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A Second Monte Carlo SAMPLER

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ABSTRACT

Methods are suggested for sampling many additional probability densities occurring in practice, as well as more general forms of some of those appearing in the first SAMPLER. Notably, the frequent restriction to half-integer exponents has been removed. As before, no claim to priority is intended, the sole object being to provide a handbook for Monte Carlo practice.

FOREWORD

In all cases, the density to be sampled is followed by a rule (R) for choice of the variable, in terms of random numbers r, uniform on (0,1), and a justification (J) for the method is indicated. The indices (D, C, R) provide "key words" which may help in locating a desired density, and usually a reference is given for further information (see REFERENCES, last page). Some of the basic densities of the first SAMPLER have been included, with the original numbering, so that the present handbook is reasonably self-contained.

D-INDEX

<u>Discrete Densities</u>

D2. $\theta^k/k!$ — Poisson (JK1/87)

D7. $C_{s-1}^{s+k-1}\ p^s q^k$ — Negative binomial, s integral (JK 1/124)

D17. λ^j/j — Log series (JK 1/166)

D18. $1/(k+1)^{\rho+1}$ — Zeta, Zipf-Estoup, word distribution (JK 1/240)

D19.	$\sum_a^b C_\mu^M C_{\mu-k}^N$ x $q^{M+N+k-2\mu} p^{2\mu-k}$	Binomial difference (JK 1/55)
D20.	$\left(C_{s-1}^M C_k^N / C_{k+s-1}^{M+N}\right)$ x $(M-s+1)/(M+N-k-s+1)$	Negative hypergeometric (JK 1/157)
D21.	$C_k^f \sum_0^k (-1)^i C_i^k (k-i)^N / f^N$	Arfwedson, occupancy (JK 1/251)
D22.	$\int_a^b dx\ f(x,k)$	Continuous-discrete marginal
D23.	$\int_a^b dx\ p(x)\ f_x\ (k)$	p(x) - compounded $f_x(k)$ density
D24.	$e^{-a} \sum_0^k a^i/i! - e^{-b} \sum_0^k b^i/i!$	Uniform-compounded Poisson (JK 1/184)
D25.	$p^s q^k\ \Gamma(s+k)/\Gamma(s)\ k!$	Negative binomial, s real >0, Γ-compounded Poisson (JK 1/125)
D26.	$C_k^N\ \Gamma(a+k)\ \Gamma(N+b-k)$ x $\{B(a,b)\ \Gamma(N+a+b)\}^{-1}$	B-compounded binomial (JK 1/79)
D27.	$\Gamma(s+k)\ B(\rho+s,\ \sigma+k)$ x $\{\Gamma/s\ k!\ B(\rho,\sigma)\}^{-1}$	B-compounded negative binomial
D28.	$\rho\ B(\rho+1,k+1) = \rho k!/(\rho+1)\ldots(\rho+1+k)$	Simon, power-compounded geometric (JK 1/245)
D29.	$\int_0^\infty ae^{-(a+b)u}\left(1-e^{-bu}\right)^k du$	Yule, exponential-compounded geometric (JK 1/245)
D30.	$\sum_J^\infty f(j,k)$	Discrete-discrete marginal
D31.	$\sum_J^\infty p(j)\ f_j(k)$	p(j)-compounded $f_j(k)$ density
D32.	$\left(\phi^k/k!\right) \sum_1^\infty j^{k-1}\left(\lambda e^{-\phi}\right)^j$	Log series-compounded Poisson (JK 1/211)

D33. $\left(e^{-\lambda}/k!\right) \sum_{0}^{\infty} \left(\lambda e^{-\phi}\right)^{j} \left(\phi j\right)^{k}/j!$ — Neyman Type A, Contagious, Poisson-compounded Poisson (JK 1/217)

D34. $e^{-\lambda} p^{k} \sum_{j \geqslant k/n} C_{k}^{nj}\, q^{nj-k} \lambda^{j}/j!$ — Poisson-compounded binomial (JK 1/190)

D35. $\exp - \lambda\left(1-p^{K}\right);\ k = 0,$ — Poisson-compounded negative binomial (JK 1/196)

$\sum_{1}^{\infty} e^{-\lambda} q^{k}\, \Gamma(Kj+k) \left(\lambda p^{K}\right)^{j}/j!k!\Gamma(Kj);$

$k>0.$

D36. $e^{-\lambda};\ \nu = 0$ — Generalized Pólya-Aeppli (JK 1/197)

$\sum_{j \leqslant \nu/K} e^{-\lambda} C_{Kj-1}^{\nu-1} \left(\lambda p^{K}\right)^{j} q^{\nu-Kj}/j!\ ;$

$\nu = K, K+1, \ldots$

D37. $e^{-\lambda};\ \nu = 0$ — Pólya-Aeppli (JK 1/197)

$\sum_{1}^{\nu} e^{-\lambda}\, C_{j-1}^{\nu-1}\, (\lambda p)^{j}\, q^{\nu-j}/j!\ ;$

$\nu = 1,2,\ldots$

D38. $\sum_{k+1}^{\infty} e^{-\theta} \theta^{j-1}/j! =$ — Poisson tail-end, Poisson's exponential binomial limit (JK 1/262)

$\int_{0}^{\theta} x^{k} e^{-x} dx/\theta k!$

D39. $\{1-1/1!+\ldots+(-1)^{N-k}/(N-k)!\}/k!$ — k - coincidences, matching (JK 1/264)

D40. $(e-1)^{2}/(e^{a+b}-1)^{2}$ — A density for fractions a/b (JK 1/31)

D41. $\sum_{j=0}^{N} C_{j}^{N}\, (\phi j)^{k} (pe^{-\phi})^{j}\, q^{N-j}/k!$ — Binomial-compounded Poisson (JK 1/186)

Discrete Densities

D2. $p(k) = e^{-\theta}\, \theta^{k}/k!\ ;\ k=0,1,2,\ldots,\theta>0$

R1. Set $k = \min\left\{n; \sum_{0}^{n} \theta^{k}/k! \geqslant r_{0}\, e^{\theta}\right\}$

R 2. Set $k = -1 + \min\left\{n;\ \Pi_1^n r_i \leqslant e^{-\theta}\right\}$

J. See Sampler I.

D7. $q(k) = C_{s-1}^{s+k-1}\ p^s q^k$; $k = 0,1,2,\ldots,0<p<1$, $q = 1-p$, s integer $\geqslant 1$.

