambda\lambda}]$$

$$+ \gamma p^{\lambda\lambda}(\nabla\cdot\underline{\xi}^\lambda)^2 + [\underline{\xi}\cdot\nabla p(\nabla\cdot\underline{\xi})]^{\lambda\lambda\lambda\lambda}\}\,d\tau + \int_{V'} \underline{\xi}^o\cdot\nabla p^{\lambda\lambda}(\nabla\cdot\underline{\xi}^\lambda)d\tau \qquad (23)$$

where V' is the perturbed region. Since changing $\underline{\xi}$ by $f\underline{B}$ only affects the γp term, by taking f of order λ , $\xi_z^\lambda(s,n)$ may be picked to make $\nabla\cdot\underline{\xi}^\lambda(s,n)$ vanish, while $\nabla\cdot\underline{\xi}^\lambda(0,n)$ vanishes already. This simplifies the integral over V and eliminates the integral over V' .

Next consider the $j_z^{\lambda\lambda}$ term in δW^4. We are only interested in $\underline{Q}^{\lambda\lambda}(s,n)$ with $s = 0$ since otherwise the term integrates to zero in z . Thus $\partial/\partial z\ (\underline{\xi}\times\underline{B})^{\lambda\lambda}(0,n) = \partial/\partial z\ (\underline{\xi}^o\times\underline{B}^o)(0,n)$ and we have for this term

$$\int \{ j_z^{\lambda\lambda} [- \xi_r^{\,o} \frac{\partial}{\partial r} (\underline{\xi} \times \underline{B})_z^{\lambda\lambda} - \frac{\xi_\theta^{\,o}}{r} \frac{\partial}{\partial \theta} (\underline{\xi} \times \underline{B})_z^{\lambda\lambda}]$$

$$+ j_z^{\lambda\lambda} B^o (\xi_r^{\,o} \frac{\partial \xi_\theta^{\,o}}{\partial z} - \xi_\theta^{\,o} \frac{\partial \xi_r^{\,o}}{\partial z}) \} \, r\, d r\, d\theta\, d z \; . \tag{24}$$

By integrating the terms in the first bracket and making use of the fact that $\nabla \cdot \underline{\xi}^o$ and $\xi_r^{\,o}(S)$ are zero, we get

$$\int \{ \frac{dj_z^{\lambda\lambda}}{dr} \xi_r^{\,o} (\underline{\xi} \times \underline{B})_z^{\lambda\lambda} + B^o j_z^{\lambda\lambda} (\xi_r^{\,o} \frac{\partial \xi_\theta^{\,o}}{\partial r} - \xi_\theta^{\,o} \frac{\partial \xi_r^{\,o}}{\partial z}) \} \, r\, d r\, d\theta\, d z \tag{25}$$

where

$$(\underline{\xi} \times \underline{B})_z^{\lambda\lambda} (0, n) = (\underline{\xi}^\lambda \times \underline{B}^\lambda)_z (0, n) + (\xi_r^o \; B_\theta^{\lambda\lambda}) (0, n) \; . \tag{26}$$

Returning to δW^4 and completing the square on $Q_z^{\lambda\lambda}$, we can now write

$$2 \delta W^4 = \int \{ \underline{Q}^{\lambda\lambda^2} + [Q_z^{\lambda\lambda} - j_\theta^{\lambda\lambda} \xi_r^{\,o}]^2 - j_\theta^{\lambda\lambda^2} \xi_r^{\,o^2} + \frac{dj_z^{\lambda\lambda}}{dr} \xi_r^{\,o} (\underline{\xi} \times \underline{B})_z^{\lambda\lambda}$$

$$+ j_z^{\lambda\lambda} B^o [\xi_r^{\,o} \frac{\partial \xi_\theta^{\,o}}{\partial z} - \xi_\theta^o \frac{\partial \xi_r^{\,o}}{\partial z}] + \xi_r^o j_\theta^{\lambda\lambda} [Q_z^{\lambda\lambda} + B^o (\nabla \cdot \underline{\xi})^{\lambda\lambda}] (0, n) \} \, d\tau . \tag{27}$$

Since

$$Q_z^{\lambda\lambda} + B^o (\nabla \cdot \underline{\xi})^{\lambda\lambda} = \underline{e}_z \cdot \nabla \times (\underline{\xi}^\lambda \times \underline{B}^\lambda) + \underline{e}_z \cdot \nabla \times (\underline{\xi}^o \times \underline{B}^{\lambda\lambda}) , \tag{28}$$

it is seen that $\underline{\xi}^{\lambda\lambda}$ only occurs in the first two positive definite terms of (27). Therefore, $\underline{\xi}^{\lambda\lambda}$ may be chosen to make

$$\underline{Q}^{\lambda\lambda}(s, n) = 0 \qquad s \neq 0 \tag{29a}$$

and

$$Q_z^{\lambda\lambda} - j_\theta^{\lambda\lambda}\,\xi_r^{o} = 0 \tag{29b}$$

since $\underline{Q}^{\lambda\lambda}$ (0, n) does not involve $\underline{\xi}^{\lambda\lambda}$. That this is possible follows from an argument similar to that used in making $\underline{Q}^{\lambda}$ vanish by choosing $\underline{\xi}^{\lambda}$ with the change that $\nabla\cdot\underline{\xi}^{\lambda\lambda}(0,n)$ is here chosen to make $Q_z^{\lambda\lambda}(0,n) + j_\theta^{\lambda\lambda}\,\xi_r^{o}(0,n)$ vanish. The positive definite terms of (27) thus become

$$\sum_n \int |\underline{Q}^{\lambda\lambda}(0,n)|^2\, d\tau \tag{30}$$

which we temporarily denote as

$$\int \overline{\underline{Q}^{\lambda\lambda 2}}\, d\tau \;, \tag{31}$$

the bar indicating an average over the rapid variation in z. With this change and (46), δW^4 can be written

$$2\delta W^4 = \int \Big\{ \overline{\underline{Q}^{\lambda\lambda}}^{\,2} + \frac{dj_z^{\lambda\lambda}}{dr}\,\xi_r^{o}(\underline{\xi}\times\underline{B})_z^{\lambda\lambda} + 2\xi_r^{o}\,\frac{\partial\xi_\theta^{o}}{\partial z}\,j_z^{\lambda\lambda}\,B^{o}$$

$$+\ \xi_r^{o}\,j_\theta^{\lambda\lambda}\,\underline{e}_z\cdot\nabla\times\overline{(\underline{\xi}^{\lambda}\times\underline{B}^{\lambda})}\Big\}\, d\tau \;, \tag{32}$$

where

$$\overline{(\underline{\xi}^{\lambda}\times\underline{B}^{\lambda})} \equiv \sum_n (\underline{\xi}^{\lambda}\times\underline{B}^{\lambda})(0,n) \;. \tag{33}$$

But $\underline{\xi}^{\lambda}$ is given in terms of $\underline{\xi}^{o}$ by equations (20a), (20b), and $\nabla\cdot\underline{\xi}^{\lambda} = 0$, (at least insofar as $\underline{\xi}^{\lambda}$ enters into δW^4 in eq. (32).) Further from eq. (14) and $\xi_z^o = 0$, δW^4 can be expressed entirely in terms of $\xi_r^o(0, n)$. It is found that all the coefficients of ξ_r^o are independent of θ and z and depend only on r. Thus we may Fourier analyze the $\xi_r^o(0, n)$ in terms of θ

$$\xi_r^o(0, n) = \sum_m \xi_r^o(m;n)\, e^{im\theta} \tag{34}$$

and δW^4 will break up into a sum over m and n with the m, n^{th} term only involving $\xi_r^o(m;n)$. Since each $\xi_r^o(m;n)$ is still at our disposal as an arbitrary function of r subject only to $\xi_r^o(m;n)\big|_S = 0$, we will have stability if and only if every one of these terms is always positive, while if a single term can be made negative we will have instability.

Let us restrict ourselves to a single term of this Fourier expansion $(m \neq 0)$ and suppress the m and n indices. Then $Q_r^{\lambda\lambda}$ becomes

$$Q_r^{\lambda\lambda} = \frac{1}{r}\frac{\partial}{\partial\theta}\overline{(\underline{\xi}^\lambda \times \underline{B}^\lambda)_z} + B^o \frac{\partial \xi_r^o}{\partial z} + \frac{1}{r}\frac{\partial}{\partial\theta}(\xi_r^o B_\theta^\eta) \tag{35a}$$

or

$$Q_r^{\lambda\lambda} = \frac{im}{r}\left\{\left(\frac{nkrB_o}{m} + B_\theta^\eta\right)\xi_r^o + \overline{(\underline{\xi}^\lambda \times \underline{B}^\lambda)_z}\right\} . \tag{35b}$$

Similarly by eq. (14) $Q_\theta^{\lambda\lambda}$ becomes

$$Q_\theta^{\lambda\lambda} = -\frac{\partial}{\partial r}\left\{\left(\frac{nkrB_o}{m} + B_\theta^\eta\right)\xi_r^o + \overline{(\underline{\xi}^\lambda \times \underline{B}^\lambda)_z}\right\} . \tag{35c}$$

Substituting for $\underline{\xi}^\lambda$ its value in terms of ξ_r^o by equations (20a) and (20b) we have

$$\overline{(\underline{\xi}^{\lambda} \times \underline{B}^{\lambda})}_z = \sum_i \frac{\xi_r^o}{2p_i h B^o} \frac{d}{dr} (\hat{B}_{r_i}^{\lambda} \hat{B}_{\theta_i}^{\lambda}) , \tag{36}$$

so that using equations (3) and (4) we can write

$$Q_r^{\lambda\lambda} = \frac{im}{r} (\nu \xi_r^o) \tag{37}$$

and

$$Q_\theta^{\lambda\lambda} = - \frac{\partial}{\partial r} (\nu \xi_r^o) , \tag{38}$$

where

$$\nu \equiv \frac{nkrB_o}{m} + B^\eta + \sum_i \frac{\ell_i p_i h \sigma_i \delta^2}{2r} B^o \left[\frac{I_{\ell_i}'(p_i hr)}{I_{\ell_i}'(p_i hS)}\right]^2 [1 - 2I^{\ell_i}(p_i hr) + (\ell_i^2 + p_i^2 h^2 r^2) I^{\ell_i 2}] \tag{39}$$

with

$$I^\ell(x) = \frac{I_\ell(x)}{x I_\ell'(x)} . \tag{40}$$

The $j_z^{\lambda\lambda}$ terms in (32) may be written, after using equation (14) and integrating by parts as,

$$\int \frac{d j_z^{\lambda\lambda}}{dr} \nu |\xi_r^o|^2 \, d\tau . \tag{41}$$

The m, n[th] term in δW^4 now reads

$$2\delta W^4 = \int d\tau \{ \frac{m^2}{r^2} \nu^2 |\xi_r^o|^2 + |\frac{d}{dr}(\nu \xi_r^o)|^2 + \frac{dj^n}{dr} \nu |\xi_r^o|^2 + \xi_r^o j_\theta^\beta \underline{e}_z \cdot \nabla \times \overline{(\underline{\xi}^\lambda \times \underline{B}^\lambda)} \} \quad (m \neq 0) \tag{42}$$

where the absolute value signs arise from the product $\xi_r^o(m;n)\,\xi_r^o(-m;-n)$ on application of the reality condition on ξ_r^o. The remaining terms $(\overline{\underline{\xi}^\lambda \times \underline{B}^\lambda})$ can be expressed in exactly the same way in terms of ξ_r^o. Before combining all these results into a final formula we must consider the case $m = 0$. Here by eq. (14) $\xi_r^o(0;n) = 0$. Therefore, $Q_r^{\lambda\lambda} = 0$ and $Q_\theta^{\lambda\lambda} = inkB_o\xi_\theta^o$ and the remaining terms are zero so that for $m = 0$, δW is positive definite and can be made zero by choosing $\xi_\theta^o(0,n)$ to be zero. Consequently, we can suppress it. Further the negative m terms are just equal to the positive terms so we can multiply our sum by two and restrict it to positive m and all n. Finally our sum depends only on the absolute value of ξ_r^o so we may write $|\xi_r^o|\,\nu = \mu$. Carrying out the integration over θ and z and restoring the subscripts we find

$$2\delta W^4 = \frac{8\pi^2}{k} \sum_{\substack{m>0 \\ n}} \int_o^S r\,dr \left\{ \left(\frac{d\mu_{m;n}}{dr}\right)^2 + \frac{\alpha_{m;n}}{r^2}\,\mu_{m;n}^{\;2} \right\} , \qquad (43a)$$

where

$$\alpha_{m;n} = m^2 + \frac{dj^\eta}{dr}\;\frac{r^2}{\nu_{m;n}}$$

$$- \frac{j^\beta B^o}{\nu_{m;n}^{\;2}} \Big[\sum_i p_i^2 h^2 \sigma_i^{\,2} \delta^2 r \left(\frac{I'_{\ell_i}(p_i hr)}{I'_{\ell_i}(p_i hS)}\right)^2 \left(1 - 2(\ell_i^2 + p_i^2 h^2 r^2) I^{\ell_i}(p_i hr) + \ell_i^2 I^{\ell_i}(p_i hr)^2\right)$$

$$+ \sum_i q_i^2 h^2 \sigma_i \epsilon_i^2 r \left(\frac{I'_o(q_i hr)}{I'_o(q_i hS)}\right)^2 \left(1 - 2q_i^{\,2} h^2 r^2 I^o(q_i hr)\right) \Big] , \qquad (43b)$$

and $\nu_{m;n}$ is given by equation (39).

We have thus reduced our stability problem to the consideration of the signs of an infinite number of one-dimensional integrals in r of single variables $\mu_{m,n}$. These are worked out in section IV and minimized over m and n there and stability criteria are established.

In the reduction of the problem of stability to equation (43), it has been tacitly assumed that $\underline{\xi}^{\lambda}$ and $\underline{\xi}^{\lambda\lambda}$ could always be chosen (under the constraints imposed on them) to satisfy $\underline{\xi} \cdot \underline{n} = 0$ on the boundary to first and second order respectively. It will now be verified that this is possible. Observe that on the boundary

$$\underline{Q} \cdot \underline{n} = 0 \tag{44}$$

through second order. This is easily seen since $\underline{Q}^{o}$ and $\underline{Q}^{\lambda}$ are zero and $\underline{Q} \cdot \underline{n} = Q_r^{\lambda\lambda}(S) \sim m\, \xi_r^{o}$ is zero since $\xi_r^{o}(S)$ is. We will drop the expansion in λ temporarily and consider the implication of $\underline{Q} \cdot \underline{n} = 0$. Expanding the triple vector product in $\underline{Q}$ we have

$$\underline{n} \cdot [\underline{B} \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B} - \underline{B}\, \nabla \cdot \underline{\xi}\,] = 0 \tag{45a}$$

or since $\underline{B} \cdot \underline{n} = 0$

$$\underline{n} \cdot (\underline{B} \cdot \nabla \underline{\xi}) = \underline{n} \cdot (\underline{\xi} \cdot \nabla \underline{B}) \tag{45b}$$

Thus

$$\underline{B} \cdot \nabla (\underline{\xi} \cdot \underline{n}) = \underline{n} \cdot (\underline{B} \cdot \nabla \underline{\xi}) + \underline{\xi} \cdot (\underline{B} \cdot \nabla \underline{n})$$

$$= \underline{n} \cdot (\underline{\xi} \cdot \nabla \underline{B}) + \underline{\xi} \cdot (\underline{B} \cdot \nabla \underline{n}) \tag{46}$$

If we write $\underline{\xi} = \underline{\xi}_{\perp} + \underline{\xi}_{\parallel}$ where $\underline{\xi}_{\perp} = \underline{n}(\underline{\xi} \cdot \underline{n})$ then

$$0 = \underline{\xi}_{\parallel} \cdot \nabla(\underline{n} \cdot \underline{B}) = (\underline{\xi}_{\parallel} \cdot \nabla\underline{n}) \cdot \underline{B} + (\underline{\xi}_{\parallel} \cdot \nabla\underline{B}) \cdot \underline{n} \tag{47}$$

so the $\underline{\xi}_{\parallel}$ terms in (46) become

$$-(\underline{\xi}_{\parallel} \cdot \nabla\underline{n}) \cdot \underline{B} + \underline{\xi}_{\parallel} \cdot (\underline{B} \cdot \nabla\underline{n})$$

$$= (\underline{B} \times \underline{\xi}_{\parallel}) \cdot \nabla \times \underline{n} = 0 \tag{48}$$

since $\underline{n} \cdot \nabla \times \underline{n} = 0$. This is easily seen by applying Stokes' theorem to the line integral $0 = \int \underline{n} \cdot \underline{d\ell}$ over an arbitrary curve lying in the surface. Therefore, (46) becomes

$$\underline{B} \cdot \nabla (\underline{\xi} \cdot \underline{n}) = [(\nabla\underline{B}) \cdot \underline{n} + \underline{B} \cdot \nabla\underline{n}] \cdot \underline{\xi}_{\perp} = (\underline{n} \cdot \nabla\underline{B} \cdot \underline{n})(\underline{\xi} \cdot \underline{n}), \tag{49}$$

since $\underline{B} \cdot \nabla\underline{n}$ is obviously perpendicular to $\underline{\xi}_{\perp}$. These results are, of course, valid only through second order for our $\underline{\xi}$'s. Expanding (49) out in λ, we find that the right hand side vanishes to first order and $(\underline{\xi} \cdot \underline{n})^{\lambda}$ is zero for $(s \neq 0)$. Since the restriction on $\underline{\xi}^{\lambda}$ $(s = 0)$ is $\nabla \cdot \underline{\xi}^{\lambda} = 0$, we can pick $\underline{\xi}^{\lambda}$ to make $(\underline{\xi} \cdot \underline{n})^{\lambda}$ independent of θ. If one integrates $\nabla \cdot \underline{\xi}^{\lambda}$ over a cross section, one sees this constant must be zero. Proceeding to the second order in (49), one sees that $(\underline{\xi} \cdot \underline{n})^{\lambda\lambda}$ is also zero for $s \neq 0$. Arguing in a similar manner, one sees that $(\underline{\xi} \cdot \underline{n})^{\lambda\lambda}$ $(s = 0)$ is independent of θ, and by integrating

$$\nabla \cdot \underline{\xi}^{\lambda\lambda} = \frac{\underline{e}_z}{B^o} \cdot \nabla \times (\underline{\xi}^{\lambda} \times \underline{B}^{\lambda}) \tag{50}$$

for $s = 0$, $m = 0$, that it is zero.

In many equilibria of interest there is a discontinuity in the plasma. One could treat the stability of such equilibria by including surface terms in the expression for δW, equation (6),[8] and carrying out the analysis we have developed including these terms. However, a surface of discontinuity can generally be regarded as a region where physical quantities vary very rapidly and one can arrive at the properties of a discontinuity by letting the thickness shrink to zero. In practice the region in question is not zero but can be treated as zero to a good approximation. Furthermore, it should not matter how we pass to this limit (if it did the approximation of the region by a surface continuity would not be a good one). Therefore, we can regard the physical quantities as continuously but rapidly varying over this region while we reduce the problem of stability to the consideration of δW^4 as given by Eq. (43) and only then pass to the limit of a surface discontinuity. However, it is necessary to make the jump in $\underline{\xi} \cdot \underline{n}$ across the discontinuity zero in order to prevent cavitation or interpenetration. This can be done by considering only $\underline{\xi}$'s which vary slowly over the region as its thickness goes to zero.

In evaluating the integral over the region of rapidly varying quantities as its thickness goes to zero, one can neglect all integrands which remain finite as giving zero contribution in the limit. There are some terms which are products of one factor which becomes large to first order in the thickness and another smoothly varying one. In the integration it is permissible to take the smoothly varying factor out of the integral and integrate the large factor. Other terms are products of two factors one of which is large of the second order, such as dj^n/dr, and the other smoothly varying. For these terms it is not permissible to factor out the smoothly varying factor but one must integrate by parts first reducing the first factor to first order of largeness and then proceed as before. Finally there occur two terms which are each products of two large factors. These terms just cancel each other.

We shall carry out the above scheme for an equilibrium which has a discontinuity in pressure, a surface longitudinal current $j^{\eta *}$, and a discontinuity in the volume longitudinal current. We define ν^* by the relation

$$\nu \equiv \nu^* + B^{\eta} \tag{51}$$

where ν^* is slowly varying and B^{η} rapidly varying. The last term in equation (43b) can be written

$$- j_{\theta}^{\beta} F = \frac{d B_z^{\beta}}{dr} F \tag{52}$$

with F slowly varying. From it we get $r F [\![B_z]\!]$ where $[\![A]\!]$ indicates the jump outward across the boundary. Since the dj^{η}/dr term in (43b) is of second order largeness, we integrate it by parts and substitute for j^{η} its value in terms of B^{η} to obtain

$$\int \frac{dj^{\eta}}{dr} \nu \xi_r^{o2} \, r dr = [\![j^{\eta}]\!] r \nu^* \xi_r^2 + [\![j^{\eta} B^{\eta}]\!] r \xi_r^2$$

$$- [\![\frac{B^{\eta 2}}{2}]\!] \xi_r^2 - \int \frac{dB^{\eta}}{dr} \frac{d(r\nu^* \xi_r^2)}{dr} dr$$

$$- \int \left(\frac{dB^{\eta}}{dr}\right)^2 \xi_r^2 \, r dr - \int \frac{d(B^{\eta 2}/2)}{dr} \frac{d(r\xi_r^2)}{dr} dr . \tag{53}$$

Note that the fifth term on the right side of (53) is of second order of largeness as already remarked. The $m^2/r^2 \, \nu^2 \xi_r^2$ term in (43a) gives nothing.

Expanding out the $[d/dr\,(\nu\xi_r)]^2$ term in (43a) in the same manner and collecting terms, we find that the two large terms cancel. Making use of eq. (51) again, assuming j^η and p^β are zero outside and denoting the radius at which the discontinuity occurs by R, we obtain finally

$$2\,\delta W^4 = \frac{8\pi^2}{k} \sum_{\substack{m>0\\ \eta}} \Big\{ \int_0^S r\,dr \Big[\Big(\frac{d\mu_{m;n}}{dr}\Big)^2 + \frac{\alpha_{m;n}\mu_{m;n}^2}{r^2} \Big] \; \frac{R j^\eta \mu_{m;n}^2}{\nu_{m;n}}\Big|_{r=R_-}$$

$$+\Big[[\![B^\beta]\!] \Big(\sum_i p_i^2 h^2 \sigma_i^0 \delta^2 B^0 \Big(\frac{I'_{\ell_i}(p_i hR)}{I'_{\ell_i}(p_i h S)}\Big)^2 \Big(1-2(\ell_i^2+p_i^2h^2R^2)\, I^{\ell_i}(p_i hR) + \ell_i^2\, I^{\ell_i}(p_i hR)^2\Big)$$

$$+ \sum_i q_i^2 h^2 \sigma_i \epsilon^2 B^0 \Big(\frac{I'_0(q_i hR)}{I'_0(q_i hS)}\Big)^2 \Big(1-2\,q_i^2h^2R^2 I^0(q_i hR)\Big)\Big)$$

$$- [\![B^\eta]\!] \sum_i \frac{\ell_i p_i h \sigma_i^0 \delta^2 B^0}{R} \Big(\frac{I'_{\ell_i}(p_i hR)}{I'_{\ell_i}(p_i hS)}\Big)^2 \Big(3-2(2+\ell_i^2+p_i^2h^2R^2) I^{\ell_i}(p_i hR)$$

$$+ (3\ell_i^2+2p_i^2h^2R^2)\, I^{\ell_i}(p_i hR)^2\Big)$$

$$- [\![B^{\eta^2}]\!]\Big] \frac{\mu_{m;n}^2}{\nu_{m;n}^2} \Big\}, \tag{54}$$

where

$$\alpha_{m;n} = m^2 + \frac{r^2}{\nu_{m;n}} \frac{dj^\eta}{dr} - \frac{j^\beta B^0}{\nu_{m;n}^2} \Big(\sum_i p_i^2 h^2 \sigma_i^0 \delta^2 r \Big(\frac{I'_{\ell_i}(p_i hr)}{I'_{\ell_i}(p_i hS)}\Big)^2 \Big(1-2(\ell_i^2+p_i^2h^2r^2) I^{\ell_i}(p_i hr)$$

$$+ \ell_i^2 I^{\ell_i}(p_i hr)^2\Big)$$

$$+ \sum q_i^2 h^2 \sigma_i \epsilon^2 r \Big(\frac{I'_0(q_i hr)}{I'_0(q_i hS)}\Big)^2 \Big(1-2\,q_i^2h^2r^2 I^0(q_i hr)\Big)\Big). \quad (0 < r < R) \tag{55}$$

and $\nu_{m;n}$ is given by (39).

In deciding the sign of δW^4 in eq. (54), one can treat μ as the arbitrary function instead of $|\xi_r^o|$ with the provision that $\mu(S)$ is zero, and further that μ vanishes at all points where ν does and μ' is continuous at these points. With this in mind, one can minimize the part of δW in the region $R < r < S$ over all μ such that $\mu(R)$ is prescribed. If ν does not vanish in this region, one gets for this part of $2\delta W$

$$m \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} \mu_{m;n}(R)^2 , \tag{56}$$

and thus can replace the limits on the integral in (54) by 0 to R and add this contribution. If, however, ν does vanish in this region, let aR be the smallest point at which it vanishes. The contribution to $2\delta W$ from the integral between R and aR has the minimum value

$$m \frac{a^{2m}+1}{a^{2m}-1} \mu_{m;n}(R)^2 \tag{57}$$

Because μ' at aR is continuous, the contribution to δW from the region aR to S is not zero, but it can be made as small as one wants, for instance, by taking

$$\mu = \frac{\mu'(aR)}{k} \sin k(r-aR) \qquad aR < r < aR + < \pi/k$$

$$\mu = 0 \qquad aR + \pi/k < r \tag{58}$$

and letting k approach infinity. Thus we may neglect any contribution from this region. Thus we may take (57) as the contribution to δW^4 from the external region.

