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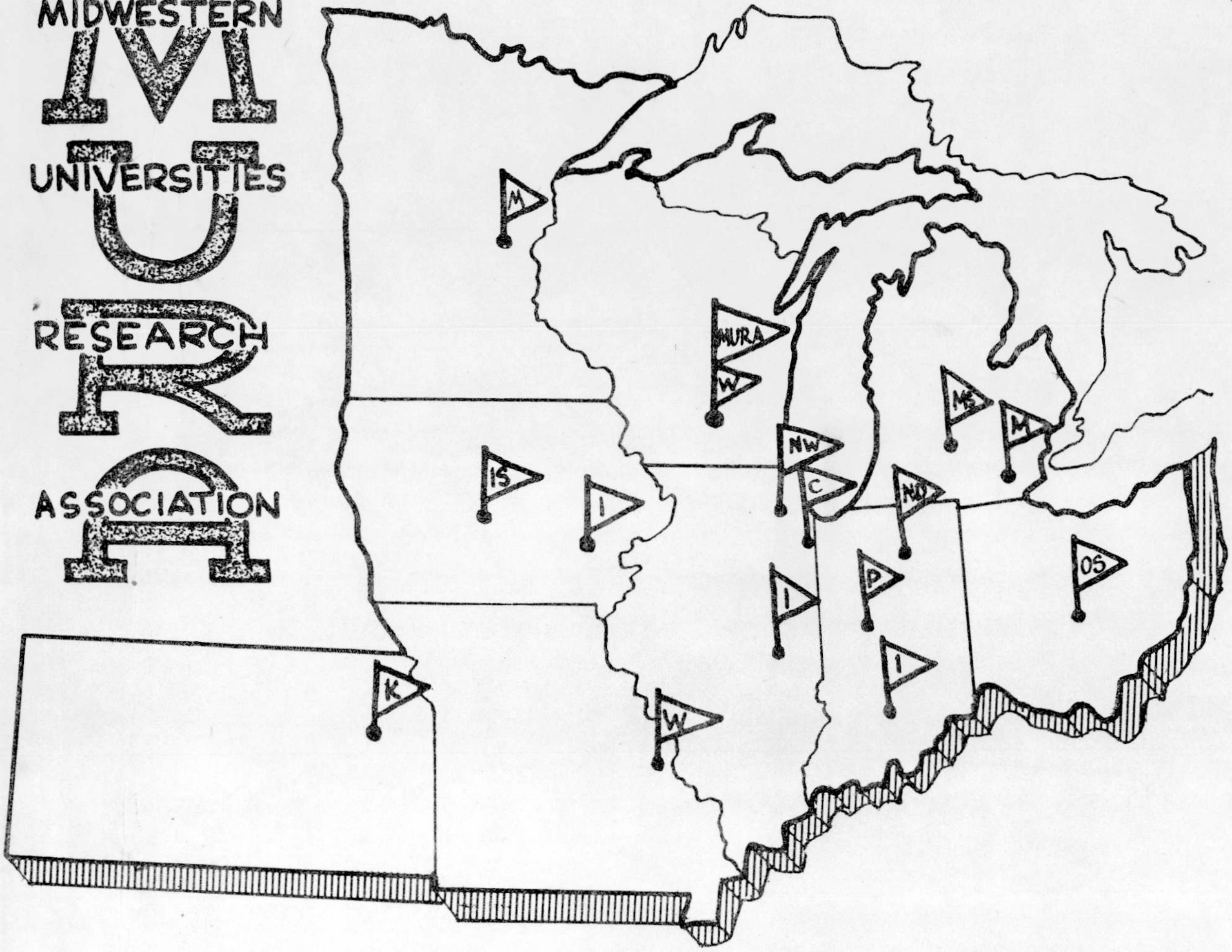

EFFECTS OF RADIAL STRAIGHT SECTIONS ON THE BETATRON OSCILLATION FREQUENCIES IN A SPIRAL SECTOR FFAG ACCELERATOR

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EFFECTS OF RADIAL STRAIGHT SECTIONS ON THE BETATRON OSCILLATION FREQUENCIES IN A SPIRAL SECTOR FFAG ACCELERATOR

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ABSTRACT

It is desirable to introduce radial straight sections into spiral sector accelerators in order to accommodate accelerating cavities of reasonable design and magnet windings. Such straight sections make the accelerator non-scaling, i.e., in general the betatron oscillation frequencies vary with energy and resonances may be crossed. These effects have been investigated analytically in the linear approximation. The equations of motion are now functions not only of the accelerator parameters, but also of the geometry of the radial straight sections and of the equilibrium orbit radius. If the number of spirals per revolution is N and the number of radial straight sections per revolution is p, then all harmonic numbers $n < \frac{1}{2} q$, where q equals p divided by the greatest common divisor of p and N, do not contribute to the change of betatron oscillation frequencies with energy. Since the contributions of a harmonic to the frequencies decrease as the harmonic number increases, it appears that the variation of betatron oscillation frequencies with energy can be kept within acceptable limits.

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I. INTRODUCTION

It is the purpose of this report to study the effect of radial straight sections on the tune (i.e., the values of ν_x and ν_y) of a spiral sector accelerator. At different radii the straight sections occur at different positions relative to the spiral magnets. This destroys the scaling properties of the accelerator, and ν_x and ν_y are no longer constant for different radii. One can study the problem of variation of tune with radius by treating the clearly equivalent problem of variation of tune with the position of the straight sections.

Define τ as the angular distance between a spiral ridge and the center of a straight section along a circle of radius r_o. Let p equal the number of straight sections and N the number of sectors. If one moves out in radius to a new radius r' where τ' the new angular distance becomes equal to $\tau + \frac{2\pi}{qN}$,* then the equilibrium orbit at r' is scaling with respect to the equilibrium orbit at r_o. Thus in order to study the variation of tune with radius, one need only consider the variation of tune between r_o and r' (i.e., between τ and $\tau + \frac{2\pi}{qN}$) because the variation is periodic with a period equal to $r' - r_o$ (i.e., $\frac{2\pi}{qN}$). As one can readily see, in order to decrease the variation of the tune, one can decrease the periodicity of the variation by choosing p and N such as to make q large. The largest value of q is obtained by making p and N relatively prime numbers in which case q equals p.

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*q is hereafter used for the number equal to p divided by the greatest common divisor of p and N.

This report obtains the linear equations of particle motion about the equilibrium orbit for two types of radial straight sections. The special case where one neglects the field harmonics greater than or equal to q/2 for a spiral sector accelerator without radial straight sections is treated in detail. It is found that for this case that there is no variation of the tune with the radius for either type of radial straight sections.

II. THEORY

A. Form of Magnetic Field in Median Plane

The form of the magnetic field in the median plane for a spiral sector design of a FFAG accelerator without radial straight sections is given by the vertical field, $[B_z]^{(1)}$ (The [ ] about B_z indicates the field in the absence of straight sections):

$$[B_z] = -B_0 \, (1+X)^k \sum_n \beta_n \, e^{in\phi}$$

where

$$(1+X) = \frac{r}{r_0}$$

$$\phi = \theta - \xi$$

$$\xi = \tan\eta \, \ln(1+X) = \frac{1}{Nw} \ln(1+X)$$

$$\beta_n = \beta^*_{-n}$$

and the index n takes only the values:

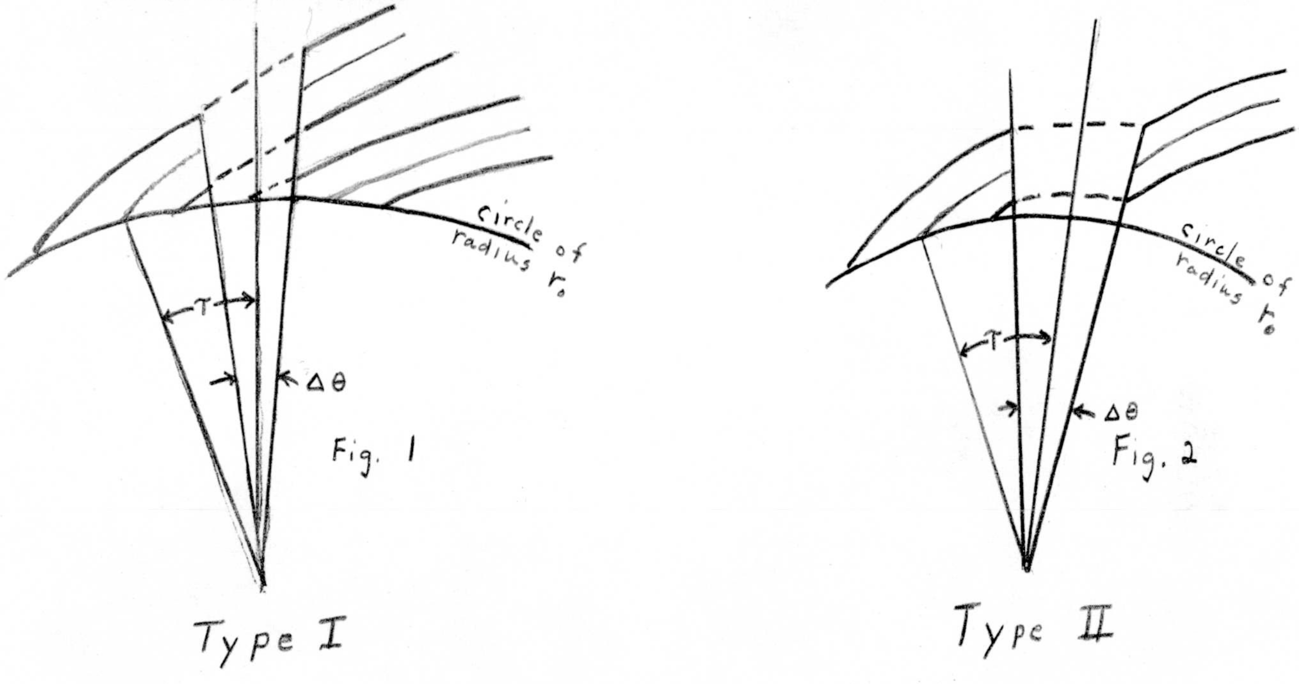
$$n = 0, \pm N, \pm 2N, \pm 3N, \cdots\cdots$$