R. Set $k = -s + \{$first n for which s of the random numbers $r_1,\ldots, r_n$ are $\leqslant p$.

J. $\left\{C_{s-1}^{s+k-1}\ p^{s-1}\ q^{s+k-1-(s-1)}\right\} p$ is the probability of exactly s successes

occurring for the first time on the $(s+k)$-th trial. See Sampler I.

D17. $p(j) = \lambda^j / jL(\lambda)$; $j = 1,2,\ldots,0<\lambda<1$, $L(\lambda) = -\ell n(1-\lambda)$.

R. Set $j = \min\left\{J;\ \sum_1^J \lambda^j/j \geqslant r_0\ L(\lambda)\right\}$

D18. $q(k) = 1/(k+1)^{\rho+1}\ \zeta(\rho+1)$; $k = 0,1,2,\ldots$, $\rho>0$.

R. Set $k = \min\left\{K;\sum_0^K 1/(k+1)^{\rho+1} \geqslant r\zeta(\rho+1)\right\}$.

D19. $p(k) = \sum_a^b C_\mu^M\, C_{\mu-k}^N\ q^{M+N+k-2\mu}\ p^{2\mu-k}$; $M,N \geqslant 1$, $-N \leqslant k \leqslant M$, $a = \max\{0,k\}$,

$b = \min\{M,N+k\}$, $0<p<1$, $q = 1-p$.

R. Set μ = number of $r_1,\ldots, r_M$ such that $r_i \leqslant p$. Set ν = number of $r_1',\ldots,$

r_N' such that $r_i' \leqslant p$. Set $k = \mu-\nu$.

J. $p(k)$ is the probability that $\mu-\nu = k$, where μ and ν have the binomial

densities $C_\mu^M\ q^{M-\mu}\ p^\mu$ and $C_\nu^N\ q^{N-\nu}\ p^\nu$ respectively. Cf. D6.

$\underline{\text{D20.}\quad p(k) = \left(C^M_{s-1}\ C^N_k / C^{M+N}_{k+s-1}\right)(M - s+1)/(M+N - k-s+1);}$ $k = 0,\ldots,N$, s integral,

$1 \leqslant s \leqslant M,\ M,N \geqslant 1.$

R. One follows the steps:

1. List the integers $1,2,\ldots, M+N$.
2. Put $0 \to \sigma$, $M+N \to b$, $1 \to \nu$.
3. Set $I = \min\{i \geqslant br_\nu\}$. $(I = 1,2,\ldots,b)$
4. Delete the I-th integer $\equiv a_I$ from the <u>remaining</u> list.
5. If $a_I \leqslant M$ go to (6). If $a_I > M$ go to (7).
6. Put $\sigma+1 \to \sigma$. If $\sigma < s$, go to (7). If $\sigma = s$ go to (8).
7. Put $b-1 \to b$, $\nu+1 \to \nu$. Return to (3).
8. Exit with $k = \nu-s$.

J. p(k) is the probability of drawing s integers $\leqslant M$ from the list $1,2,\ldots, M,\ldots,M+N$ for the first time on the (s+k)-th drawing (without replacement). Note that $C^M_{s-1}\ C^N_k/C^{M+N}_{k+s-1}$ is the probability of drawing exactly s-1 of these in (k+s-1) drawings (Hypergeometric, D12). The "negative hypergeometric" of D20 is the dependent analogue of the negative binomial (D7).

$\underline{\text{D21.}\quad p(k) = C^f_k \sum_{i=0}^{k} (-1)^i\ C^k_i \left(\frac{k-i}{f}\right)^N;}$ $k = 1,2,\ldots, \min(f,N)$.

R. One follows the steps:

1. Put $0 \to n_1,\ldots,0 \to n_f$, $1 \to t$.
2. Set $K = \min\{k;\ k \geqslant fr\}$. Put $1+n_k \to n_k$.
3. If $t < N$, put $t + 1 \to t$, return to (2). If $t = N$ go to (4).
4. Set k = number of positive components n_i of $[n_1,\ldots,n_f]$

J. p(k) is the probability of exactly k of f boxes being occupied if N particles are assigned to f equally likely boxes. This may be seen from the inclusion-exclusion principle

$$\#(S_1 \cup \ldots \cup S_k) = \sum_{C^k_1} \#(S_{i_1}) - \sum_{C^k_2} \#(S_{i_1}\ S_{i_2}) + \ldots + (-1)^{k-1}\ \#(S_1 \ldots S_k)$$

where S_i denotes the set of all assignments forbidding the boxes i and k+1,...,f. Note that $p(k) = (1/f)^N \cdot \{\#S - \#(S_1 \cup \ldots \cup S_k)\}$, where S is the set forbidding boxes k+1,...,f.

D22. $q(k) = \int_a^b dx\, f(x,k)$; k = K, K+1,..., f(x,k) density for a<x<b, k = K,K+1,...

R. Sample the marginal density $p(x) \equiv \sum_K^\infty f(x,k)$ for x on (a,b). For this x, sample the x-dependent discrete k-density $f_x(k) \equiv f(x,k)/p(x)$ for k on K, K+1,...

