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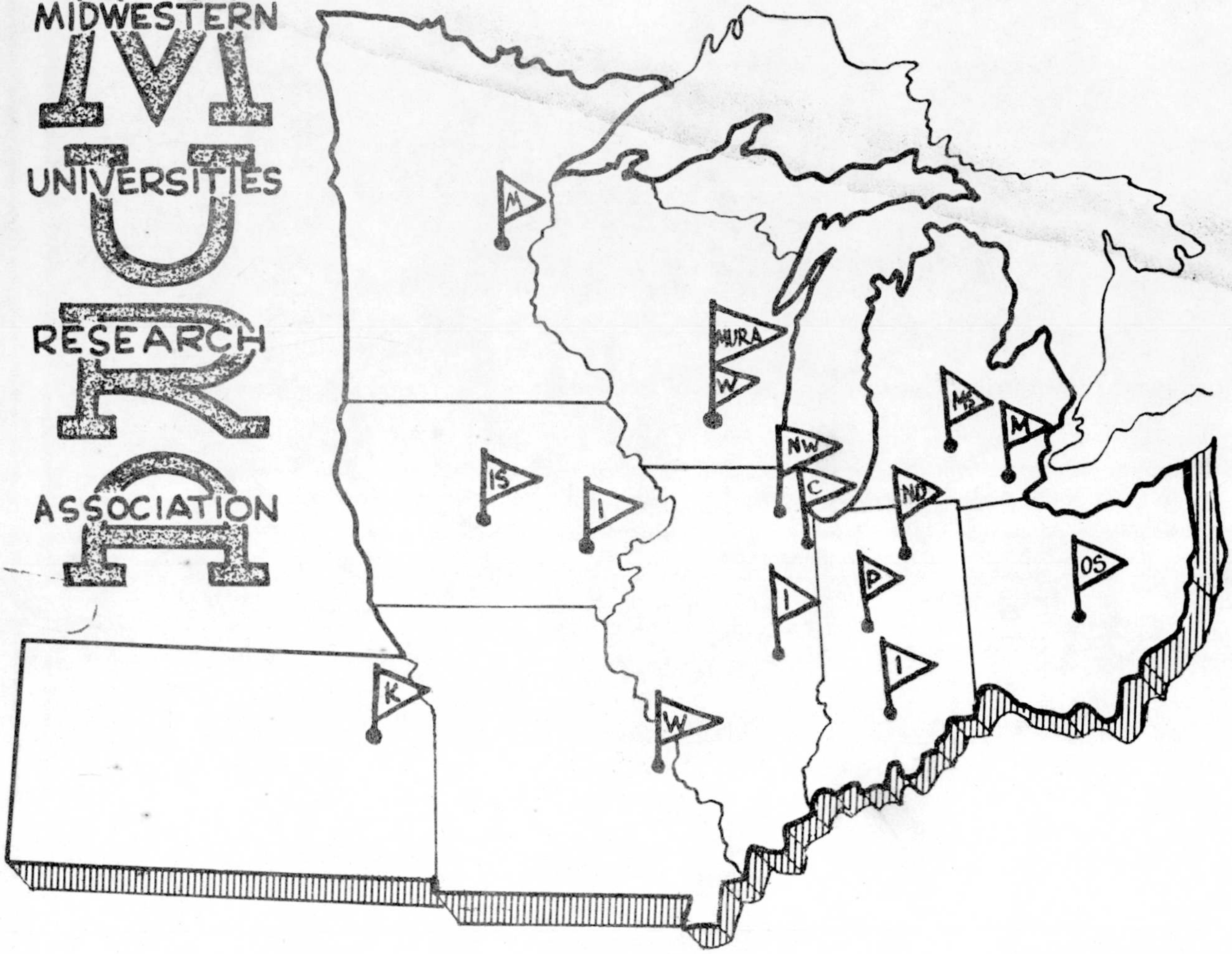

SCALING SPIRAL SECTOR FFAG ACCELERATORS WITHOUT MEDIAN PLANE SYMMETRY

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ABSTRACT

The treatment of an earlier report (MURA-406) of particle orbits in FFAG accelerators with radial magnetic fields in the median plane is extended to included the effects of magnet spiraling. It is found that a two-way accelerator with a circumference factor lower than that for conventional fields and with reasonable focusing properties can be made in this way. Methods of producing such magnetic fields are not treated.

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**On leave from the State University of Iowa.

I. INTRODUCTION

In an earlier report,* the linear motion in scaling radial sector accelerators without median plane symmetry was discussed. The vertical focusing, already weak in large radial sector accelerators, was weakened further by the introduction of radial median plane fields. This report extends the treatment to the scaling spiral sector case. A reasonable amount of vertical focusing can be achieved by spiraling, at the expense of complication in magnet construction.