Up to this point we have considered equilibria in which the region $R < r < S$ was filled with pressureless plasma. For equilibria in which this region is a true vacuum the energy principle can be generalized.[8] Instead of minimizing $\underline{Q}^2$ in this region, one minimizes $(\nabla \times \underline{A})^2$, subject to the boundary conditions $\underline{n} \times \underline{A} = (\underline{n} \cdot \underline{\xi}) \underline{B}$ at the plasma interface and $\underline{n} \times \underline{A} = 0$ at the external boundary. The minimum $\underline{A}$ satisfies

$$\nabla \times (\nabla \times \underline{A}) = 0 \tag{59}$$

But $\mu \underline{e}_z$ with μ given by equation (56) satisfies precisely these conditions, and yields the contribution (56) to δW. Thus in the case of a vacuum, we take the external contribution to be precisely (56) while in the case of a pressureless plasma one chooses (56) or (57) according to whether ν has a root in this region or not. The difference in the two cases may be seen by observing that the pressureless plasma develops a sheet current at aR which is not possible in a vacuum. It should be noted that the vacuum case is always unstable if the corresponding pressureless plasma case is, although the converse need not be valid. This is in agreement with the First Comparison Theorem of the energy principle paper.[8]

It is possible to rewrite eq. (54) entirely in terms of the rotational transform angle ι which was discussed in Appendix II A and the quantity $V''_{(vacuum)}$ which was obtained in Appendix II B ,

$$2\delta W^4 = \frac{8\pi^2}{k} \sum_{\substack{m>0 \\ n}} \{ \int_0^R r dr [(\frac{d\mu_{m;n}}{dr})^2 + \frac{\alpha_{m;n}}{r^2} (\mu_{m;n})^2]$$

$$- \frac{1}{R} \frac{d(r^2 \iota^\eta)}{dr} \frac{(\mu_{m;n})^2}{\bar{\nu}_{m;n}} \Big|_{r=R_-} + [- [\![\iota^{\eta^2}]\!] + [\![\iota^\eta]\!] \sum_i R \frac{d\iota_i^{\delta\delta}}{dr}$$

$$- \frac{4\pi^2 B^o}{k} [\![B^\beta]\!] V''_{(vacuum)}] \frac{(\mu_{m;n})^2}{(\bar{\nu}_{m;n})^2} \Big|_{r=R_-} + \Lambda_{m;n} \} , \qquad (60)$$

where

$$\mu_{m;n} = \frac{kr}{2\pi} B^o \bar{\nu}_{m;n} | \xi_r^o (m;n) | , \qquad (60a)$$

$$\bar{\nu}_{m;n} = \frac{2\pi n}{m} + \iota^\eta + \sum_i \iota_i^{\delta\delta} , \qquad (60b)$$

$$\alpha_{m;n} = m^2 + \frac{1}{r\bar{\nu}_{m;n}} \frac{d}{dr} (r^3 \frac{d\iota^\eta}{dr}) + \frac{4\pi^2 r B^o j^\beta}{k(\bar{\nu}_{m;n})^2} V''_{(vacuum)} \qquad (60c)$$

and $\Lambda_{m;n}$ is the contribution to $2\delta W$ from the external region given by equation (56) if the external region is a vacuum or by equation (56) or (57) if it is a pressureless plasma depending on whether $\bar{\nu}_{m;n}$ is zero in that region or not. Here the rotational transforms, $\iota_i^{\delta\delta}$ from the helically symmetric field depending on u_i, ι^η from the axial current, and the term $V''_{(vacuum)}$ are computed over the length of the machine $(2\pi/k)$.

The terms in j^β and $[\![B^\beta]\!]$ are similar to those which in the axially symmetric case[8] represent the energy released by the expansion of the gas.

The first two terms in δW represent the change in magnetic field energy produced by the perturbation. This energy cannot be avoided since in expanding the gas it is always necessary to twist the lines to some extent because ι varies from radius to radius and it is impossible to interchange these surfaces exactly. The terms in ι^{η} represent the work done by the force term $\underline{j} \times \underline{\delta B}$ (computed at a fixed point). It is only present if there is some voltage driving the longitudinal current j^{η}, and it is generally destabilizing.

Section IV - Stability of Equilibria II, Stability Criteria

In the last section the minimization of δW with respect to all components of $\underline{\xi}$ except $\xi_r^o(m;n)\, e^{i(m\theta + nkz)}$ was carried out. (In this section to "minimize" a quantity will be used to mean "reduce it to its smallest value", or when no such "minimum" exists, "to reduce it so as to approach its lowest bound".) Since the final minimization is too difficult to do in general, it will be done for several special cases in this section. "Interchange" instabilities which are due to a plasma pressure gradient will first be treated. "Kink" type instabilities due to the presence of axial currents in the plasma will then be discussed.

A. Interchange Instabilities

One main class of instabilities can be understood as an interchange of lines of force so as to carry plasma outward. It has been shown for axially symmetric situations that such an interchange tends to decrease the potential energy by an amount essentially proportional to $M'' V''$ where M and V are the mass and volume contained inside a surface of constant ψ. The prime represents differentiation in respect to ψ. If the magnetic lines must be distorted in order to make an interchange, energy must be given to the magnetic field. If this "twist energy" just balances the "destabilizing energy", the system is neutral in respect to such instabilities.

Probably the simplest case in which this effect can be seen is in the inherent stability of a stellarator with no stabilizing windings, for example B-1. The confining field in this machine is modified by a series of "bulges" due to the finite spacing of the coils which are used to maintain the longitudinal field. In this case the $m;n$ term in δW can be obtained from equations (54) and (55) of Section III as

$$2\delta W^{4}_{m;n} = \frac{8\pi^2}{k} \left\{ \int_0^R r dr \left\{ \frac{\alpha}{r^2} \mu^2 + \left(\frac{d\mu}{dr}\right)^2 \right\} + m\mu^2 \Big|_{r=R} \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} \right. \tag{1}$$

where

$$\alpha = m^2 \left\{ 1 - \frac{j^\beta}{B^o} \left(\frac{\sigma^\epsilon}{S}\right)^2 \frac{h^2 S^2}{n^2 k^2 r} \left(\frac{I_1(hr)}{I_1(hS)}\right)^2 (1 - 2h^2 r^2 I^o(hr)) \right\} , \tag{1a}$$

and

$$\mu = \frac{nkrB_o}{m} \xi_r^o . \tag{1b}$$

Since μ has no singularities the treatment is the same if the external region is a pressureless plasma as it would be if it were a vacuum. If the pressure is parabolic so that

$$\beta \equiv 2p_o/(B^o)^2 \doteq - \frac{j^\beta R^2}{B^o r} , \tag{1c}$$

where p_o is the pressure at the center of the plasma, and hR is small enough that higher order terms in the expansion of the Bessel functions can be neglected, Eq. (1a) reduces to

$$\alpha = m^2 - \sqrt{A} r^2 \tag{1d}$$

with

$$\sqrt{A} = \beta \, 3m^2 \left(\frac{\sigma_\epsilon}{R}\right)^2 \frac{h^2}{n^2 k^2 R^2} . \tag{1e}$$

The solution of the Euler equation for minimization in the internal region is, therefore,

$$\mu = \text{const.} \; J_m(\sqrt{A} r) . \tag{1f}$$

It may be assumed that $\mu(R) \neq 0$ since it can be seen easily that if δW is negative with $\mu(R) = 0$, it can be made negative with $\mu(R) \neq 0$. To see this let $\overline{\mu}$ make δW negative with $\overline{\mu}(R) = 0$. Then $\Lambda\overline{\mu} + \hat{\mu}$ will make δW negative for any $\hat{\mu}$ and Λ sufficiently large.

Multiplying the Euler equation by μ and integrating, one obtains

$$2\,\delta W^4_{m;n} = \frac{8\pi^2}{k}\left(\frac{\sqrt{AR}\,J_m'(AR)}{J_m(AR)} + m\,\frac{1+(R/S)^{2m}}{1-(R/S)^{2m}}\right)(\mu)^2_{r=R} \quad . \tag{1g}$$

If γ_m is the lowest solution of the trancendental equation

$$\frac{J_m(x)}{xJ_m'(x)} = -\frac{1}{m}\,\frac{1-(R/S)^{2m}}{1+(R/S)^{2m}} \quad , \tag{1h}$$

$\beta_{critical}$ is given by

$$\beta_{m;n} = \frac{\gamma_m^2}{3m^2}\,\frac{n^2 k^2}{h^2(\sigma^{\epsilon}/S)^2} \quad . \tag{1i}$$

Kruskal[14] has pointed out that in a closed system such as a stellarator only certain wave lengths are allowed. If the machine has a transform ι^M, it is necessary to require the matching condition

$$\xi(\theta_o, z_o) = \xi(\theta_o + \iota^M, z_o + L), \tag{1j}$$

where L is the length of the machine. Thus only wave numbers nk can be considered such that

$$n = N - \frac{m\,\iota^M}{2\pi} \tag{1k}$$

where N can be any integer. Then Eq. (1i) becomes

$$\beta_{m;n} = \frac{\gamma_m^{\;2} \; (L_B/L_M)^2}{12\pi^2 m^2 \; (\sigma^{\epsilon}/S)^2} \; (2\pi N - m\iota^M)^2 \quad , \tag{1l}$$

where L_B is the length of a bulge and L_M is the length of the machine. In the case of B-1, $\iota = 164^o$, and for values of m less than thirty, β is most severely restricted by m = 2 and m = 11. Higher m's can probably be ignored as the instabilities which they represent would be localized in a region smaller than the ion Larmor radius for which the present theory may not apply.

If (R/S) is allowed to approach 1, it is clear from Eq. (1h) that γ_M is the lowest root of the equation

$$J_m(\gamma_m) = 0 \quad . \tag{1m}$$

The effect of terms of the next order can be determined by simple perturbation theory. If

$$\alpha = m^2 - Ar^2 - \bar{A}r^4 \tag{1n}$$

and

$$\mu(R) = 0 \quad , \tag{1o}$$

β can easily be shown to be given by

$$\beta = \beta_o (1 - \frac{\bar{A}\int_o^R J_m^{\;2}(\sqrt{A}r)\, r^3 dr}{A\int_o^R J_m^{\;2}(\sqrt{A}r)\, r dr} \quad , \tag{1p}$$

where β_o is the lowest order critical β .

The determination of the critical β when stabilizing fields are present can be achieved in a similar way, but the presence of singularities in the Euler equation which is employed, complicate the determination.

Again it will be assumed that there is no axial current, i.e. $\underline{j}^{\eta} = 0$. It will be shown later in this section that helical fields with $\ell = 3$ provide optimum stabilization. This can be seen crudely from the fact that $d\iota^{\delta\delta}/dr$ is much larger for $\ell = 3$ than for $\ell = 1$ or 2 (see Appendix II A) while higher ℓ's require more power to produce. At first, therefore, only helical fields with $\ell = 3$ and bulges ($\ell = 0$) will be considered. Further, hR will be taken to be small.

If the plasma occupies the entire tube ($R = S$), the boundary condition on ξ_r^o at R will require that $\mu(R) = 0$. If two regions are considered ($R < S$), the condition that ξ_r^o is continuous at the boundary requires that μ be continuous at R. For the most part the two cases can be treated at the same time. Here the case where μ is zero at R will be treated. The minimization will be carried out for an arbitrary pressure distribution, which will then be selected to give the maximum value of $\beta_{critical}$.

The expression (Eq. (54) of Section III) for $\delta W_{m;n}$ can be written for $\mu(R) = 0$ as

$$2\delta W^4_{m;n} = \frac{8\pi^2}{k} \int_0^1 \left(\frac{\alpha}{t^2}\mu^2 + (\mu')^2\right) t\,dt \,, \tag{2}$$

where

$$\alpha = m^2 - \frac{\beta f'(t) \sum_i (2 p_i^2 \delta_i^2 t^3 + \frac{3}{2} q_i^2 \epsilon_i^2 t)}{(\frac{n}{m} + \sum_i 2 p_i \delta_i^2 t^2)^2} \,, \tag{2a}$$

and

$$\mu = kRB^o \left(\frac{\eta}{m} + \sum_i 2 p_i \delta_i^2 t^2\right) t\, \xi_r^o \,. \tag{2b}$$

Here

$$t = \frac{r}{R}, \ \delta_i = \rho_i^{\delta}/R \ , \ \epsilon_i = \rho_i^{\epsilon}/R \quad ; \tag{2c}$$

the prime represents differentiation with respect to t, and ρ_i^{δ} and ρ_i^{ϵ} are respectively the amplitudes of the Fourier components of the first order distortion of the plasma surface with wave lengths such that p_i helical and q_i bulge periods can fit into the machine. Thus any superposition of helical fields which have a bounding magnetic surface of the form $\sum_i \rho_i^{\delta} \cos(3\theta - p_i kz)$ is considered. Similarly, the $\ell = 0$ fields also can represent the superposition of many bulges with different wave lengths. (Recall that the restriction $p_i \neq q_i$ was made in Section II.) The function f(t), which is related to the pressure by

$$p^{\beta} = p^{\beta}(0)\,(1 - f(t)) \ , \tag{2 d}$$

is an arbitrary given function of t, such that

$$f(0) = 0 \ , \tag{2 e}$$

and

$$f(1) = 1 \ . \tag{2 f}$$

First f(t) will be taken to be monatonic. (Diffusion of the plasma would probably insure this.) The diamagnetic current is related to f(t) and β, where

$$\beta \equiv \frac{2p(0)}{B^{o2}} \ , \tag{2 g}$$

by

$$j^{\beta} = - \frac{\beta f'(t) B^{o}}{2R} \ . \tag{2 h}$$

Clearly μ must satisfy the conditions

$$\mu(0) = 0 \ , \tag{2 i}$$

and

$$\mu(1) = 0 \quad . \tag{2j}$$

If a is defined by

$$a^2 \equiv - n/2m \sum_i p_i \; \delta_i^2 \quad , \tag{2k}$$

$\mu(t)$ must go to zero as t approaches a at least as fast as $t - a$. (That is,

$$|\mu(t)| \leq |(t-a) \frac{2\,nk\,R\,B_o}{m} \xi_M| \quad , \tag{2ℓ}$$

where ξ_M is the maximum value of ξ_r^o in the vicinity of a.) $\mu'(a)$ must be continuous.

Let

$$\Gamma(t) \equiv \frac{\sum_i (2 p_i^2 \delta_i^2 t^3 + \frac{3}{2} q_i^2 \epsilon_i^2 t)}{(2 \sum_i p_i \delta_i^2)^2} \quad . \tag{2m}$$

Then

$$2\,\delta W^4_{m;\,n} = \frac{8\pi^2}{k} \int_o^1 \{ (\frac{m^2}{t^2} - \frac{\beta f'(t) \Gamma(t)}{t^2 (t^2 - a^2)^2}) \mu^2 + (\mu')^2 \} \; t \; dt \quad . \tag{2n}$$

The critical β will be determined first for those values of n and m for which a is in the range $0 \leq a \leq 1$. The minimizing μ must satisfy the Euler equation

$$\mu'' + \frac{1}{t} \mu' + (\frac{\beta f'(t) \Gamma(t)}{t^2 (t^2 - a^2)^2} - \frac{m^2}{t^2}) \mu = 0 \; . \tag{2o}$$

(The normality condition does not have to be carried explicitly in this discussion.) In the vicinity of the singularity at a, Eq. (2o) behaves like

$$\mu'' + \frac{1}{a}\,\mu' + \left(\frac{\beta f'(a)\Gamma(a)}{4a^4(t-a)^2} - \frac{m^2}{a^2} \right) \mu = 0 \ . \tag{2 p}$$

The contributions to δW from the two regions $t < a$ and $t > a$ will be considered separately.

For $t > a$ the solutions of Eq. (2 p) are

$$\mu = \sqrt{(t-2)}\ e^{\pm \frac{1}{2} \sqrt{1-A}\ \ell n\ (t-a)} \ , \tag{2 q}$$

where

$$A = \frac{\beta f'(a)\Gamma(a)}{a^4} \ . \tag{2 r}$$

It now will be shown that for the values of m and n under consideration, $\beta_{critical}$ is determined by setting A equal to 1 .

If $A > 1$, the solution μ of Eq. (2 o) which vanishes at $t = 1$, varies for t sufficiently near a , as

$$\mu = \sqrt{(t-a)}\ \cos\left(\tfrac{1}{2} \sqrt{A-1}\ \ell n\ (t-a) + \gamma\right) . \tag{2 s}$$

It, therefore, must possess at least one zero for $t > a$. For a particular value of A, $A_1 > 1$, let t_1 be the largest zero below 1 . Consider for this A_1 a function μ_1 , defined to be identically zero for $t < t_1$ and to be a solution of Eq. (2o) for $t > t_1$. It follows from Eqs. (2 n) and (2 o) that for this μ_1 ; $\delta W = 0$ if $A = A_1$ and $\delta W < 0$ for $A > A_1$. Therefore, the critical β corresponds to a value of $A \leq 1$.

In cases where $A < 1$, first consider any μ which is identically zero for all t less than some $t_1 > a$, and vanishes at $t = 1$. It will now be shown that for this t_1 , an $A_1 > 1$ exists such that δW is positive if A

is less than A_1 and consequently positive for all A less than one. Consider the pressure distribution given by Eq. (3d) which will later be found to lead to be largest value of $\beta_{critical}$. It can be seen that for this pressure distribution with m and $A = \beta/\beta_{critical}$ set equal to one, the general solution of the Euler equation, Eq. (2o), is

$$\mu = [1-(a^2/t^2)]^{1/2} [C + D \ln (t^2-a^2)] \quad . \tag{2t}$$

This μ cannot vanish for t between a and 1 since it must vanish at $t = 1$. Since δW, given by Eq. (2n) is a continuous function of β, t_1, the position at which the minimizing μ must first vanish for a given $A_1 > 1$, must approach a as A_1 approaches one. It can, therefore, be seen that for any $t_1 > a$, an $A_1 > 1$ exists below which $\delta W > 0$ for any μ which is identically zero for $t < t_1$ and vanishes at $t = 1$.

It is still necessary to show that no other μ which vanishes at $t = a$ and $t = 1$ (i.e., $t_1 = a$) can cause instability with $A < 1$. To do this, assume that such a μ ($\overline{\mu}$) exists which makes δW negative, say $\delta W = -\epsilon$. It will first be shown that for any δ sufficiently small, the contribution to δW from the region between a and $a + \delta$ is positive so that the integral from $t = a + \delta$ to $t = 1$ must be more negative than $-\epsilon$. This integral then will be shown to differ from a positive integral by an amount which can be made as small as desired by taking δ sufficiently small. This contradiction will complete the proof that $\beta < \beta_{critical}$ if $A < 1$.

The contribution to δW from the region between a and $a + \delta$ is found from Eq. (2n), to be

$$I = \frac{8\pi^2}{k} a \int_a^{a+\delta} \{(\overline{\mu}')^2 + (\frac{m^2}{a^2} - \frac{A}{4(t-a)^2})\mu^2\} dt . \tag{2u}$$

Let

$$x = t - a \tag{2v}$$

and

$$\overline{y} = \overline{\mu}/\sqrt{x} . \tag{2w}$$

Then

$$I = \frac{8\pi^2}{k} a \int_0^{\delta} \{x(\overline{y}')^2 + \frac{1+(2mx/a)^2 - A}{4x} \overline{y}^2 + \frac{1}{2}(\overline{y}^2)'\} dx , \tag{2x}$$

where the prime indicates differentiation in respect to x . The first two terms are positive since $A < 1$. Eqs. (2 ℓ) and (2 w) show that $\overline{y}(0) = 0$ so that the last term on integration is obviously positive for any $\overline{y}$.

Now consider the value of δW which corresponds to a particular μ ($\hat{\mu}$) which is zero for $t < a + \delta/2$, increases linearly over the region between $a + \delta/2$ and $a + \delta$ and from there to $t = 1$ is the same as the previous $\overline{\mu}$, which was assumed to make δW negative. Since A is less than the A_1 which determines the solution of the Euler equation which vanishes at $t_1 = a + \delta/2$, this value of δW must be positive. It differs from the one for $\overline{\mu}$, which had to be less than $-\epsilon$ by the amount

$$I = \frac{8\pi^2}{k} a \int_{\delta/2}^{\delta} \{\hat{\mu}'^2 + (\frac{m^2}{a^2} - \frac{A}{4x^2})\hat{\mu}^2\} dx . \tag{2y}$$

When this integral is evaluated, it is found to be less than $\frac{64\pi^2}{k} \frac{n^2k^2R^2}{m^2} B_o^2 \xi_M^2 \delta$ (Use Eq. (2 ℓ).) and can be made smaller than ϵ by a suitable choice of δ .

This completes the proof that no μ exists which can make δW negative if $A < 1$, and, therefore, that $\beta_{critical}$ is determined in the region $t > a$ by setting $A = 1$ in Eq. (2 r).

Eq. (2 q) can be replaced by

$$\mu = \sqrt{(a-t)}\; e^{\pm \frac{1}{2}\sqrt{1-A}\; \ln\,(a-t)} \tag{2z}$$

in the region where t is less than a, and the entire argument can be repeated to show that $A = 1$ determines the critical β.

If a is zero or one, the argument can still be carried through. Values of n and m for which a^2 is not in the range $0 \leq a^2 \leq 1$ must still be considered. Before considering these, the pressure distribution which maximizes the critical β with a^2 in this range will be determined. It will then be shown that for this optimum pressure distribution, instabilities for which a^2 is not in this range lead to higher critical values of β.

If $0 \leq a^2 \leq 1$, the critical β is determined by setting

$$\beta_{critical} = \min_{a} \frac{a^4}{f'(a)\Gamma(a)} \quad . \tag{3}$$

It is now necessary to determine the pressure distribution, i.e., f (t), so that for the worst value of a, $\beta_{critical}$ is as large as possible. For any particular pressure distribution f (t), Eq. (3) requires that

$$\beta_{critical} \leq \frac{a^4}{f'(a)\Gamma(a)} \tag{3a}$$

or

$$\beta_{critical}\, f'(a) \leq \frac{a^4}{\Gamma(a)} \; . \tag{3b}$$

If large enough values of n and m are considered, a can take on essentially any value, so that it may be treated as a continuous variable. Integrating both sides of Eq. (3 b) with respect to a, and using the boundary conditions on f(t) given in Eqs. (2 e) and (2 f), one finds that for any f

$$\beta_{\text{critical}} \leq \int_0^1 \frac{a^4}{\Gamma(a)} \, da \quad . \tag{3c}$$

Now consider a pressure distribution defined by

$$f'(t) = \frac{t^4/\Gamma(t)}{\int_0^1 (t^4/\Gamma(t))\,dt} \quad . \tag{3d}$$

Then, inserting this f(t) into Eq. (3),

$$\beta_{\text{critical}} = \min_a \int_0^1 \frac{a^4}{\Gamma(a)} \, da = \int_0^1 \frac{t^4}{\Gamma(t)} \, dt \quad . \tag{3e}$$

Since $\Gamma(t)$ is given by Eq. (2 m), the optimum pressure is given by

$$f(t) = \frac{t^2 - \frac{3\phi}{4} \ln\left(1 + \frac{4t^2}{3\phi}\right)}{1 - \frac{3\phi}{4} \ln\left(1 + \frac{4}{3\phi}\right)} \,, \tag{3f}$$

and the critical β is

$$\beta_{\text{critical}} = \frac{(\Sigma\, p_i \delta_i^2)^2}{\Sigma\, p_i^2 \delta_i^2} \left(1 - \frac{3\phi}{4} \ln\left(1 + \frac{4}{3\phi}\right)\right) \quad . \tag{3g}$$

Here

$$\phi = \frac{\Sigma\, q_i^2 \epsilon_i^2}{\Sigma\, p_i^2 \delta_i^2} \quad . \tag{3h}$$

If no bulges are present, the optimum pressure distribution is parabolic. If, in addition, only one helical field is present, the inherent stability is given by

$$\beta_{critical} = \left(\frac{\rho^{\delta}}{R}\right)^2 , \tag{3i}$$

for the case of $\ell = 3$.

Eq. (3 g) can be somewhat simplified[15] for use in numerical computations by an application of Parseval's theorem. In particular, since to first order the distortion of the plasma surface due to the bulges (ℓ=0) is

$$\frac{\rho}{R} = \sum_s \epsilon_s \cos(q_s kz + \alpha_s) , \tag{3j}$$

so that

$$\frac{1}{R}\frac{d\rho}{dz} = - \sum_s \epsilon_s \; q_s k \sin(q_s kz + \alpha_s) , \tag{3k}$$

it can be seen by squaring both sides of Eq. (3 k) and integrating over the length of the machine that

$$\sum q_s^2 \epsilon_s^2 = \frac{1}{\pi k R^2} \int_0^{2\pi/k} (d\rho/dz)^2 \, dz . \tag{3ℓ}$$

Since the field along the magnetic axis due to the bulges ($\ell = 0$) is

$$B_z^{\epsilon} = - \sum_s 2 B_o \epsilon_s \cos(q_s kz + \alpha_s) \tag{3m}$$

it can be seen in the same way that

$$\sum q_s^2 \epsilon_s^2 = \frac{1}{4\pi k B_o^2} \int_0^{2\pi/k} (dB_z^{\epsilon}/dz)^2_{R=0} \, dz . \tag{3n}$$

Helical fields with different values of ℓ can be treated in exactly the same way. In particular,

$$\sum_s \delta_s^2 = \frac{k}{\pi R^2} \int_0^{2\pi/k} \rho^2 \, dz \, , \tag{3o}$$

$$\sum_s p_s^2 \, \delta_s^2 = \frac{1}{\pi k R^2} \int_0^{2\pi/k} (d\rho/dz)^2 \, dz \, , \tag{3p}$$

where ρ is the lowest order distortion of the plasma surface due to the helical fields. These expressions could be expressed in terms of the helical contribution to the magnetic field along the magnetic axis as was done in Eq. (3n) for bulges.

The expression, $\sum_s p_s \delta_s^2$, cannot be converted into such a form. A perspicuous form of it is

$$\left(\sum_s p_s \delta_s^2\right)^2 = \left(\sum_s \delta_s^2\right)\left(\sum_s p_s^2 \delta_s^2\right) \cos^2 \Theta \tag{3q}$$

where

$$\cos \Theta = 1 - \frac{\sum_s (p_s - \underline{P})^2 \delta_s^2}{2 \sum_s p_s^2 \delta_s^2} \, , \tag{3r}$$

and

$$\underline{P}^2 = \frac{\sum_s p_s^2 \delta_s^2}{\sum_s \delta_s^2} \, . \tag{3s}$$

If the p_s' do not differ by much from each other, $\cos \Theta \sim 1$.

It is still necessary to show that instabilities for which a^2 lie outside the range which has been considered, do not lead to lower values of $\beta_{critical}$.

For the pressure distribution, Eq. (3d), δW is given by Eq. (2n) as

$$2\delta W = \frac{8\pi^2}{k}\int_0^1 \{(\mu')^2 + (\frac{m^2}{t^2} - \frac{\beta}{\overline{\beta}_c}\frac{t^2}{(a^2-t^2)^2})\mu^2\}\, t\,dt \qquad (3t)$$

where $\overline{\beta}_c$ is the $\beta_{critical}$ defined by Eq. (3e). Obviously δW is positive if a^2 is less than zero and β is less than $\overline{\beta}_c$. In order to consider cases where a^2 is greater than 1, one can make the transformation

$$\mu = \sqrt{(a-t)}\; y \; . \qquad (3u)$$

in Eq. (3t). Then

$$2\delta W = \frac{8\pi^2}{k}\int_0^1 \{(a-t)(y')^2 - \frac{1}{t}(\frac{ty^2}{2})'$$

$$+ (\frac{1}{2t} + \frac{1}{4(a-t)} + \frac{m^2(a-t)}{t^2} - \frac{\beta}{\overline{\beta}_c}\frac{t^2}{(a-t)(a+t)^2})y^2\}\, t\,dt\,. \qquad (3v)$$

It can easily be seen that if $\beta/\overline{\beta}_c$ is less than 1 the last term in Eq. (3v) is less than $(1/4(a-t))y^2 t$. Since $y(1) = 0$, δW is positive.

For equilibria in which the plasma does not fill the entire tube ($R < S$), the analysis can be carried through in the same way as before for values of m and n which make $a^2 < 1$. The argument which shows that all μ's which vanish at t_1 ($0 < a < t_1 < 1$) make δW positive if A is less than one can be carried through as before since it is not changed by replacing the boundary condition $\mu(1) = 0$ with $\mu'(1)/\mu(1) \leq -m$. When a pressureless plasma exists between the plasma boundary R and the walls of the system and $1 < a^2 < (S/R)^2$, $2\delta W$ differs from Eq. (3v) by the term

$$+ \frac{8\pi^2}{k} m \frac{a^{2m}+1}{a^{2m}-1} (a-1) y(1)^2 ,$$

the contribution to δW from the external region. (This term is obtained from Eq. (57) of Section III.) Thus, if $\beta < \overline{\beta}_c$,

$$2\delta W > \frac{8\pi^2}{k} \left(m \frac{(a^{2m}+1)(a-1)}{(a^{2m}-1)} - \frac{1}{2} \right) y(1)^2 \qquad (3\,w)$$

and, therefore, greater than zero for any a which is greater than or equal to one. This completes the proof that the optimum pressure distribution for stability is given by Eq. (3 f) and the critical β by Eq. (3 g).