J. The probability of choosing the integer k is $\int_a^b dx\, p(x)\, f_x(k) = q(k)$.

D23. $q(k) = \int_a^b dx\, p(x)\, f_x(k)$; k = K,K+1,..., p(x) density on (a,b), $f_x(k)$ discrete k-density for each value of a parameter x on (a,b).

R. Sample p(x) for x on (a,b). For this x, sample density $f_x(k)$ for k.

J. Corollary of D22.

D24. $q(k) = (b-a)^{-1}\left\{e^{-a}\sum_0^k a^i/i! - e^{-b}\sum_0^k b^i/i!\right\}$; k = 0,1,2,..., 0<a<b.

R. Set $x = a+(b-a)r$. Sample $e^{-x}x^k/k!$ for k by D2.

J. For $p(x) = 1/(b-a)$ on (a,b), and $f_x(k) = e^{-x}x^k/k!$ on {0,1,2,...} one has $\int_a^b (b-a)^{-1} x^k e^{-x} dx/k! = q(x)$, using the basic formula (F3) $\int_0^y x^{n-1}e^{-Bx}dx = (n-1)!\, B^{-n}\left\{1-e^{-By}\sum_0^{n-1}(By)^i/i!\right\}$. The rule follows from D23.

D25. $q(k) = p^s q^k \Gamma(s+k)/\Gamma(s)k!$; $k = 0,1,2,\ldots,0<p<1$, $q = 1-p$, s real >0.

R. Sample $U^{s-1}e^{-u}/\Gamma(s)$ for u on $(0,\infty)$ by C22, C32 or R18. For $x = uq/p$, sample $e^{-x}x^k/k!$ for k on $\{0,1,2,\ldots\}$ by D2. (See D7 for s integral.)

J. For $p(x) = (p/q)^s\, x^{s-1}e^{-xp/q}/\Gamma(s)$ on $(0,\infty)$ and $f_x(k) = e^{-x}x^k/k!$; $k = 0,1,2,\ldots$, $x>0$, one finds that $\int_0^\infty p(x)f_x(k)dx = q(k)$ as given. Moreover, $p(x)dx = u^{s-1}e^{-u}du/\Gamma(s)$ for $x = uq/p$. The rule follows from D23.

Note. $1 = p^s(1-q)^{-s} = \sum_{k=0}^\infty \frac{(-s)(-s-1)\ldots(-s-k+1)}{k!} p^s(-q)^k =$

$$\sum_{k=0}^\infty \frac{(s+k-1)\ldots s}{k!} p^s q^k = \sum_{k=0}^\infty \frac{\Gamma(s+k)}{\Gamma(s)k!} p^s q^k = \sum_0^\infty q(k).$$

D26. $q(k) = C_k^N\, \Gamma(a+k)\Gamma(b+N-k)/B(a,b)\, \Gamma(a+b+N)$; $k = 0,1,\ldots,N$; $a,b>0$, N integer ≥ 1.

R. Sample $v^{a-1}(1-v)^{b-1}/B(a,b)$ for v on $(0,1)$ by C35 or R19 ($b\neq 1$), or by C13 or C13A ($b=1$).
Set k = number of $r_1,\ldots,r_N$ such that $r_i \leq v$.

J. For $p(v) = v^{a-1}(1-v)^{b-1}/B(a,b)$, and $f_v(k) = C_k^N\, v^k(1-v)^{N-k}$, one has $\int_0^1 dv\, p(v)f_v(k) = q(k)$. Cf. D23.

D27. $q(k) = \Gamma(s+k)\; B\,(\rho+s,\sigma+k)/\Gamma(s)k!\; B(\rho,\sigma)$; $k = 0,1,2,\ldots;\rho,\sigma,s >$

R. Sample $p(x) = x^{\rho-1}\,(1-x)^{\sigma-1}/B(\rho,\sigma)$ for x on $(0,1)$ by C35 or R19 if $\sigma \neq 1$, or by C13 or C13A if $\sigma = 1$. For this x, sample $f_x(k) = x^s(1-x)^k\; \Gamma(s+k)/\Gamma(s)k!$ for k on $\{0,1,2,\ldots\}$ by D7 or D25 ($p=x$).

J. For, $\int_0^1 dx\; p(x)\; f_x(k) = q(k)$ as above. (D23)

Note. Included are the special cases: Beta-compounded geometric ($s = 1$), power-compounded negative binomial ($\sigma = 1$), and power-compounded geometric ($\sigma = 1 = s$) Cf. D28, 29.

D28. $q(k) = \rho\, B(\rho+1,k+1) = \rho k!/(\rho+1)(\rho+2)\ldots(\rho+1+k)$; $k = 0,1,2,\ldots;\rho > 0$.

℞. Sample $p(x) = \rho x^{\rho-1}$ for x on $(0,1)$ by C13 or C13A. For this x, sample $f_x(k) = x(1-x)^k$ for k on $\{0,1,2,\ldots\}$ by D7 ($p = x$, $s = 1$).

J1. Case $\sigma = 1$, $s = 1$ of D27.

℞2. Choose any $a,b > 0$ such that $\rho = a/b$. Sample $\tilde{q}(k)$ for k as in D29, ℞1.

J2. Under the substitutions $\rho = a/b$, $x = e^{-bu}$ one finds that $q(k) =$

$$\rho \int_0^1 x^{\rho}(1-x)^k\, dx = \tilde{q}(k) \text{ as in D29.}$$

Note the case of integral ρ, e.g., $\rho = 1$, gives $q(k) = 1/(k+1)(k+2)$; $k = 0,1,2,\ldots$ (uniform-compounded geometric).

D29. $\tilde{q}(k) = \int_0^\infty a\, e^{-(a+b)u} \left(1-e^{-bu}\right)^k du$; $k = 0,1,2,\ldots$; $a,b>0$.