II. FIELDS AND POTENTIALS

The field expansions corresponding to (A 2.1) and (A 2.3) are written

$$\left\{\begin{aligned} B_r &= -B_0\left(\frac{r}{r_0}\right)^k \sum_{m,n} \xi_{m,n}\, e^{in\varphi}\left(\frac{y}{r}\right)^m \\ B_\theta &= -B_0\left(\frac{r}{r_0}\right)^k \sum_{m,n} \eta_{m,n}\, e^{in\varphi}\left(\frac{y}{r}\right)^m \\ B_y &= -B_0\left(\frac{r}{r_0}\right)^k \sum_{m,n} \zeta_{m,n}\, e^{in\varphi}\left(\frac{y}{r}\right)^m , \end{aligned}\right. \tag{2.1}$$

where

$$\left\{\begin{aligned} \varphi &= \theta - Z \ln \frac{r}{r_0} \\ Z &= \tan \zeta \\ 0 &\leq m < \infty \\ -\infty &< n < \infty , \end{aligned}\right. \tag{2.2}$$

where ζ is the angle of a spiral with respect to a radius.

From the definition of φ,

$$e^{in\varphi} = e^{in\theta} e^{-inZ \ln \frac{r}{r_0}} = e^{in\theta}\left(\frac{r}{r_0}\right)^{-inZ},$$

*MURA-406, which will be referred to as A. Equations from A will be referred to prefixed by A.

so that effectively k is replaced by $k_n = k - in$ Z. The recursion relations corresponding to (A 2. 5) are

$$
\left\{
\begin{aligned}
\zeta_{m+2,n} &= \frac{1}{(m+1)(m+2)}\left[n^2-(k_n-m)^2\right]\zeta_{m,n}\\
\xi_{m+1,n} &= \frac{k_n-m}{m+1}\,\zeta_{m,n}\\
\eta_{m+1,n} &= \frac{in}{m+1}\,\zeta_{m,n}
\end{aligned}
\right.
\qquad (2.\ 3)
$$

For fields of Type I, while the relations corresponding to (A 2. 6) are

$$
\left\{
\begin{aligned}
\xi_{m+2,n} &= \frac{1}{(m+1)(m+2)}\,\frac{k_n-1-m}{k_n+1-m}\left[n^2-(k_n+1-m)^2\right]\xi_{m,n}\\
\eta_{m,n} &= \frac{in}{k_n+1-m}\,\xi_{m,n}\\
\zeta_{m+1,n} &= \frac{1}{(m+1)(k_n+1-m)}\left[n^2-(k_n+1-m)^2\right]\xi_{m,n}
\end{aligned}
\right.
\qquad (2.\ 4)
$$

for fields of Type II.

The vector potential components are treated in an analogous way, with expansion coefficients $\varphi_{m,n}$, $\psi_{m,n}$ and $\chi_{m,n}$ for the r, θ and z components respectively. The recursion relations corresponding to (A 2. 10) are

$$
\left\{
\begin{aligned}
&(m+1)(m+2)\varphi_{m+2,n}+\left[(k_n+2-m)^2-n^2\right]\varphi_{m,n} = 2\zeta_{m,n}\\
&(m+1)(m+2)\psi_{m+2,n}+\left[(k_n+2-m)^2-n^2\right]\psi_{m,n} = \frac{2in}{k_n+2-m}\,\zeta_{m,n}\\
&(m+1)(m+2)\chi_{m+2,n}+\left[(k_n+1-m)^2-n^2\right]\chi_{m,n} = 0,
\end{aligned}
\right.
\qquad (2.\ 5)
$$

where (2. 3) and (2. 4) have been used on the right-hand side in each case. Eqs. (A 2. 11) give the vector potential coefficients in terms of the field coefficients with k replaced by k_n throughout.