One can expand the form of the vertical magnetic field in the median plane with radial straight sections, B_z, in a Fourier series:

$$B_z = -B_0 \sum_n \lambda_n e^{in\theta}$$

where λ_n is a function of not only the geometrical properties of the straight sections but also of the variable x (i.e., $\lambda_n = \lambda_n(x)$).

Consider the two types of radial straight sections shown below. Type I is where two radial slices are made and the portion of the magnet between the two slices is removed. Type II is where a series of slices are made and the magnets on each side of a slice are moved apart from the slice while each magnet (and associated normal straight section) is shortened or moved outward radially so that there are still N sectors in the accelerator.



In the following development p is the number of straight sections, $\Delta\theta$ the angular width of each straight section, and τ the angular distance between a spiral ridge and the center of a straight section along a circle of radius r_o.

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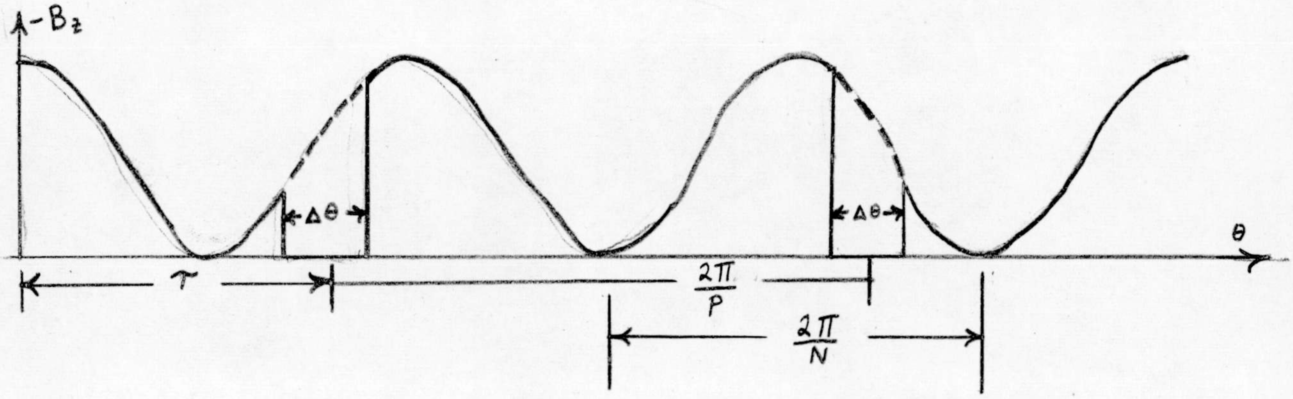
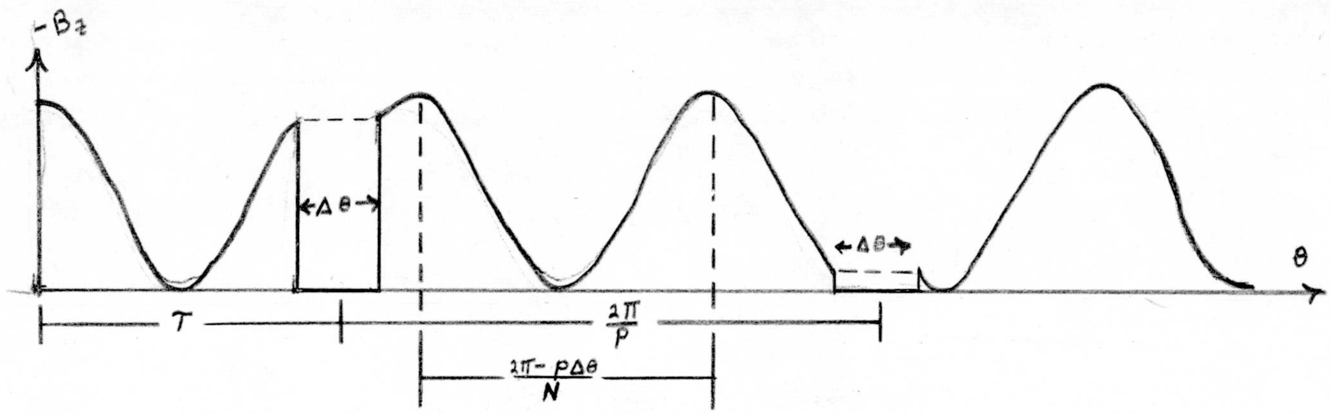
Below are diagrams showing B_z as a function of θ when $r = r_o$.





Several things are apparent from the diagrams. First, Type I field is of the form:

$$B_z = -B_o (1+x)^k \sum_n \beta_n e^{in\phi}$$

except for a region of width $\frac{\Delta\theta}{2}$ on each side of a radial straight section. Secondly, except for a region of width $\frac{\Delta\theta}{2}$ on each side of the center of the radial straight section, Type II field is of the form:

$$B_z = -B_o (1+x)^k \sum_n \beta_n e^{in\gamma(\phi - r\Delta\theta)}$$

where

$$\gamma = \frac{2\pi}{2\pi - p\Delta\theta}$$

and where r is an integer equal to the number of straight sections between the region under consideration and the origin. The factor $r\,\Delta\theta$ is needed because the field must slip in phase every time a straight section is crossed. The factor γ is needed because n waves must fit into a width $2\pi - p\Delta\theta$.

The results of Appendix I, where expressions for the λ_n's are derived, are given below.

For Type I field:

$$\lambda_n = (1+x)^{\left(k - i\frac{n}{Nw}\right)} \beta_n \left(1 - \frac{p\Delta\theta}{2\pi}\right) +$$

$$- \sum_{\substack{s \\ s \neq 0}} \left[(1+x)^{\left(k - i\frac{n+sp}{Nw}\right)} \beta_{n+sp} \left(\frac{e^{ispT}}{s\pi} \right) \left(\sin \frac{sp\Delta\theta}{2} \right) \right]$$

For Type II field:

$$\lambda_n = (1+x)^{\left(k - i\frac{n}{Nw}\right)} \delta_{n,rp}\, \delta_{\frac{n}{\gamma},tp}\, \beta_{\frac{n}{\gamma}} \left[1 - \frac{p\Delta\theta}{2\pi}\right] +$$

$$- \sum_{\substack{s \\ s \neq \frac{n-n\gamma}{p\gamma}}} \left\{ (1+x)^{\left[k - i\frac{(n+sp)\gamma}{Nw}\right]} \beta_{n+sp} \left[\frac{p\, e^{i[(n+sp)\gamma - n][T - \frac{\Delta\theta}{2}]}}{2\pi\,[(n+sp)\gamma - n]} \right] \left[1 - e^{-in(1-\gamma)\left(\Delta\theta - \frac{2\pi}{p}\right)} \right] \right.$$

From the expressions for the λ_n's it is readily apparent that the form of the magnetic field in the median plane is:

For Type I field:

$$B_z = -B_o \sum_{n,s} \left[(1+X)^{\left[k - i\frac{(n+sp)}{N\omega}\right]} g_{n,s} \right] e^{in\theta}$$

where

$$g_{n,o} = \left[1 - \frac{p\Delta\theta}{2\pi}\right] \beta_n$$

$$g_{\substack{n,s \\ s\neq 0}} = \left(- \frac{\sin \frac{sp\Delta\theta}{2}}{s\pi} \right) \beta_{n+sp}\, e^{isp\tau}$$