℞1. Sample $p(u) = a\, e^{-au}$ for $u = -a^{-1}\ell nr$ on $(0,\infty)$ by C17. For this u, sample $f_u(k) = e^{-bu}\,(1-e^{-bu})^k$ for k on $\{0,1,2,\ldots\}$ by D7 ($p = e^{-bu}$, $s = 1$).

J1. For, $\int_0^\infty du\, p(u)\, f_u(k) = \tilde{q}(k)$ as above. (D23).

℞2. Define $\rho = a/b$ and sample $q(k)$ for k as in D28, ℞1.

J2. Under the substitution $e^{-bu} = x$, one sees that $\tilde{q}(k) = (a/b)\, B\left((a/b)+1,K+1\right)$.

D30. $q(k) = \sum_{j=J}^{\infty} f(j,k)$; $k = K, K+1, \ldots$; $f(j,k)$ density for $j \geqslant J$, $k \geqslant K$.

℞. Sample the marginal density $p(j) \equiv \sum_{k=K}^{\infty} f(j,k)$, $j \geqslant J$, for $j \geqslant J$. For this j, sample the j-dependent k-density $f_j(k) \equiv f(j,k)/p(j)$ for $k \geqslant K$.

J. Cf. D22.

D31. $q(k) = \sum_{j=J}^{\infty} p(j)\ f_j(k)$; $k = K, K+1, \ldots, p(j)$ density for $j = J, J+1, \ldots,$

$f_j(k)$ density for $k = K, K+1, \ldots,$ for each $j \geqslant J$.

℞. Sample $p(j)$ for $j \geqslant J$. For this j, sample $f_j(k)$ for $k \geqslant K$.

J. Corollary of D30. Cf. D23.

D32. $q(k) = L^{-1}(\lambda)\left(\phi^k/k!\right)\sum_{j=1}^{\infty} j^{k-1}\left(\lambda e^{-\phi}\right)^j$; $k = 0,1,2,\ldots, 0<\lambda<1$, $\phi>0$,

$L(\lambda) = -\ell n(1-\lambda)$.

℞. Sample $p(j) = \lambda^j/j\ L(\lambda)$ for j on $\{1,2,\ldots\}$ by D17. For this j, sample $f_j(k) = e^{-j\phi}\ (j\phi)^k/k!$ for k on $\{0,1,\ldots\}$ by D2.

J. One has $\sum_{j=1}^{\infty} p(j)\ f_j(k) = q(k)$ (D31).

D33. $q(k) = \left(e^{-\lambda}/k!\right)\sum_{j=0}^{\infty}\left(\lambda e^{-\phi}\right)^j\ (\phi j)^k/j!$; $k = 0,1,\ldots$; λ, $\sigma>0$, and

(N.B.!) $(\phi j)^k \equiv 1$ for $j = 0$, $k = 0$.

℞. Sample $p(j) = e^{-\lambda}\ \lambda^j/j!$ for j on $\{0,1,2,\ldots\}$ by D2. If $j = 0$, set $k = 0$.

For $j \geqslant 1$, sample $f_j(k) = e^{-\phi j}(\phi j)^k/k!$ for k on $\{0,1,2,\ldots\}$ by D2.

J. For, $\sum_{j=0}^{\infty} p(j)\ f_j(k) = q(k)$ (D31).

D34. $q(k) = e^{-\lambda}\ p^k \sum_{j \geq k/n} C_k^{nj}\ q^{nj-k} \lambda^j / j!$; $k = 0,1,2,\ldots;\lambda>0,\ 0<p<1,\ q = 1-p,$

n positive integer.

℞. Sample $e^{-\lambda}\ \lambda^j/j!$ for j on $\{0,1,2,\ldots\}$ by D2. For this j, set k = number of $r_1, \ldots, r_{nj}$ such that $r_i \leq p$.

J. $f(j,k) = \left(e^{-\lambda}\ \lambda^j/j!\right)\left(C_k^{nj}\ p^k\ q^{nj-k}\right)$ is a doubly-discrete density on the lattice points (j,k) with $j = 0,1,2,\ldots,\ k = 0,1,\ldots,\ nj$. Its marginal densities are

$$\sum_{j \geq \frac{k}{n}} f(j,k) = q(k) \text{ as above, and } p(j) = \sum_{k=0}^{nj} f(j,k) = e^{-\lambda}\lambda^j/j!$$

Moreover, $f_j(k) \equiv f(j,k)/p(j) = C_k^{nj}\ p^k\ q^{nj-k}$. The rule is an obvious modification of that in D31, and is an analogue of C139.

D35. $q(k) = \begin{cases} \exp - \lambda\left(1-p^k\right);\ k = 0 \\ \sum_{j=1}^{\infty} \left(e^{-\lambda}\ q^k/k!\right)\ \Gamma\left(Kj+k\right)\ \left(\lambda p^K\right)^j/j!\ \Gamma\left(Kj\right);\ k \geq 1 \end{cases}$

λ,K real >0, $0<p<1$, $q = 1-p$.

℞. Sample the Poisson density $e^{-\lambda}\ \lambda^j/j!$ for j on $\{0,1,\ldots\}$ by D2. If $j = 0$, set $k = 0$. If $j \geq 1$, sample the negative binomial density $\Gamma(Kj+k)\ p^{Kj}\ q^k/\Gamma(Kj)k!$ for k on $\{0,1,2,\ldots\}$ by D7 or D25 (with $s = Kj$).