If the external region is really a vacuum, $\mu(a)$ is not necessarily zero so that the stabilizing term due to the external region, (by Eq. (56) of Section III)

$$+ \frac{8\pi^2}{k} m \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} (a-1) y(1)^2 ,$$

is negligibly small if a is near 1 . It can be shown that the system is unstable for some values of m and n which make a sufficiently near one, if $f'(1)$ does not vanish. The pressure distribution would then be expected to adjust itself so as to satisfy the condition $f'(1) = 0$. The critical β would therefore be somewhat lower if the external region is a vacuum, than that given by Eq. (3 g) which was obtained by treating the external region as a pressureless plasma.

The function $f(t)$ has been assumed to be monatonic. Consider some non-monatonic pressure distribution $\overline{f}(t)$. Then by Eqs. (2 e) and (2 f), $\overline{f}'(t)$ must be greater than $f'(t)$ given by Eq. (3 d) for some value of t . The critical β corresponding to instabilities centered at this point would be lower than the one in Eq. (3 g) so that such a pressure distribution would not be the optimum one. This follows from the fact that for the optimum f, the right hand side of Eq. (3 a) is the constant of Eq. (3 e).

The problem can be carried through in exactly the same way for the case where hR is finite and any combination of helical fields (with any ℓ) amd "bulges" are present (subject to the condition that no two fields with different values of ℓ have the same wave length). Again consider the case where all surface currents are zero and j^{η} and therefore ι^{η} is zero. From Eq. (6 0) of Section III

$$2\,\delta W^{4}_{m;n} = \frac{8\pi^2}{k}\left\{\int_0^1 \left((\mu')^2 + \frac{\alpha}{t^2}\mu^2\right) t\,dt + m\,\frac{a_+^{2m}+1}{a_+^{2m}-1}\,\mu(1)^2\right\}, \qquad (4)$$

$$\alpha = m^2 - \frac{2\pi^2 \beta B^{o\,2}\, t f'(t)\, V''(\psi(t))_{(vacuum)}}{k\left(\frac{2\pi n}{m} + \sum_s \iota_s(t)\right)^2}\,, \qquad (4\,a)$$

and

$$\mu = \frac{kRB^o}{2\pi}\left(\frac{2\pi n}{m} + \sum_s \iota_s(t)\right) t\,\xi_r^{\,o}\,. \qquad (4\,b)$$

Again j^{β} has been expressed in terms of the pressure distribution given by Eq. (2 d) by means of Eq. (2 h). Here $\iota_s(t)$, the transform associated with a helical field which depends on θ and z as $\cos(\ell_s\theta - p_s hz)$, and $V''_{(vacuum)}$ are computed over the length of the machine $(2\pi/k)$. Note that bulge fields are included here by setting $\ell_s = 0$ for some values of s. The numbers a are defined by

$$\frac{2\pi n}{m} + \sum_s \iota_s(a) = 0\,. \qquad (4\,c)$$

The number a_+ is the smallest root of Eq. (4 c) such that $1 < a_+ < S/R$ or if no roots exist in this range, $a_+ = S/R$.

For any a, α can then be written as

$$\alpha = m^2 - \frac{\beta f'(t)\Gamma(t)}{(a-t)^2} \tag{4d}$$

where

$$\Gamma(t) = \frac{2\pi^2 B_o^2 \, t \, V''_{(vacuum)}}{k[\sum_s ((\iota_s(a))' + (t-a)(\iota_s(a))''/2! + \ldots)]^2} . \tag{4e}$$

Here the $\iota_s(t)$ has been expanded as a Taylor series about the point, $t = a$.

As in the previous analysis, the worst instability for values of a between zero and one lead to critical β's given by

$$\beta_{critical} = \min_a \frac{a^2}{4f'(a)\Gamma(a)} . \tag{4f}$$

Continuing in the same way one finds that for these values of a, the cricital β is given by

$$\beta_{critical} = \frac{k}{8\pi^2 B_o^2} \int_0^1 \frac{t(\sum_s d\iota_s^{\delta}(t)/dt)^2 \, dt}{V''(\psi(t))_{(vacuum)}} , \tag{4g}$$

and the optimum pressure distribution is

$$f(t) = \frac{k}{8\pi^2 B_o^2 \beta_{critical}} \int_0^t \frac{t(\sum_s d\iota_s^{\delta}(t)/dt)^2 \, dt}{V''(\psi(t))_{(vacuum)}} . \tag{4h}$$

The argument which was made in connection with Eq. (2t) in the preceding calculation has not yet been carried through completely for this case. The demonstration that values of a outside the range between zero and one do not lead to a lower critical β if the external region is a pressureless plasma goes through in exactly the same way as before.

In minimizing δW with β just slightly above β_c, we find that the $\underline{\xi}$'s which make δW negative change extremely rapidly over a small distance, and thus describe motions for which the present theory may not apply if this distance is as small as the ion Larmor radius. It is, therefore, interesting to assume that such motions are stable and ask if β_c is raised appreciably. We determine the new β_c by minimizing δW over all $\underline{\xi}$ such that $d\underline{\xi}/dx < \underline{\xi}\ max/\lambda$ where $x = r/R$, $\underline{\xi}_{max}$ is the maximum value of $\underline{\xi}$, and λ is the ion Larmor radius in units of the radius of the plasma. A rough idea of the result of minimizing δW may be obtained by picking ξ_r^o to be a constant ξ_{max} in the neighborhood of a, $a < x < a + \alpha$, and to be a solution of the Euler equation in the region $a + \alpha < x < 1$, joined to $\xi_r^o = \xi_{max}$ at $a + \alpha$ so that ξ_r^o and its derivative are continuous. A similar function is chosen for $x < a$. We pick α so that the maximum slope attained by the solution of the Euler equation is ξ_{max}/λ. One may then conclude that if the solution of the Euler equation vanishes before the boundary of the plasma is reached ($x = 1$), then the system is unstable since it can easily be seen that δW is negative for this trial function. It is not clear that this trial function gives the lowest value for δW for our restriction but it is expected that it will give a good approximation to the β for which the lowest value of δW first becomes negative.

This program for determining the new value of β_c is carried out by a further approximation which makes use of the fact that the rapid behavior of ξ_r^o occurs for x close to a. We, therefore, approximate the solution of the Euler equation by a series expansion in the neighborhood of a keeping only the first two terms. For the trial function and in the situation corresponding to Eq. (3 t) we have

$$\delta W \sim \frac{\beta}{4} \alpha\, a \xi^2_{max} + a \int_{\alpha}^{1-a} (t^2 \xi'^2 + \frac{\beta}{4} \xi^2 + \frac{m^2}{a^2} \xi^2 t^2)\, dt \tag{4i}$$

where $t = x - a$, x has been replaced by a wherever possible, and ξ_r^o has been replaced by ξ. The Euler equation for ξ is

$$\xi'' + \frac{2\xi'}{t} + \frac{\beta}{4} \frac{\xi}{t^2} - \frac{m^2}{a^2} \xi = 0 \tag{4j}$$

and its solution is

$$\xi \sim t^n + \gamma t^{n+2} + \ldots \tag{4k}$$

with

$$n = -\frac{1}{2} \pm i \frac{\sqrt{\beta - 1}}{2} , \tag{4ℓ}$$

$$\gamma = \frac{m^2/a^2}{4 \pm 2i \sqrt{\beta - 1}} , \tag{4m}$$

or

$$\xi = A t^{-1/2} \sin(\frac{\sqrt{\beta-1}}{2} \ln t + \delta) + A \gamma_o t^{3/2} \sin(\frac{\sqrt{\beta-1}}{2} \ln t + \delta_1) \tag{4n}$$

where δ and A are arbitrary,

$$\gamma_o = \frac{m^2/a^2}{2 \sqrt{3 + \beta}} \tag{4o}$$

and

$$\tan(\delta - \delta_1) = \sqrt{\beta - 1}/2 . \tag{4p}$$

At $t = t_o \equiv \alpha$, we require $d\xi/dt = 0$ and $\xi = \xi_{max}$, which gives (using only the first term in Eq. (4n),

$$\xi_{max} = A t_o^{-1/2} \sin \theta_o , \tag{4q}$$

$$\tan \theta_o = \sqrt{\beta - 1} \quad , \tag{4r}$$

where $\theta = \sqrt{(\beta - 1)/2}\, \ln t + \delta$ and θ_o is its value for $t = t_o$. The maximum slope occurs when $t = \bar{t}$ where again using only the first term

$$\tan \bar{\theta} = \frac{4\sqrt{\beta - 1}}{4 - \beta} \tag{4s}$$

and setting

$$|\xi'| = \xi_{max}/\lambda \quad , \tag{4t}$$

or

$$\xi_{max}/\lambda = A \, \frac{\sin \bar{\theta} - \sqrt{\beta - 1} \cos \bar{\theta}}{2\bar{t}^{\,3/2}} \quad . \tag{4u}$$

Eqs. (4q), (4r), and (4u) serve to determine t_o, δ, and A. To settle the question whether or not the solution (4n) crosses the axis before $t = 1\text{-}a$, we note that such a trial function as we have chosen could always be made to cross the axis as close to zero as one pleases by removing the restriction on ξ'_{max} and choosing t_o sufficiently small. Further the first term alone would vanish at t_1 where $\theta_1 = \pi$, and it is expected that t_1 is quite small compared with $1 - a$. Therefore, instead of giving an exact answer to our question which would require numerical integration, we demand only that the second term in (4n) be significant at t_1, or that

$$\gamma_o \, t_1^2 = C(\beta) \, \frac{m^2 \lambda^2}{a^2} \sim 1 \tag{4v}$$

be the critical condition for instability. Here

$$C(\beta) = \frac{B^2}{2\sqrt{3+\beta}} \, \exp \frac{4}{\sqrt{\beta - 1}} \, (\pi - \theta_o) \; , \tag{4w}$$

where

$$B = \frac{\sin\bar{\theta} - \sqrt{\beta-1}\cos\bar{\theta}}{2\sin\theta_o D^{3/2}} \tag{4x}$$

and

$$D = \exp\frac{2(\bar{\theta}-\theta_o)}{\sqrt{\beta-1}}\ , \tag{4y}$$

with θ_o and $\bar{\theta}$ given by Eqs. (4 r) and (4 s). A table of $C(\beta)$ is adjoined. On inspection of this table and assuming reasonable values of λ/a it is seen that the critical value of β may be raised from 1 to at least 2 for all but the lowest values of m .

β	C
1.5	300
2.0	40
4.0	1.3

B. Kink Instabilities

A current in the direction of the magnetic field is employed in an early stage of heating of the plasma to thermonuclear temperatures. The instabilities associated with this current[16] have been studied by Kruskal and Tuck[14,17], Roberts[18], Taylor[19], and Shafranov[20], by means of normal mode calculations and by Rosenbluth and Longmire[21,22], using an individual particle picture as well as by the Matterhorn group who have used the energy principle. In this section it will be shown how these instabilities (which are usually called Kink Instabilities) can be found easily by means of the energy principle, and the stabilization which can be obtained by applying a helical field will be discussed.

If no helical fields (or bulges) are present, Eqs. (54) and (55) of section III reduce to

$$2\delta W = \frac{8\pi^2}{k} \sum_{\substack{m \geq 0 \\ n}} \Bigg\{ \int_0^R \left(\frac{\alpha}{r^2} \mu^2 + \left(\frac{d\mu}{dr}\right)^2 \right) r\,dr + m \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} (\mu)^2_{r=R_+} - \frac{(R\, j^\eta \mu^2)_{r=R_-}}{\left(\frac{nkr\,B^o}{m} + B^\eta\right)_{r=R_-}} - \frac{[\![B^{\eta 2}]\!] (\mu)^2_{r=R_-}}{\left(\frac{nkr\,B^o}{m} + B^\eta\right)^2_{r=R_-}} \Bigg\} , \tag{5}$$

where

$$\alpha = m^2 + \frac{r^2 \frac{dj^\eta}{dr}}{\left(\frac{nkr\,B^o}{m} + B^\eta\right)} , \tag{5a}$$

and

$$\mu = \left(\frac{nkr\,B^o}{m} + B^\eta\right) \xi^o_r . \tag{5b}$$

Notice that j^{β} does not enter this expression. The external region is considered to be a vacuum.

If the current density j^{η} is uniform in the plasma, the volume terms can be minimized easily and $\delta W^4_{m;n}$ can be put into the form

$$2\,\delta W^4_{m;n} = \frac{8\pi^2}{k} (\mu)^2_{r=R_-} \Big\{ m + m \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} \frac{\left(\frac{nkr\,B^o}{m} + B^{\eta}\right)^2_{r=R_+}}{\left(\frac{nkr\,B^o}{m} + B^{\eta}\right)^2_{r=R_-}}$$

$$- \frac{(R\,j^{\eta})_{r=R_-}}{\left(\frac{nkr\,B^o}{m} + B^{\eta}\right)_{r=R_-}} - \frac{[\![{B^{\eta}}^2]\!]}{\left(\frac{nkr\,B^o}{m} + B^{\eta}\right)^2_{r=R_-}} \Big\} . \qquad (5\,c)$$

If only a surface current is present, this reduces to

$$2\,\delta W = \frac{8\pi^2}{k} (\xi_r^o)^2 \Big\{ m \left(\frac{nkr\,B^o}{m}\right)^2 + m \left(\frac{nkr\,B^o}{m} + B^{\eta}\right)^2 \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}}$$

$$- {B^{\eta}}^2 \Big\} \qquad (5\,d)$$

and is stable if for each n and m,

$$\frac{n^2 k^2 R^2 {B^o}^2}{m} + m \left(\frac{nkR\,B^o}{m} + B^{\eta}\right)^2 \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} - {B^{\eta}}^2 > 0 \ . \qquad (5\,e)$$

The inequality is satisfied for $n > 0$. For $m = 2$, the inequality (5 e) is satisfied for any finite S. If S is finite, the $m = 2$ mode is neutral for some values of B^{η} and n. All higher m's are clearly stable for any value of S. If $m = 1$, the inequality (5 e) reduces to

$$n^2k^2R^2{B^o}^2 + nkRB^oB^\eta(1+(R/S)^2)+{B^\eta}^2(R/S)^2 > 0 . \qquad (5f)$$

This shows that an instability occurs for negative n if

$$1 < \frac{B^\eta}{|n|kRB^o} < (\frac{S}{R})^2 . \qquad (5g)$$

The first inequality of (5 g) defines the usual Kruskal limit for stability; the second inequality is utilized in the stabilized pinch where the stabilizing effect of the conducting boundary must be employed.

Kruskal and Tuck , and Rosenbluth considered situations in which all the fields are finite whereas in this treatment kR and B^η/B^o are infinitesimal quantities of the same order. Their results reduce to the inequality (5 g) in the limit of small kR . Since B_z is finite no "sausage" type (m = 0) instability can exist here.

Now consider the case where the volume current density j^η is uniform in the plasma and no surface current is present. Then, from Eq. (5 c) ,

$$2\delta W = \frac{16\pi^2}{k}\left(\frac{nkRB^o}{m}+B^\eta\right)^2({\xi_r}^o)^2\left\{\frac{m}{1-(R/S)^{2m}} - \frac{B^\eta}{\left(\frac{nkRB^o}{m}+B^\eta\right)}\right\}, \qquad (5h)$$

where all quantities are evaluated at R . This can be reduced to

$$2\delta W = \frac{16\pi^2}{k}({\xi_r}^o)^2\,\frac{(m-1+(R/S)^{2m})}{1-(R/S)^{2m}}\left(B^\eta+\frac{nkRB^o}{m}\right)\left(B^\eta+\frac{nkRB^o}{m-1+(R/S)^{2m}}\right) . \qquad (5i)$$

Again the system is stable for all positive n . It is unstable for negative n if

$$\frac{1}{m} < \frac{B^{\eta}}{|n| kR B^{o}} < \frac{1}{m - 1 + (R/S)^{2m}} \quad . \tag{5j}$$

For $m = 1$ the stability criterion is the same as in the case where only a surface current exists. Here, however, instabilities exist for all higher m's. If one introduces the matching condition by Eq. (1k) and recognizes that

$$\iota^{\eta}(R) = \frac{LB^{\eta}(R)}{R B^{o}} \tag{5k}$$

is the transform over the length of the machine produced by the heating current, the condition for instability can be written as

$$\frac{m \iota^{M} - 2\pi N}{m} < \iota^{\eta} < \frac{m \iota^{M} - 2\pi N}{m - 1 + (R/S)^{2m}} \quad . \tag{5ℓ}$$

It is clear from Eq. (5ℓ) for any ι^{η}, no matter how small, values of m and N exist such that an instability can occur. However, if ι^{η} is small the range of ι^{η} for which the system will be unstable for given m and N will also be small. A rough estimate shows that the rates of growth of these large m instabilities are small when compared with that for $m = 1$. Considerable evidence has been found for the existence of the $m = 1$ instability (i.e., the Kruskal limit) in experiments with the B-1 stellarator; other instabilities have not yet been specifically identified. It is possible that they do not grow rapidly enough to disturb the plasma before the heating current has been increased out of the unstable range.

The results which have been obtained here for the case where the axial current is confined to surface of the plasma, and the case where it is distributed uniformly in the plasma apply to situations in which the external region is really a vacuum incapable of sustaining a current. If the external region is a pressureless plasma, which can support a current, the same results apply for only those values of m and N for which $-2\pi N/m + \iota^M$ is greater than $\iota^\eta(R)$ so that μ need not necessarily vanish in the external region. Otherwise μ must vanish at some point a, in the external region, and δW is obtained by replacing S in Eq. (5d) by $a = (-mRB^\eta(R)/nkB^o)^{1/2}$. If only a surface current is present, and if $mB^\eta(R)/nkRB^o$ is large enough that μ vanishes for $m = 1$ in the external region, the system is neutral for $m = 1$ to this order and stable for higher values of m. It is, therefore, necessary to expand δW to a higher order in λ in order to determine the stability condition. It is shown in Eq. (A 31) of Appendix IV A that for this case the system is unstable. In the case where a uniform axial current is in the interior region but no surface current is present, the calculation must again have to be carried to a higher order, if the external region is considered a pressureless plasma rather than a vacuum, to show instability.

In order to understand the effects of an arbitrary axial current distribution, calculations are now carried through assuming that the radial dependence of j^η is either $(r/R)^p$ or $(1-(r/R)^p)$ where p can have any positive value. The external region is again a vacuum.

If $j^\eta \sim (r/R)^p$ it follows from Eqs. (5), (5a), and (5b) that

$$2\,\delta W = \frac{8\pi^2}{k} \sum_{\substack{m>0\\n}} \Big\{ \int_0^1 \Big(\mu'^2 + \Big(\frac{m^2}{t^2} + \frac{p(p+2)t^{p-2}}{t^p - a^p}\Big)\mu^2\Big)\, t\, dt + \Big(m - \frac{(p+2)}{1-a^p}\Big)\,\mu(1)^2 \Big\}, \tag{6}$$

where

$$\mu = kRB^o \xi_r^{\ o}(m;n)\, t\,(a^p - t^p) \;, \tag{6 a}$$

$$a^p = -\frac{2\pi n}{m\iota^\eta} \;. \tag{6 b}$$

Here, t equals r/R, and ι^η is the transform at the plasma boundary produced by the axial current in the plasma which will be taken to be positive. The external region is infinite ($S = \infty$) in Eq. (6). The consideration of finite S leads to nothing basically new. The Euler equation which the minimizing μ must satisfy is

$$\mu'' + \frac{1}{t}\mu' - \left(\frac{m^2}{t^2} + \frac{p(p+2)t^{p-2}}{t^p - a^p}\right)\mu = 0 \;. \tag{6 c}$$

The solution of Eq. (6 c) which is zero at the origin and finite at 1, is, for $m = 1$,

$$\mu = t\,(a^p - t^p) \;, \tag{6 d}$$

so that after multiplying Eq. (6 c) by $t\mu$ and integrating with respect to t, one finds

$$2\,\delta W = \frac{8\pi^2}{k}\left\{(a^p - 1)(a^p - (p+1)) + (a^p - 1)^2\left(1 - \frac{p+2}{1-a^p}\right)\right\} \;, \tag{6 e}$$

or

$$2\,\delta W = \frac{8\pi^2}{k}\left\{2\,a^p(a^p - 1)\right\} \;. \tag{6 f}$$

This is clearly positive unless

$$0 < a^p < 1 \;, \tag{6 g}$$

or

$$\frac{\iota^{\eta}}{2\pi} > (-n) \qquad (6\,h)$$

where n is negative. This result is independent of p and is the same as the usual Kruskal limit for the case either where the axial current is confined to a thin sheet at the surface as in Eq. (5 g) or where it is uniform as in Eq.(5j). Since the surface terms in Eq. (6) are positive if $a^p > 1$, it is still necessary to show that the system is stable with respect to μ's which vanish when t equals one. Eq. (6 c) is now replaced by

$$\mu'' + \frac{1}{t}\,\mu' - \left(\frac{1}{t^2} + \frac{p(p+2)t^{p-2}}{t^p - a^p} + \Lambda\, f^2(t)\right)\mu = 0 \ . \qquad (6\,i)$$

The Lagrange multiplier Λ is introduced to guarantee that the perturbation has a finite norm. Comparing Eqs. (6 c) and (6 i), one sees that Λ must be positive in order to enable μ to become zero before t reaches 1. Therefore, the system is also stable with respect to perturbations which do not move the boundary.

It is clear from Eq. (6) that the system is stable for all m if a^p does not lie in the region defined by Eq. (6 g) since it is stable for m = 1. If a^p lies in the region defined by Eq. (6 g), the contribution to δW from the region between t = 0 and t = a is positive definite, and can be minimized by making μ negligible small in this region by properly chosing ξ. To show that such a minimization can be made, consider as a trial function

$$\xi_r^o = 0 \ , \qquad (t \leq a - \epsilon)$$

$$\xi_r^o = \xi_r^o(a)\ \frac{t - a + \epsilon}{\epsilon} \ , \qquad (a - \epsilon \leq t \leq a) \qquad (6\,j)$$

where ϵ is small. The integral from zero to a in Eq. (6) is then proportional to ϵ and can be made negligible by making ϵ sufficiently small. This minimization requires that $\underline{\xi}$ be allowed to change rapidly over a very small region. It is possible that other considerations may prevent the acceptability of such rapid changes in $\underline{\xi}$. For example, one might argue that $\underline{\xi}$ could not change appreciably over the distance of an ion Larmor radius if the present theory is to be applicable.

After multiplying Eq. (6c) by $t\mu$, integrating with respect to t, and introducing the transformation $y = t/a$, so that $y_1 = 1/a$ is related to ι^{η} by

$$\frac{\iota^{\eta}}{2\pi} = \frac{(-n)}{m} \; y_1^{p} \, , \tag{6k}$$

one sees that

$$2\,\delta W = \frac{8\pi^2}{k} \left\{ \frac{y_1 \mu'(y_1)}{\mu(y_1)} + m - \frac{(p+2)\, y_1^{p}}{y_1^{p} - 1} \right\} \mu(y_1)^2 \, , \tag{6ℓ}$$

where $\mu(y)$ must satisfy the equation

$$\mu'' + \frac{1}{y}\mu' - \left(\frac{m^2}{y^2} + \frac{p(p+2)\, y^{p-2}}{y^{p} - 1} \right) \mu = 0 \tag{6m}$$

in the region between 1 and y_1, with $\mu(1)$ equal zero. Eq. (6 m) can be integrated numerically and the system is found to be unstable if

$$\frac{(-n)}{m} < \frac{\iota^{\eta}}{2\pi} < (-n)\, \Xi \tag{6n}$$

where $\Xi \equiv \frac{1}{m} y_1^{p}$ is given as a function of p on the right half of Figure 1. Only the indicated points on the figure have been calculated. The other

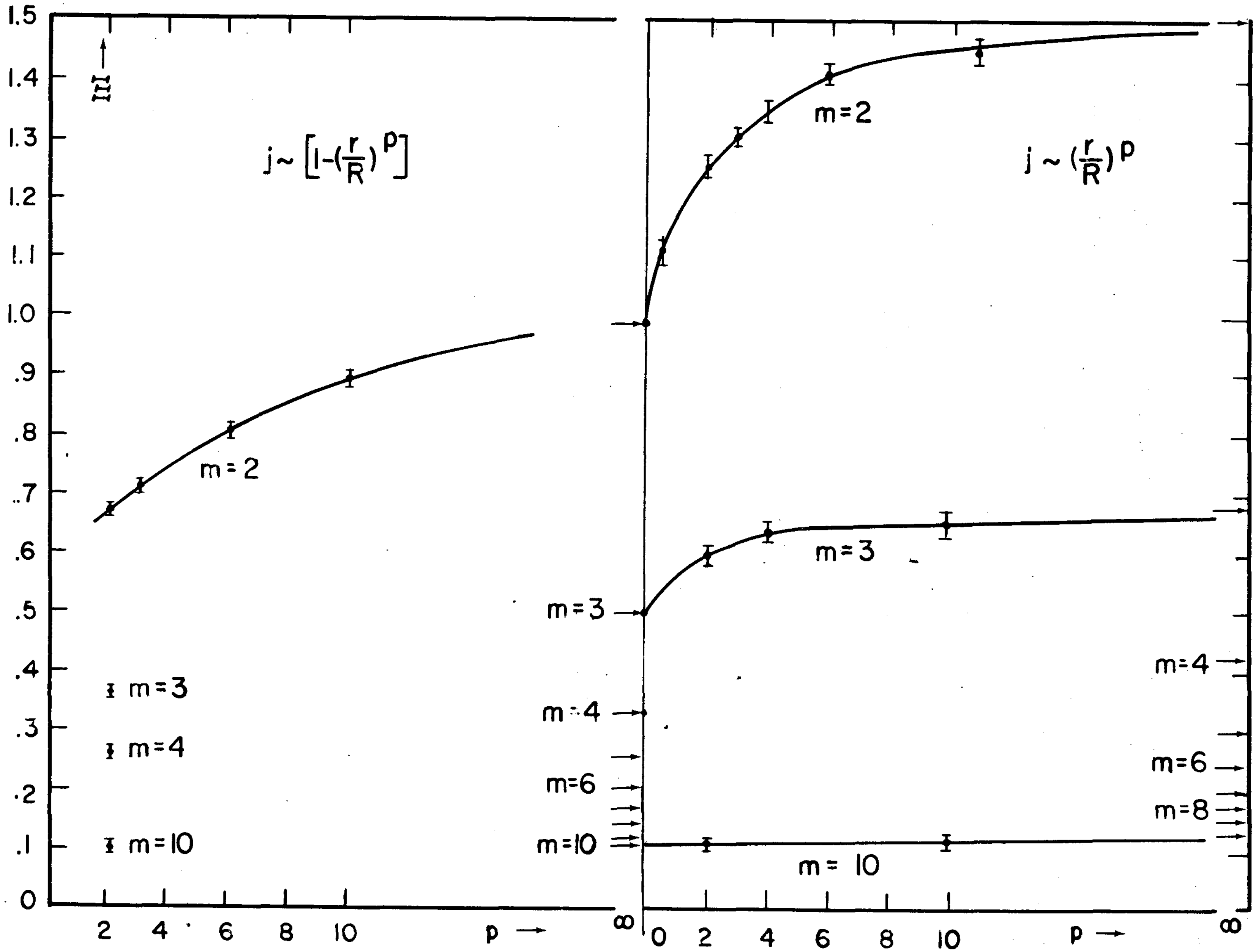

Figure 1. Upper limit of range of instability.

curves have extrapolated from their values in the limiting cases where p is very small or very large.