For Type II field:

$$B_z = -B_o \sum_{n,s} \left[(1+X)^{\left[k - i\frac{(n+sp)\gamma}{N\omega}\right]} \ell_{n,s} \right]$$

where

$$\ell_{n,\frac{n-n\gamma}{p\gamma}} = \delta_{n,rp}\, \delta_{\frac{n}{\gamma},tp}\, \beta_{\frac{n}{\gamma}} \left[1 - \frac{p\Delta\theta}{2\pi}\right]$$

$$\ell_{\substack{n,s \\ s\neq\frac{n-n\gamma}{p\gamma}}} = \left(\frac{p e^{-i[(n+sp)\gamma - n]\frac{\Delta\theta}{2}}}{2\pi\,[(n+sp)\gamma - n]} \right) \left(1 - e^{in(\gamma-1)(\Delta\theta - \frac{2\pi}{p})}\right) \beta_{n+sp}\, e^{i[(n+sp)\gamma - n]\tau}$$

It will be advantageous for the calculation to have the magnetic field in the form:

$$B_z = -B_o \sum_n \left\{ A_{o,n} + A_{1,n} X + A_{2,n} X^2 + \cdots \right\} e^{in\theta}$$

This can be accomplished by expanding

$$(1+X)^\rho = \left[1 + \rho X + \frac{\rho(\rho-1)}{2} X^2 + \frac{\rho(\rho-1)(\rho-2)}{6} X^3 + \cdots \right],$$

which gives for Type I field

$$A_{0,n} = \sum_s g_{n,s}$$

$$A_{1,n} = \sum_s \left(k - i\,\frac{n+sp}{N\omega}\right) g_{n,s}$$

$$A_{2,n} = \frac{1}{2}\sum_s \left(k - i\,\frac{n+sp}{N\omega}\right)\left(k-1-i\,\frac{n+sp}{N\omega}\right) g_{n,s}$$

and in general

$$A_{m,n} = \frac{1}{m!}\sum_s \left(k - i\,\frac{n+sp}{N\omega}\right)\left(k-1-i\,\frac{n+sp}{N\omega}\right)\cdots\left(k-m+1-i\,\frac{n+sp}{N\omega}\right) g_{n,s}$$

and for Type II field

$$A_{0,n} = \sum_s \ell_{n,s}$$

$$A_{1,n} = \sum_s \left(k - i\,\frac{(n+sp)\gamma}{N\omega}\right) \ell_{n,s}$$

$$A_{2,n} = \frac{1}{2}\sum_s \left(k - i\,\frac{(n+sp)\gamma}{N\omega}\right)\left(k-1-i\,\frac{(n+sp)\gamma}{N\omega}\right) \ell_{n,s}$$

and in general

$$A_{m,n} = \frac{1}{m!}\sum_s \left(k - i\,\frac{(n+sp)\gamma}{N\omega}\right)\left(k-1-i\,\frac{(n+sp)\gamma}{N\omega}\right)\cdots\left(k-m+1-i\,\frac{(n+sp)\gamma}{N\omega}\right) \ell_{n,s}$$

B. The Equilibrium Orbit

The first step in determining particle motion in the accelerator is to find the equilibrium orbit, i.e., find a periodic solution $x = x(\theta)$.

One starts with the Lagrangian describing the particle motion in the median plane[2]

$$L = r_0 p \left\{ \sqrt{(1+x)^2 + x'^2} + \frac{\alpha}{r_0 B_0}\left[x' A_r + (1+x) A_\theta\right] \right\},$$

where $\alpha = \frac{e r_0 B_0}{cp}$, $X = \frac{r - r_0}{r_0}$

$$X' = \frac{dx}{d\theta}$$

Expanding the Lagrangian in terms of x and x' gives[2]

$$L \approx 1 + X + \tfrac{1}{2} X'^2 - \tfrac{1}{2} X X'^2 + \frac{\alpha}{r_0 B_0} \left[X' A_r + (1+X) A_\theta \right]$$

The approximate equation of motion derivable from this approximate Lagrangian is

$$X'' = 1 + \tfrac{1}{2} X'^2 + X X'' + \frac{\alpha}{B_0} (1+X) B_z$$

Inserting the expression for B_z gives:

For Type I field:

$$X'' = 1 + \tfrac{1}{2} X'^2 + X X'' - \alpha \sum_n \left\{ (1+X)^{\left(k+1-i\frac{n+sp}{Nw}\right)} g_{n,s} \right\} e^{in\theta}$$

For Type II field:

$$X'' = 1 + \tfrac{1}{2} X'^2 + X X'' - \alpha \sum_n \left\{ (1+X)^{\left[k+1-i\frac{(n+sp)\gamma}{Nw}\right]} \ell_{n,s} \right\} e^{in\theta}$$

Expanding $(1+X)^\rho = \left[1 + \rho X + \frac{\rho(\rho-1)}{2} X^2 + \cdots \right]$

gives for the equation of motion

$$X'' = 1 + \tfrac{1}{2} X'^2 + X X'' - \alpha \sum_n \left[B_{0,n} + B_{1,n} X + B_{2,n} X^2 + \cdots \right] e^{in\theta}$$

where for Type I field

$$B_{0,n} = \sum_s g_{n,s}$$

$$B_{1,n} = \sum_s \left(k+1-i\frac{n+sp}{Nw}\right) g_{n,s}$$

$$B_{2,n} = \frac{1}{2}\sum_s \left(k+1-i\frac{n+sp}{Nw}\right)\left(k-i\frac{n+sp}{Nw}\right) g_{n,s}$$

and in general

$$B_{m,n} = \frac{1}{m!}\sum_s \left(k+1-i\frac{n+sp}{Nw}\right)\left(k-i\frac{n+sp}{Nw}\right)\cdots\left(k-m+2-i\frac{n+sp}{Nw}\right) g_{n,s}$$

and for Type II field

$$B_{0,n} = \sum_s l_{n,s}$$

$$B_{1,n} = \sum_s \left(k+1-i\frac{(n+sp)\gamma}{Nw}\right) l_{n,s}$$

$$B_{2,n} = \frac{1}{2}\sum_s \left(k+1-i\frac{(n+sp)\gamma}{Nw}\right)\left(k-i\frac{(n+sp)\gamma}{Nw}\right) l_{n,s}$$

and in general

$$B_{m,n} = \frac{1}{m!}\sum_s \left(k+1-i\frac{(n+sp)\gamma}{Nw}\right)\left(k-i\frac{(n+sp)\gamma}{Nw}\right)\cdots\left(k-m+2-i\frac{(n+sp)\gamma}{Nw}\right) l_{n,s}$$

For the equilibrium orbit $x = x(\theta)$ must be a periodic function, so that we may expand x in a Fourier series,

$$X = \sum_{n=-\infty}^{\infty} X_n e^{in\theta}$$

Inserting this into the equation of motion and the use of harmonic balance yields

$$-n^2 X_n = \delta_{0,n} - \alpha B_{0,n} - \alpha \sum_m B_{1,m} X_{n-m} +$$

$$-\frac{1}{2}\sum_m m(n+m) X_{n-m} X_m - \alpha \sum_{m,r} B_{2,m} X_{n-m-r} X_r + \cdots$$

In order to arrive at an approximate solution,[2] we will substitute the i^{th} approximation in the right hand side and determine the $(i + 1)^{th}$ approximation from the left hand side; also we will choose the zeroth approximation equal to zero. We will choose $x_o = 0$ thus specifying the value of α and hence the value of r_o. This gives for the first approximation $x_{n,\,1}$

$$X_{n,1} = \alpha \frac{B_{o,n}}{n^2}$$

Since non-linear terms in x_n are small, we will neglect them in making the second approximation $x_{n,\,2}$ and then it follows that

$$X_{n,2} = \frac{\alpha}{n^2} \left\{ B_{o,n} + \alpha \sum_{m \neq n} \frac{B_{1,m} B_{o,n-m}}{(n-m)^2} \right\}$$

If the process of successive apprxomation is continued and more and more terms included, then higher and higher powers of α enter the solution. There are therefore an infinite number of values for α. However, most of these are not of practical interest, since $|\alpha| >> 1$ and consequently the circumference factor is very large. Thus keeping only terms through α^2 gives the quadratic equation

$$\alpha^2 \sum_{\substack{m \\ m \neq 0}} \frac{B_{o,-m}}{m^2} \left[B_{1,m} + \tfrac{1}{2} B_{o,m} \right] + \alpha B_{o,o} - 1 = 0$$

C. Linear Equations of Motion

Let the radius vector r be given by

$$\vec{r} = \vec{r}_e + x\vec{n} + z\vec{b}$$

where $\vec{r}_e$ is the radius vector of the equilibrium orbit, $\vec{n}$ is a unit vector in the median plane perpendicular to the equilibrium orbit, and $\vec{b}$ is a unit vector perpendicular to the median plane. Then the linearized equations of motion are[3]

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{n}{\rho^2}\right]x = 0$$

$$\frac{d^2z}{ds^2} - \frac{n}{\rho^2}\, z = 0$$

where $\frac{1}{\rho} = -\left(\frac{eB_z}{cp}\right)$ evaluated at $\vec{r} = \vec{r}_e$

$$n = \left(\frac{\rho}{B_z}\frac{\partial B_z}{\partial x}\right)$$ evaluated at $\vec{r} = \vec{r}_e$