J. Define $p(j) = e^{-\lambda} \lambda^j/j!$, $j = 0,1,\ldots$, and

$$f_j(k) = \begin{cases} 1 \text{ for } j = 0,\ k = 0 \\ 0 \text{ for } j = 0,\ k \geqslant 1 \\ \Gamma(Kj+k)\ p^{Kj}\ q^k/\Gamma(Kj)k! \text{ for } j \geqslant 1,\ k \geqslant 0. \end{cases}$$

Then one verifies that $\sum_{j=0}^{\infty} p(j)\ f_j(k) = q(k)$ as above and the rule follows as in D31.

Note. If one defines "$\Gamma(k)/\Gamma(0)$" = δ_k^0 (Kronecker delta) then for all $k = 0,1,2,\ldots$, one may formally write $q(k) =$

$$\sum_{j=0}^{\infty} \left(e^{-\lambda}\ q^k/k!\right)\ \Gamma\left(Kj+k\right)\left(\lambda p^K\right)^j/j!\Gamma\left(Kj\right).$$

D36. $$q(\nu) = \begin{cases} e^{-\lambda}\ ; \quad \nu = 0 \\ \sum_{1 \leqslant j \leqslant \nu/K} e^{-\lambda}\ C_{Kj-1}^{\nu-1}\left(\lambda p^K\right)^j\ q^{\nu-Kj}/j!;\ \nu = K,K+1\ldots \end{cases}$$

λ real > 0, K integer $\geqslant 1$, $0 < p < 1$, $q = 1-p$.

R. Sample $e^{-\lambda} \lambda^j/j!$ for j on $\{0,1,\ldots\}$ by D2. If $j = 0$, set $\nu = 0$. If $j \geqslant 1$, sample $C_{Kj-1}^{\nu-1}\ p^{Kj}\ q^{\nu-Kj}$ for ν on $\{jK, jK+1,\ldots\}$ by D7 ($s = Kj$, set $\nu = jK+k$).

J. Define $p(j) = e^{-\lambda} \lambda^j/j!$, $j = 0,1,2,\ldots$, and

$$F_j(\nu) = \begin{cases} 1 \text{ for } j = 0,\ \nu = 0 \\ 0 \text{ for } j = 0,\ \nu>0 \\ C_{Kj-1}^{\nu-1}\ p^{Kj}\ q^{\nu-Kj} \text{ for } j > 0,\ \nu \geqslant Kj \end{cases}$$

the domain of (j,ν) being all lattice points with $j \geq 0$, $\nu \geq Kj$. Then the above $q(\nu) = \sum_{0 \leq j \leq \nu/K} p(j)\, F_j(\nu)$. The method is an obvious modification of D31, and a discrete analogue of C69.

D37. $q(\nu) = \begin{cases} e^{-\lambda} ; & \nu = 0 \\ \sum_{j=1}^{\nu} e^{-\lambda}\, C_{j-1}^{\nu-1} (\lambda p)^j q^{\nu-j}/j! ; & \nu = 1,2,\ldots; \lambda \text{ real } >0,\ 0<p<1, \end{cases}$

$q = 1-p$.

℞. Sample $e^{-\lambda} \lambda^j/j!$ for j on $\{0,1,\ldots\}$ by D2. If $j = 0$, set $\nu = 0$. If $j > 0$, sample $C_{j-1}^{\nu-1} p^j q^{\nu-j}$ for ν on $\{j,j+1,\ldots\}$ by D7 ($s = j$, set $\nu = j+k$).

J. Case $K = 1$ of D36.

D38. $q(k) = \sum_{j=k+1}^{\infty} e^{-\theta} \theta^{j-1}/j! = \int_0^{\theta} x^k e^{-x}\, dx/\theta k!$; $k = 0,1,2,\ldots;\theta>0$.

℞. For $x = r\theta$, sample $e^{-x}x^k/k!$ for k on $\{0,1,2,\ldots\}$ by D2.

J. The integral form of the Poisson "tail-end" density follows from the formula F3 (See D24.J). Using the marginal method of D30 on $f(x,k) = x^k e^{-x}/\theta k!$, $0<x<\theta$, $k = 0,1,2,\ldots$, one has $\int_0^{\theta} dx\, f(x,k) = q(k)$ as above, and $p(x) = \sum_0^{\infty} f(x,k) = 1/\theta$, $f_x(k) = f(x,y)/p(x) = e^{-x}x^k/k!$

D39. $p(k) = (k!)^{-1}\{1-1/1! + 1/2! \ldots + (-1)^{N-k}/(N-k)!\}$; $k = 0,1,2,\ldots,N$.

Note $p(N-1) = 0$.

℞. Follow the steps:

1. Put $1 \to A_1, \ldots, 1 \to A_N$ (A_i storage positions)
2. Put $1 \to t$, $N \to D$.
3. Set $K = \min\{k;\ k \geq Dr_t,\ k = 1,2,\ldots,D\}$
4. If $K \neq D$, interchange contents of A_K and A_D. Go to (5).
5. If $t < N$, put $t + 1 \to t$, $D - 1 \to D$, return to (3)

If $t = N$, one has a random permutation $(C_1, \ldots C_N)$ of $1,\ldots,N$, C_i = content of A_i. Set k = number of i for which $C_i = i$.

J. $p(k)$ is the probability of exactly k coincidences. From the inclusion-exclusion principle, the probability of no coincidence on a random permutation on any n digits is

$$1 - 1/1! + 1/2! - \ldots + (-1)^n/n!$$

Hence the probability of exactly k coincidences on $1,\ldots,N$ is

$$C_k^N \frac{(N-k)!}{N!} \{1 - 1/1! + \ldots + (-1)^{N-k}/(N-k)!\} = p(k) \text{ as above.}$$

D40. $p(a/b) = (e-1)^2/(e^{a+b}-1)^2$; $a,b \geq 1$, $(a,b) = 1$.

℞. For $m = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$, define $\phi(m) = m\ \Pi_1^k (1 - 1/p_i)$, Euler's ϕ-function.