If a^p is not equal to 1, the term, $p(p+2)t^{p-2}/(t^p - a^p)$, in Eq. (6) goes to zero as p goes to zero with a^p fixed, so that these results agree with the previous calculation for a uniform axial current distribution, i.e.

$$\mathcal{Z} = \frac{1}{m-1} \quad . \tag{6 o}$$

In order to investigate the stability when p is large, let $z = y^p$, so that Eqs. (6 ℓ) and (6 m) become

$$2\delta W = \frac{8\pi^2}{k} p \left\{ \frac{z_1 \mu'(z_1)}{\mu(z_1)} + \frac{m}{p} - \frac{(1+\frac{2}{p})z_1}{z_1 - 1} \right\} \mu (z_1^{1/p})^2 \tag{6 p}$$

and

$$\mu'' + \frac{1}{z}\mu' - (\frac{(1+2/p)}{z(z-1)} + \frac{m^2}{p^2 z^2}\mu = 0 . \tag{6 q}$$

To zeroth order in $\frac{1}{p}$,

$$\mu''_{(o)} + \frac{1}{z} u'_{(o)} - \frac{\mu_{(o)}}{z(z-1)} = 0 , \tag{6 r}$$

so that

$$\mu_{(o)} = z - 1 . \tag{6 s}$$

(The other solution does not vanish if $z = 1$.) The system is clearly neutral to this order. Keeping terms of order $1/p$, one obtains

$$\mu''_{(1)} + \frac{1}{z} \mu'_{(1)} - \frac{\mu_{(1)}}{z(z-1)} = \frac{2\mu_{(o)}}{z(z-1)} , \tag{6 t}$$

so that

$$\mu_{(1)} = (z-1) \ln z \;, \tag{6u}$$

and

$$2\delta W = \frac{8\pi^2}{k}\left\{ \frac{z\mu'_{(1)}}{\mu_{(o)}} - \frac{z\mu'_{(o)}\mu_{(1)}}{\mu^2_{(o)}} + m - \frac{2z}{z-1} \right\} \mu^2_{(o)}, \tag{6v}$$

or

$$2\delta W = \frac{8\pi^2}{k}\left\{ \frac{(m-1)z - (m+1)}{z-1} \right\} \mu^2_{(o)} \;. \tag{6w}$$

Thus the system is unstable if

$$1 < z < \frac{m+1}{m-1} \;, \tag{6x}$$

or

$$\frac{(-n)}{m} < \frac{\iota^{\eta}}{2\pi} < \frac{(-n)(m+1)}{m(m-1)} \;. \tag{6y}$$

For $m > 1$ this is a completely different result from that which was obtained by assuming in advance that the axial current is confined to a thin sheet at the surface of the plasma. In that case the system cannot be unstable for $m > 1$. In that calculation it was assumed that ξ_r^o was continuous across the sheet in which the current is confined so that the contribution to δW from the region where $t < a$ is large. The difference between the results of the two cases shows that <u>care must be taken when any current distribution is mocked up by a model in which it is confined to a current sheet.</u>

Now consider the case where $j^{\eta} \sim 1 - (r/R)^p$. Eqs. (5), (5a), and (5b) can be written as

$$2\delta W = \frac{8\pi^2}{k} \sum_{\substack{m \\ n} > 0} \left\{ \int_o^1 \left(\mu'^2 + \left(\frac{m^2}{t^2} + \frac{p(p+2)t^{p-2}}{t^p - a^p}\right)\mu^2\right) t\,dt + m\mu(1)^2 \right\} \tag{7}$$

where

$$\mu = kRB^{o}\xi_r^{o}(m;n)\, t\,(a^p - t^p)\ , \tag{7a}$$

$$a^p = 1 + \frac{p}{2}\left(\frac{2\pi n}{m\iota^{\eta}} + 1\right)\ . \tag{7b}$$

Here, as before, t equals r/R, and ι^{η}, the transform at the plasma boundary, is positive, and the external region is a vacuum bounded by conducting walls infinitely far away. The Euler equation for the minimizing μ is again Eq. (6c).

For $m = 1$,

$$\mu = t\,(a^p - t^p) \tag{7c}$$

is again a solution, so that

$$2\,\delta W = \frac{8\pi^2}{k}\left\{(a^p-1)(2a^p-(p+2))\right\}\ . \tag{7d}$$

This is positive unless

$$1 < a^p < \frac{p+2}{2}\ . \tag{7e}$$

When a^p is expressed in terms of ι^{η} by means of Eq. (7b), the second inequality in (7e) requires that n be negative, and the first that

$$\frac{\iota^{\eta}}{2\pi} > (-n) \tag{7f}$$

for the system to be unstable, so that the usual Kruskal limit is obtained. Again if $a^p > 1$, it can be seen that the Lagrange multiplier, which must be introduced to guarantee that the perturbation has a finite norm, must be positive if μ is to become zero before t reaches 1. However, for values of a between zero and 1, it is possible to consider a trial function

$$\xi_r^o = \xi_r^o(a) \qquad 0 \leq t \leq a$$

$$\xi_r^o = \frac{t - a - \epsilon}{\epsilon} \qquad a \leq t \leq a + \epsilon$$

$$\xi_r^o = 0 \qquad a + \epsilon \leq t \tag{7 g}$$

for which δW approaches zero as ϵ is made small. The system is, therefore, neutral to this order if

$$\frac{(-n)\,p}{(2+p)} < \frac{\iota\eta}{2\pi} < (-n) \quad . \tag{7 h}$$

If such a sharp discontinuity in ξ_1, as given by Eq. (7 g), were not allowed, the system would be stable in this order. Therefore, the calculation of δW to a higher order has not been carried out.

It is clear from Eq. (7) that the system is stable for all values of $m > 1$ unless a is in the region defined by Eq. (7 e) since for any trial function δW is greater than if m were 1. One now considers the case in which a lies in the region given by Eq. (7 e). If the transformation $y = t/a$ is made, and the Euler equation (6 c) is again multiplied by $y\mu$ and integrated with respect to y, one finds

$$2\delta W = \frac{8\pi^2}{k}\left\{ y_1 \frac{\mu'(y_1)}{\mu(y_1)} + m \right\} \mu(y_1)^2 \ , \tag{7 i}$$

where μ must satisfy the equation

$$\mu'' + \frac{\mu'}{y} - \left(\frac{m^2}{y^2} + \frac{p(p+2)\,y^{p-2}}{y^p - 1}\right) \mu = 0 \ , \tag{7 j}$$

in the region between 0 and y_1 with $\mu(0)$ equal to zero. The transform ι^{η} is related to y_1 by

$$\frac{\iota^{\eta}}{2\pi} = \frac{(-n)\ p}{m\ (p+2-2y_1^{-p})} \ . \tag{7 k}$$

One can show by integrating Eq. (7 j) numerically that the system is unstable if

$$\frac{(-n)}{m} < \frac{\iota^{\eta}}{2\pi} < (-n)\ \Xi \ , \tag{7 ℓ}$$

where $\Xi = p/m(p+2-2y_1^{-p})$ is given as a function of p on the left side of Figure 1. Again only the indicated points have been calculated.

It still must be shown that this situation reduces to the case of a uniform axial current distribution in the limit as p becomes infinitely large with a^p kept fixed. Since the system is stable for values of a^p less than 1 (if the sharply defined perturbations which lead to neutrality to this order for $m=1$ are ignored), only values of $a^p > 1$ need be considered. For p sufficiently large the term

$$-\int_0^1 \frac{p(p+2)\,t^{p-1}}{a^p - t^p}\ \mu^2\, dt$$

is negligibly small unless t is nearly 1. Consider any trial function $\mu(t)$

which is nearly constant in the narrow region between $1-\epsilon$ and 1. This term can then be integrated, becoming approximately

$$(p+2)\,\mu(1)^2 \ln \frac{a^p - 1}{a^p}$$

or

$$-(p+2)\,\mu(1)^2 \left\{ \frac{1}{a^p - 1} - \frac{1}{2}\left(\frac{1}{a^p - 1}\right)^2 + \frac{1}{3}\left(\frac{1}{a^p - 1}\right)^3 - \ldots \right\} .$$

When a^p is expressed in terms of ι^η by means of Eq. (7 b), this becomes

$$-\frac{2\,\iota^\eta\,\mu(1)^2}{\frac{2\pi\eta}{m} + \iota^\eta}$$

plus higher order terms in $\frac{1}{p}$, so that

$$2\,\delta W = \frac{8\pi^2}{k}\left\{ \int_o^1 \left(\mu'^2 + \frac{m^2}{t^2}\mu^2\right) t\,dt + \left(m - \frac{2\,\iota^\eta}{\frac{2\pi\eta}{m} + \iota^\eta}\right)\mu(1)^2 \right\} , \qquad (7\,m)$$

as one would expect. Since the minimizing μ varies in this case as t^m, it is clear that for any m, a region ϵ can be defined over which μ can be taken out of the integral sign, and then a large enough value of p may be found that $(1-\epsilon)^p$ can be made sufficiently small.

It should be mentioned that for an arbitrary axial current distribution the minimizing perturbation for $m = 1$ is obtained by making ξ_r^o constant so that the Kruskal limit can be shown to apply. Of course, this result cannot be extrapolated to the case where B^η and kR are finite.

Now consider the effect of a helical field on these instabilities. In the stellarator the axial current will be applied in the first stage of heating of the plasma. At this time, since the plasma is cold, β will be quite small. By the time β has increased to a reasonable value the heating will be done by other means (e.g. magnetic pumping) and j^{η} will be small. It is, therefore, possible to set $\beta = 0$ in this discussion. It is shown in Appendix IV B that the results are continuous as β goes to zero. For simplicity it will be assumed that $R/S = 0$, i.e., the confining walls are infinitely far away, and that the external region is a pressureless plasma.

When helically symmetric fields and a uniform axial current in the plasma are present, Eqs. (60) of Section III become

$$2\delta W^{4}_{m;n} = \frac{8\pi^2}{k}\left\{\int_0^S \left(\frac{m^2}{r^2}\mu^2 + \mu'^2\right) r\,dr - \left(\frac{2\,\iota^{\eta}\,\mu^2}{\frac{2\pi n}{m} + \iota^{\eta} + \sum\limits_{s>0} \iota^{\lambda\lambda}(s)}\right)\right\}, \quad (8)$$

where

$$\mu = \frac{krB^{o}}{2\pi}\left(\frac{2\pi\eta}{m} + \iota^{\eta} + \sum_{s>0} \iota^{\lambda\lambda}(s)\right)\xi_r^{o} \quad . \quad (8a)$$

(Here j^{β} has been set equal to zero). It has been shown that helices with $\ell = 3$ stabilize the highest β for a given power input. The question of the stabilization of the Kruskal instability will, therefore, be investigated using a single helical field with $\ell = 3$ and the wave length hR small enough that higher order terms in the expansions of the Bessel functions can be ignored. For notational simplicity, one may absorb the positive coefficient $4\pi^2/k$ into δW, suppress the summation sign, abbreviate $\iota^{\lambda\lambda}(s)$ to ι^{δ} and introduce the parameter

$$q \equiv -\frac{2\pi n}{m} \quad . \quad (8b)$$

Then, minimizing the volume regions, one finds

$$2\delta W = \{ (m \frac{1 + a_-^{2m}}{1 - a_-^{2m}} (-q + \iota^\eta + \iota^\delta)^2 + m \frac{a_+^{2m} + 1}{a_+^{2m} - 1} (-q + \iota^\eta + \iota^\delta)^2$$

$$- 2 \iota^\eta (-q + \iota^\eta + \iota^\delta)) \left(\frac{kRB^o \xi_r^o}{2\pi} \right)^2 \}_{r=R} \quad , \qquad (8c)$$

where Ra_- and Ra_+ represent the values of r nearest to R at which μ must vanish because $q + \iota^\eta(r) + \iota^\delta(r)$ does, a_- being less than 1 and a_+ greater than 1. If no such points exist a_- is defined to be zero and a_+ is defined to be infinite. Henceforth, the positive definite factor $(kRB^o \xi_r^o / 2\pi)^2$, will not be written. Since ι^δ is proportional to r^2, and ι^η is constant (The current is uniform in the plasma.) in the interior region and proportional to r^{-2} in the external region, it can be seen that, when they exist,

$$a_-^2 = \frac{q - \iota^\eta}{\iota^\delta} \quad , \qquad (0 \le a_-^2 < 1) \qquad (8d)$$

$$a_+^2 = \frac{q \pm (q^2 - 4\iota^\delta \iota^\eta)^{1/2}}{2\iota^\delta} \quad , \qquad (1 < a_+^2) \qquad (8e)$$

where if the value for a_+^2 is greater than 1 with the negative sign, that sign is used, otherwise the plus sign. If neither sign leads to a value of $a_+^2 > 1$, or if a_+^2 is complex, it must be set equal to infinity. It is clear that if the external region were really a vacuum, the minimization in respect to μ would be carried out for μ's which need not vanish when $\iota^\delta + \iota^\eta - q$ does, so that a_+^2 must be set equal to infinity in Eq. (8c).

Eq. (8 c) can be written as

$$\delta W \doteq \left\{ m \frac{1+a_-^{2m}}{1+a_-^{2m}} + m \frac{a_+^{2m}+1}{a_+^{2m}-1} - \frac{2\iota^\eta}{-q+\iota^\eta+\iota^\delta} \right\} \tag{9}$$

where the positive factors, $\frac{1}{2}(-q+\iota^\delta+\iota^\eta)^2$ and $(kRB^o\xi_r^o/2\pi)^2$ have been absorbed into δW. Clearly each of the first two terms in Eq. (9) is always greater than or equal to m.

First consider cases where $0 < \iota^\eta < \iota^\delta$. If q is negative, or if $0 < q < \iota^\delta$ the last term in Eq. (9) is less than 2 so that δW is positive. If q is in the range $\iota^\delta < q < \iota^\delta + \iota^\eta$, a_-, given by Eq. (8 d), is not zero. Therefore, for $m = 1$,

$$\delta W \geq 2 \frac{\iota^\delta - \iota^\eta}{-q+\iota^\delta+\iota^\eta} > 0, \tag{9 a}$$

where the equal sign is used if a_+^2 is equal to infinity. If $m > 1$,

$$\delta W \geq 2 \frac{m\,\iota^{\delta m} - \iota^\eta(\iota^{\delta m-1} + \iota^{\delta m-2}(q-\iota^\eta) + \ldots + \iota^\delta(q-\iota^\eta)^{m-2} + (q-\iota^\eta)^{m-1})}{\iota^{\delta m} - (q-\iota^\eta)^m} \tag{9 b}$$

is also greater than zero. If $\iota^\delta + \iota^\eta < q$ all three terms in Eq. (9) are positive. The system has thus been shown to be stable for all values of m if $0 < \iota^\eta < \iota^\delta$.

Next consider cases in which $\iota^\delta < \iota^\eta < 4\iota^\delta$. As before δW is clearly positive unless q is in the range $\iota^\delta < q < \iota^\delta + \iota^\eta$. If $\iota^\delta < q < \iota^\eta$ it follows from Eqs. (8 d) and (8 e) that $a_-^2 = 0$ and $a_+^2 = \infty$, so that

$$\delta W = 2\left\{ \frac{m(-q+\iota^{\delta})+(m-1)\,\iota^{\eta}}{-q+\iota^{\delta}+\iota^{\eta}} \right\} \tag{9 c}$$

is unstable for $m = 1$ for all values of q in this range. For higher values of m this is unstable for values of q in this range only if

$$q > \iota^{\delta} + \frac{m-1}{m}\,\iota^{\eta} \,. \tag{9 d}$$

If $\iota^{\eta} < q < 2\sqrt{\iota^{\delta}\,\iota^{\eta}}$, a_- is not zero, but $a_+ = \infty$, so that δW is

$$\delta W = \left\{ m\,\frac{\iota^{\delta\,m}+(q-\iota^{\eta})^{m}}{\iota^{\delta\,m}-(q-\iota^{\eta})^{m}} + m - \frac{2\,\iota^{\eta}}{-q+\iota^{\delta}+\iota^{\eta}} \right\} \,. \tag{9 e}$$

For $m = 1$,

$$\delta W = 2\left\{ \frac{\iota^{\delta} - \iota^{\eta}}{-q+\iota^{\delta}+\iota^{\eta}} \right\} \tag{9 f}$$

is negative. The conditions for the system to be stable with respect to these values of q can be obtained from Eq. (9 e) for higher values of m with more work. If $2\sqrt{\iota^{\delta}\,\iota^{\eta}} < q < \iota^{\delta} + \iota^{\eta}$, δW is given by

$$\delta W = \left\{ m\,\frac{\iota^{\delta\,m}+(q-\iota^{\eta})^{m}}{\iota^{\delta\,m}-(q-\iota^{\eta})^{m}} + m\,\frac{(q-\sqrt{\ })^{m}+(2\,\iota^{\delta})^{m}}{(q-\sqrt{\ })^{m}-(2\,\iota^{\delta})^{m}} - \frac{2\,\iota^{\eta}}{-q+\iota^{\delta}+\iota^{\eta}} \right\}, \tag{9 g}$$

where $\sqrt{\ } = (q^2 - 4\iota^{\delta}\,\iota^{\eta})^{1/2}$. This expression can be rationalized so that, for $m = 1$,

$$\delta W = \left\{ \frac{q+\sqrt{\ }-2\,\iota^{\eta}}{-q+\iota^{\delta}+\iota^{\eta}} \right\} \tag{9 h}$$

is negative for these values of q . Again the conditions for $m > 1$ are not so easily obtained. Therefore, for $\iota^{\delta} < \iota^{\eta} < 4\iota^{\delta}$ the system has been shown to be stable for all q's not in the range $\iota^{\eta} < q < \iota^{\delta} + \iota^{\eta}$. It is unstable for $m = 1$ for these q's . Conditions for instability for higher values of m have not been obtained.

Now consider cases in which $0 < 4\iota^{\delta} < \iota^{\eta}$. Again only values of q in the range $\iota^{\delta} < q < \iota^{\delta} + \iota^{\eta}$ could make the system unstable. Eq. (9 c) applies if $\iota^{\delta} < q < 2\sqrt{\iota^{\delta}\iota^{\eta}}$ so that the system is unstable for $m = 1$ for all values of q in this range and for higher m for values of q in this range if Eq. (9 d) is satisfied. In the range $2\sqrt{\iota^{\delta}\iota^{\eta}} < q < \iota^{\eta}$,

$$\delta W = \left\{ m + m \frac{(q - \sqrt{\Gamma})^m + (2\iota^{\delta})^m}{(q - \sqrt{\Gamma})^m - (2\iota^{\delta})^m} - \frac{2\iota^{\eta}}{-q + \iota^{\delta} + \iota^{\eta}} \right\} \tag{9 i}$$

where, as usual, $\sqrt{\Gamma} = (q^2 - 4\iota^{\delta}\iota^{\eta})^{1/2}$. This reduces, for $m = 1$, to

$$\delta W = \left\{ \frac{-q + \sqrt{\Gamma}}{-q + \iota^{\delta} + \iota^{\eta}} \right\} \tag{9 j}$$

which is clearly negative. Eq. (9 g) and, for $m = 1$, Eq. (9 h) apply if $\iota^{\eta} < q < \iota^{\delta} + \iota^{\eta}$ so that δW is still negative. Thus, if $0 < 4\iota^{\delta} < \iota^{\eta}$, the system is stable unless q is in the range $\iota^{\delta} < q < \iota^{\delta} + \iota^{\eta}$. It is unstable for $m = 1$ for all these q's. Again conditions for instabilities to occur have not been obtained for higher values of m .

Now consider the situation if $\iota^{\eta} < 0 < \iota^{\delta}$. It is clear from Eq. (9) that the system is stable for all values of q which are not in the range, $\iota^{\delta} + \iota^{\eta} < q < \iota^{\delta}$ (ι^{η} is negative), since they would not make the last term more negative than -2 . It follows from Eqs. (8 d) and (8 e), that $a_-^2 = 0$ and $a_+^2 = [q + (q^2 - 4\iota^\delta \iota^\eta)^{1/2}]/2\iota^\delta$ over this entire range of q, so that, from Eq. (9),

$$\delta W = \left\{ m + m \frac{(q+\sqrt{\Gamma})^m + (2\iota^\delta)^m}{(q+\sqrt{\Gamma})^m - (2\iota^\delta)^m} - \frac{2\iota^\eta}{-q + \iota^\delta \iota^\eta} \right\} \tag{9k}$$

where $\sqrt{\Gamma} = (q^2 - 4\iota^\delta \iota^\eta)^{1/2}$. For $m = 1$,

$$\delta W = \left\{ \frac{-q - \sqrt{\Gamma}}{-q + \iota^\delta + \iota^\eta} \right\} \tag{9ℓ}$$

is clearly positive. To show that the system is stable for all values of m if $\iota^\eta < 0 < \iota^\delta$, it will be shown that the second term in Eq. (9) is a monotonically increasing function of m. The logarithmic derivative of this term in respect to m is

$$\frac{a_+^{4m} - 1 - 2a_+^{2m} \ln a_+^{2m}}{m(a_+^{4m} - 1)}$$

Since the denominator of this term is positive ($a_+ > 1$) it is necessary to show that the numerator is positive for all $a_+ > 1$. It is zero if $a_+ = 1$. The first and second derivatives of the numerator with respect to a^{2m} are

$$2a^{2m} - 2\ln a^{2m} - 2 \quad ,$$

and

$$2 - 2/a^{2m} \quad ,$$

respectively. Since these derivatives vanish when $a_+ = 1$ and the second derivative is positive for all $a_+ > 1$, the logarithmic derivative is positive and the second term in Eq. (9) increases with m. The system is, therefore, stable for all values of m if $\iota^{\eta} < 0 < \iota^{\delta}$.

The expression for δW in Eq. (9) is not changed if ι^{δ} is set equal to $-\iota^{\delta}$, ι^{η} to $-\iota^{\eta}$ and q to -q. Thus the stability of systems in which ι^{δ} and ι^{η} are both negative or $\iota^{\delta} < 0 < \iota^{\eta}$ can be determined from the preceding results.

It has been shown that the system is stable for all m if $|\iota^{\delta}|>|\iota^{\eta}|$ and ι^{δ} and ι^{η} have the same sign and if $\iota^{\eta} < 0 < \iota^{\delta}$ or $\iota^{\delta} < 0 < \iota^{\eta}$. If $0 < \iota^{\eta} < \iota^{\delta}$ or $\iota^{\delta} < \iota^{\eta} < 0$, it is unstable in respect to values of q which lie between ι^{δ} and $\iota^{\delta} + \iota^{\eta}$ and stable for all other q's. It should be remembered that these results apply to the case where the external region is a pressureless plasma.

It can be seen from Eq. (8 b) that q is limited to the values $-2\pi n/m$ where n can be any integer. These results are exhibited in Figure 2. The unshaded region is stable for <u>m = 1</u>. The regions denoted by left diagonal lines are unstable for m = 1 if n = ± 1; those with vertical markings are unstable if n = ± 2; those with right diagonal markings are unstable if n = ± 3; etc. Instabilities due to higher values of m can occur only in part of the first and third quadrants for which $|\iota^{\delta}| < |\iota^{\eta}|$.

In the previous treatment the external region has been assumed to be a pressureless plasma. If it is a vacuum, the same discussion can be carried through as before except that a_+ must always be infinite. If ι^{δ} and ι^{η} have the same sign, the same results are obtained for m = 1 (i.e. stable if $|\iota^{\delta}| > |\iota^{\eta}|$, otherwise unstable only if q is between $\iota^{\delta} + \iota^{\eta}$ and ι^{δ}.).
If ι^{δ} and ι^{η} have opposite signs the system is now unstable if q is between $\iota^{\delta} + \iota^{\eta}$ and ι^{δ} and otherwise stable. Calculations for m > 1 have not been carried through.

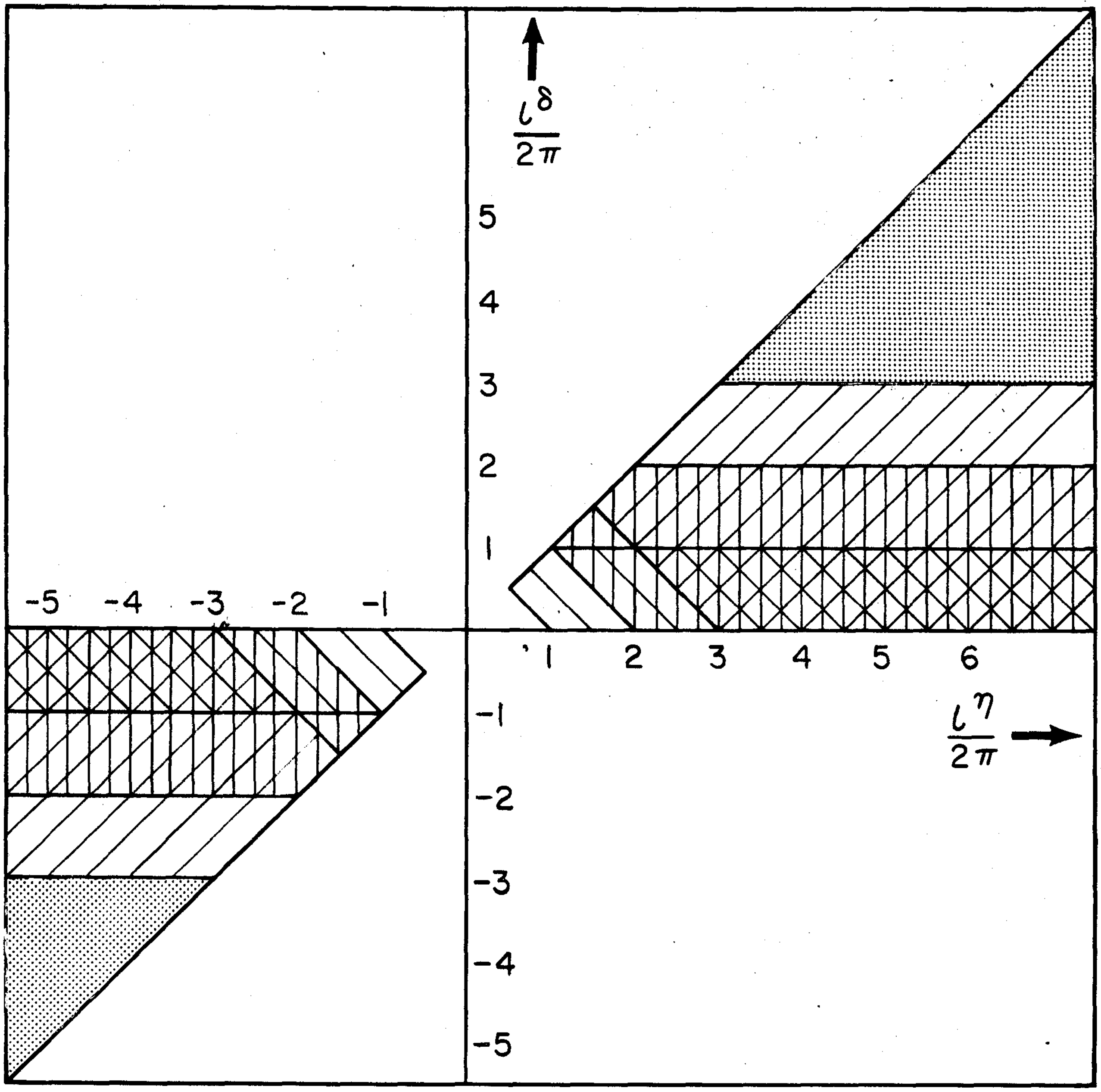

Figure 2. Stability diagram.
External region a pressureless plasma. No machine ι.