III. EQUILIBRIUM ORBIT

Eqs. (A 4. 4) are replaced by

$$
\left\{
\begin{aligned}
-n^2 x_n ={}& \delta_{n0} - \alpha\, \zeta_{0,n} - \alpha \sum_m{}' (k_m+1)\, \zeta_{0,m}\, x_{n-m} + i\alpha \sum_m{}' m\, y_m\, \eta_{0,n-m} \\
& - \alpha \sum_m{}' \zeta_{1,m}\, y_{n-m} - \tfrac{1}{2} \sum_m{}' m(n+m)\, x_m\, x_{n-m} + \tfrac{1}{2} \sum_m{}' m(n-m)\, y_m\, y_{n-m} \\
& - \tfrac{1}{2}\alpha \sum_{m,p}{}' (k_m+1)\, k_m\, \zeta_{0,m}\, x_p\, x_{n-m-p} - \alpha \sum_{m,p}{}' k_m\, \zeta_{1,m}\, x_p\, y_{n-m-p} \\
& - \alpha \sum_{m,p}{}' y_m\, y_p\, \zeta_{2,n-m-p} + i\alpha \sum_{m,p}{}' m\, y_m\, k_p\, \eta_{0,p}\, x_{n-m-p} \\
& + i\alpha \sum_{m,p}{}' m\, y_m\, \eta_{1,p}\, y_{n-m-p} + \cdots \qquad (3.1) \\
-n^2 y_n ={}& \alpha\, \xi_{0,n} + \alpha \sum_m{}' \xi_{1,m}\, y_{n-m} + \alpha \sum_m{}' (k_m+1)\, \xi_{0,m}\, x_{n-m} \\
& - i\alpha \sum_m{}' m\, x_m\, \eta_{0,n-m} - \sum_m{}' n\, m\, y_m\, x_{n-m} \\
& + \alpha \sum_{m,p}{}' y_m\, y_p\, \xi_{2,n-m-p} - i\alpha \sum_{m,p}{}' p\, k_m\, \eta_{0,m}\, x_p\, x_{n-m-p} \\
& + \tfrac{1}{2}\alpha \sum_{m,p}{}' (k_m+1)\, k_m\, \xi_{0,m}\, x_p\, x_{n-m-p} + \alpha \sum_{m,p}{}' k_m\, \xi_{1,m}\, x_p\, y_{n-m-p} \\
& - i\alpha \sum_{m,p}{}' m\, x_m\, \eta_{1,p}\, y_{n-m-p} + \cdots
\end{aligned}
\right.
$$

Again we choose α so that $x_o = 0$. The first approximation for $n \neq 0$, given in (A 4. 5),

$$\left\{\begin{aligned} x_n^{(1)} &= \frac{\alpha\, \zeta_{0,n}}{n^2} \\ y_n^{(1)} &= -\frac{\alpha\, \xi_{0,n}}{n^2} \end{aligned}\right. \tag{3.2}$$

is unchanged by the addition of spiraling. This approximation is sufficiently accurate for the work of later sections. For the case $\xi_{0,0} = 0$, it is easy to see that $y_0 = 0$. The case $\xi_{0,0} \neq 0$ requires special treatment, which will not be given here.

The equation determining α, (A 4.7), becomes

$$0 = 1 - \alpha\, \zeta_{0,0} - \alpha^2 \sum_{m \neq 0}{}' \frac{k_m \frac{3}{2}}{m^2} \left(\zeta_{0,m}\, \zeta_{0,m} + \xi_{0,m}\, \xi_{0,-m} \right) \tag{3.3}$$

The sum

$$i Z \sum_{m \neq 0}{}' \left(\frac{\zeta_{0,m}\, \zeta_{0,-m} + \xi_{0,m}\, \xi_{0,-m}}{m} \right)$$

vanishes identically, since the summand is odd and is summed over an even interval in m. There is therefore no contribution to the value of α from the magnet spiraling. Then (3.3) is

$$0 = 1 - \alpha\, \zeta_{0,0} - \alpha^2 \left(k + \tfrac{3}{2}\right)\left(G_1^2 + G_2^2\right), \tag{3.4}$$

where, as in A,

$$\left\{\begin{aligned} G_1^2 &= \sum_{m \neq 0}{}' \frac{\zeta_{0,m}\, \zeta_{0,-m}}{m^2} \\ G_2^2 &= \sum_{m \neq 0}{}' \frac{\xi_{0,m}\, \xi_{0,-m}}{m^2} \end{aligned}\right. \tag{3.5}$$

For a two-way accelerator, $\zeta_{0,0} = 0$ and

$$\alpha = \pm \left[\left(k + \tfrac{3}{2}\right)\left(G_1^2 + G_2^2\right)\right]^{-\frac{1}{2}}, \tag{3.6}$$

as in A.