Set: $m = \min\{m; \sum_{i=2}^{m} (e^i - 1)^2 \geq r_1/(e-1)^2$, $f = \phi(m)$, $j = \min\{j;\ j \geq r_2 f\}$.

List the integers a_i prime to m on $\{1,\ldots,m\}$ as $a_1, a_2, \ldots, a_f$. Set $a = a_j$, $b = m - a_j$.

J. Classify all fractions a/b according to the sum $m = a+b$, $(a,m) = 1$. Those belonging to the same m are equally likely, and the probability of the subset with sum m is

$$\sum_{a+b=m} (e-1)^2/(e^{a+b}-1)^2 = (e-1)^2\ \phi(m)/(e^m - 1)^2.$$

D41. $q(k) = \sum_{j=0}^{N} C_j^N (\phi j)^k (pe^{-\phi})^j q^{N-j}/k!$; $k = 0,1,2,\ldots$, $\phi>0$, $0<p<1$,

$q = 1-p$, $(\phi j)^k \equiv 1$ for $j = k = 0$.

R. Set j = number of $r_1,\ldots, r_N$ such that $r_i \leqslant p$. For this j, sample $e^{-j\phi}(j\phi)^k/k!$ for k on $\{0,1,2,\ldots\}$ by D2. (If $j = 0$, set $k = 0$)

J. For $p(j) = C_j^N p^j q^{N-j}$ on $j = 0,1,\ldots,N$ and $f_j(k) = e^{-j\phi}(j\phi)^k/k!$ on $k = 0,1,2,\ldots$, one has $\sum_{j=0}^{N} p_j f_j(k) = q(k)$ above.

C-INDEX

Continuous Densities

C1.	$p(\nu)$	General density, continuous on open interval, finite or infinite
C3.	$\sum_1^J a_i(\nu)$	Sum of positive functions, interpolated densities, discrete-continuous marginal.
C13, 13A.	u^{m-1}	Power, $m > 0$
C15, 15A.	v^{-m-1}	Power, $m > 0$
C17.	$e^{-a\nu}$	Exponential
C22.	$u^{n-1}e^{-u}$, $n = 1,2,3,\ldots$	Gamma
C25.	$\nu^{2n-1}e^{-\nu^2}$, $n = 1,2,3,\ldots$	Gauss type, $2n-1 = 1,3,5,\ldots$
C26.	Re^{-R^2}	Gauss type ($n = 1$)
C27.	$e^{-\nu^2}$, $(0,\infty)$	Error function

C28. $e^{-\nu^2}$, $(-\infty,\infty)$ — Normal

C29. $u^{2n-1}e^{-u^2}$, $n = 1/2, 3/2, 5/2, \ldots$ — Gauss type, $2n-1 = 0,2,4\ldots$

C32. $\nu^{n-1}e^{-\nu}$, $n = 1/2, 3/2, 5/2, \ldots$ — Gamma

C35. $v^{m-1}(1-v)^{n-1}$ — Beta; $m,n \in \{1/2, 1, 3/2, 2, \ldots\}$.

$z^{m-1}/(1+z)^{m+n}$

$\sin^{2m-1}\theta \cos^{2n-1}\theta$

C43. $1\Big/\left(1+\frac{t^2}{N}\right)^{\frac{N+1}{2}}$ — Student's t

C62. $\sum_0^\infty a_j \nu^j$ — Power series, Butler

C63. $\int_a^b dx\ f(x,y)$ — Marginal, Composition, Butler

C63A. $\int_a^b dx\ p(x)\ f_x(y)$ — Marginal, Composition, Butler

C64. $\int_0^1 dx\ x^{m-3/2} \exp\left(-y^2/2bx\right)$ — Romanowski, modulated normal, equi-normal (m=1), radico-normal (m=3/2), lineo-normal (m=2). (JK3/276)

C65. $\int_0^\infty dx\ x^N\ e^{-x^2/2} \times$ — Non-central t (JK3/204)

$$\exp\left\{-\left(\frac{xy}{\sqrt{N}} - \delta\right)^2/2\right\}$$

C66. $\int_0^\infty dx\ x^{\frac{n-4}{2}}\ e^{-x/2H^2} \times$ — Sample covariance (JK3/231)

$$\exp\left\{-\left(y - \frac{\rho K x}{H}\right)^2\Big/2K^2(1-\rho^2)\ x\right\}$$

C67.	$\left(e^{-ay} - e^{-by}\right)\Big/ y$	Exponential marginal (n = 1)
C68.	$\left\{\exp\left(-ay^{1/n}\right) - \exp\left(-by^{1/n}\right)\right\} \times y^{-1/n}$	Exponential marginal (n ≠ 1)
C69.	$\int_a^y dx\ f(x,y)$	Triangular marginal, composition
C70.	$1/2\left(g(w) + g(-w)\right)$	Symmetric sum
C71.	$\Phi(w-\rho)e^{-\rho w} + \Phi(-w-\rho)e^{\rho w}$	Compound Laplace (JK3/32)
C72.	$e^{\frac{1}{2}(\sigma/\phi)^2} \times \left\{\Phi\left(\frac{z-\zeta}{\sigma} - \frac{\sigma}{\phi}\right)e^{-\frac{z-\zeta}{\phi}} + \Phi\left(-\frac{z-\zeta}{\sigma} - \frac{\sigma}{\phi}\right)e^{\frac{z-\zeta}{\phi}}\right\}$	3-parameter compound Laplace (JK3/32)
C73.	$1/(1+\beta x)$	Truncated Type VI, Bradford (JK3/89)
C74.	$(1+\theta x)\ \exp\left\{-\left(x+\frac{1}{2}\theta x^2\right)\right\}$	Linear failure rate, life-times (JK3/268)
C75.	$\left[1 + \theta\left(1-e^{-x}\right)\right] \times \exp\left\{-\left[x + \theta\left(x+e^{-x} - 1\right)\right]\right\}$	Life-times (JK3/268)
C76.	$\exp\left(-y-e^{-y}\right)$	Extreme value (JK2/277)
C77.	$e^{-(z-\zeta)/\theta} \times \exp\left\{-e^{-(z-\zeta)/\theta}\right\}$	2-parameter extreme value (JK2/277).
C78.	$e^{-\phi t/\sigma}\ \exp\left(-\rho e^{-t/\sigma}\right)$	Gompertz (JK3/271)
C79.	$\left(1 + \frac{x}{a}\right)^{ab} e^{-bx}$	Transition Type III (EJ/78)
C80.	$1/x^{n+1}\ e^{b/x}$	Transition Type V (EJ/81)