Denoting B_z on the equilibrium orbit by B_z^e

$$\frac{1}{\rho} = -\frac{e}{cp} B_z^e = \frac{e}{cp} B_0 \sum_n \mathfrak{z}_{0,n}\, e^{in\vartheta}$$

where $\vartheta = \frac{2\pi s}{s_0}$ and s is the length measured along the equilibrium orbit, and s_o is the total length of the equilibrium orbit around the machine. Observe that

$$\frac{n}{\rho^2} = -\frac{e}{cp}\frac{\partial B_z^e}{\partial x} = \frac{e}{cp}\frac{B_0}{r_0}\sum \mathfrak{z}_{1,n}\, e^{in\vartheta}$$

and hence that

$$\mathfrak{z}_{0,n} = -\frac{1}{2\pi B_0}\int_0^{2\pi} B_z^e\, e^{-in\vartheta}\, d\vartheta$$

$$\mathfrak{z}_{1,n} = -\frac{r_0}{2\pi B_0}\int_0^{2\pi}\frac{\partial B_z^e}{\partial x}\, e^{-in\vartheta}\, d\vartheta$$

1. Relationship Between $d\vartheta$ and $d\theta$

$$d\vartheta = \frac{d\vartheta}{ds}\frac{ds}{d\theta}\,d\theta$$ substituting for $\frac{d\vartheta}{ds}$ and $\frac{ds}{d\theta}$

yields:

$$d\vartheta = \frac{2\pi r_o}{s_o}\sqrt{(1+x)^2 + x'^2}\;d\theta\,.$$

Expanding $\sqrt{(1+x)^2 + x'^2}$ in terms of x and x' yields:

$$d\vartheta = \frac{2\pi r_o}{s_o}\left\{1 + x + \tfrac{1}{2}x'^2 + \cdots\right\} d\theta$$

or

$$\frac{d\vartheta}{d\theta} \approx \frac{2\pi r_o}{s_o}\left\{1 + x + \tfrac{1}{2}x'^2\right\}.$$

Substituting for x the sum $\sum_n X_n\, e^{in\theta}$ yields

$$\frac{d\vartheta}{d\theta} \approx \frac{2\pi r_o}{s_o}\left\{1 - \frac{1}{2}\sum_m m^2 X_{-m}X_m + \sum_{n\neq 0}\left(X_n - \frac{1}{2}\sum_m n(n-m)X_n X_{n-m}\right)e^{in\theta}\right\}.$$

Since

$$s_o = r_o\int_0^{2\pi}\sqrt{(1+x)^2 + x'^2}\;d\theta$$

then

$$s_o \approx r_o\int_0^{2\pi}\left(1 + x + \tfrac{1}{2}x'^2\right)d\theta$$

or

$$s_o \approx 2\pi r_o\left\{1 - \frac{1}{2}\sum_m m^2 X_{-m}X_m\right\}$$

one obtains

$$\frac{2\pi r_o}{s_o} \approx \frac{1}{\left[1 - \frac{1}{2}\sum_m m^2 X_{-m}X_m\right]}\,,$$

and hence

$$\frac{d\vartheta}{d\theta} \approx 1 + \frac{\sum\limits_{n \neq 0} \left(x_n - \frac{1}{2} \sum\limits_m n(n-m)\, X_n X_{n-m} \right) e^{in\theta}}{1 - \frac{1}{2} \sum\limits_m m^2 X_{-m} X_m} \approx 1$$

since x_n is small compared to one it can be neglected. Notice that while x_n is neglected compared to one, neither $k\,x_n$ or $\frac{1}{w}\,x_n$ is neglected compared to one.

Hence
$$\frac{d\vartheta}{d\theta} \approx 1 ,$$

$$\vartheta \approx \theta$$

and

$$z_{0,n} \approx -\frac{1}{2\pi B_0} \int_0^{2\pi} B_z^e \, e^{-in\theta} \, d\theta$$

$$z_{1,n} \approx -\frac{r_0}{2\pi B_0} \int_0^{2\pi} \frac{\partial B_z^e}{\partial \mathcal{X}} \, e^{-in\theta} \, d\theta$$

2. Relationship between $\mathcal{X}$ and x

The geometry of the equilibrium orbit is indicated in Fig. 3, from which we obtain:

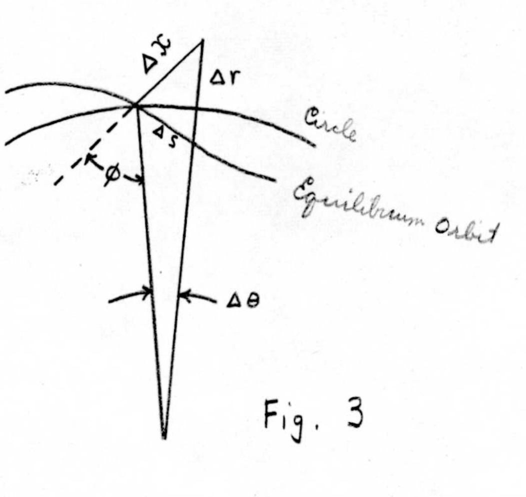


Fig. 3

$$\left\{ \begin{aligned} \frac{\partial r}{\partial \mathcal{X}} &= \cos\phi \\ \frac{\partial \theta}{\partial \mathcal{X}} &= \frac{\sin\phi}{r_e} \\ \frac{ds}{d\theta} &= r_e \cos\phi \end{aligned} \right.$$

Since $r_e \approx r_o$ and $\frac{ds}{d\theta} \approx r_o \sqrt{(1+x)^2 + x'^2} \approx r_o$

$\frac{ds}{d\theta} \approx r_e$ and $\cos\phi \approx 1$. Hence ,

$\frac{\partial\theta}{\partial x} \approx 0$, $\frac{\partial r}{\partial x} \approx 1$, and

$$\frac{\partial B_z^e}{\partial x} = \frac{\partial B_z^e}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial B_z^e}{\partial\theta}\frac{\partial\theta}{\partial x} \approx \frac{\partial B_z^e}{\partial r}$$

or

$$\frac{\partial B_z^e}{\partial x} \approx \frac{1}{r_o}\frac{\partial B_z^e}{\partial X}$$

Consequently

$$\mathfrak{z}_{1,n} \approx -\frac{1}{2\pi B_o}\int_0^{2\pi}\frac{\partial B_z^e}{\partial X} e^{in\theta}\, d\theta$$

$$B_z = -B_o \sum_n \left\{ A_{0,n} + A_{1,n} X + A_{2,n} X^2 + \cdots \right\} e^{in\theta}$$

$$\frac{\partial B_z}{\partial X} = -B_o \sum_n \left\{ A_{1,n} + 2A_{2,n} X + 3A_{3,n} X^2 + \cdots \right\} e^{in\theta}$$

Substituting for x the sum $\sum_n X_n e^{in\theta}$ yields the magnetic field on the equilibrium orbit, B_z^e and the $\frac{\partial B_z}{\partial X}$ evaluated on the equilibrium orbit, $\frac{\partial B_z^e}{\partial X}$.

Using these values for B_z^e and $\frac{\partial B_z^e}{\partial X}$ yields,

$$\mathfrak{z}_{0,n} = \left\{ A_{0,n} + \sum_l A_{1,n-l} X_l + \sum_{l,r} A_{2,n-l-r} X_l X_r + \cdots \right\};$$

and

$$\mathfrak{z}_{1,n} = \left\{ A_{1,n} + 2\sum_{\ell} A_{2,n-\ell} X_\ell + 3\sum_{\ell,r} A_{3,n-\ell-r} X_\ell X_r + \cdots \right\}$$