The results for the case where the external region is a vacuum are exhibited in Figure 3. The unshaded region is stable for m = 1. The regions in the second and fourth quadrants which are denoted by horizontal lines are unstable when $n = 0$. Those regions denoted by left diagonal lines are unstable if $n = \pm 1$; those with vertical markings are unstable if $n = \pm 2$; those with right diagonal markings are unstable if $n = \pm 3$; etc.

In order to estimate the effect of a machine transform ι^M, one can again express n in terms of ι^M by means of Eq. (1k). It should be pointed out, however, that with such an identification there need not be an integral number of periods of the helical field in the machine. It would be necessary to restrict values of the wave number of the helical field to

$$h = \left(n - \frac{\ell \iota^M}{2\pi}\right) k \tag{9m}$$

where n is an integer. The treatment which has been carried through is, therefore, not clearly applicable. Nevertheless, Eq. (1k) has been applied to the situation when ι^M is 164° (Model B-1 stellarator) and the results are presented for m = 1 in Figure 4 with the same system of markings as before. It should be noticed that the shape of the stable region is altered slightly if ι^M is changed. In both cases the parts of the unshaded regions in the first and third quadrants for which $|\iota^{\delta}| > |\iota^{\eta}|$ are stable in respect to all m. The other regions are riddled with instabilities due to higher values of m.

If the axial current had been confined to a thin sheet at the surface of the plasma, Eq. (8c) would have been replaced by

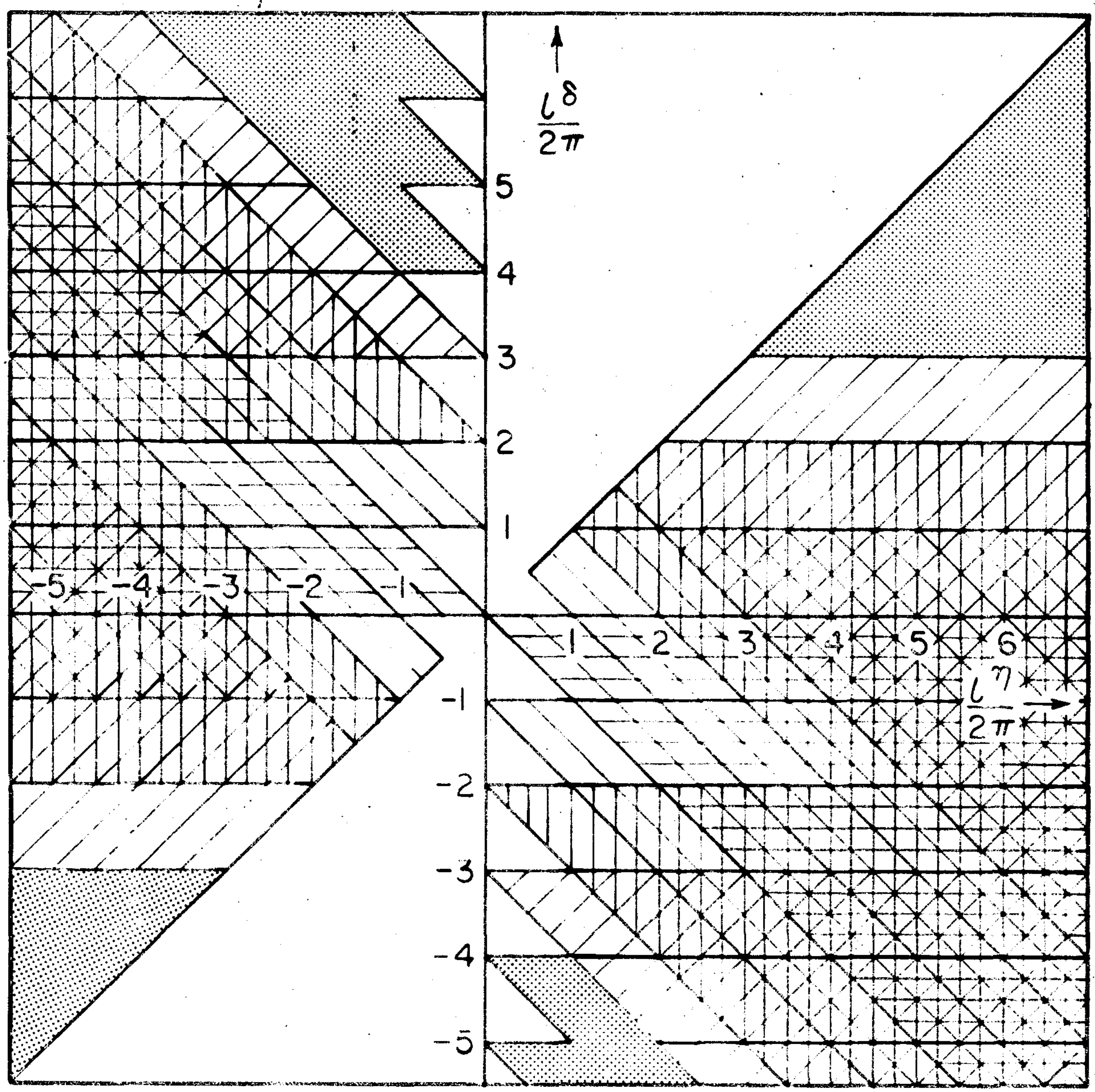

Figure 3. Stability diagram.

External region a vacuum. No machine ι

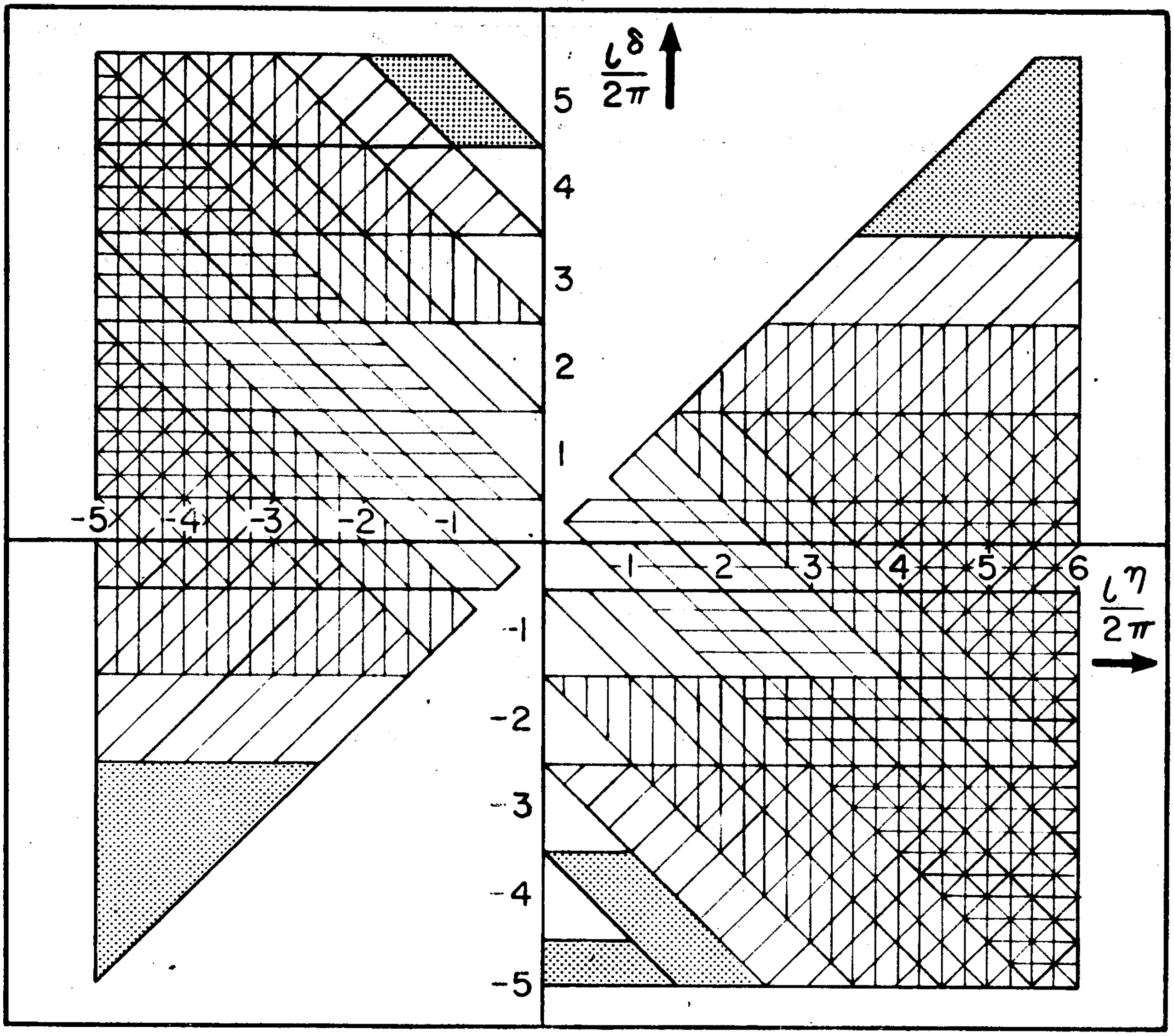

Figure 4. Stacility diagram.

External region a vacuum. Machine : present.

$$2\delta W = \{ (m \frac{1+a_-^{2m}}{1-a_-^{2m}} (-q+\iota^\delta)^2 + m \frac{a_+^{2m}+1}{a_+^{2m}-1} (-q+\iota^\delta+\iota^\eta)^2$$

$$- \iota^{\eta^2} + 2\,\iota^\delta\,\iota^\eta) (\frac{kRB^o \xi_r^o}{2\pi})^2 \}_R \quad . \qquad (10)$$

where a_-^2 is given by Eq. (8 d) with ι^η set equal to zero and a_+^2 is given by Eq. (8 e).

The determination of the conditions for which the system is stable can be carried through as in the previous discussion. The considerations are much more difficult, however, so it will merely be shown here that if $|\iota^\eta| < 4|\iota^\delta|$, where ι^η and ι^δ have the same sign, the system is stable for all values of m.

Clearly,

$$2\delta W \geq \{ m(-q+\iota^\delta)^2 + m(-q+\iota^\delta+\iota^\eta)^2 - \iota^{\eta^2} + 2\,\iota^\delta\,\iota^\eta \} (\frac{kRB^o\xi_r^o}{2\pi})^2 . \qquad (10\,a)$$

It can be seen by taking the first and second derivatives of Eq. (10 a) in respect to q , that the minimum value of δW is obtained if

$$q = \iota^\delta + \iota^\eta/2 \quad , \qquad (10\,b)$$

so that

$$2\delta W \geq \{ \frac{m-2}{2} \iota^{\eta^2} + 2\cdot\iota^\eta\,\iota^\delta \} . \qquad (10\,c)$$

If ι^δ and ι^η have the same sign, this is clearly positive if $|\iota^\eta| < 4|\iota^\delta|$.

One might expect the current distribution to be $j^\eta = \gamma(1 - r^2/R^2)$ during the heating phase. If no helical field is present the results which were obtained for an arbitrary current distribution do not differ qualitatively from those for a uniform current distribution. Since the calculation of the effect of stabilization windings would have to be done for a specific current distribution using numerical techniques, it does not seem to be worthwhile to do such a calculation at the present time.

Section V - Conclusion

The basic results of the preceding sections are summarized in Part A and discussed in Part B of this section.

Part A - Summary

Equilibrium situations are calculated in Section II by means of an expansion technique, by considering, (a) fields which arise from helically invariant current distributions, (b) bulge fields, ($\ell = 0$), for example those due to gaps between the confining field coils, (c) fields set up by axial currents in the plasma, for example, ohmic heating currents, and (d) fields due to diamagnetic currents, all superimposed on a large axial magnetic field. The fields are determined for convenience subject to the condition that the normal component of $\underline{B}$ be zero on a perfectly conducting rigid surface

$$r = S + \sum_s \sigma_s^{\delta} \cos(\ell_s \theta - s h z) \quad , \qquad (1)$$

rather than from a given external current distribution. Solutions are explicitly obtained for the case where the pressure distribution is a parabolic function of r, and techniques for determining the fields for arbitrary pressure distributions are given. Superpositions of helical fields with the same wavelength but different values of ℓ are not considered since they involve complicated interference effects.

The rotational transform angle ι (over one helical field period $2\pi/h$) is calculated in Appendix II A. The functions ι and $d\iota/dr$ are given to second order in δ by Eqs. (A22) and (A23), and their small hR limits are tabulated for various values of ℓ in Table 1. The function ι is given to fourth order in δ for $\ell = 3$ in the small hR limit in Eq. (A27).

The quantity V'', where V is the volume of length $2\pi/k$ (the length of the machine) inside a surface of constant flux ψ and the prime represents a derivative with respect to ψ, is given in Appendix IIB by Eq. (B21) in terms of the distortion σ^{δ} of the boundary, and by Eq. (B22) in terms of $\iota^{\delta\delta}$ and $d\iota^{\delta\delta}/dr$. Here $\iota^{\delta\delta}$ is the lowest order term in the expansion of ι.

In Section III the minimization of δW with respect to all components of $\underline{\xi}$ except ξ_r^o is carried through. This minimized δW is given by Eq. (54) and alternatively by Eq. (60).

The final minimization of δW is done and critical conditions for stability are obtained in Section IV for several cases.

For any given m the critical β for the stability of an axially symmetric system, which represents an idealization of the Figure-8 stellarator (which possesses a rotational transform), is given by Eq. (1ℓ).

Eq. (4g) gives the critical β optimized over the pressure distribution, for a system in which there is a superposition of helically invariant fields with arbitrary ℓ's (including $\ell = 0$), all having different periods, and in which no axial current is present. This optimum pressure distribution is given by Eq. (4h). In this calculation the external region is treated as a pressureless plasma. If it were treated as a vacuum, the above pressure distribution would not be optimum and the critical β would be smaller. For the special case where only $\ell = 0$ and 3 fields are present and the periods of these fields are large, the optimum critical β is given by Eq. (3g) and the optimum pressure is given by Eq. (3f). Eq. (3g) reduces to Eq. (3i) that is,

$$\beta_{critical} = (\rho^{\delta}/R)^2 \qquad (2)$$

for the case where only one $\ell = 3$ field is present.

For the case where no helically invariant (or bulge) fields are present, and where there is present an axial current proportional to $(r/R)^p$ or $1 - (r/R)^p$, the system is unstable with respect to the minimizing $\underline{\xi}(m;n)$ is and only if

$$(-n)/m < \iota^{\eta}/2\pi < (-n)\Xi . \tag{3}$$

Here, the $\underline{\xi}(m;n)$ vary as $e^{i(m\theta + nkz)}$ and Ξ is a function of p and m represented in Figure 1 for both the case where $j^{\eta} \sim (r/R)^p$ and where $j^{\eta} \sim 1-(r/R)^p$. These results are independent of β for the assumed ordering of the parameters ($\beta \sim \eta$). For the case $j^{\eta} \sim 1-(r/R)^p$, Ξ approaches $1/(m-1)$ as p approaches ∞, which agrees with the uniform axial current case. Also, Ξ approaches $1/(m-1)$ for the case $j^{\eta} \sim (r/R)^p$ as p approaches 0 which again agrees with the uniform axial current case.

As p becomes large, in the case $j^{\eta} \sim (r/R)^p$, the current distribution approaches that of a sheet current at R . However, Ξ approaches $(m+1)/m(m-1)$ which is in disagreement with the earlier results of Kruskal's treatment of this problem, where m = 2 was found to be neutral and all higher m's were found to be stable. The disagreement is explained by noting that the normal component of the minimizing $\underline{\xi}$ in the present treatment approaches a function which is discontinuous across the surface while in Kruskal's treatment the usual assumption that the normal component of $\underline{\xi}$ be continuous is made. This discrepancy points out the need for exercising care in the choice of models in which sheet currents are used to represent volume current distributions confined to small regions. In obtaining these results, the external region has been treated as a vacuum with conducting walls infinitely far away.

It is found that for $\beta = 0$, a helically invariant field with $\ell = 3$ and with small hR can stabilize the instability associated with a uniform axial current. The stability diagram is given in Figure 2 for the case where the external region is treated as a pressureless plasma, and in Figures 3 and 4 for the case where it is treated as a vacuum. In particular in both cases, the system is stable if $0 < \iota^{\eta} < \iota^{\delta}$ or $\iota^{\delta} < \iota^{\eta} < 0$. If the axial current is a sheet current rather than a uniform volume current, the system is again stable for both cases if $0 < \iota^{\delta} < 4\iota^{\delta}$ or $4\iota^{\delta} < \iota^{\eta} < 0$. Again, the external conducting walls are infinitely far away.

Part B - Physical Interpretation

In this section we wish to give some of the intuitive background which leads us to consider helically invariant fields for the purpose of stabilization. We also would like to form a simple physical picture of their stabilizing action both on the "interchange" instability and the "kink" instability.

In any axisymmetric equilibrium in which the plasma and the magnetic field are imbedded in each other and in which the magnetic lines of force lie entirely in r, z planes (where r, θ, and z are cylindrical coordinates with the z axis along the axis of symmetry), it is possible to carry out a displacement of the plasma which interchanges lines of force in such a way that the magnetic field and its magnetic energy are unchanged. This displacement may be constructed by first specifying, on a cross-section with $z = \text{constant}$, a mapping of the magnetic lines of force into themselves, and then extending this displacement throughout the volume by specifying that any line of force continues to go into the same line as that assigned on this particular cross-section. Since the magnetic field strength is given by the density of lines, the magnetic field is clearly unchanged. However, the state of the plasma is changed and its energy will increase or decrease according to whether $(M''/M')V''$ is positive or negative[8], where M is related to the mass contained in a flux tube with flux ψ and is given up to a constant by

$$\frac{M''}{M'} = \frac{V''}{V'} + \frac{p'}{\gamma p} , \tag{4}$$

where V is the volume (over some length) in this flux tube, p is the pressure on its surface, γ is the ratio of specific heats and primes denote differentiation with respect to ψ. The quantities V' and p' are always positive and, in general, V'' is positive while p' is negative. If $p = 0$ (i.e. containment) on the surface of the plasma, $(M''/M')V''$ is negative, and the equilibrium is unstable to interchanges.

This argument (for instability) applies in general to more complicated situations, since nothing would prevent us from carrying out the "interchange" displacement unless the system is such that the lines return on themselves. In this latter case our original assignment of the mapping of lines into lines would not match when we bring the lines around onto themselves. Nevertheless, one might ask whether even in re-entrant systems, one can construct interchanges which lower the energy of the plasma, and leave the magnetic field unchanged. The answer can be given in terms of the concept of the rotational transform angle ι, which can be defined on each surface as 2π times the average number of turns about the magnetic axis which a line of force makes as one follows it once around the system. (The magnetic axis is that line of force which closes on itself.)

We consider first a system in which ι is constant over the cross-section, i.e., is independent of the flux ψ within each magnetic surface. If in some such system every line returns on itself after once around the system, or every line returns on itself after n times around the system (i.e., $\iota = 2\pi m$ for every ψ, or $\iota = 2\pi m/n$ respectively), an interchange can be selected so localized that it need not match until the nth time around, when it matches perfectly. If ι is not a rational multiple of 2π, the matching of the interchange can never be achieved exactly, but ι can be so closely approximated by a rational multiple of 2π that the matching comes arbitrarily close. If one does not demand a perfect interchange, one can achieve matching with an accompanying change in the field. However, one can choose the displacement sufficiently close to an interchange to make the change in the magnetic energy negligible. Thus for any ι constant on all flux tubes, an effective interchange can be carried out with respect to which the system is unstable.

The situation is different if ι depends on ψ. In this case if we try to construct an interchange as we carry the mapping around the tube, the matching becomes worse and worse. This is made clear from Figure 1.

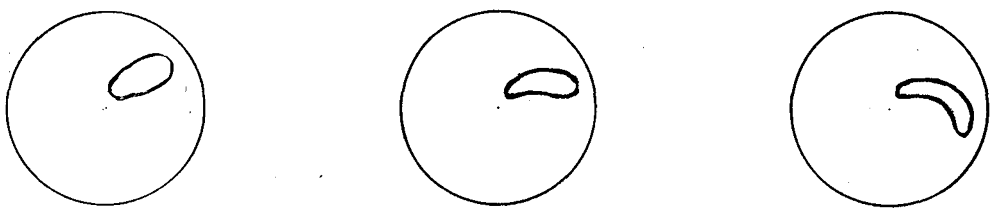

Figure 1.

The first diagram indicates an attempt at assigning on a cross-section, a mapping of the lines into each other indicated by the flow pattern. After carrying this mapping along the lines once around the system, the flow pattern becomes that of the second diagram. Here, it is assumed that ι

is larger on the outer surfaces then on the inner ones, and since the lines are the same in the flow pattern, the outer lines move farther, shearing the flow pattern. After a second time around the system, the flow pattern (for the mapping) becomes that illustrated in the third diagram. It is clear that the possibility of matching becomes more and more hopeless as we carry the mapping around more and more times. Further, even by carrying out displacements differing from the interchange which manage to produce matching, the magnetic energy cannot become negligible. Thus the presence of an ι which depends on ψ introduces an inhibiting effect on the unstable interchanges and might tend to make the system stable.

One can produce (or increase) the dependence of the rotational transform on ψ by adding externally produced multipolar helically invariant fields to the main magnetic field. These can be produced by wrapping wires in a helical fashion around the tube and passing currents of alternate sign through them. The case of six wires is illustrated in Figure 2.

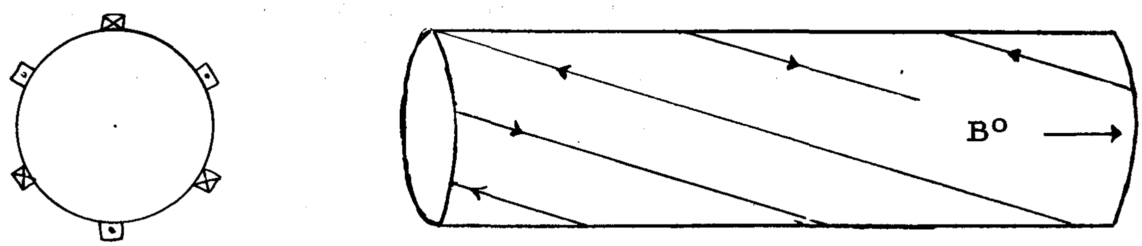

Figure 2 .

In the simple case in which these currents are small compared to those producing the main uniform field B^o, and in which the pitch of the wires is long compared to the radius of the tube, the field due to 2ℓ wires will be nearly proportional to $r^{\ell - 1}$. That the field in this case produces a varying rotational transform can be seen in the following way: Considering for definiteness the case of six wires ($\ell = 3$) we have

$$B_r \sim r^2 \cos(3\theta - hz); \qquad B_\theta \sim r^2 \sin(3\theta - hz); \qquad B_z = B^o \quad ; \tag{5}$$

where the amplitudes of B_r and B_θ are small compared to B^o. The lines of force will, to first order in the amplitudes of B_r and B_θ be small helices about a line parallel to the axis, whose radii are proportional to r^2. But, since B_θ is larger on the outer part of this helix and smaller on the inner part, the lines will drift in the second order, in the θ direction by an amount proportional to r^3 and thus produce an ι proportional to r^2. The fields due to two wires ($\ell = 1$) will produce no ι, while if $\ell = 2$, ι is independent of r. It is clear that larger ℓ produce steeper dependences of ι on r for the same ι but the advantage for stability of this steeper dependence is off-set by the steeper dependence of the fields on r ($B \sim r^{\ell-1}$) so that if the wires are placed some distance from the surface of the plasma the fields in the plasma will be correspondingly smaller. In consideration of these factors, it seems reasonable to suppose that either the $\ell = 3$ or the $\ell = 4$ fields are most favorable for stability, although there are indications that the $\ell = 2$ fields produce a large rotational transform, which may have advantages in the equilibrium situation. In the case where the currents are not small or their pitch is not large, the arguments go through essentially the same way.

Since the imposition of externally produced helically invariant fields makes the lines of force into small helices about their original position, they also enhance $p' V''$ (the change in energy of the plasma), and contribute a destabilizing influence as well. As a result, if one puts a small helically invariant field on a system with finite pressure, it will be made less stable, and if it is already neutral or unstable, it will become or remain unstable.

However, as the field is increased, it will eventually become stable at a value depending on the pressure.

For the further analysis the closed tube with helical windings is replaced by a long straight tube of length L, whose ends are to be identified at points with equal values of r and θ. If the change of potential energy is minimized over all displacements $\underline{\xi}$, which to lowest order have a given radial dependence ξ_r^o and θ and z dependences $e^{i(m\theta - nkz)}$ the result, which is given in Eq. (60) of Section III, can be written in the form

$$-\int_o^S r\,dr \left\{\left[\frac{d}{dr}((\iota^\delta - \iota_o)r\xi_r^o)LB^o\right]^2 + \frac{m^2}{r^2}\left[(\iota^\delta - \iota_o)r\xi_r^o LB^o\right]^2 + (2\pi LB^o)^3 p'V''r^2\xi_r^{o2}\right\} \tag{6}$$

Here S is the outer boundary of the system, and $\iota_o = -2\pi n/m$ represents the angle through which the displacement "turns" over the length of the tube. The last term is the energy released by the plasma as in the first part of this section, since in this case $M''/M' = p'/\gamma p$ because the system does not differ much from a cylinder, and V'' includes the destabilizing effect of the helically invariant fields. The first two terms represent the energy increase due to the change of the magnetic field under the approximate interchange. It is clear from their form that these terms can be made negligibly small if ι^δ is constant, by choosing m and n sufficiently large, in agreement with our original picture. If ι^δ is not constant they can never be reduced to zero but have a nonzero minimum so that if the pressure is made sufficiently small, the last term can be made smaller than this minimum. Thus we see that any dependence of ι^δ on r except a constant dependence leads to stability for sufficiently small pressure.

The first two terms of expression (6) for δW can be made more transparent as follows: The quantity $(\iota^{\delta} - \iota_o)$ represents the average rate at which the lines turn compared to the displacement and thus it represents an effective field across the displacement. Consider as a model a straight tube with large uniform B_z, with a B_θ, independent of θ and z, just sufficient to produce a rotational transform angle equal to $\iota^{\delta} - \iota_o$, but with no current in the z direction, so that one of Maxwell's equations is violated. Consider further a displacement ξ_r, also independent of z but depending on θ as $e^{im\theta}$ and depending on r in the same way as the ξ_r^o in expression (6), and a displacement ξ_θ, so that $\nabla \cdot \underline{\xi} = 0$. Then one computes easily that

$$\delta B_r = \frac{im}{r} (\iota^{\delta} - \iota_o) r \xi_r L B^o , \tag{7}$$

$$\delta B_\theta = -\frac{\partial}{\partial r} (\iota^{\delta} - \iota_o) r \xi_r L B^o . \tag{8}$$

Hence the first term in (6) is just $(\delta B_\theta)^2$ while the second is $(\delta B_r)^2$. The situation and its model are analogous, the model being obtained by simply smoothing out the ripples in the lines of force (which the displacements in the primary situation automatically take care of), and untwisting it so that the displacement is "untwisted". The origin of the terms in the model is made pictorial by considering δB_r as being due to compression of the lines by the displacement, and δB_θ as being due to shearing of the lines.