C81. $\left\{1 + \left(\frac{x-\xi}{\lambda}\right)^2\right\}^{-\frac{1}{2}} \times$ S_U curves (EJ/126)

$$\exp -\tfrac{1}{2}\left\{\gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right)\right\}^2$$

C82. $\left(\frac{x-\xi}{\lambda}\right)^{-1} \left(1 - \frac{x-\xi}{\lambda}\right)^{-1} \times$ S_B curves (EJ/130)

$$\exp -\tfrac{1}{2}\left\{\gamma + \delta\ \ell n \frac{x-\xi}{\xi+\lambda-x}\right\}^2$$

C83. $\exp\left(-\ \ell n^2\ u\right)$ Pseudo log-normal

C84. $(x-\theta)^{-1} \times$ 3-parameter log-normal,

$\exp\left\{-\left[\ell n(x-\theta) - \zeta\right]^2/2b\right\}$ Cobb, Douglas (JK2/113).

C85. $\exp\left\{-2^{-1}\left(1-\rho^2\right)^{-1} \times\right.$ General 2-variable normal.

$$\left.\left[\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{y_1-\mu_1}{\sigma_1}\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right) + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

C85A. $\exp -\sum x_i\ a_{ij}\ x_j$ n-variable normal

C86. $g(x) + g(-x)$ Folded density

C87. $\cosh\left(\xi x/\sigma^2\right) \times$ Normal symmetric sum (JK3/136)

$$\exp\left\{-\left(x^2+\xi^2\right)/2\sigma^2\right\}$$

C88.	$\cosh(\xi x/\sigma^2) \times \exp\left\{-\left(x^2+\xi^2\right)/2\sigma^2\right\}$	Folded normal (JK3/136)
C89.	$1/\left(e^x + b + e^{-x}\right),\ b\varepsilon(-2,2)$	Symmetric exponential I (JK3/15)
C90.	$1/\left(e^x + e^{-x}\right)$	Hyperbolic secant (JK3/15)
C91.	$1/\left(e^x + 2 + e^{-x}\right)$	Logistic, sech-square, growth-curve, symmetric exponential II (JK2/244)., (JK3/3)
C92.	$1/\left(e^x + b + e^{-x}\right),\ b > 2$	Symmetric exponential III(JK3/15)
C93.	$1/\left(\lambda + \cosh\alpha\left(y-y_o\right)\right)$	Champernowne, income, Perks (JK2/242).
C94.	$1/t\left\{\frac{1}{2}\left(\frac{t}{t_o}\right)^{\alpha} + \lambda + \frac{1}{2}\left(\frac{t}{t_o}\right)^{-\alpha}\right\}$	Champernowne, income (JK2/243).
C95.	$e^x/\left(\beta + e^x\right)^{m+1}$	Generalized logistic I (JK3/17)
C96.	$e^{-y}/\left(1 + \beta^{-1}e^{-y}\right)^{m+1}$	Generalized logistic II (JK3/17)
C97.	$(x-a)^{m-1}\ (b-x)^{n-1}$	Pearson Types I, II, general Beta (JK3/37)
C98.	$(x-b)^r/(x-a)^q$	Pearson type VI (JK2/13), (JK3/87)
C99.	$E^{M-1}/\left(1 + \frac{M}{N}E^2\right)^{\frac{M+N}{2}}$	Square root of Snedecor's F, rms/rms
C100.	$1/\left(e^{x/2} + e^{-x/2}\right)^{2m}$	Logistic power, power of sech-square (JK3/5,17)
C101.	$e^{-mx/\sigma}/\left(1 + \rho e^{-x/\sigma}\right)^{m+n}$	4-parameter generalized logistic (JK3/271)

C102. $e^{-mx/\sigma} \cdot \left(1 - \rho e^{-x/\sigma}\right)^{n-1}$ — 4-parameter generalized exponential (JK3/271)

C103. $\left(1 - \frac{x^2}{a^2}\right)^{n-1}$ — Transition Type II (EJ/74)

C104. $x^{m-1}/(1 + x)$ — Restricted Beta (0 < m < 1)

C105. $x(x-a)^{m-1}\ (b-x)^{n-1}$ — x-Beta

C106. $x^{m-1}(1-x)^{n-1}/(x+a)^{m+n}$ — Modified Beta

C107. $(a+x)^{m-1}\ (a-x)^{n-1}$ — Centered Beta

C108. $F(x) + x^{-2}\ F(x^{-1})$ — Reflected density

C109. $\left(x^{m-1} + x^{n-1}\right)/(1+x)^{m+n}$ — Reflected Beta

C110. $1/\left[c^2 + (\zeta-\zeta_0)^2\right]^m$ — Pearson Type VII (JK2/13), (JK3/114)

C111. $1/\left[1 + \left(\frac{t-\theta}{\lambda}\right)^2\right]$ — 2-parameter Cauchy (JK2/154)