The linear equations of motion now become:

$$\frac{d^2 x}{ds^2} + \sum_n \left[\frac{e^2 B_0^2}{c^2 p^2} \sum_m \mathfrak{z}_{0,m} \mathfrak{z}_{0,n-m} + \frac{e B_0}{c p r_0} \mathfrak{z}_{1,n} \right] e^{in\theta} x = 0$$

$$\frac{d^2 z}{ds^2} - \sum_n \frac{e B_0}{c p r_0} \mathfrak{z}_{1,n} z = 0$$

Since

$$\frac{2\pi r_0}{S_0} \approx \frac{1}{1 - \frac{1}{2}\sum_m m^2 X_{-m} X_m} \approx 1$$

$$\frac{d^2 x}{ds^2} = \frac{1}{r_0^2} \frac{d^2 x}{d\vartheta^2} \approx \frac{1}{r_0^2} \frac{d^2 x}{d\theta^2}$$

$$\frac{d^2 z}{ds^2} = \frac{1}{r_0^2} \frac{d^2 z}{d\vartheta^2} \approx \frac{1}{r_0^2} \frac{d^2 z}{d\theta^2}$$

then

$$\frac{d^2 x}{d\theta^2} + \sum_n D_n e^{in\theta} x = 0$$

and

$$\frac{d^2 z}{d\theta^2} + \sum_n E_n e^{in\theta} z = 0$$

where $D_n = \left[\alpha^2 \sum_m \mathfrak{z}_{0,m} \mathfrak{z}_{0,n-m} + \alpha\, \mathfrak{z}_{1,n} \right]$

and $E_n = -\alpha\, \mathfrak{z}_{1,n}$

D. Treatment of the Special Case for Type I Field

The special case treated is the one for which one can neglect the field harmonics greater than or equal to q/2 for a spiral sector accelerator without radial straight sections, where q is equal to p divided by the greatest common divisor of p and N. The results of Appendix II show that for Type I field and the special case $A_{m, n}$ and $B_{m, n}$ equal zero unless $n = rN - sp$ where r and s are zero, or positive or negative integers. It also shows that:

$$A_{m, rN-sp} = f_m(p, \Delta\theta, r, s)\, e^{isp\tau}$$

and

$$B_{m, rN-sp} = F_m(p, \Delta\theta, r, s)\, e^{isp\tau}$$

The equation for α is:

$$\alpha^2 \sum_n \frac{B_{o,-n}}{n^2}\left[B_{1,n} + \tfrac{1}{2} B_{o,n}\right] + \alpha B_{o,o} - 1 = 0$$

Since $B_{m, n} = 0$ unless $n = rN - sp$

$$\sum_n \frac{B_{o,-n}}{n^2}\left[B_{1,n} + \tfrac{1}{2} B_{o,n}\right] = \sum_{r,s} \frac{B_{o,-rN+sp}}{(rN-sp)^2}\left[B_{1,rN-sp} + \tfrac{1}{2} B_{o,rN-sp}\right]$$

$$= \sum_{r,s} \frac{F_o(p, \Delta\theta, -r, -s)}{(rN-sp)^2}\left[F_1(p, \Delta\theta, r, s) + \tfrac{1}{2} F_o(p, \Delta\theta, r, s)\right]$$

Since $\sum_n \frac{B_{o,-n}}{n^2}\left[B_{1,n} + \frac{1}{2} B_{o,n}\right]$ is independent of τ and $B_{o,o} = F_o(p, \Delta\theta, 0, 0)$ is independent of τ, it necessarily follows that α is independent of τ and $\alpha = \alpha(p, \Delta\theta)$

The expression for x_n is:

$$X_n = \frac{\alpha}{n^2}\left\{ B_{o,n} + \alpha \sum_{m \neq n} \frac{B_{1,m} B_{o,n-m}}{(n-m)^2} \right\}$$

Since $B_{m,n} = 0$ unless $n = rN - sp$

$$\sum_{m \neq n} \frac{B_{1,m} B_{o,n-m}}{(n-m)^2} = \sum_{t,\mu} \frac{B_{1,tN-\mu p} B_{o,n-(tN-\mu p)}}{[n-(tN-\mu p)]^2}$$

and since $B_{o,n-(tN-\mu p)} = 0$ unless

$$n-(tN-\mu p) = r'N - s'p \quad \text{or} \quad n = rN - sp,$$

and also $B_{o,n} = 0$ unless $n = rN - sp$, therefore $x_n = 0$ unless $n = rN - sp$

and

$$X_{rN-sp} = \frac{\alpha}{(rN-sp)^2}\left\{ B_{o,rN-sp} + \alpha \sum_{t,\mu} \frac{B_{1,tN-\mu p} B_{o,(r-t)N-(s-\mu)p}}{[(r-t)N-(s-\mu)p]^2} \right\}$$

or, substituting for the B's yields

$$X_{rN-sp} = \frac{\alpha}{(rN-sp)^2}\left\{ F_o(p,\Delta\theta,r,s)\, e^{ispT} + \right.$$

$$\left. + \alpha \sum_{t,\mu} \frac{F_1(p,\Delta\theta,t,u)\, e^{iupT} F_o(p,\Delta\theta,r-t,s-\mu)\, e^{i(s-u)pT}}{[(r-t)N-(s-\mu)p]^2} \right\}$$

and hence

$$X_{rN-sp} = H(p,\Delta\theta,r,s)\, e^{ispT}$$

Since $X_\ell = 0$ unless $\ell = tN - \mu p$

then

$$\sum_{\ell} A_{1,n-\ell} X_\ell = \sum_{t,\mu} A_{1,n-(tN-\mu p)} X_{tN-\mu p}$$

and since $A_{1,\, n-(tN-\mu p)} = 0$ unless $n = rN - sp$

therefore $\sum_{\ell} A_{1,\, n-\ell}\, X_{\ell} = 0$ unless $n = rN - sp$

Similarly

$$\sum_{\ell} A_{2,\, n-\ell}\, X_{\ell} = 0$$

unless

$$n = rN - sp$$
$$\ell = tN - \mu p \, ,$$

$$\sum_{\ell, m} A_{2,\, n-\ell-m}\, X_{\ell}\, X_m = 0$$

unless

$$n = rN - sp$$
$$\ell = tN - \mu p$$
$$m = gN - v p$$

and

$$\sum_{\ell, m} A_{3,\, n-\ell-m}\, X_{\ell}\, X_m = 0$$

unless

$$n = rN - sp$$
$$\ell = tN - \mu p$$
$$m = gN - v p$$

Therefore $\mathfrak{z}_{0,n}$ and $\mathfrak{z}_{1,n}$ equal zero unless n = r N - sp.

Substituting for $A_{m,\, n}$ and for x_n yields

$$\mathfrak{z}_{0,\, rN-sp} = \Big\{ f_0(p, \Delta\theta, r, s) + \sum_{t,\mu} f_1(p, \Delta\theta, r-t, s-\mu)\, H(p, \Delta\theta, t, \mu) +$$

$$+ \sum_{t,\mu,g,v} f_2(p, \Delta\theta, r-t-g, s-\mu-v)\, H(p, \Delta\theta, t, \mu)\, H(p, \Delta\theta, g, v) + \cdots \Big\}\, e^{ispT}$$

and

$$\mathfrak{z}_{1,rN-sp} = \Big\{ f_1(p, \Delta\theta, r, s) + 2\sum_{t,u} f_2(p, \Delta\theta, r-t, s-u)\, H(p, \Delta\theta, t, u) +$$

$$+ 3 \sum_{t,u,q,v} f_3(p, \Delta\theta, r-t-q, s-u-v)\, H(p, \Delta\theta, t, u)\, H(p, \Delta\theta, q, v) + \cdots \Big\} e^{isp\tau}$$

and hence

$$\mathfrak{z}_{0,rN-sp} = J(p, \Delta\theta, r, s)\, e^{isp\tau}$$

and

$$\mathfrak{z}_{1,rN-sp} = L(p, \Delta\theta, r, s)\, e^{isp\tau}$$

It thus follows that:

$$D_n = \Big[\alpha^2 \sum_m \mathfrak{z}_{0,m}\, \mathfrak{z}_{0,n-m} + \alpha\, \mathfrak{z}_{1,n}\Big] = 0$$

unless

$$n = rN - sp,$$

$$E_n = -\alpha\, \mathfrak{z}_{1,n} = 0$$

unless

$$n = rN - sp,$$

$$D_{rN-sp} = M(p, \Delta\theta, r, s)\, e^{isp\tau},$$

and

$$E_{rN-sp} = Q(p, \Delta\theta, r, s)\, e^{isp\tau}.$$

From Vogt-Nilsen's formulas:[4]

$$\cos \nu_x^* = \cos 2\pi\sqrt{D_0} - \frac{\pi \sin 2\pi\sqrt{D_0}}{2\sqrt{D_0}} \sum_{k\neq 0} \frac{D_k\, D_{-k}}{k^2 - 4D_0}$$

$$\cos \nu_y^* = \cos 2\pi\sqrt{E_0} - \frac{\pi \sin 2\pi\sqrt{E_0}}{2\sqrt{E_0}} \sum_{k\neq 0} \frac{E_k\, E_{-k}}{k^2 - 4E_0}$$

where σ_x^* is the phase change per revolution for radial betatron oscillations, σ_y^* is the phase change per revolution for vertical betatron oscillations, $\sigma_x^* = 2\pi\nu_x$ and $\sigma_y^* = 2\pi\nu_y$

Since D_k and E_k equal zero unless $k = rN - sp$, then it follows that

$$\cos\sigma_x^* = \cos 2\pi\sqrt{D_0} - \frac{\pi \sin 2\pi\sqrt{D_0}}{2\sqrt{D_0}} \sum_{r,s} \frac{D_{rN-sp}\, D_{-rN+sp}}{(rN-sp)^2 - 4D_0}$$

and

$$\cos\sigma_y^* = \cos 2\pi\sqrt{E_0} - \frac{\pi \sin 2\pi\sqrt{E_0}}{2\sqrt{E_0}} \sum_{r,s} \frac{E_{rN-sp}\, E_{-rN+sp}}{(rN-sp)^2 - 4E_0}$$

Since D_0 and E_0 are independent of $\mathcal{T}$,

$$\sum_{r,s} \frac{D_{rN-sp}\, D_{-rN+sp}}{(rN-sp)^2 - 4D_0} = \sum_{r,s} \frac{M(p,\Delta\theta,r,s)\, M(p,\Delta\theta,-r,-s)}{[rN-sp]^2 - 4D_0}$$

is independent of $\mathcal{T}$ and

is independent of $\mathcal{T}$, it necessarily follows that σ_x^* and σ_y^* are independent of $\mathcal{T}$. Since $d\mathcal{T} = -\frac{r_0}{Nw}\, dr$, the change in the values of σ_x^* and σ_y^* due to a change in $\mathcal{T}$ are proportional to the change in σ_x^* and σ_y^* due to a change in the radius r. Therefore it must follow that σ_x^* and σ_y^* and hence ν_x and ν_y are independent of the radius r.