If expression (6) is minimized over all allowable ξ_r^o for a given pressure distribution, one finds a critical value for β (the ratio of the pressure at the center to the zeroth order magnetic pressure) above which the system becomes unstable. Then one can select the shape of the pressure distribution to make this critical value β_c a maximum to find $(\beta_c)_{optimum}$, which is given in Eq. (4 g) of Section IV as

$$(\beta_c)_{optimum} = \frac{1}{4\pi B_0^2} \int \overline{(\frac{d\iota}{dr})^2} / V''_{vac} \;\; rdr \qquad (9)$$

where $\overline{(d\iota/dr)^2}$ is the average of $(d\iota/dr)^2$ per unit length and V'' is evaluated over the length of the machine. This form illustrates clearly the balance between the destabilizing effect $p' V''$ and the stabilizing effect $\overline{(d\iota/dr)^2}$.

When expression (6) is minimized over ξ_r^o it is found, especially when β is only slightly bigger than β_c, that the worst $\underline{\xi}$ takes on its maximum near the radius for which $\iota^\delta = \iota_o$. It is clear from the definition of ι_o that such a radius exists for finite m and n only if ι^δ is finite and would only exist for infinitesimal ι^δ if m were infinite. Since we wish to avoid infinitely large values of m in treating the stability of the system with infinitely small helically invariant fields, it is necessary to consider the tube identified over an infinitely large length so that ι^δ can remain finite.

Up to this point we have assumed that the magnetic lines are everywhere imbedded in an infinitely conducting plasma, and even in regions exterior to the main bulk of plasma we have imagined a zero pressure plasma to be present. This has the consequence that during any displacement the lines of force preserve their identity and cannot be broken. It was on this fact that we based most of our arguments for stability. In a situation where the external region is a true vacuum the lines in the vacuum do not possess this stabilizing effect, so that if n and m are picked to make the radius at which $\iota = \iota_o$ very close to the surface of the plasma, it is actually possible to construct a displacement near an interchange, with the result that any distribution for which dp/dr is not zero at the surface of the plasma, is unstable. However, the stable equilibria with dp/dr zero at the surface are roughly similar to the case of a plasma surrounded by a pressureless plasma, so that one might argue that the assumption of a pressureless plasma at least in the immediate neighborhood of the main plasma is justified.

In minimizing expression (6) it is found for β greater than β_c, that as $\beta - \beta_c$ is made smaller the minimizing $\underline{\xi}$'s become more singular in the neighborhood of the radius at which $\iota = \iota_o$, and eventually change appreciably over a region very small compared to a Larmor radius. The minimizing $\underline{\xi}$ thus represents a motion to which the theory no longer applies, since it is based on equations which assume that the ion Larmor radius is the smallest length in the system under description. Thus we cannot assert that these systems, which are only unstable to such $\underline{\xi}$'s, are really unstable, and it is of interest to ask whether the smallest β which is unstable on the basis of our equations with the condition that $\underline{\xi}$ varies slowly over an ion Larmor radius, is much bigger than β_c derived allowing any continuous $\underline{\xi}$. This question can be answered by minimizing expression (6) over all ξ_r^o subject to the restriction that $d\xi_r^o/dr < \xi_{r\,max}^o/\lambda$ where λ is a length of the size of the Larmor radius. It is found in Section IV that the β_c under this restriction is, under certain conditions, appreciably bigger than β_c derived allowing unrestricted ξ_r^o. Whether the systems with β between these two β_c's are stable or unstable must be settled by a more refined theory.

It should be emphasized in looking for stable systems by application of equation (9) that the quantities there involved are per unit length. For example as the machine is made longer by adding long straight sections which contribute nothing to $p'V''$ for the entire machine, one might suppose that if $d\iota/dr$ is kept fixed the stability would be unchanged. On the contrary, it is necessary to increase $d\iota/dr$ in proportion to the length of the machine, $p'V''$ always being the same, to preserve the same β_c. This is also clear from our earlier intuitive picture of the interchanges, since the interchanges may make use of this long straight section to unwrap themselves, after being curled up by $d\iota/dr$, and can match with less increase in the change in magnetic energy. With this caution in mind, one can produce the $d\iota/dr$ shear by wrapping helical wires over only parts of the tube, and can still obtain stability provided the $(d\iota/dr)$ total is large enough (according to equation (9)).

The kink instability and its stabilization according to the results of Section IV are not so well understood as the interchange instability and its stabilization. However, the existence of the kink instability may be made plausible by a simple force picture based on the fact that the lines of force are tied to the matter. Further, on the basis of this picture the stabilization mechanism of the helically invariant fields may be suggested.

We consider a long cylinder (see Fig. 3) of pressureless plasma of radius R and length L imbedded in a large axial uniform magnetic field B_o, and in which a small axial current of uniform density j is flowing. As usual, the ends are to be identified. The current j produces a field B_θ and an ι, given in the plasma by

$$B_\theta = 2jr , \qquad \iota = \frac{B_\theta}{krB^o} , \qquad k \equiv \frac{2\pi}{L} . \tag{10}$$

Note that ι is constant in the plasma.

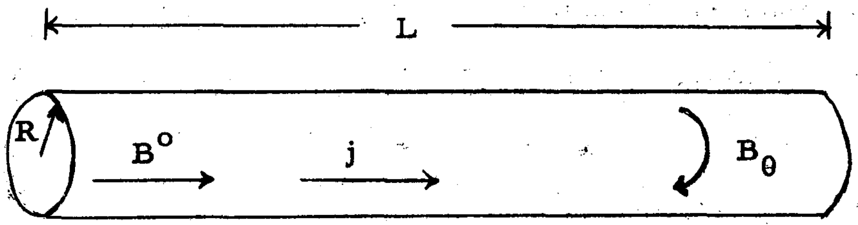

Figure 3 .

Let us subject this plasma to a displacement $\underline{\xi}$ given by

$$\xi_r = \hat{\xi} \cos(\theta - kz); \quad \xi_\theta = \hat{\xi} \sin(\theta - kz); \xi_z = 0 ; \tag{11}$$

which moves each z = constant cross section rigidly a distance $\hat{\xi}$ perpendicular to the axis, so that the tube of plasma is distorted into a helix whose pitch is L . Consider two cross sections α and β a distance $\iota/4$ apart which are, therefore, displaced in perpendicular directions. If $\iota = 2\pi$, lines of force rotate through $\pi/2$ between these cross sections. That is, any line of force through a point S in α passes through a point T in β such that QT makes an angle $\pi/2$ with QS, (see Fig. 4). In these circumstances, it is clear that any line

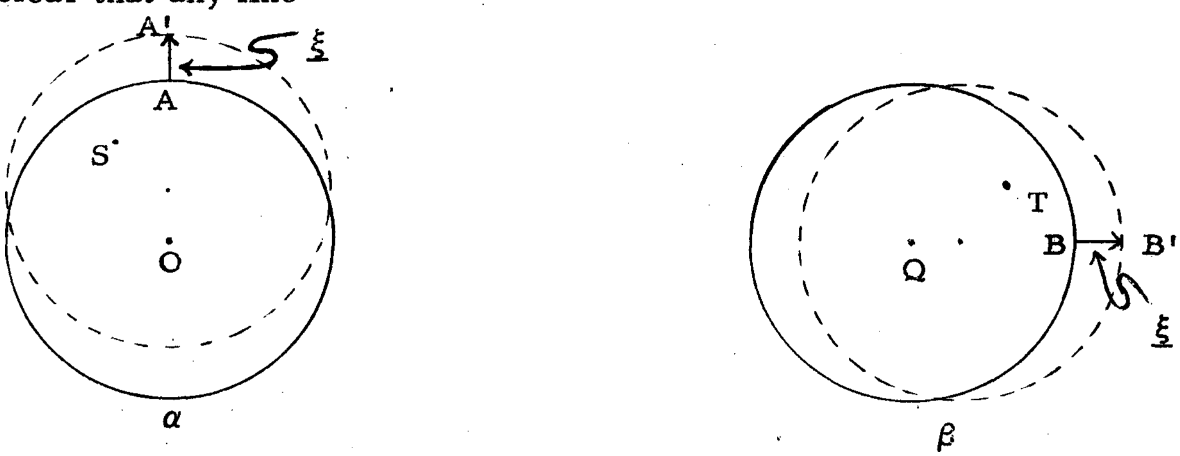

Figure 4 .

of force passing through A in α and B in β is displaced to a line of force passing through A' in α and β' in β where OA' and QB' make a right angle with each other. But this means that the line of force through AB is displaced into the position of a line of force which passed through A', B' in the undisplaced equilibrium. Since further the density of lines is unchanged because the displacement of each cross section is rigid, the field is is unchanged by the displacement. Thus the situation characterized by $\iota = 2\pi$ is neutral with respect to the perturbation (11).

Let us now consider the case in which $\iota - 2\pi$ is positive but small, subject it to the same perturbation (11), and examine the same cross sections α and β, L/4 apart. Then any line of force of the equilibrium passing through a point S in α will pass through a point T in β such that QT makes an angle $\iota/4 > \pi/2$ with OS (See Fig. 5). A line of force through A in α and B in β, where OA and QB make an angle of $\iota/4$ with each other, will then be displaced to a position passing through A' and B' where OA' and QB' no longer make an angle of $\iota/4$ but make a slightly smaller angle. Thus the displaced line of force is rotating about the axis

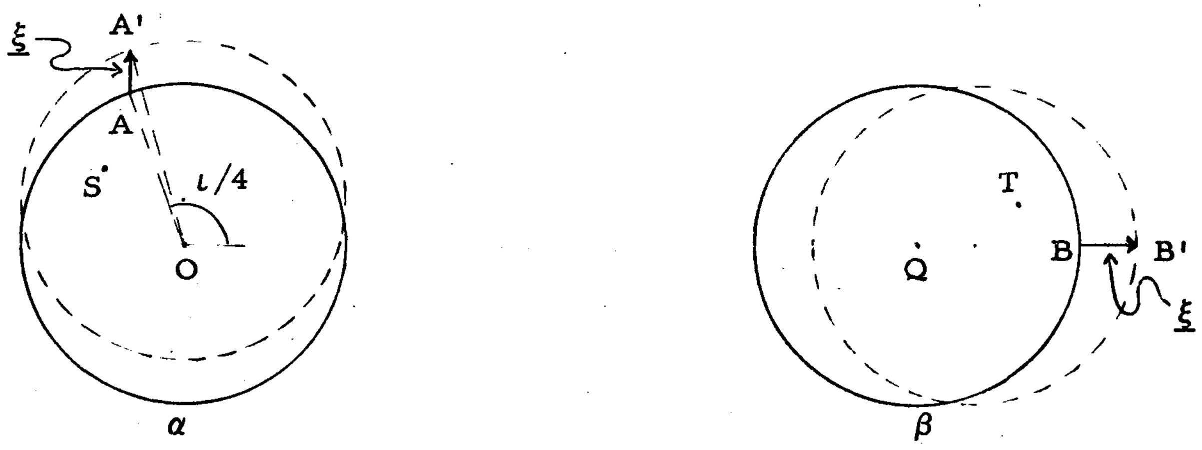

Figure 5 .

OQ at a slower rate (in z) than the line of force which passed through B' in the undisplaced equilibrium. Hence we see that B_θ is weakened by an amount proportional to $(\iota/4 - \pi/2)\ \hat{\xi}$. If we consider other points in the cross section we find that $\delta \underline{B}$ is constant in each cross section. Since the cylinder is long, $\delta \underline{j} = \nabla \times \delta \underline{B}$ is negligible and $\delta \underline{F} = \underline{j} \times \delta \underline{B} + \delta \underline{j} \times \underline{B} \cong \underline{j} \times \delta \underline{B}$. We see, therefore, that $\delta \underline{F}$ is in the same direction as $\underline{\xi}$ and tends to enhance the perturbation. Thus for $\iota > 2\pi$ the system is unstable with respect to perturbation (11).

It is found in Section IV that helically invariant fields can, in certain circumstances, stabilize the kink instability for $\iota^\eta > 2\pi$, where we have now denoted the ι produced by the current by ι^η. The helical fields have three effects on the kink instability which might lead to an understanding of the ι^δ vs ι^η stability diagrams of Section IV. Before describing these it should be remarked that the effect of the helically invariant fields may be obtained from a model in which the same ι^δ as that produced by the helically invariant fields, is produced by a radially dependent B_θ field. (Again in the model the B_θ field is produced by no current).

Accordingly, the first effect (1) is to increase the effective ι by ι^δ (which may be negative) so that for m = 1 the kink is made more unstable if ι^δ is positive (or less if ι^δ is negative). (2) The helically invariant fields affect the stability by introducing a shear $(d\iota/dr)$ in ι inside the plasma, and according to the sign of $d\iota/dr$ by increasing or decreasing the shear outside the plasma. Finally, (3) the helically invariant fields may cause a surface to exist on which the effective $\iota = \iota^\delta + \iota^\eta$ is 2π. On this surface the electric field $\underline{E}$, due to the perturbation, is parallel to $\underline{B}$ and along a line of force is always in the same direction. Therefore, this electric field leads to large currents on the surface so that the surface acts like a rigid perfectly conducting wall to the displacement. This effect can lead to increased stability.

In connection with this third remark it should be noted that in cases where the axial current is confined to so narrow a region, that for many purposes it may be considered as a surface current, the critical surface may occur inside this narrow region, leading to $\underline{\xi}$'s which vanish on one side of the region. Such $\underline{\xi}$'s are not considered in the usual method of treating the stability of surfaces in which it is demanded that the normal component of $\underline{\xi}$ be continuous. Which assumption about the $\underline{\xi}$'s is the correct one depends on the actual physical size of the region which is to be approximated by a surface, presumably compared with the ion Larmor radius. This remark explains to some extent the disagreement between the result of the investigation in Section IV in which the stability is calculated for an axial current which depends on r as r^p where p is large and the result of the usual treatment of the stability of a surface current. For this reason the stability of a pinch with stabilizing B_z field is lowered by considering these more general $\underline{\xi}$'s which have a discontinuity in their normal component.

Acknowledgment

The authors wish to express their deepest gratitude to Dr. Lyman Spitzer, Jr. for suggesting most of the problems treated in this paper and to Dr. Martin Kruskal for his constant advice and encouragement without which the present work hardly could have been accomplished. We wish to thank Dr. Ira Bernstein for many helpful suggestions. We owe special thanks to Miss Marlene DiDonato for capably carrying out the arduous task of typing this complicated manuscript.

Appendix II - A Rotation Transform for Helically Invariant Field

At the heart of the Matterhorn effort lies the concept of rotational transform[10]. We wish to exhibit the structure of the rotational transform (ι) for the case of helically invariant magnetic fields and we shall see how the stability criteria of later chapters are stated most tersely in terms of ι.

We shall find ι in the blunt, but straightforward and pictorial, way by finding the average angle of rotation[11,12] of a magnetic line of force about the magnetic axis as one proceeds along the z-axis. Since the length over which we specify the average angle of rotation is arbitrary we shall choose it to be a helical field period or $\frac{2\pi}{h}$. We first compute the angle of rotation $\Delta\theta$ for a length $\tilde{z}$ such that a line of force shall have sampled once every point in a constant z cross section of a Ψ surface, so that

$$\frac{h}{\ell}\tilde{z} - \Delta\theta = 2\pi , \tag{A1}$$

The constant Ψ surfaces have cross sections perpendicular to the z axis which rotate about the z-axis at the constant rate $d\theta/dz = h/\ell$ as we proceed along the z-axis, whereas the lines of force rotate around the z-axis at a much slower average rate $(\overline{d\theta/dz} \sim \delta^2)$ and fall behind the Ψ surfaces. Hence we must proceed in the z - direction until a line of force first comes back to the same relative position in a cross section that it originally occupied. We are now assured that the average angle of rotation for all lines in a Ψ surface is the same as that of any one line. We thus have

$$\iota = \frac{\Delta\theta}{\tilde{z}} \cdot \frac{2\pi}{h} + 2n\pi, \quad n = 0, \pm 1, \pm 2 \ldots \tag{A2a}$$

[12]

or, making use of (A1)

$$\iota + \frac{4\pi^2}{h\,\tilde{z}} = \frac{2\pi}{\ell} \tag{A2b}$$

where we have adopted the convention $n = 0$. We are now able to compute ι with equations (A1) and (A2b). Since the fields are known as a power series in δ, the equations of lines of force are power series in δ, and thus ι will be given as a power series in δ. We consider only vacuum fields.

We write

$$\tilde{z} = z^{o} + z^{\delta} + z^{\delta\delta} + \ldots \tag{A3}$$

and hence

$$\iota = \frac{2\pi}{\ell} - \frac{4\pi^2}{h\,z^{o}} \left(1 - \frac{z^{\delta}}{z^{o}} - \frac{z^{\delta\delta}}{z^{o}} + \frac{{z^{\delta}}^2}{{z^{o}}^2} + \ldots \right) \tag{A4}$$

We then have

$$\iota^{o} = \frac{2\pi}{\ell} \left(1 - \frac{2\pi\,\ell}{h\,z^{o}} \right) , \tag{A5a}$$

$$\iota^{\delta} = \frac{4\pi^2}{h{z^{o}}^2}\, z^{\delta} , \tag{A5b}$$

$$\iota^{\delta\delta} = \frac{4\pi^2}{h z^o} \left(\frac{z^{\delta\delta}}{z^o} - \frac{z^{\delta 2}}{z^{o2}} \right) , \tag{A5c}$$

etc.

If we expand (A1) we have (taking $\theta(0) = \theta^o(0)$)

$$\theta^o (z^o) - \theta^o(0) - h z^o/\ell = -2\pi , \tag{A6a}$$

$$\theta^\delta (z^o) + z^\delta \frac{d\theta^o}{dz} (z^o) - h z^\delta/\ell = 0 , \tag{A6b}$$

$$\theta^{\delta\delta}(z^o) + z^\delta \frac{d\theta^\delta(z^o)}{dz} + z^{\delta\delta} \frac{d\theta^o(z^o)}{dz} + \frac{z^{\delta 2}}{2} \frac{d^2\theta^o(z^o)}{dz^2} - h z^{\delta\delta}/\ell = 0 , \tag{A6c}$$

etc.

It remains to write the equations of the lines of force as an expansion in δ :

$$\frac{dr}{dz} = \frac{B_r(r, \theta, z)}{B_z} ; \tag{A7a}$$

$$\frac{d\theta}{dz} = \frac{B_\theta}{r B_z} . \tag{A7b}$$

If we write

$$r = r^o + r^\delta + r^{\delta\delta} + \ldots \tag{A8a}$$

and

$$\theta = \theta^o + \theta^\delta + \theta^{\delta\delta} + \ldots , \tag{A8b}$$

and remember

$$\underline{B} = \underline{B}^{o}\ \underline{e}_z + \underline{B}^{\delta}(r, \theta, z) + \underline{B}^{\delta\delta}(r, \theta, z) + \ldots \quad , \tag{A9}$$

then we obtain, after expanding both sides of (A7a) and (A7b) into power series in δ, to zero order in δ

$$\frac{dr^{o}}{dz} = 0 \ , \ \therefore \ r^{o} = \text{const.} \tag{A10a}$$

$$\frac{d\theta^{o}}{dz} = 0 \ , \ \therefore \ \theta^{o} = \text{const.} \tag{A10b}$$

To first order in δ ,

$$\frac{dr^{\delta}}{dz} = \frac{B_r^{\delta}(r^{o}, \theta^{o}, z)}{B^{o}} \quad , \tag{A11a}$$

$$\frac{d\theta^{\delta}}{dz} = \frac{B_\theta^{\delta}(r^{o}, \theta^{o}, z)}{r^{o}B^{o}} \tag{A11b}$$

and therefore

$$r^{\delta}(z; r^{o}, \theta^{o}) = \frac{B_r^{\delta}(r^{o})}{h\ B^{o}}\left[\cos(\ell\theta^{o} - hz) - \cos \ell\theta^{o}\right] \tag{A12a}$$

and

$$\theta^{\delta}(z\ ; r^{o}, \theta^{o}) = -\frac{B_\theta^{\delta}(r^{o})}{h r^{o}\ B^{o}}\left[\sin(\ell\theta^{o} - hz) - \sin \ell\theta^{o}\right] \quad , \tag{A12b}$$

when use is made of our equilibrium vacuum fields. To second order in δ we have

$$\theta^{\delta\delta}(z\,;\,r^o,\theta^o) = \int_o^z dz\ \Big[\frac{r^\delta}{r^oB^o}\ \frac{\partial B_\theta^\delta(r^o,\theta^o,z)}{\partial r} + \frac{\theta^\delta}{r^oB^o}\ \frac{\partial B_\theta^\delta}{\partial\theta}$$

$$- \frac{B_\theta^\delta}{r^oB^o}\Big(\frac{r^\delta}{r^o} + \frac{B_\theta^\delta}{B^o}\Big) + \frac{B_\theta^{\delta\delta}(r^o,\ \theta^o,\ z)}{r^o\ B^o}\Big]\ , \tag{A13}$$

or

$$\theta^{\delta\delta}(z^o;\,r^o,\theta^o) = \frac{\pi\ell^2}{h^2 r^{o2} B^{o2}}\Big[B_r^\delta(r^o)^2 + \frac{2B_z^\delta(r^o)\ B_r^\delta(r^o)}{h\,r} + \frac{(\ell^2+h^2\,r^2)}{h^2\,r^2}B_z^\delta(r^o)^2\Big]\ , \tag{A14}$$

where, from (A6a) and (A10b), $z^o = 2\pi\ell/h$. Thus

$$\theta^{\delta\delta}(z^o;\,r^o,\theta^o) = \pi\ell^2\Big(\frac{\rho^\delta}{R}\Big)^2\frac{X^2}{x^2}\ \frac{I_\ell'(x)^2}{I_\ell'(X)^2}\ \big[1 - 2\,I^\ell(x) + (\ell^2 + (x)^2)\ (I^\ell(x))^2\big]\ , \tag{A15}$$

where $x = h\,r$, $X = hR$, and $I^\ell(x) = \dfrac{I_\ell(x)}{xI_\ell'(x)}$.

Now, from (A5a), (A6a) and (A10b) we have

$$\iota^{(o)} = 0\ . \tag{A16}$$

From (A6b) and (A10b)

$$\theta^{\delta}(z^{o}) = h\, z^{\delta}/\ell \tag{A17}$$

and from (A12b)

$$\theta^{\delta}(z^{o}) = 0 \quad . \tag{A18}$$

Therefore from (A17) and (A5b)

$$\iota^{\delta} = 0 \quad . \tag{A19}$$

From (A6c), (A10b), and (A18),

$$\theta^{\delta\delta}(z^{o}) = h\, z^{\delta\delta}/\ell \quad , \tag{A20}$$

and from (A5c), (A17), (A18) and (A20)

$$\iota^{\delta\delta} = \frac{4\pi^2}{h z^{o\,2}} \quad z^{\delta\delta} = \frac{h\, z^{\delta\delta}}{\ell^2} = \frac{\theta^{\delta\delta}}{\ell} \tag{A21}$$

since $z^{o} = 2\pi\ell/h$. Therefore

$$\iota^{\delta\delta} = \pi\ell\left(\frac{\rho^{\delta}}{R}\right)^2 \frac{X^2}{x^2} \frac{I'_{\ell}(x)^2}{I'_{\ell}(X)^2}\left[1 - 2 I^{\ell}(x) + (\ell^2 + x^2)(I^{\ell}(x))^2\right], \tag{A22}$$

Also

$$R\frac{d\iota^{\delta\delta}}{dr} = -2\pi\ell\left(\frac{\rho^{\delta}}{R}\right)^2\left(\frac{X}{x}\right)^3 \frac{I'_{\ell}(x)^2}{I'_{\ell}(X)^2}\left[3 - 2(2 + \ell^2 + x^2) I^{\ell}(x) + (3\ell^2 + 2x^2)(I^{\ell}(x))^2\right] . \tag{A23}$$

If we choose $x \leq X << 1$, we then have

$$\iota^{\delta\delta} = \pi \left(\frac{\rho^{\delta}}{R}\right)^2 \frac{1}{X^{2\ell-4}} \frac{\left[2(\ell-1)\; x^{2\ell-4} + x^{2\ell-2} + \frac{(2\ell+3)}{8\ell(\ell+1)}\; x^{2\ell} + \dots\right]}{\left[1 + \frac{(\ell+2)}{2\ell(\ell+1)}\; X^2 + \frac{\ell(\ell+1)(\ell+4) + (\ell+2)^3}{16\ell^2\,(\ell+1)^2\,(\ell+2)}\; X^4 + \dots\right]}, \tag{A24}$$

and

$$\frac{R\,d\iota^{\delta\delta}}{dr} = \pi X \left(\frac{\rho^{\delta}}{R}\right)^2 \frac{1}{X^{2\ell-4}} \frac{\left[4(\ell-1)(\ell-2)\, x^{2\ell-5} + 2(\ell-1)\, x^{2\ell-3} + \frac{(2\ell+3)}{4(\ell+1)} x^{2\ell-1} + \dots\right]}{\left[1 + \frac{(\ell+2)}{2\ell(\ell+1)}\; X^2 + \dots\right]}. \tag{A25}$$

For convenience we construct Table 1 which gives $\iota^{\delta\delta}$ and $R\frac{d\iota^{\delta\delta}}{dr}$ for the small X limit.

To next order in δ we find

$$\iota^{\delta\delta\delta} = 0 \quad , \tag{A26}$$

and, after a very tedious calculation, we find for $\ell = 3$, and in the limit $x \leq X << 1$,

$$\iota^{\delta\delta\delta\delta} = 3\pi\delta^4 \left(-10 + 12\left(\frac{r}{R}\right)^2 + 36\left(\frac{r}{R}\right)^4 - 100\left(\frac{r}{R}\right)^6 + \frac{125}{4}\left(\frac{r}{R}\right)^8\right) . \quad (A27)$$

Here this $\iota^{\delta\delta\delta\delta}$ is evaluated on a surface of constant Ψ such that to fourth order the value of $\Psi = \Psi^o + \Psi^\delta + \ldots$ is equal to $\Psi^o(r)$. Calculations of ι for equilibria in which pressure is present have been made but are not included here.

ℓ	$\iota^{\delta\delta}$ (per length $2\pi/h$)	$R\frac{d\iota^{\delta\delta}}{dr}$ (per length $2\pi/h$)
1	$\pi\delta^2 X^2$	$\frac{5}{8}\pi\delta^2 x X^3$
2	$2\pi\delta^2$	$2\pi\delta^2 x X$
3	$4\pi\delta^2 \frac{x^2}{X^2}$	$8\pi\delta^2 \frac{x}{X}$
4	$6\pi\delta^2 \frac{x^4}{X^4}$	$24\pi\delta^2 \frac{x^3}{X^3}$

Table 1.

Appendix II - B Expressing V'' in Terms of ι .

It is of interest to express some of the equilibrium quantities which were introduced by Kruskal and Kulsrud[12] in terms of the transform ι for the helically invarient fields. We will first find the relation that exists between the Ψ which was introduced in Part A of Section II and the usual fluxes ψ and χ . The equilibrium quantity V'' , where $V(\psi_o) = \int_{\psi < \psi_o} d\tau$ and the prime denotes a derivative in respect to ψ , will then be related to ι and $d\iota/dr$ to the lowest significant order for the more general equilibria of Part B of Section II. Finally the function, $\Lambda = \gamma p \left[\frac{V''}{V'} + \frac{p'}{\gamma p}\right] \frac{[V'' - p' L']}{[V' + \gamma p L']}$ [8] , which must be positive for stability of axial symmetric systems, will be expressed in terms of the $\iota^{\delta\delta}$ for these fields.