IV. LINEAR OSCILLATIONS ABOUT THE EQUILIBRIUM ORBIT

We make the same approximations as in A, namely, we expand the $a_{i,m}$ only through first order in $\alpha k_m \gamma / m^2$, we neglect $1/k$ compared to unity and we drop many small terms of essentially kinematic origin. Then the equations of motion are (as in (A 5.8))

$$\begin{cases} u'' = Ku + Lv \\ v'' = Mu + Nv, \end{cases} \tag{4.1}$$

with

$$\begin{cases} K_n = a_8 - a_4' = \sum_n K_n e_n \\ L = a_9 - a_6' = \sum_n L_n e_n \\ M = a_9 - a_5' = \sum_n M_n e_n \\ N = a_{10} - a_7' = \sum_n N_n e_n, \end{cases} \tag{4.2}$$

where the a_i are defined in (A 5.3). Expansion of (A 5.3) and substitution of (3.2) yields

$$\begin{cases} a'_{4,n} = \alpha \zeta_{0,n} \\ a'_{5,n} = -\alpha \xi_{0,n} \\ a'_{6,n} = \dfrac{\alpha n^2}{k_n} \xi_{0,n} \\ a'_{7,n} = -\dfrac{\alpha n^2}{k_n} \xi_{0,n} \\ a_{8,n} = -\alpha k_n \zeta_{0,n} - \alpha^2 \sum_{m \neq n}' \dfrac{k_m}{(n-m)^2} \left(\zeta_{0,m} \zeta_{0,n-m} + \xi_{0,m} \xi_{0,n-m} \right) \end{cases} \tag{4.3}$$

$$
\begin{cases}
a_{9,n} = \alpha k_n \xi_{0,n} + \alpha^2 \sum\limits_{m \neq n}' \dfrac{k_m^2}{(n-m)^2} \xi_{0,m} \zeta_{0,n-m} \\
\quad + \alpha^2 \sum\limits_{m \neq 0}' \dfrac{n-m}{m} \left[\xi_{0,n-m} \zeta_{0,m} + \xi_{0,m} \zeta_{0,n-m} \right] \\
\quad + \alpha^2 \sum\limits_{m \neq n}' \dfrac{m^2 - k_m^2}{(n-m)^2} \zeta_{0,m} \xi_{0,n-m} \\
\\
a_{10,n} = -\alpha \dfrac{n^2 - k_n^2}{k_n} \zeta_{0,n} - \alpha^2 \sum\limits_{m \neq n}' \dfrac{m^2 - k_m^2}{(n-m)^2} \left(\zeta_{0,m} \zeta_{0,n-m} + \xi_{0,m} \xi_{0,n-m} \right)
\end{cases}
$$

Then

$$
\begin{cases}
K_0 = \alpha k \zeta_{0,0} - \alpha^2 \sum\limits_{m \neq 0}' \dfrac{k_m^2}{m^2} \left(\zeta_{0,m} \zeta_{0,-m} + \xi_{0,m} \xi_{0,-m} \right) \\
\quad = -\alpha k \zeta_{0,0} - \alpha^2 k^2 (G_1^2 + G_2^2) + \alpha^2 Z^2 (F_1^2 + F_2^2) \\
K_n = -N_n = -\alpha k_n \zeta_{0,n} = -\alpha (k - inZ) \zeta_{0,n} \\
L_0 = M_0 = \alpha k \xi_{0,0} - \alpha^2 \sum\limits_{m \neq 0} \xi_{0,m} \zeta_{0,-m} = \alpha k \xi_{0,0} - \alpha^2 F_{12}^2 \\
L_n = \alpha k_n \xi_{0,n} - \dfrac{\alpha n^2}{k_n} \xi_{0,n} = \alpha \left(\dfrac{k_n^2 - n^2}{k_n} \right) \xi_{0,n} \\
M_n = \alpha k_n \xi_{0,n} \\
N_0 = \alpha k \xi_{0,0} - \alpha^2 \sum\limits_{m \neq 0}' \left(1 - \dfrac{k_m^2}{m^2} \right) \left(\zeta_{0,m} \zeta_{0,-m} + \xi_{0,m} \xi_{0,-m} \right) \\
\quad = \alpha k \zeta_{0,0} + \alpha^2 k^2 (G_1^2 + G_2^2) - \alpha^2 (1 + Z^2)(F_1^2 + F_2^2).
\end{cases}
\qquad (4.4)
$$

Note that in K_0 and N_0 the cross terms in $(k - i m Z)^2$ vanish because they are odd in m and are summed over an even interval.

We calculate for substitution into (A 5. 15) the quantities

$$\sum_{m\neq 0}' \frac{K_m K_{-m}}{m^2} = \alpha^2 \sum_{m\neq 0}' \frac{k^2 + m^2 Z^2}{m^2} \zeta_{0,m} \zeta_{0,-m}$$

$$= \alpha^2 k^2 G_1^2 + \alpha^2 Z^2 F_1^2$$

and

$$\sum_{m\neq 0}' \frac{L_m M_{-m}}{m^2} = \alpha^2 \sum_{m\neq 0}' \frac{k_{-m}}{k_m} \frac{(k_m^2 - m^2)}{m^2} \xi_{0,m} \xi_{0,-m}$$

$$\cong \alpha^2 k^2 G_2^2 + \alpha^2 Z F_2^2 - \alpha^2 F_2^2 ,$$

where we have used $mZ \gg k$ in the last term.