E. Treatment of the Special Case for Type II Field

The results of Appendix II show that for Type II field and the special case where one neglects harmonics greater than or equal to q/2, $B_{m,n}$

equal zero unless $n = rN - sp$ where r and s are zero or positive or negative integers. It shows that:

$$A_{m, rN-sp} = R_m\,(\rho, \Delta\theta, r, s)\, e^{irN(\gamma-1)\tau}\, e^{isp\tau}$$

$$B_{m, rN-sp} = T_m\,(\rho, \Delta\theta, r, s)\, e^{irN(\gamma-1)\tau}\, e^{isp\tau}$$

The equation for α is:

$$\alpha^2 \sum_n \frac{B_{0,-n}}{n^2}\left[B_{1,n} + \tfrac{1}{2} B_{0,n}\right] + \alpha B_{0,0} - 1 = 0$$

Since $B_{m,\,n} = 0$ unless $n = rN - sp$

$$\sum_n \frac{B_{0,-n}}{n^2}\left[B_{1,n} + \tfrac{1}{2} B_{0,n}\right] = \sum_{r,s} \frac{B_{0,-rN+sp}}{(rN-sp)^2}\left[B_{1,rN-sp} + \tfrac{1}{2} B_{0,rN-sp}\right]$$

$$= \sum_{r,s} \frac{T_0\,(\rho, \Delta\theta, -r, -s)}{(rN-sp)^2}\left[T_1\,(\rho, \Delta\theta, r, s) + \tfrac{1}{2} T_0(\rho, \Delta\theta, r, s)\right]$$

Since

$$\sum_n \frac{B_{0,-n}}{n^2}\left[B_{1,n} + \tfrac{1}{2} B_{0,n}\right]$$

is independent of τ and $B_{0,0} = T_0\,(\rho, \Delta\theta, 0, 0)$ is independent of τ, it necessarily follows that α is independent of τ and $\alpha = \alpha(\rho, \Delta\theta)$

The expression for x_n is:

$$X_n = \frac{\alpha}{n^2}\left\{B_{0,n} + \alpha \sum_{m \neq n} \frac{B_{1,m}\, B_{0,n-m}}{(n-m)^2}\right.$$

Since $B_{m,n} = 0$ unless $n = rN - sp$

$$\sum_{m \neq n} \frac{B_{1,m} B_{0,n-m}}{(n-m)^2} = \sum_{t,\mu} \frac{B_{1,tN-\mu p} B_{0,n-(tN-\mu p)}}{[n-(tN-\mu p)]^2}$$

and since $B_{0,n-(tN-\mu p)} = 0$ unless $n-(tN-\mu p) = r'N - s'p$

or $n = rN - sp$ and $B_{0,n} = 0$ unless $n = rN - sp$, therefore $x_n = 0$

unless $n = rN - sp$ and

$$X_{rN-sp} = \frac{\alpha}{(rN-sp)^2} \left\{ B_{0,rN-sp} + \alpha \sum_{t,\mu} \frac{B_{1,tN-\mu p} B_{0,(r-t)N-(s-\mu)p}}{[(r-t)N-(s-\mu)p]^2} \right\}$$

or substituting for the B's

$$X_{rN-sp} = \frac{\alpha}{(rN-sp)^2} \left\{ T_0(p, \Delta\theta, r, s)\, e^{irN(\gamma-1)\tau} e^{isp\tau} \right.$$

$$\left. + \alpha \sum_{t,\mu} \frac{T_1(p,\Delta\theta,t,\mu)\, e^{itN(\gamma-1)\tau} e^{i\mu p\tau}\, T_0(p,\Delta\theta,r-t,s-\mu)\, e^{i(r-t)N(\gamma-1)\tau} e^{i(s-\mu)p\tau}}{[(r-t)N-(s-\mu)p]^2} \right\}$$

hence

$$X_{rN-sp} = U(p,\Delta\theta,r,s)\, e^{irN(\gamma-1)\tau} e^{isp\tau}$$

Since X_ℓ unless $\ell = tN - \mu p$, then

$$\sum_\ell A_{1,n-\ell} X_\ell = \sum_{t,\mu} A_{1,n-(tN-\mu p)} X_{tN-\mu p}$$

and since $A_{1,n-(tN-\mu p)} = 0$ unless $n = rN - sp$, therefore

$\sum_\ell A_{1,n-\ell} X_\ell = 0$ unless $n = rN - sp$.

Similarly

$$\sum_{\ell} A_{2,\,n-\ell}\, X_\ell = 0$$

unless

$$n = rN - sp$$
$$\ell = tN - \mu p \,,$$

$$\sum_{\ell,\,m} A_{2,\,n-\ell-m}\, X_\ell\, X_m = 0$$

unless

$$n = rN - sp$$
$$\ell = tN - \mu p$$
$$m = gN - \nu p \,,$$

and

$$\sum_{\ell,\,m} A_{3,\,n-\ell-m}\, X_\ell\, X_m = 0$$

unless

$$n = rN - sp$$
$$\ell = tN - \mu p$$
$$m = gN - \nu p$$

therefore $\mathfrak{z}_{0,n}$ and $\mathfrak{z}_{1,n}$ equal zero unless $n = rN - sp$.

Substituting for $A_{m,\,n}$ and for x_n yields

$$\mathfrak{z}_{0,\,rN-sp} = \Big\{ R_0(p, \Delta\theta, r, s) + \sum_{t,\mu} R_1(p, \Delta\theta, r-t, s-\mu)\, U(p, \Delta\theta, t, \mu) +$$

$$+ \sum_{t,\mu,g,\nu} R_2(p, \Delta\theta, r-t-g, s-\mu-\nu)\, U(p, \Delta\theta, t, \mu)\, U(p, \Delta\theta, g, \nu) + \cdots \Big\}\, e^{irN(\gamma-1)\tau}\, e^{isp\tau}$$

and

$$\mathfrak{z}_{1,rN-sp} = \Big\{ R_1(p,\Delta\theta, r, s) + 2\sum_{t,u} R_2(p,\Delta\theta, r-t, s-u)\, U(p,\Delta\theta, t, u) + $$

$$+ 3\sum_{t,u,q,v} R_3(p,\Delta\theta, r-t-q, s-u-v)\, U(p,\Delta\theta,t,u)\, U(p,\Delta\theta, q, v) + \cdots \Big\} e^{irN(\gamma-1)\tau} e^{isp\tau}$$

and hence

$$\mathfrak{z}_{0,rN-sp} = V_0(p,\Delta\theta, r, s)\, e^{irN(\gamma-1)\tau} e^{isp\tau}$$

and

$$\mathfrak{z}_{1,rN-sp} = V_1(p,\Delta\theta, r, s)\, e^{irN(\gamma-1)\tau} e^{isp\tau}$$

it thus follows that

$$D_n = \Big[\alpha^2 \sum_m \mathfrak{z}_{0,m}\mathfrak{z}_{0,n-m} + \alpha\, \mathfrak{z}_{1,n}\Big] = 0$$

unless n = r N - sp,

$$E_n = \big[-\alpha\, \mathfrak{z}_{1,n}\big] = 0$$

unless n = r N - sp,

$$D_{rN-sp} = W(p,\Delta\theta, r, s)\, e^{irN(\gamma-1)\tau} e^{isp\tau}$$

and

$$E_{rN-sp} = Y(p,\Delta\theta, r, s)\, e^{irN(\gamma-1)\tau} e^{isp\tau}$$

From Vogt-Nielsen's formulas:[4]

$$\cos \nu_x^* = \cos 2\pi\sqrt{D_0} - \frac{\pi \sin 2\pi\sqrt{D_0}}{2\sqrt{D_0}} \sum_k \frac{D_k D_{-k}}{k^2 - 4D_0}$$

$$\cos \nu_y^* = \cos 2\pi\sqrt{E_0} - \frac{\pi \sin 2\pi\sqrt{E_0}}{2\sqrt{E_0}} \sum_k \frac{E_k E_{-k}}{k^2 - 4E_0}$$