The Ψ which has been used in Part A of Section II can be shown to be related to the flux crossing a ribbon one side of which lies on the axis, the other in a surface Ψ = constant, and depends on θ and z as $u = \ell\,\theta - hz =$ const. The flux the long way inside a constant Ψ surface, ψ , is given by

$$\psi(\Psi) = \int_0^{2\pi} d\theta \int_0^{r(\Psi\,;\,u)} B_z r\,dr \qquad \text{(B1)}$$

where z is a constant. Since $\nabla \cdot \underline{B} = 0$, ψ is independent of this constant. Since B_z and Ψ are periodic functions of u ,

$$\psi(\Psi) = \frac{1}{\ell} \int_0^{2\pi\ell} du \int_0^{r(\Psi,\,u)} B_z r\,dr \quad . \qquad \text{(B2)}$$

The flux the short way, χ is defined only up to an arbitrary integral multiple of ψ[12] . In particular, we define it to be

$$\chi(\Psi) = \int_0^{2\pi/h} dz \int_0^{r(\Psi, u)} B_\theta dr , \qquad \text{(B3)}$$

the flux through a surface of constant θ over one wave length of the helically symmetric magnetic field, and inside a constant Ψ surface, or

$$\chi(\Psi) = \int_0^{2\pi} \frac{du}{h} \int_0^{r(\Psi, u)} B_\theta \, dr \quad . \qquad \text{(B4)}$$

Then

$$\psi - \ell\chi = \frac{1}{h} \int_0^{2\pi} du \int_0^{r(\Psi, u)} (h r B_z - \ell B_\theta) \, dr \quad , \qquad \text{(B5)}$$

or, using Eq. (18d) of Section II ,

$$\psi - \ell \chi = \frac{2\pi}{h} \Psi \quad . \qquad \text{(B6)}$$

Since the transform is given by $\iota = 2\pi \frac{d\chi}{d\psi}$,

$$\frac{d\Psi}{d\psi} = \frac{h}{2\pi} \left(1 - \frac{\ell}{2\pi} \iota \right) \quad . \qquad \text{(B7)}$$

The volume V enclosed by a surface of constant Ψ and its derivatives in respect to ψ will be computed for the equilibria of Part B of Section II.

We consider a surface of constant ψ determined by the condition

$$f \equiv r - r^{o} - r^{\delta} - r^{\delta\delta} - r^{\beta} - r^{\eta} - \ldots = 0 \tag{B8}$$

where the r^{δ}, $r^{\delta\delta}$, etc., are functions of r^{o}, θ, and z, and must be chosen to satisfy the condition

$$\underline{B} \cdot \Delta f = 0 . \tag{B9}$$

The equilibria which are considered are given by

$$\underline{B} = \underline{B}^{o} + \underline{B}^{\delta} + \underline{B}^{\delta\delta} + \underline{B}^{\beta} + \underline{B}^{\beta\delta} + \underline{B}^{\eta} + \ldots \tag{B10}$$

where : $\underline{B}^{o}$ is a constant in the z direction ;

$$\begin{aligned} \underline{B}^{\delta} = \sum_{s>o} \{ & \underline{e}_r \, A_s \, sh \, I'_{\ell_s}(shr) \sin u_s \\ & + \underline{e}_\theta \, A_s \frac{\ell_s}{r} \, I_{\ell_s}(shr) \cos u_s \\ & - \underline{e}_z \, A_s \, sh \, I_{\ell_s}(shr) \cos u_s \} , \end{aligned} \tag{B11}$$

and

$$\mu_s = (\ell_s \, \theta - shz + \phi_s) , \tag{B12}$$

$$A_s = \sigma_s^{\delta} \quad B^{o} / I'_{\ell_s} \ (shS) \quad ; \tag{B13}$$

$\underline{B}^{\beta}$ is in the z direction and depends only on r so as to support an abitrary pressure distribution ; $\underline{B}^{\eta}$ is in the θ direction, depends only on r , and is determined by an arbitrary axial current ; $\underline{B}^{\delta\delta}$ depends on z trigonometrically with average value zero ; etc. Note that terms in $\underline{B}^{\delta}$ with $\ell = 0$ correspond to bulge fields .

A sufficient condition to satisfy Eq. (B9) to zeroth order is

$$r^{o} = \text{constant} \quad . \tag{B13}$$

Eq. (B9) is satisfied to order δ by

$$r^{\delta} = \sum_{s>o} \sigma_s^{\delta} \frac{I'_{\ell_s}(shr^{o})}{I'_{\ell_s}(shS)} \quad \cos \quad \mu_s \quad . \tag{B14}$$

In the $\delta\delta$ order it is necessary to note only that $r^{\delta\delta}$ has a trigonometric dependence on z with average value zero. The β and η orders can be satisfied by setting

$$r^{\beta} = r^{\eta} = 0 \quad . \tag{B15}$$

The flux ψ in the long way inside the surface defined by Eq. (B8) is

$$\psi = \frac{k}{2\pi} \int_0^{2\pi/k} dz \int_0^{2\pi} d\theta \int_0^{r^o + r^\delta + r^{\delta\delta} + \dots} (B_z^o + B_z^\delta + B_z^{\delta\delta} + B_z^\beta + \dots)\, r\, dr \, . \quad \text{(B16)}$$

Since ψ is independent of z it has been averaged over z. Averaging over z eliminates terms in $r^{\delta\delta}$ and $B_z^{\delta\delta}$ in a trivial manner. Eq. (B16) can be integrated to get

$$\psi = \pi {r^o}^2 B^o \Big\{ 1 + \frac{1}{2{r^o}^2} \sum_s {\sigma_s^\delta}^2 \left[\frac{I_{\ell s}'(shr^o)}{I_{\ell_s}'(shS)} \right]^2 (1 - 2 s^2 h^2 {r^o}^2 I^{\ell_s}(shr^o)) + \frac{2}{{r^o}^2 B^o} \int_0^{r^o} B_z^\beta \, r\, dr + \dots \Big\} \, , \quad \text{(B17)}$$

where $I^\ell(x) = I_\ell(x)/x\, I_\ell'(x)$. There is no term of order η in ψ.

The volume enclosed by the surface over the machine length $(2\pi/k)$ is

$$V = \int_0^{2\pi/k} dz \int_0^{2\pi} d\theta \int_0^{r^o + r^\delta + r^{\delta\delta} + \dots} r\, dr \quad , \quad \text{(B18)}$$

or

$$V = \frac{2\pi^2 {r^o}^2}{k} \Big\{ 1 + \frac{1}{2{r^o}^2} \sum_s {\sigma_s^\delta}^2 \left[\frac{I_{\ell_s}'(shr^o)}{I_{\ell_s}'(shS)} \right]^2 + \dots \Big\} \, . \quad \text{(B19)}$$

We can now compute $V'(\psi)$ and $V''(\psi)$:

$$V' = (dV/dr^o)/(d\psi/dr^o)$$

$$= \frac{2\pi}{kB^o}\left\{1 + \sum_s \frac{s^2h^2\sigma^{\delta 2}}{2}\left[\frac{I'_{\ell_s}(shr^o)}{I_{\ell_s}(shS)}\right]^2 [1+(\ell_s^2+s^2h^2r^{o2})I^{\ell_s 2}_s(shr^o)] - \frac{B_z^\beta}{B^o} + \dots\right\}; \quad \text{(B20)}$$

$$V'' = (dV'/dr^o)/(d\psi/dr^o)$$

$$= -\sum_s \frac{s^2h^2\sigma_s^{\delta 2}}{kr^{o2}B^{o2}}\left[\frac{I'_{\ell_s}(shr^o)}{I'_{\ell_s}(shS)}\right]^2 [1-2(\ell_s^2+s^2h^2r^{o2})I^{\ell_s}(shr^o)+\ell_s^2 I^{\ell_s 2}(shr^o)]$$

$$+ \frac{j_\theta^\beta}{kr^oB^{o3}}, \quad \text{(B21)}$$

since $j_\theta^\beta = -dB_z^\beta/dr$.

We next use Eqs. (A22) and (A23) of Appendix II A to get

$$V'' = \frac{1}{2\pi B^{o2}} \sum_s \frac{sh}{\ell_s r^{o3}} \frac{d}{dr^o}(r^{o4}\iota_s^{\delta\delta}) + \frac{j_\theta^\beta}{kr^oB^{o3}} . \quad \text{(B22)}$$

In this expression $\iota_s^{\delta\delta}$ represents the value of the rotational transform over the length of the machine $(2\pi/k)$ arising from the helical field identified with u_s. If the expression for $\iota^{\delta\delta}/\ell$ given by Eq. (A22) of Appendix II A is generalized to include $\ell = 0$, (B22) can be extended to include Bulge $(\ell = 0)$ fields.

A necessary condition for axially symmetric systems to be stable is that the sign of

$$\Lambda = \gamma p \left(\frac{V''}{V'} + \frac{p'}{\gamma p} \right) \frac{(V'' - p'L')}{(V' + \gamma p L')} \tag{B23}$$

be positive. Here V is defined by Eq. (B18), p is the pressure, and

$$L(\psi) = \int_0^{2\pi/k} dz \int_0^{2\pi} d\theta \int_0^{r^o + r^\delta + r^{\delta\delta} + \ldots} \frac{r\,dr}{B^2} \quad . \tag{B24}$$

To the lowest significant order

$$p' = (dp/dr^o)/(d\psi/dr^o) = j_\theta^\beta / 2\pi r^o \quad ,$$

$$L' = \frac{d(2\pi^2 r^{o\,2}/h\,B^{o\,2})/dr^o}{(d\psi/dr^o)} = 2\pi/k\,B^{o\,3} \quad , \tag{B26}$$

so that

$$V'' = p'L' = \frac{1}{2\pi B^{o\,2}} \sum_s \frac{s\,h}{\ell_s r^{o\,3}} \frac{d}{dr^o} (r^{o\,4} \iota_s^{\delta\delta}) \quad . \tag{B27}$$

and

$$\Lambda = \frac{j_\theta^\beta \quad k}{8\pi^3 r^{o\,4} B^o} \sum_s \frac{s\,h}{\ell_s} \frac{d}{dr} (r^4 \iota_s^{\delta\delta}) \quad . \tag{B28}$$

Appendix III A - Demonstration of Uniformity of Minimization of δW for Trancendentally Small hR .

The stability analysis in Section III is carried through for a particular ordering of the parameters: $\beta \sim \delta^2 \sim \epsilon^2 \sim \eta \sim kR \sim \lambda^2$; $hR \sim 1$. Eq. (54) of Section III is thus valid for any equilibrium in which this ordering holds. Moreover, it yields results for equilibria in which other choices of the ordering are assumed. As an example consider equilibria with $\beta \sim \lambda$ and all the other parameters the same as above. The stability calculation was carried out from the beginning and it was found that the system is always unstable. Yet this result is contained in Eq. (54) of Section III since if β is taken of order λ there, the expression is always negative. As another example consider $kR \sim \lambda$, all other parameters the same. The stability of this system was also examined and the results found were identical with those obtained by setting $n = 0$ in Eq. (54) of Section III. This is the obvious way to extend Eq. (54) to the case of $kR \sim \lambda$ since $n \neq 0$ leads to positive $\delta W_{m;n}$. Further it is impossible to go backwards and obtain Eq. (54) from the results of the stability calculation with $kR \sim \lambda$ from the beginning. In the same way the stability of equilibria with other orderings of the parameters can be determined directly from Eq. (54) so that we speak of Eq. (54) as being "uniformly" valid for any choice of orders of the parameters.

If $hR \sim 1$, it is clear from the discussion of the preceding examples and the nature of the calculation of Section III that Eq. (54) is "uniformly" valid. However, if hR is of a different order, it is no longer obvious that Eq. (54) can be applied. Calculations have been carried through which show that Eq. (54) is valid for $hR \sim \lambda^p$ with p any integer. In this appendix the calculation of the extreme case in which hR is smaller than any power of λ

(i.e., trancendentally small) is carried out and it is shown that Eq. (54) is still valid.

It will be assumed for simplicity that the plasma occupies the entire volume, i.e., R = S. Further, only a diamagnetic current j_θ^β independent of θ exists. Generalization to the situation which was treated in Section II is straightforward. Quantities will be written with two superscripts of which the first denotes its order as a power of the parameter h and the second as a power of λ. That is, $A^{u,v} \sim h^u \lambda^v$. The procedure will consist of minimizing $\delta W^{u,v}$ for u = 0 and ascending values of v through $v = \infty$, then for u = 1 as v increases from 0 to ∞, and finally for u = 2 from v = 0 to that order for which δW is not trivially positive. The parameters will be chosen so that $\rho^\lambda/R \sim \lambda$, $\beta \sim \lambda^2$, and $kR \sim h\lambda^2$.

The equilibrium situation which is investigated can be obtained from Section II (and an extension of it to higher order terms) by expanding the Bessel functions as power series in hr and identifying each order of h. In describing the helical fields we have used the notation introduced in Eq. (B11) of Appendix II B. One can convince oneself that such a procedure is acceptable by solving the equilibrium equations directly for small values of v and utilizing the conditions

$$\underline{B} \cdot \nabla \underline{j} = \underline{j} \cdot \nabla \underline{B} \tag{A1}$$

and

$$\underline{B} \cdot \underline{n} = 0 \qquad (r = R + \rho^{0,1} \cos u) \tag{A2}$$

to investigate higher values of v. In particular, one can show that

$$B^{0,v} = 0 \tag{A3}$$

for all v .

The minimization procedure can be simplified for this calculation by writing Eq. (7) of Section III in a different form. Substituting for $B_z\, \nabla \cdot \underline{\xi}$ in terms of Q_z, expanding $\underline{Q}$, and using the identity,

$$\underline{\xi} \cdot \nabla \underline{B} = \underline{j} \times \underline{\xi} + (\nabla \underline{B}) \cdot \underline{\xi} \; , \tag{A4}$$

one gets

$$2\,\delta W = \int \Big\{ \underline{Q}^2 + (Q_z + \underline{j} \times \underline{\xi} \cdot \underline{e}_z)^2 - \underline{j} \times \underline{\xi} \cdot \underline{e}_z (\underline{B} \cdot \nabla \xi_z - (\tfrac{\partial}{\partial z} \underline{B}) \cdot \underline{\xi})$$

$$+ \underline{j} \times \underline{e}_z \xi_z \cdot (\underline{B} \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B} - 2 \underline{B}\, \nabla \cdot \underline{\xi}) + \underline{e}_z j_z \times \underline{\xi} \cdot (\underline{Q} - \underline{B}\, \nabla \cdot \underline{\xi})$$

$$+ \gamma p\, (\nabla \cdot \underline{\xi})^2 \Big\}\, d\tau \; . \tag{A5}$$

By using the identity,

$$(\underline{B} \cdot \nabla \underline{j}) \times \underline{\xi} \cdot \underline{e}_z \xi_z - \underline{j} \times \underline{e}_z \xi_z \cdot (\underline{\xi} \cdot \nabla \underline{B}) = [(\underline{j} \cdot \nabla \underline{B}) \times \underline{\xi} + \underline{j} \times (\underline{\xi} \cdot \nabla \underline{B})] \cdot \underline{e}_z \xi_z$$

$$= - \, \underline{j} \times \underline{\xi} \cdot \frac{\partial \underline{B}}{\partial z}\, \xi_z \; , \tag{A6}$$

one can put Eq. (A 5) into the form

$$2\delta W = \int \Big\{ \underline{Q}^2 + (Q_z + \underline{j} \times \underline{\xi} \cdot \underline{e}_z)^2 - \nabla \cdot (\underline{B}\, \underline{j} \times \underline{\xi} \cdot \underline{e}_z \xi_z) + \gamma p (\nabla \cdot \underline{\xi})^2$$

$$+ \underline{j} \times \underline{\xi} \cdot (\underline{e}_z \frac{\partial}{\partial z} \underline{B}) \cdot \underline{\xi} - \underline{j} \times \underline{e}_z \xi_z \cdot (\frac{\partial \underline{B}}{\partial z} \xi_z + 2 \underline{B}\, \nabla \cdot \underline{\xi})$$

$$+ j_z \underline{e}_z \times \underline{\xi} \cdot (\underline{Q} - \underline{B}\, \nabla \cdot \underline{\xi} - \frac{\partial \underline{B}}{\partial z} \xi_z) \Big\}\, d\tau \; . \tag{A7}$$

The third term vanishes on integration, with Eq. (A 2) . The term in δW

proportional to $h^u \lambda^v$ is

$$2\,\delta W^{u,v} = \int_0^{2\pi} d\theta \int_0^{2\pi/k} dz \int_0^R r\,dr \,\{ \underline{Q}^{b,c} \cdot \underline{Q}^{u-b,v-c}$$

$$+ (Q_z^{b,c} + \underline{j}^{d,e} \times \underline{\xi}^{b-d,c-e} \cdot \underline{e}_z)(Q_z^{u-b,v-c} + \underline{j}^{f,g} \times \underline{\xi}^{u-b-f,v-c-g} \cdot \underline{e}_z)$$

$$+ \gamma p^{u-b-d,v-c-e} (\nabla \cdot \underline{\xi}^{b,c})(\nabla \cdot \underline{\xi}^{d,e}) + \underline{j}^{u-b-d-f,v-c-e-g} \times \underline{\xi}^{b,c} \cdot \underline{e}_z \frac{\partial \underline{B}^{d,e}}{\partial z} \cdot \underline{\xi}^{f,g}$$

$$- \underline{j}^{u-b-d-f,v-c-e-g} \times \underline{e}_z \xi_z^{b,c} \cdot \left(\frac{\partial \underline{B}^{d,e}}{\partial z} \xi_z^{f,g} + 2 \underline{B}^{d,e} \nabla \cdot \underline{\xi}^{f,g} \right) \qquad (A\,8)$$

$$+ j_z^{u-b-d,v-c-e} \underline{e}_z \times \underline{\xi}^{b,c} \cdot \left(\underline{Q}^{d,e} - \underline{B}^{d-f,e-g} \nabla \cdot \underline{\xi}^{f,g} - \frac{\partial \underline{B}^{d-f,e-g}}{\partial z} \xi_z^{f,g} \right) \} ,$$

plus the contribution from the region between R and $R + \rho^{0,1}$ from integrands of lower order. Here

$$\underline{Q}^{u,v} = \sum_{s,n} \underline{Q}^{u,v}(s,n)\, e^{i(sh+nk)z} \qquad (A\,9)$$

where

$$\underline{Q}^{u,v}(s,n) = i(s-s')hB_z^{b,c}(s')\underline{\xi}^{u-b-1,v-c}(s-s',n) + i\,nk\,B_z^{b,c}(s')\underline{\xi}^{u-b-1,v-c-2}(s-s',n)$$

$$+ \underline{B}^{b,c}(s') \cdot \nabla \underline{\xi}^{u-b,v-c}(s-s',n) - \underline{\xi}^{u-b,v-c}(s-s',n) \cdot \nabla \underline{B}^{b,c}(s')$$

$$- \underline{B}^{b,c}(s') \nabla \cdot \underline{\xi}^{u-b,v-c}(s-s',n) - i\,s\,h \underline{B}^{b,c}(s') \xi_z^{u-b-1,v-c}(s-s',n)$$

$$- i\,n\,k\, \underline{B}^{b,c}(s')\, \xi_z^{u-b-1,v-c-2}(s-s',n) \qquad (A\,10)$$

and

$$Q_z^{u,v}(s,n) = B^{b,c}(s') \cdot \nabla \xi_z^{u-b,v-c}(s-s',n) - \xi^{u-b,v-c}(s-s',n) \cdot \nabla B_z^{b,c}(s')$$

$$- i s' h B_z^{b,c}(s') \xi_z^{u-b-1,v-c}(s-s',n) - B_z^{b,c}(s') \nabla \cdot \xi^{u-b,v-c}(s-s',n) \; .$$

(A 11)

The summation convention has been employed, i.e., b, c, s', etc., take on all possible meaningful values.

Since $\underline{j} = \nabla \times \underline{B}$ and $B^{o,v} = 0$, $\underline{Q}^{o,v}$ is given by

$$\underline{Q}^{o,v}(s,n) = 0 \; , \tag{A 12}$$

and

$$Q_z^{o,v}(s,n) = - \underline{j}^{o,c}(s') \times \underline{\xi}^{o,v-c}(s-s',n) \cdot \underline{e}_z - B_z^{o,c}(s') \nabla \cdot \underline{\xi}^{o,v-c}(s-s',n). \tag{A 13}$$

Thus

$$2\,\delta W^{o,o} = \int_o^{2\pi} d\theta \int_o^{2\pi/k} dz \int_o^R r dr \{ (Q_z^{o,o})^2 \} \tag{A 14}$$

is minimized by setting

$$\nabla \cdot \underline{\xi}^{o,o} = 0 \; . \tag{A 15}$$

This automatically makes $\delta W^{0,1}$ zero. Continuing, we have

$$2\,\delta W^{0,2} = \int_o^{2\pi} d\theta \int_o^{2\pi/k} dz \int_o^R r dr \{ (Q_z^{0,1})^2 \} \tag{A 16}$$

which requires that

$$\nabla \cdot \underline{\xi}^{0,1} = 0 \; . \tag{A 17}$$

In general, from Eq. (A 8)

$$2\delta W^{o,v} = \int_o^{2\pi} d\theta \int_o^{2\pi/k} dz \int_o^R r dr \{(Q_z^{o,c} + j^{o,e} \times \xi^{o,c-e} \cdot \underline{e}_z)(Q_z^{o,v-c} + j^{o,g} \times \xi^{o,v-c-g} \cdot \underline{e}_z)$$

$$+ \gamma p^{o,v-c-e} (\nabla \cdot \xi^{o,c})(\nabla \cdot \xi^{o,e})\} \ . \qquad \text{(A 18)}$$

Thus, expressing $Q_z^{o,c}$ by means of Eq. (A 13), we have

$$2\delta W^{o,v} = \int_o^{2\pi} d\theta \int_o^{2\pi/k} dz \int_o^R r dr \{(-B_z^{o,e} \nabla \cdot \xi^{o,c-e})(-B_z^{o,g} \nabla \cdot \xi^{o,v-c-g})$$

$$+ \gamma p^{o,v-c-e} (\nabla \cdot \xi^{o,c})(\nabla \cdot \xi^{o,e})\} \ , \qquad \text{(A 19)}$$

or when we replace c-e by c and v-c-g by e in the first term

$$2\delta W^{o,v} = \int_o^{2\pi} d\theta \int_o^{2\pi/k} dz \int_o^R r dr \{(B_z^{o,v-c-e-g} B_z^{o,g} + \gamma p^{o,v-c-e})(\nabla \cdot \xi^{o,c})(\nabla \cdot \xi^{o,e})\} \ . \qquad \text{(A 20)}$$

Now, if v is even let $v = 2w$; if odd, $v = 2w - 1$. Assume that

$$\nabla \cdot \xi^{o,x} = 0 \qquad \text{(A 21)}$$

for $x \leq w - 1$. Then

$$2\delta W^{0,2w-1} = 0 \ , \qquad \text{(A 22)}$$

$$2\delta W^{0,2w} = \int_o^{2\pi} d\theta \int_o^{2\pi/k} dz \int_o^R r dr \{(B^{o,o})^2 (\nabla \cdot \xi^{o,w})^2\} \ , \qquad \text{(A 23)}$$

so that the minimizing $\xi^{o,w}$ must satisfy

$$\nabla \cdot \xi^{o,w} = 0 . \qquad \text{(A 24)}$$

Since Eq. (A 21) is satisfied for $w = 1$ or 2 (Eqs. (A 15) or (A 17) by induction both $\nabla \cdot \xi^{o,v}$ and $\delta W^{o,v}$ are zero for all v.

Since $(\nabla \cdot \underline{\xi})^{o,v}$ and $j_z^{o,v}$ are zero, it is clear from Eq. (A 8) that

$$2\delta W^{1,v} = 0 \qquad \text{(A 25)}$$

for all v.

Proceeding to the h^2 orders, we get

$$2\delta W^{2,0} = \int_0^{2\pi} d\theta \int_0^{2\pi/k} dz \int_0^R r dr \{ (\underline{Q}^{1,0})^2 \} , \qquad \text{(A 26)}$$

where

$$Q^{1,0}(s,n) = i\, s\, h\, B^{o,o} \xi^{o,o}(s,n) \qquad \text{(A 27)}$$

and

$$Q_z^{1,0}(s,n) = - B^{o,o}\, \nabla \cdot \xi^{1,0}(s,n) \qquad \text{(A 28)}$$

Minimization, therefore, requires that

$$\xi^{o,o}(s,n) = 0 \qquad (s \neq 0) \qquad \text{(A 29)}$$

and

$$\nabla \cdot \xi^{1,0} = 0 . \qquad \text{(A 30)}$$

Since

$$2\delta W^{2,1} = 0 , \qquad \text{(A 31)}$$

nothing new is obtained from the $h^2 \lambda$ order.

Continuing, we get

$$2\delta W^{2,2} = \int_0^{2\pi} d\theta \int^{2\pi/k} dz \int_0^R r dr \{ (\underline{Q}^{1,1})^2 + \gamma p^{0,2} ((\frac{\partial \xi_z}{\partial z})^{1,0})^2 \} , \qquad \text{(A 32)}$$

which requires that

$$\underline{Q}^{1,1} = 0 \ . \tag{A 33}$$

and

$$\left(\frac{\partial \xi_z}{\partial z}\right)^{1,0} = 0 \ . \tag{A 34}$$

It can be shown in exactly the same way as was done in Section III that by making the transformation

$$\underline{\xi} \rightarrow \underline{\xi} + \underline{B}\ f \tag{A 35}$$

one can set

$$\xi_z^{\ 0,0} = 0 \tag{A 36}$$

without any loss of generality. Then

$$\underline{Q}^{1,1}(s,n) = ish\ B^{0,0}\underline{\xi}^{0,1}(s,n) + \underline{B}^{1,1}(s)\cdot\nabla\underline{\xi}^{0,0}(0,n) - \underline{\xi}^{0,0}(0,n)\cdot\nabla\underline{B}^{1,1}(s), \tag{A 37}$$

$$Q_z^{\ 1,1}(s,n) = -\ B^{0,0}\ \nabla\cdot\underline{\xi}^{1,1}(s,n), \tag{A 38}$$

and it is necessary to set

$$ishB^{0,0}\underline{\xi}^{0,1}(s,n) = -\ \underline{B}^{1,1}(s)\cdot\nabla\underline{\xi}^{0,0}(0,n) + \underline{\xi}^{0,0}(0,n)\cdot\nabla\underline{B}^{1,1}(s), \tag{A 39}$$

$$\nabla\cdot\underline{\xi}^{1,1}(s,n) = 0 \ . \tag{A 40}$$

The next order is

$$2\,\delta W^{2,3} = \int_0^{2\pi} d\theta \int_0^{2\pi/k} dz \int_0^R r\,dr\left\{-j_\theta^{\ o,2}\ \frac{\partial \underline{B}^{1,1}}{\partial z}\cdot\underline{\xi}^{0,0}\xi_r^{\ 0,0}\right\} . \tag{A 41}$$

This vanishes on integration by parts with respect to z. This integrand in the region between R and $R + \rho^{0,1}$ will not contribute to $\delta W^{2,4}$ since $\xi_r^{0,0}$ must be zero at R.