Then, from (A 5. 15),

$$\nu^2 = \alpha^2 k^2 (G_1^2 + G_2^2) + \alpha^2 Z^2 (F_1^2 + F_2^2) + \frac{\alpha^2}{2} (F_1^2 - F_2^2)$$
$$\pm \sqrt{\left[\alpha k \zeta_{0,0} + \alpha^2 k^2 (G_1^2 + G_2^2) - \alpha^2 (Z^2 + \tfrac{1}{2})(F_1^2 + F_2^2)\right]^2 + \left[\alpha k \zeta_{0,0} - \alpha^2 F_{12}^{-2}\right]^2} \qquad (4.5)$$

In the conventional case of a one-way accelerator with median plane symmetry, an accurate solution of (3. 4) is

$$\alpha \cong \frac{1}{\zeta_{0,0}} \left[1 - \left(k + \frac{3}{2}\right) \frac{G_1^2}{\zeta_{0,0}^2}\right]$$

and the two roots of (4. 5) become

$$\begin{cases} \nu_x^2 = k + \dfrac{k^2 G_1^2}{\zeta_{0,0}^2} \\ \nu_y^2 = -k + \dfrac{k^2 G_1^2}{\zeta_{0,0}^2} + \dfrac{F_1^2}{\zeta_{0,0}^2} (2Z^2 + 1) \end{cases} \qquad (4.6)$$

The quantity $F_1/\zeta_{0,0}$ is the flutter, so that ν_y^2 contains the usual spiraling and Thomas focusing terms. The second terms of ν_x^2 and ν_y^2 are the "A. G." terms of the smooth approximation[1] (the usual factor 1/2 is hidden in G_1^2 and F_1^2).

For a two-way accelerator with median plane symmetry $\zeta_{0,0} = 0$ and, by using (3. 6) for α ,

$$\begin{cases} \nu_x^2 = 2k \\ \nu_y^2 = \frac{1}{k}\frac{F_1^2}{G_1^2}\left(1 + 2Z^2\right), \end{cases} \tag{4. 7}$$

which is in agreement with Ohkawa's result.[2]

V. THE TWO-WAY ACCELERATOR

We add spiraling to the special case treated in A. That is, we take

$$\begin{cases} \zeta_{0,n} = \frac{1}{2} f\left(\delta_{n,N} + \delta_{n,-N}\right) \\ \xi_{0,n} = \frac{1}{2} g\left(\delta_{n,N} + \delta_{n,-N}\right). \end{cases} \tag{5. 1}$$

Then

$$\nu^2 = k + \frac{Z^2 N^2}{k} + \frac{1}{2}\frac{N^2}{k}\frac{f^2 - g^2}{f^2 + g^2}$$

$$\pm \sqrt{\left[k - \left(Z^2 + \frac{1}{2}\right)\frac{N^2}{k}\right]^2 + \left[\frac{N^2}{k}\frac{fg}{f^2 + g^2}\right]^2} \tag{5. 2}$$

The circumference factor C given on page 23 of A is not changed by spiraling. With k = 200, $\frac{k}{N^2}$ = 0.04 and $Z^2 = 5$ (ζ = 63. 25°), we calculate C, ν_1, and ν_2 for various values of g/f.

$\frac{g}{f}$	C	ν_1	ν_2
0	9.07	20	16.6
0.5	8.80	19.9	16.4
1	8.47	19.7	16.2
2	8.02	19.5	15.9
∞	7.28	19.4	15.8

C and ν_1 are unchanged from A. (Note, however, that ν_1 for $g/f = \infty$ was given incorrectly in A.)

The circumference factor can be lowered by the addition of higher harmonics to the fields. It is believed from digital computer experience that an improvement of 30% can be achieved while still keeping ν_1 comfortably below N/3 ($\sigma_1 = 2\pi/3$). Thus it appears that a two-way accelerator with a circumference factor of 5 and reasonable oscillation frequencies can be designed. By raising k (and therefore raising ν_1 above N/3), even lower values of C can be obtained.

References

1. K. R. Symon et. al., Phys. Rev. 103, 1837 (1956) (Appendix A).

2. T. Ohkawa, Rev. Sci. Instr. 29, 108 (1958).