Since D_k and E_k equal zero unless $k = rN - sp$, then it follows that:

$$\cos \nu_x^* = \cos 2\pi\sqrt{D_0} - \frac{\pi \sin 2\pi\sqrt{D_0}}{2\sqrt{D_0}} \sum_{r,s} \frac{D_{rN-sp} D_{-rN+sp}}{(rN-sp)^2 - 4D_0}$$

and

$$\cos \nu_y^* = \cos 2\pi\sqrt{E_0} - \frac{\pi \sin 2\pi\sqrt{E_0}}{2\sqrt{E_0}} \sum_{r,s} \frac{E_{rN-sp} E_{-rN+sp}}{(rN-sp)^2 - 4E_0}$$

Since D_0 and E_0 are independent of $\mathcal{T}$,

$$\sum_{r,s} \frac{D_{rN-sp} D_{-rN+sp}}{(rN-sp)^2 - 4D_0} = \sum_{r,s} \frac{W(p,\Delta\theta,r,s)\, W(p,\Delta\theta,-r,-s)}{(rN-sp)^2 - 4W(p,\Delta\theta,0,0)}$$

is independent of $\mathcal{T}$, and

$$\sum_{r,s} \frac{E_{rN-sp} E_{-rN+sp}}{(rN-sp)^2 - 4E_0} = \sum_{r,s} \frac{Y(p,\Delta\theta,r,s)\, Y(p,\Delta\theta,-r,-s)}{(rN-sp)^2 - 4Y(p,\Delta\theta,0,0)}$$

is independent of $\mathcal{T}$, it necessarily follows that ν_x^* and ν_y^* are independent of $\mathcal{T}$. Since $d\mathcal{T} = -\frac{r_0}{Nw}\, dr$ the change in the values of ν_x^* and ν_y^* due to a change in $\mathcal{T}$ are proportional to the change in ν_x^* and ν_y^* due to a change in the radius r. Therefore it must follow that ν_x^* and ν_y^* and hence ν_x and ν_y are independent of the radius r.

III. REFERENCES

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V. APPENDICES

A. Magnetic Field in Median Plane

For Type I field $B_z = -B_0 (1+x)^k \sum_m \beta_m e^{im\phi}$ except for a region $\frac{\Delta\theta}{2}$ width on both sides of the center of a radial straight section where $B_z = 0$. We expand B_z in the Fourier series

$$B_z = -B_0 \sum_n \lambda_n e^{in\theta},$$

where

$$\lambda_n = \frac{1}{2\pi}(1+x)^k \sum_m \beta_m \left\{ \int_0^{\tau - \frac{\Delta\theta}{2}} e^{im\phi} e^{-in\theta}\, d\theta + \right.$$

$$\left. + \sum_{r=1}^{p-1} \int_{\tau + \frac{2\pi}{p}(r-1) + \frac{\Delta\theta}{2}}^{\tau + \frac{2\pi r}{p} - \frac{\Delta\theta}{2}} e^{im\phi} e^{-in\theta}\, d\theta + \int_{\tau + \frac{2\pi}{p}(p-1) + \frac{\Delta\theta}{2}}^{2\pi} e^{im\phi} e^{-in\theta}\, d\theta \right\}.$$

Since

$$\phi = \theta - \frac{1}{N\omega} \ln(1+x),$$

$$e^{im\phi} = (1+x)^{-\frac{im}{N\omega}} e^{im\theta},$$

and therefore

$$\lambda_n = \frac{1}{2\pi} \sum_m (1+x)^{\left(k - i\frac{m}{N\omega}\right)} \beta_m \left\{ \int_0^{\tau - \frac{\Delta\theta}{2}} e^{i(m-n)\theta}\, d\theta + \right.$$

$$\left. + \sum_{r=1}^{p-1} \int_{\tau + \frac{2\pi}{p}(r-1) + \frac{\Delta\theta}{2}}^{\tau + \frac{2\pi}{p} r - \frac{\Delta\theta}{2}} e^{i(m-n)\theta}\, d\theta + \int_{\tau + \frac{2\pi}{p}(p-1) + \frac{\Delta\theta}{2}}^{2\pi} e^{i(m-n)\theta}\, d\theta \right\}$$

Performing the indicated integrations yields

$$\lambda_n = (1+x)^{k - i\frac{n}{N\omega}} \beta_n \left[1 - \frac{p\Delta\theta}{2\pi}\right] +$$

$$+ \sum_{m \neq n} (1+x)^{k - i\frac{m}{N\omega}} \beta_m \left\{ \frac{e^{i(m-n)\tau}}{2\pi i (m-n)} \left[e^{-i(m-n)\frac{\Delta\theta}{2}} - e^{i(m-n)\left(\frac{\Delta\theta}{2} - \frac{2\pi}{p}\right)} \right] \times \right.$$

$$\left. \times \left[\frac{1 - e^{i(m-n)2\pi}}{1 - e^{i(m-n)\frac{2\pi}{p}}} \right] \right\} .$$

Since

$$\left[\frac{1 - e^{i(m-n)2\pi}}{1 - e^{i(m-n)\frac{2\pi}{p}}} \right] = 0 \qquad \text{if} \quad m - n \neq sp$$

and

$$\left[\frac{1 - e^{i(m-n)2\pi}}{1 - e^{i(m-n)\frac{2\pi}{p}}} \right] = p \qquad \text{if} \quad m - n = sp$$

for positive or negative integer s, one obtains for Type I field

$$\lambda_n = (1+x)^{\left(k - i\frac{n}{N\omega}\right)} \beta_n \left[1 - \frac{p\Delta\theta}{2\pi}\right] +$$

$$- \sum_{s \neq 0} (1+x)^{\left(k - i\frac{(n+sp)}{N\omega}\right)} \beta_{n+sp} \left(\frac{\sin \frac{sp\Delta\theta}{2}}{s\pi} \right) e^{isp\tau}$$

For Type II field $B_z = -B_0 (1+x)^k \sum_m \beta_m e^{im\gamma(\phi - r\Delta\theta)}$

where $\gamma = \frac{2\pi}{2\pi - p\Delta\theta}$ and r is an integer equal to number of straight sections between the region under consideration and the origin.

Expanding B_z in a Fourier series yields

$$B_z = -B_0 \sum_n \lambda_n e^{in\theta}$$

where

$$\lambda_n = \frac{1}{2\pi}(1+x)^k \sum_m \beta_m \left\{ \int_0^{\tau - \frac{\Delta\theta}{2}} e^{im\gamma\phi} e^{-in\theta} \, d\theta \, + \right.$$

$$\left. + \sum_{r=1}^{p-1} \int_{\tau + \frac{2\pi}{p}(r-1) + \frac{\Delta\theta}{2}}^{\tau + \frac{2\pi}{p} r - \frac{\Delta\theta}{2}} e^{im\gamma\phi} e^{-in\theta} e^{-im\gamma r\Delta\theta} \, d\theta + \int_{\tau + \frac{2\pi}{p}(p-1) + \frac{\Delta\theta}{2}}^{2\pi} e^{im\gamma\phi} e^{-in\theta} e^{-im\gamma p\Delta\theta} \, d\theta \right\}.$$

Since

$$\phi = \theta - \frac{1}{Nw} \ln(1+x),$$

$$e^{im\gamma\phi} = (1+x)^{-i\frac{m\gamma}{Nw}} e^{im\gamma\theta}$$

and therefore

$$\lambda_n = \frac{1}{2\pi} \sum_m (1+x)^{\left(k - i\frac{m\gamma}{Nw}\right)} \beta_m \left\{ \int_0^{\tau - \frac{\Delta\theta}{2}} e^{i(m\gamma - n)\theta} \, d\theta \, + \right.$$

$$\left. + \sum_{r=1}^{p-1} \int_{\tau + \frac{2\pi}{p}(r-1) + \frac{\Delta\theta}{2}}^{\tau + \frac{2\pi}{p} r - \frac{\Delta\theta}{2}} e^{i(m\gamma - n)\theta} e^{-imr\gamma\Delta\theta} \, d\theta + \int_{\tau + \frac{2\pi}{p}(p-1) + \frac{\Delta\theta}{2}}^{2\pi} e^{i(m\gamma - n)\theta} e^{-imp\gamma\Delta\theta} \, d\theta \right\}.$$