Finally,

$$2\delta W^{2,4} = \int_0^{2\pi} d\theta \int_0^{2\pi/k} dz \int_0^R r dr \{ (\underline{Q}^{1,2})^2 + (Q_z^{1,2} - j_\theta^{0,2} \xi_r^{1,0})^2 + \gamma p^{0,2} (\frac{\partial \xi_z^{0,1}}{\partial z})^2$$

$$+ \underline{j}^{0,4-c-e-g} \times \underline{\xi}^{0,c} \cdot \underline{e}_z (\frac{\partial}{\partial z} \underline{B}^{1,e}) \cdot \underline{\xi}^{0,g} \} . \qquad \text{(A 42)}$$

Since $\underline{\xi}^{0,2}$, $\underline{\xi}^{1,2}$, and $\xi_z^{0,1}$ enter $\delta W^{2,4}$ only through positive definite terms, $\delta W^{2,4}$ can be minimized by setting

$$\underline{Q}^{1,2}(s,n) = 0 , \qquad (s \neq 0) \qquad \text{(A 43)}$$

$$Q_z^{1,2} = j_\theta^{0,2} \xi_r^{1,0} , \qquad \text{(A 44)}$$

$$\xi_z^{0,1}(s,n) = 0 . \qquad (s \neq 0) \qquad \text{(A 45)}$$

Then

$$2\delta W^{2,4} = \int_0^{2\pi} d\theta \int^{2\pi/k} dz \int_0^R r dr \{ |\underline{Q}^{1,2}(0,n)|^2$$

$$+ (j_r^{0,3} \xi_\theta^{0,0} - j_\theta^{0,3} \xi_r^{0,0} - j_\theta^{0,2} \xi_r^{0,1}) \frac{\partial \underline{B}^{1,1}}{\partial z} \cdot \underline{\xi}^{0,0} - j_\theta^{0,2} \xi_r^{0,0} \frac{\partial \underline{B}^{1,1}}{\partial z} \cdot \underline{\xi}^{0,1} \} . \qquad \text{(A 46)}$$

We now integrate the terms which contain $\partial \underline{B}^{1,1}/\partial z$ by parts in respect to z and substitute for $\partial \underline{j}^{0,3}/\partial z$ and $\partial \underline{\xi}^{0,1}/\partial z$ by means of Eqs. (A 1) and (A 39).

Making use of Maxwell's equations and recognizing that some terms vanish on integration ($\xi_r^{0,0}(R) = 0$), we find that

$$2\delta W^{2,4} = \frac{4\pi^2}{k} \int_0^R r dr \left\{ |\mathcal{Q}^{1,2}(0,n)|^2 + \frac{j_\theta^{0,2}}{B^o} |\xi_r^{0,0}(0,n)|^2 \frac{d}{dr} |\mathcal{B}^{1,1}(s,0)|^2 \right\}. \tag{A 47}$$

From Eq. (A 10)

$$\mathcal{Q}^{1,2}(0,n) = i n k B^{0,0} \xi^{0,0}(0,n) + \nabla \times [(\xi^{0,1}(-s,n) \times \mathcal{B}^{1,1}(s,0))] . \tag{A 48}$$

Eqs. (A 47) and (A 48) are the same as one would have obtained by expanding Eq. (54) of Section III, keeping only the lowest order term in hR, illustrating the uniformity of that result as hR becomes infinitesimal.

It can be shown that the boundary condition

$$\underline{\xi} \cdot \underline{n} = 0 \qquad (r = R + \rho^{0,1} \cos u) \tag{A 49}$$

is satisfied, in the same way as it is shown in Section III.

Appendix III - B The Fourier Analysis of the ξ's

In the main part of Section III the ξ's were Fourier analyzed in z and the Fourier components were expanded in λ. It has been assumed that in the Fourier analysis of ξ only those components of ξ with wave numbers independent of λ and those with wave number proportional to λ^2 were important while all the others were assumed to be zero. For instance, the component whose wave number is proportional to λ was neglected. It will be shown in this appendix, that if any of these components were non-zero δW would necessarily be positive, so that in testing δW for sign it is actually permissible for one to neglect such components.

In carrying out this justification we shall not always carry the Fourier analysis explicitly, but we will speak of the rate of variation in z of ξ by speaking of the order of $\frac{\partial \xi}{\partial z}$. We make use of the order notation O and o where f = o(g) means f goes to zero faster than g and f = O(g) means z goes to zero at the same rate as g.

We know from $\Theta^o = 0$ that $\frac{\partial \xi^o}{\partial z} = o(1)$. Further it can easily be seen that $\frac{\partial \xi^o}{\partial z} = O(\lambda)$, since $\frac{\partial \xi^o}{\partial z} = O(\lambda^\alpha)$ with $0 < \alpha < 1$, would imply $Q^\alpha \neq 0$ and therefore $\delta W^{2\alpha} > 0$. Next if $\frac{\partial \xi^o}{\partial z} = O(\lambda)$ we would have a part of $\Theta^\lambda \sim e^{i\lambda z}$ which has a different wave number than the rest of Θ^λ has, and which must thus vanish. Therefore $\frac{\partial \xi^o}{\partial z} = o(\lambda)$. Of course ξ^λ might have components whose wave number varies as different powers of λ than we consider in the main Section, but these only enter Q_z^λ and we must have $\nabla \cdot \xi^\lambda = 0$.

Further, from consideration of $\delta W^{\lambda\lambda}$ we have $\frac{\partial \xi_z^o}{\partial z} = o(1)$ and it is clear that $\frac{\partial \xi_z^o}{\partial z} = O(\lambda)$, since the $\gamma p (\nabla \xi)^2$ term would otherwise be larger than λ^4 and lead to stability (we are not yet assuming $\xi_z^o = 0$ in this appendix). Finally, in $\delta W^{\lambda\lambda\lambda\lambda}$ the $\gamma p \left(\frac{\partial \xi_z^o}{\partial z}\right)^2$ term acts independently of the remainder of the $\gamma p (\nabla \cdot \underline{\xi})^2$ term because of its different dependence on z. Combining this fact with our argument that changing $\underline{\xi}$ by $f \underline{B}$ only affects this term we can conclude that $\frac{\partial \xi_z^o}{\partial z} = o(\lambda)$. We may now apply our argument that $\xi_z^o = 0$.

It now follows easily that $\frac{\partial \xi_z^o}{\partial z} = O(\lambda^2)$ since otherwise Θ^2 would be larger than λ^4. The other components of $\underline{\xi}^\lambda$ would also lead to Θ^2 larger than λ^4 and so much vanish. The other components of $\underline{\xi}^{\lambda\lambda}$ never enter. It is thus seen that the choice of $\underline{\xi}$'s which only have Fourier components whose wave numbers are proportional to 1 or λ^2 is justified.

Appendix IV - A Kruskal Instability if only a Surface Current is Present.

The situation in which the external region is a pressureless plasma and in which the axial current is confined to the surface of the plasma is treated in Section IV. It is shown there for currents larger than the Kruskal limit the system is neutral. It is therefore necessary to carry the minimization to a higher order in λ .

Since no fields of odd order in λ are present, we carry the calculation through again from the beginning, assuming that the expansion parameter λ is of the same order as kR and B_θ / B_z . For simplicity, β will be set equal to zero and S equal to infinity. The equilibrium which is treated here is defined by

$$\underline{B} = \underline{e}_z \; B^o , \qquad (r < R)$$

$$B = \underline{e}_\theta \; B^\lambda + \underline{e}_z \; (B^o + B^{\lambda\lambda}) , \qquad (r > R)$$

where B^o is a constant externally applied field, $B^\lambda = B^\lambda(R) \; R/r$ is the field due to the axial surface current, and $B^{\lambda\lambda} = - \dfrac{B^\lambda(R)^2}{2 B^o}$ is the second order field necessary to satisfy the continuity condition on the total pressure at the surface.

To lowest order δW is[8]

$$2 \; \delta W^o = \int_o^\infty \underline{Q}^{o2} \; d\tau \tag{A1}$$

where

$$\underline{Q}^{o} = - \underline{e}_z \ B^{o} \ \Delta \cdot \underline{\xi}^{o} \quad . \tag{A2}$$

This is mimized by setting

$$\Delta \cdot \underline{\xi}^{o} = 0 \ . \tag{A3}$$

The next order vanishes trivially and

$$2 \delta W^{\lambda\lambda} = \int_{o}^{\infty} \underline{Q}^{\lambda 2} \ d\tau - \int \xi_r^{o} (R)^2 \ B^{\lambda} (R)^2 \ d\theta \, dz \quad , \tag{A4}$$

where

$$\underline{Q}^{\lambda} = B^{o} \frac{\partial}{\partial z} \underline{\xi}^{o} - \underline{e}_z \ B^{o} \ \nabla \cdot \underline{\xi}^{\lambda} \ , \quad (r < R) \tag{A5}$$

$$\underline{Q}^{\lambda} = B^{o} \frac{\partial}{\partial z} \underline{\xi}^{o} + B^{\lambda} \frac{\partial}{r \partial \theta} \underline{\xi}^{o} - \underline{\xi}^{o} \cdot \nabla \ \underline{B}^{\lambda} - \underline{e}_z \ B^{o} \nabla \cdot \underline{\xi}^{\lambda} \ . \quad (r > R) \tag{A6}$$

With the usual Fourier analysis $\underline{Q}^{\lambda}$ can be expressed in terms of

$$\underline{Q}^{\lambda} (m, n) = \underline{e}_r \frac{im}{r} \ \mu^{\lambda}(m, n) - \underline{e}_{\theta} \frac{\partial}{\partial r} \ \mu^{\lambda}(m, n) \tag{A7}$$

where

$$\mu^{\lambda} = \frac{n k r B^{o}}{m} \xi_r^{o}(m, n) \quad , \quad (r < R) \quad (A8)$$

$$\mu^{\lambda} = \frac{k r B^{o}}{2\pi} \left(\frac{2\pi n}{m} + \iota^{\eta}\right) \xi_r^{o}(m, n) \quad . \quad (r > R) \quad (A9)$$

Here $\iota^{\eta} = \frac{2\pi B^{\eta}(r)}{k r B^{o}}$ is the transform over the length of the tube due to the axial current. Since ξ^{λ} enters only in the positive definite term $Q_z^{\lambda 2}$, $\delta W^{\lambda\lambda}$ can be minimized in respect to it by setting

$$\nabla \cdot \xi^{\lambda} = 0 \quad , \quad (r < R) \quad (A10)$$

$$B^{o} \nabla \cdot \xi^{\lambda}(m, n) = \frac{i m}{r} B^{\lambda} \xi_z^{o}(m, n) \quad (r > R) \quad (A11)$$

Now each $\mu^{\lambda}(m, n)$ must satisfy the Euler equation

$$\frac{1}{r} \frac{d}{dr} r \frac{d\mu^{\lambda}}{dr} - \frac{m^2}{r^2} \mu^{\lambda} = 0 , \quad (A12)$$

or

$$\mu^{\lambda}(m, n) = A r^{|m|} \quad (r < R) \quad (A13)$$

and

$$\mu^{\lambda}(m, n) = B\left(r^{|m|} - a^{2|m|}\, r^{-|m|}\right) , \qquad (R < r < a) \qquad \textbf{(A14)}$$

so that

$$\delta W^{\lambda\lambda} = \frac{8\pi^2}{k} \sum_{\substack{m > 0 \\ n}} \left\{ m\left(\frac{2\pi n}{m}\right)^2 + m\left(\frac{2\pi n}{m} + \iota^{\eta}\right)^2 \frac{a^{2m}+R^{2m}}{a^{2m}-R^{2m}} - \iota^{\eta^2} \right\} \cdot$$

$$\cdot \left(\frac{k R B^{o}}{2\pi}\right)^2 \left|\overset{o}{\xi}_r (m, n)\right|^2 , \qquad \text{(A15)}$$

where all quantities are evaluated at R , and a is that radius at which $\mu^{\lambda}(m, n)$ must vanish , *i.e* .

$$a^2 / R^2 = m\, \iota^{\eta}(R) \,/\, (2\pi n) \quad . \qquad \text{(A16)}$$

If this expression is less than one, a is equal to infinity.

Eq. (A15) is the same as Eq. (5d) of Section IV except that S has been replaced by a . This minimization was reproduced here for completeness since the notation is a little different from that used in Section III. It is clear that δW as given by Eq. (A15) is definitely larger for any given values of n , m and ι^{η} than that given by Eq. (5d) of Section IV so that since the latter predicts stability for all m> 1 , no m> 1 instabilities can exist in this calculation. As before, only negative values of n can lead to instabilities. If $\iota^{\eta}(R) < 2\pi |n|$, a is infinite and the situation is the same as that discussed in Section IV , so that the system is stable in respect to such purturbations. If $\iota^{\eta}(R) > 2\pi |n|$, δW is zero , the system is neutral to this order, and the calculation must be continued.

To third order

$$2\delta W^{\lambda\lambda\lambda} = \frac{8\pi^2}{k} \sum_n \{ 2\int_0^\infty \Theta^{\lambda}(1,n) \cdot \Theta^{\lambda\lambda}(-1,-n) \, r dr - 2(\xi_r^{o}(1,n)\xi_r^{\lambda}(-1,-n)B^{\lambda 2})_R \} \; . \tag{A17}$$

By straightforward expansion

$$\Theta^{\lambda\lambda}(m,n) = \underline{e}_r \frac{i m}{r} \mu^{\lambda\lambda}(m,n) - \underline{e}_\theta \frac{\partial}{\partial r} \mu^{\lambda\lambda}(m,n) \quad , \tag{A18}$$

where

$$\mu^{\lambda\lambda}(m,n) = \frac{n k r B^{o}}{m} \xi_r^{\lambda}(m,n) \qquad (r < R) \tag{A19}$$

$$\mu^{\lambda\lambda}(m,n) = \frac{k r B^{o}}{2\pi} \left(\frac{2\pi n}{m} + \iota^{\eta}\right) \xi_r^{\lambda}(m,n) \; . \qquad (r > R) \tag{A20}$$

Therefore

$$2\delta W^{\lambda\lambda\lambda} = \frac{8\pi^2}{k} \sum_n \{ 2 \int_0^\infty (\frac{1}{r^2} \mu^\lambda(1,n)\, \mu^{\lambda\lambda}(-1,-n) + \frac{d\mu^\lambda(1,n)}{dr} \frac{d\mu^{\lambda\lambda}(-1,-n)}{dr}) \, r\, dr$$

$$- (2 B^{\lambda^2} \xi_r^o (1,n)\, \xi_r^\lambda (-1,-n)_R \} \quad . \qquad \text{(A21)}$$

Integrating the term which contains $\frac{d\mu^{\lambda\lambda}(-1,-n)}{dr}$ by parts, recognizing that the coefficient of $\mu^{\lambda\lambda}(-1,-n)$ in the integrand is zero due to Eq. (A12), and evaluating the integrated term by means of Eqs. (A13) and (A14), one finds that

$$2\delta W^{\lambda\lambda\lambda} = \frac{8\pi^2}{k} \sum_n 2 \{ \mu_{in}^\lambda (1,n)\, \mu_{in}^{\lambda\lambda} (-1,-n) + \mu_{out}^\lambda (1,n)\, \mu_{out}^{\lambda\lambda} (-1,-n) \frac{a^2 + R^2}{a^2 - R^2}$$

$$- B^{\lambda 2} \xi_r^o (1,n)\, \xi_r^\lambda (-1,-n) \}_R \qquad \text{(A22)}$$

and $\delta W^{\lambda\lambda\lambda}$ vanishes trivially when the μ's are expressed in terms of the ξ's.

In the fourth order

$$2\delta W^{\lambda\lambda\lambda\lambda} = \frac{8\pi^2}{k} \sum_n \{ \int_0^\infty (2\, \mathcal{Q}^\lambda (1,n) \cdot \mathcal{Q}^{\lambda\lambda\lambda} (-1,-n) + |\mathcal{Q}^{\lambda\lambda}(1,n)|^2) \, r\, dr$$

$$- (B^{\lambda 2})_R \, (2\, \xi_r^o (1,n)\, \xi_r^{\lambda\lambda} (-1,-n) + |\xi_r^\lambda(1,n)|^2)_R \} \quad . \qquad \text{(A23)}$$

Here

$$Q_z^{\lambda\lambda}(m,n) = - B^o \, \nabla \cdot \underset{\sim}{\xi}^{\lambda\lambda}(m\,,\,n) + \frac{i\, m\, B^{\lambda}}{r} \, \xi_z^{\lambda}(m\,,\,n) \tag{A24}$$

and

$$\underset{\sim}{Q}^{\lambda\lambda\lambda}(m\,,\,n) = \underset{-}{e}_r \, \frac{im}{r} \left(\mu^{\lambda\lambda\lambda}(m\,,\,n) + \nu^{\lambda\lambda\lambda}(m\,,\,n) \right) \,,$$

$$- \underset{-}{e}_\theta \left(\frac{d}{dr} \left(\mu^{\lambda\lambda\lambda}(m\,,\,n) + \nu^{\lambda\lambda\lambda}(m\,,n) \right) + \frac{nkr}{m} Q_z^{\lambda\lambda}(m\,,\,n) \right) \,, \tag{A25}$$

where

$$\mu^{\lambda\lambda\lambda}(m,n) = \frac{nkrB^o}{m} \, \xi_r^{\lambda\lambda}(m,n) \;\; ; \;\; \nu^{\lambda\lambda\lambda}(1,n) = 0 \; ; \qquad (r < R) \tag{A26}$$

$$\mu^{\lambda\lambda\lambda}(m,n) = \frac{krB^o}{2\pi} \left(\frac{2\pi n}{m} + \iota^{\eta} \right) \xi_r^{\lambda\lambda}(m,n) \; ; \; \nu^{\lambda\lambda\lambda}(m,n) = \frac{nkrB^{\lambda\lambda}}{m} \, \xi_r^{o}(m,n) \,. \quad (r > R) \tag{A27}$$

The terms in Eq. (A23) which contain $\mu^{\lambda\lambda\lambda}$ cancel the first surface term in exactly the same way that $\delta W^{\lambda\lambda\lambda}$ was shown to vanish. Completing the square in $Q_z^{\lambda\lambda}$, one obtains

$$2\delta W^{\lambda\lambda\lambda\lambda} = \frac{8\pi^2}{k} \sum_n \Big\{ \int_0^\infty \Big(\frac{1}{r^2} |\mu^{\lambda\lambda}(1,n)|^2 + |\frac{d\mu^{\lambda\lambda}(1,n)}{dr}|^2$$

$$+ |Q_z^{\lambda\lambda}(1,n) + nkr\frac{d\mu^{\lambda}(1,n)}{dr}|^2 - n^2k^2r^2 \, |\frac{d\mu^{\lambda}(1,n)}{dr}|^2 + \frac{2\mu^{\lambda}(1,n)\,\nu^{\lambda\lambda\lambda}(-1,-n)}{r^2}$$

$$+ 2\frac{d\mu^{\lambda}(1,n)}{dr}\,\frac{d\nu^{\lambda\lambda\lambda}(-1,-n)}{dr}\Big) r dr - (B^{\lambda 2}\, |\xi_r^{\lambda}(1,n)|^2)_R \Big\} \quad . \tag{A28}$$

Since $\xi^{\lambda\lambda}$ enters $\delta W^{\lambda\lambda\lambda\lambda}$ only through $Q_z^{\lambda\lambda}$ in a positive definite term the minimization in respect to it requires that

$$Q_z^{\lambda\lambda}(1,n) = - nkr \frac{d\mu^{\lambda}(1,n)}{dr} \quad . \tag{A29}$$

Since $\mu^{\lambda\lambda}$ must satisfy the same Euler equation as μ^{λ}, the first two terms just cancel the remaining surface term. Integrating the term containing $\frac{d\nu^{\lambda\lambda\lambda}}{dr}$ by parts and recognizing from Eq. (A12) that the coefficient of $\nu^{\lambda\lambda\lambda}$ in the integrand is zero, one finds

$$2\delta W^{\lambda\lambda\lambda\lambda} = \frac{8\pi^2}{k} \sum_n \Big\{ -\int_0^\infty n^2k^2r^2 \, |\frac{d\mu^{\lambda}(1,n)}{dr}|^2 \, r dr + 2r\frac{d\mu^{\lambda}(1,n)}{dr}\,\nu^{\lambda\lambda\lambda}(-1,-n)\Big|_R^a \Big\} \quad . \tag{A30}$$

Since μ^λ is given by Eqs. (A13) and (A14) the first term can be integrated directly. Then expressing μ^λ and $\nu^{\lambda\lambda\lambda}$ in terms of ξ_r^o and $B^{\lambda\lambda}$ in terms of B^λ and therefore ι^η and using Eq. (A16) one can write

$$2\delta W^{\lambda\lambda\lambda\lambda} = -\frac{8\pi^2}{k} \sum_n \left\{ n^4 + \frac{n^2 \iota^{\eta 2}}{4\pi^2 a^4}(a^2 - R^2)(5a^2 + R^2) + 2a^4 \ln \frac{a^2}{R^2}) - \frac{n \iota^{\eta 3}}{2\pi^3 a^2}(a^2 - R^2)\right\} \frac{k^4 R^4 B^{o\,2}}{4} |\xi_r^o(1, n)|^2 \qquad \text{(A31)}$$

which is clearly negative for all ι^η in the range where the calculation is applicable. The system is therefore unstable if ι^η is greater than the Kruskal limit.

Appendix IV - B Kruskal Instability with Helically Invarient Field and Small β

Since the axial current is present only during ohmic heating when the plasma is relatively cool, it is assumed for simplicity that $\beta = 0$ in the treatment of the Kruskal instability when helical fields are present. However β enters Eq. (54) of Section III through a term which can have a singularity at some point in the plasma. It is, therefore, possible that the presence of even a small value of β can change the situation completely. Here it is shown that the results one obtains by calculating δW assuming β is present and then taking the limit as β approaches zero are the same as those one finds by setting β equal to zero at the start of the calculation.

Assuming that: only a single helical field with $\ell = 3$ is present, the axial current is uniform in the plasma region, the pressure distribution is parabolic, and the helical period is long enough that only the leading terms in the expansions of the Bessel functions in terms of hr need be considered, one finds from Eq. (60) of Section III

$$2\delta W = \frac{8\pi^2}{k} \sum_{\frac{m}{n} > 0} \left\{ \int_0^R \left(\frac{\alpha \mu^2}{r^2} + \mu'^2\right) r dr + m \, \frac{1+(R/S)^{2m}}{1-(R/S)^{2m}} \, \mu^2\Big|_R - \frac{2\,\iota^{\eta} R^2}{\iota^{\delta}(R^2-a^2)} \, \mu^2\Big|_R \right\} \quad \text{(B1)}$$

where

$$\alpha = m^2 - \frac{4\pi\beta}{\iota^{\delta}} \frac{r^4}{(r^2-a^2)^2} , \quad \text{(B2)}$$

$$\mu = \frac{k r B^o \iota^\delta}{2\pi R^2} (r^2 - a^2) \xi_r^o \quad , \tag{B3}$$

and

$$a^2 = - \left(\frac{2\pi n}{m} + \iota^\eta\right) \frac{R^2}{\iota^\delta} \quad , \tag{B4}$$

ι^δ and ι^η being evaluated at R. The minimum value of δW is clearly uniform as $\beta \to 0$ for those values of m and n which do not make a lie in the region $0 < a < R$ since no singularity would exist in Eq. (B1). It is shown in the discussion of the intrinsic stability of the helix $(j^\eta = 0)$ that the contribution to δW from the region $0 < r < a$ is positive if $\beta < \delta^2$ and can be made as small as one desires by making ξ_r^o go to zero close enough to a.

The problem can therefore be redefined as follows: Consider the minimization of the function

$$I = \int_a^R \left(\frac{\alpha}{r^2}\mu^2 + \mu'^2\right) r dr \tag{B5}$$

where α, μ, and a are given by Eqs. (B2), (B3) and (B4), subject to the condition that $\mu(R)$ be prescribed and different from zero. Is the minimum value of I calculated with $\beta = 0$ the same as the limit of the minimum value of I calculated with β present as β approaches zero?

The Euler equation corresponding to Eq. (B5) is

$$\frac{1}{r}(r\mu')' - \left(\frac{m^2}{r^2} - \frac{4\pi\beta r^2}{\iota^\delta (r^2 - a^2)^2}\right)\mu = 0 \quad . \tag{B6}$$

Let $\beta << 1$. In the region where

$$\frac{r^2 - a^2}{r^2} >> \left(\frac{4\pi\beta}{m^2 \iota^\delta}\right)^{1/2} \tag{B7}$$

the last term in Eq. (B6) is negligible and the Euler equation reduces to

$$\frac{1}{r}(r\mu')' - \frac{m^2}{r^2}\mu = 0 , \tag{B8}$$

so that μ must be given by

$$\mu = \mu(R) \frac{r^m + \lambda r^{-m}}{R^m + \lambda R^{-m}} \tag{B9}$$

where the arbitrary constant λ must still be chosen. If

$$r - a << a \tag{B10}$$

Eq. (B6) becomes

$$\mu'' + \frac{\mu'}{a} - \left(\frac{m^2}{a^2} - \frac{\pi\beta}{\iota^\delta (r-a)^2}\right)\mu = 0 . \tag{B11}$$

It follows from Eqs. (B7) and (B10) that for β sufficiently small there exists a region

$$\left(\frac{\pi\beta}{\iota^\delta}\right)^{1/2} \frac{a}{m} << r - a << a \tag{B12}$$

in which μ must satisfy both Eqs. (B8) and (B11) .

Introducing the parameter

$$\chi \equiv (r - a)/a \tag{B13}$$

and the function

$$y \equiv \mu e^{\chi/2} \tag{B14}$$

and letting the prime represent $^{d}/d\chi$, one writes Eq.(B11) as

$$y'' + \left(\frac{\pi\beta}{\iota^{\delta}\chi^{2}} - m^{2} - \frac{1}{4}\right) y = 0 . \tag{B15}$$

Assume a solution of the form

$$y = \chi^{s} (1 + \alpha_2 \chi^2 + \alpha_4 \chi^4 + \ldots + \alpha_n \chi^n + \ldots). \tag{B16}$$

Then

$$s = \frac{1}{2} \pm \frac{1}{2} \left(1 - \frac{4\pi\beta}{\iota^{\delta}}\right)^{1/2} \tag{B17}$$

and

$$\alpha_n = \frac{(m^2 + \frac{1}{4})\,\alpha_{n-2}}{(s+n)(s+n-1) + \frac{\pi\beta}{\iota^{\delta}}} . \tag{B18}$$

For large values of n and small β

$$\alpha_n \sim \frac{\alpha_{n-2}}{(n)(n-1)} \sim \frac{\alpha_{n-4}}{(n)(n-1)(n-2)(n-3)} \sim \frac{1}{n!} \tag{B19}$$

so that the series representation in Eq. (B16) for y is uniformly convergent with respect to χ and β. The series obtained by differentiating Eq. (B16) term by term is also uniformly convergent and represents y'. Since each term of the series is continuous in β at β equal to zero, y and y', and therefore the logarithmic derivative of y are continuous functions of β at $\beta = 0$.

The constant λ in Eq. (B9) is determined by the condition

$$\frac{r\,d\mu}{\mu\,dr}\Big|_{r=r_1} = \frac{\chi\,dy}{y\,d\chi}\Big|_{\chi = (r_1 - a)/a} \tag{B20}$$

where r_1 must lie in the region defined by Eq. (B12). Since the right hand side of this equation is a continuous function of β at $\beta = 0$, the left hand side, and therefore λ, must also be. The integral in Eq. (B5) can be evaluated by multiplying Eq. (B6) by μ and integrating, so that

$$I = r\mu \frac{d\mu}{dr}\Big|_R = m\,\mu(R)^2 \frac{R^{2m} - \lambda}{R^{2m} + \lambda} \tag{B21}$$

must be continuous at $\beta = 0$.

It should be noted from Eq. (B3) that μ must go to zero at a as fast as χ, whereas the minimizing function, given by Eqs. (B14), (B16) and (B17), does not vanish as rapidly. Since for any value of β and any allowable μ, the integral, I, is greater than or equal to that value which is given by Eq. (B21) where λ is between the value which is obtained from Eq. (B20) and a^{2m}, the value it would have if β were exactly zero, it is clear that it is a continuous function of β as β goes to zero.

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