Performing the indicated integrations,

$$\lambda_n = \delta_{n,rp}\,\delta_{\frac{n}{\gamma},tp}\,(1+x)^{\left(k-i\frac{n}{N\omega}\right)}\,\beta_{\frac{n}{\gamma}}\left[1-\frac{p\Delta\theta}{2\pi}\right] +$$

$$+\sum_{m\neq\frac{n}{\gamma}}(1+x)^{\left(k-i\frac{m\gamma}{N\omega}\right)}\beta_m\left(\frac{e^{i(m\gamma-n)\left(\tau-\frac{\Delta\theta}{2}\right)}}{2\pi i\,(m\gamma-n)}\right)\left(1-e^{i(m\gamma-n)\left(\Delta\theta-\frac{2\pi}{p}\right)}\right)\times$$

$$\times\left(\frac{1-e^{i(m-n)2\pi}}{1-}\right.$$

where $\delta_{i,j}$ is the Kronecker delta $\delta_{i,j} = \begin{cases} 0 & \text{if } i\neq j \\ 1 & \text{if } i=j \end{cases}$

and r and t equal zero or any positive or negative integer, also since

$$\left(\frac{1-e^{i(m-n)2\pi}}{1-e^{i(m-n)\frac{2\pi}{p}}}\right) = \begin{cases} 0 & \text{if } m-n\neq sp \\ p & \text{if } m-n=sp \end{cases}$$

for zero, positive, or negative integer s then

$$\lambda_n = \delta_{n,rp}\,\delta_{\frac{n}{\gamma},tp}\,(1+x)^{k-i\frac{n}{N\omega}}\,\beta_{\frac{n}{\gamma}}\left[1-\frac{p\Delta\theta}{2\pi}\right] +$$

$$\sum_{s\neq\frac{n-n\gamma}{p\gamma}}(1+x)^{\left(k-i\frac{(n+sp)\gamma}{N\omega}\right)}\beta_{n+sp}\left(\frac{p\,e^{i[(n+sp)\gamma-n]\left[\tau-\frac{\Delta\theta}{2}\right]}}{2\pi\,[(n+sp)\gamma-n]}\right)\times$$

$$\times\left(1-e^{in[\gamma-1]\left[\Delta\theta-\frac{2\pi}{p}\right]}\right)$$

B. Special Case

In the following argument s, n, r, t, and u are integers or zero.

Given

$$g_{n,0} = \left[1 - \frac{p\Delta\theta}{2\pi}\right] \beta_n$$

$$g_{n,s} = \left(\frac{-\sin\frac{sp\Delta\theta}{2}}{s\pi}\right) e^{isp\tau} \beta_{n+sp}$$

and $\beta_n = 0$ unless $n = rN$.

If one assumes that $\beta_{rN} = 0$ for $|r| \geq \frac{q}{2}$ where q equals p divided by the greatest common divisor of p and N, then it follows that if $g_{n,s'} \neq 0$ then $g_{n,s''} = 0$ unless $s' = s''$.

Proof:

$g_{n,s'} \neq 0$ implies that $\beta_{n+s'p} \neq 0$

$\beta_{n+s'p}$ implies that $n + s'p = rN \qquad |r| < \frac{q}{2}$

Similarly $g_{n,s''} \neq 0$ implies that $n + s''p = tN \qquad |t| < \frac{q}{2}$

$$\left.\begin{aligned} n + s'p &= rN \\ n + s''p &= tN \end{aligned}\right\} \text{ implies that } (s'' - s')p = (t - r)N$$

If ν is the greatest common divisor of p and N, then

$$(s'' - s')\frac{p}{\nu} = (t - r)\frac{N}{\nu}$$

where $\frac{p}{\nu}$ and $\frac{N}{\nu}$ are relatively prime numbers.

Since $\frac{P}{v}$ and $\frac{N}{v}$ are relatively prime numbers,

$$(t-r) = \mu \frac{P}{v} = \mu q$$

Since $|t| < q/2$, $|r| < q/2$; $(t-r) < q$ hence $\mu = 0$

$$\therefore \quad s'' - s' = 0 \quad \text{and} \quad s'' = s'$$

For Type I field

$$A_{o,n} = \sum_s g_{n,s} = \left\{ \left[1 - \frac{p\Delta\theta}{2\pi}\right] \beta_n + \sum_{s\neq 0} \left(\frac{-\sin \frac{sp\Delta\theta}{2}}{s\pi} \right) e^{ispT} \beta_{n+sp} \right\}$$

Since for any particular value of n at the most only one value of $g_{n,s} \neq 0$ then the $\sum_s g_{n,s}$ contains only one term. Also if for $g_{n,s'} \neq 0$, then n + sp = r N and therefore n = r N - sp if $A_{o,n} \neq 0$.

From the expression for $A_{o,n}$, it is obvious that

$$A_{o,rN-sp} = f_o(p, \Delta\theta, r, s)\, e^{ispT}$$

It can similarly be shown that

$$A_{1,rN-sp} = f_1(p, \Delta\theta, r, s)\, e^{ispT}$$

$$A_{2,rN-sp} = f_2(p, \Delta\theta, r, s)\, e^{ispT}$$

$$A_{3,rN-sp} = f_3(p, \Delta\theta, r, s)\, e^{ispT}$$

$$B_{o,rN-sp} = F_o(p, \Delta\theta, r, s)\, e^{ispT}$$

$$B_{1,rN-sp} = F_1(p, \Delta\theta, r, s)\, e^{ispT}$$

and in general

$$A_{m,rN-sp} = f_m(p, \Delta\theta, r, s)\, e^{ispT}$$

$$B_{m,rN-sp} = F_m(p, \Delta\theta, r, s)\, e^{ispT}$$

In the following argument s, n, r, t, and u are integers or zero.

Given

$$\ell_{n, \frac{n-n\gamma}{p\gamma}} = \delta_{n, rp}\, \delta_{\frac{n}{\gamma}, tp}\, \beta_{\frac{n}{\gamma}} \left[1 - \frac{p\Delta\theta}{2\pi}\right]$$

$$\ell_{n,s} = \left(\frac{p\, e^{-i[(n+sp)\gamma - n]\frac{\Delta\theta}{2}}}{[(n+sp)\gamma - n]}\right)\left(1 - e^{i n(\gamma-1)(\Delta\theta - \frac{2\pi}{p})}\right)\beta_{n+sp}\, e^{i[(n+sp)\gamma - n]\tau}$$

$$s \neq \frac{n-n\gamma}{p\gamma}$$

and $\beta_n = 0$ unless n = r N.

If we assume $\beta_{rN} = 0$ for $|r| \geq q/2$ where q equals p divided by the greatest common divisor of p and N, then it follows that if $\ell_{n,s'} \neq 0$ then $\ell_{n,s''} = 0$ unless $s' = s''$.

Proof:

$$\ell_{n,s} = \Psi_0 (p, \Delta\theta, n, s)\, \beta_{n+sp}\, e^{i[(n+sp)\gamma - n]\tau}$$

$\ell_{n,s'} \neq 0$ implies that $\beta_{n+s'p} \neq 0$

$\beta_{n+s'p} \neq 0$ implies that $n + s'p = rN \qquad |r| < q/2$

Similarly $\ell_{n,s''} \neq 0$ implies that $n + s''p = tN \qquad |t| < q/2$

$$\left.\begin{array}{l} n + s'p = rN \\ n + s''p = tN \end{array}\right\} \text{ implies that } (s'' - s')p = (t - r)N$$

If v is the greatest common divisor of p and N, then

$$(s'' - s')\frac{p}{v} = (t - r)\frac{N}{v}$$

where $\frac{p}{v}$ and $\frac{N}{v}$ are relatively prime numbers.

Since $\frac{P}{v}$ and $\frac{N}{v}$ are relatively prime numbers,

$$(t-r) = \mu \frac{P}{v} = \mu q$$

Since $|t| < q/2$ and $|r| < q/2$, then $(t-r) < q$ hence $\mu = 0$

$$\therefore \quad s'' - s' = 0 \qquad \text{and} \qquad s'' = s'$$

For Type II field

$$A_{o,n} = \sum_s \ell_{n,s} = \sum_s \Psi_o(p, \Delta\theta, n, s)\, \beta_{n+sp}\, e^{i[(n+sp)\gamma - n]\tau}$$

Since for any particular value of n at the most only one value of $\ell_{n,s} \neq 0$, then the $\sum_s \ell_{n,s}$ contains only one term. Also if for $\ell_{n,s'} \neq 0$, then n + sp = r N and therefore if $A_{o,n} \neq 0$, n = r N - sp. From the expression for $A_{o,n}$ it is obvious that

$$A_{o,rN-sp} = R_o(p, \Delta\theta, r, s)\, e^{i[rN\gamma - rN + sp]\tau}$$

It can similarly be shown that in general

$$A_{m,rN-sp} = R_m(p, \Delta\theta, r, s)\, e^{i[rN\gamma - rN + sp]\tau}$$

and

$$B_{m,rN-sp} = T_m(p, \Delta\theta, r, s)\, e^{i[rN\gamma - rN + sp]\tau}$$

or $$A_{m,rN-sp} = R_m(p, \Delta\theta, r, s)\, e^{irN(\gamma-1)\tau}\, e^{isp\tau}$$

$$B_{m,rN-sp} = T_m(p, \Delta\theta, r, s)\, e^{irN(\gamma-1)\tau}\, e^{isp\tau}$$