C112. $\left[1 + \left(\frac{t^2+\theta^2}{\lambda^2}\right)\right] \times$ — Cauchy symmetric sum (JK2/163)

$$\left[1 + 2\left(\frac{t^2+\theta^2}{\lambda^2}\right) + \left(\frac{t^2-\theta^2}{\lambda^2}\right)^2\right]^{-1}, \quad (-\infty,\infty)$$

C113. $2 \times$ C112 on $(0,\infty)$ — Folded Cauchy (JK2/163)

C114. $1/\left[1 + \left|\frac{x-\theta}{\lambda}\right|^{1/m}\right]^{m+n}$ — Generalized Cauchy, Rider (JK2/162)

C115. $\left[1 - P(x)\right]^{N-k}\ P^{k-1}(x)\ p(x)$ — General order statistics

C116. $\left[1 - P(x)\right]^{N-1} p(x)$, $P^{N-1}(x)\ p(x)$, $\left[\left(1 - P(x)\right) P(x)\right]^{M} p(x)$ — Min, Max, Median statistics (JK2/3).

C117. $(x-a)^{k-1} (b-x)^{N-k}$ — Order statistics (uniform)(JK3/38)

C118. $x^{k-1} (1-x)^{N-k}$ — Order statistics (random numbers) (JK3/38)

C119. $\left(1 - \exp(-e^{-x})\right)^{N-k} e^{-x} \times \exp(-ke^{-x})$ — Order statistics (extreme value) (JK2/279)

C120. $e^{-(N-k+1)x}/(1+e^{-x})^{N+1}$ — Order statistics (logistic) (JK3/8)

C121. $\left[\frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{t-\theta}{\lambda}\right)\right]^{N-k} \times \left[\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{t-\theta}{\lambda}\right)\right]^{k-1} \times \left[1 + \left(\frac{t-\theta}{\lambda}\right)^{2}\right]^{-1}$ — Order statistics (Cauchy) (JK2/157)

C122. $x^{b-1} \left(1 - e^{-x^{b}}\right)^{k-1} e^{-(N-k+1)x^{b}}$ — Order statistics (Weibull) (JK2/254)

C123. $e^{-(N-k+1)x} \left(1-e^{-x}\right)^{k-1}$ — Order statistics (exponential)(JK2/214)

C124. $\left[1 - \Gamma_{x}(n)/\Gamma(n)\right]^{N-k} \times \left[\Gamma_{x}(n)/\Gamma(n)\right]^{k-1} x^{n-1} e^{-x}$ — Order statistics (Gamma) (JK2/191)

C125. $(a/x)(k/x)^{a(N-k+1)} \times \left[1 - (k/x)^{a}\right]^{k-1}$ — Order statistics (Pareto) (JK2/241)

C126. $\begin{cases} 4(x-a)/(b-a)^2 \\ 4(b-x)/(b-a)^2 \end{cases}$ Symmetric triangular, tine (JK3/64)

C127. $1 - |x|$ Centered triangular (JK3/64)

C128. $\begin{cases} a_1 \ (x) \\ a_2 \ (x) \end{cases}$ Composite, symmetric q(-x) = q(x)

C129. $\begin{cases} h(x-a)/(b-a) \\ h(c-x)/(c-b) \end{cases}$ General triangular

C130. $\begin{cases} e^{-ax} \\ e^{bx} \end{cases}$ Asymmetric Laplace (JK3/31)

C131. $a_i(x), \ [x_i, x_{i+1}], \quad i = 0,1,2,\ldots$ General composite

C132. $\begin{cases} px/a^2 \\ pq^{i-1}\{(1+ip)\ a-px\}/a^2 \end{cases}$ Binomial-uniform, traffic flow (JK3/70)

C133. $\exp\left[-\lambda(t-\mu)^2/2\mu^2 t\right]$ Inverse Gaussian, first passage time (Brownian motion with drift) (JK2/138)

C134. $x^{m-1}\ y^{n-1}\ F(x+y)$ Bivariate with marginal Beta

C135. $x^{m-1}\ y^{n-1}/(1-x-y)^n$ Bivariate with Beta marginals

C136. $\{(1+ay)e^{-ay} - (1+by)e^{-by}\}\ y^{-2}$ Time between calls (B/69) uniform-compounded exponential

C137. $e^{-ay^2} - e^{-by^2}$ Marginal normal

C138. $e^{-y}\int_0^y dx\ x^{n-1}/(y-x)^n$ Marginal Γ

C139. $\int_y^b dx\ f(x,y)$ Marginal, triangular region

C140. $\int_y^b dx\ t(x)/t_1$ — Tail-end density

C141. $e^{-By}\sum_0^{n-1} (By)^i/i!$ — Gamma tail-end, $n = 1,2,3,\ldots$

C141A. $\int_y^\infty dx\ x^{n-1}\ e^{-Bx}$ — General Gamma tail-end

C142. $b^m - y^m$ — Power tail-end

C143. $e^{-y}/(1-\lambda e^{-y})$ — Log series-compounded exponential

Continuous Densities

C1. $p(\nu)$; (a,b)

R. Define $P(\nu) = \int_a^\nu p(\nu)d\nu$, $P_1(\nu) = \int_\nu^b p(\nu)d\nu$. Set $\nu = P^{-1}(r_0)$ or $\nu = P_1^{-1}(r_1)$

C3. $p(\nu) = \sum_1^J a_j(\nu)$; (a,b), $a_j(\nu) \geq 0$.

R. Define $A_j = \int_a^b a_j(\nu)d\nu$. Set $K = \min\left\{k; \sum_1^k A_j \geq r_0\right\}$. Sample density $a_K(\nu)/A_K$ for ν.

Note 1. For $J = 2$, this provides an elegant way of sampling an interpolated density $\alpha_1 p_1(\nu) + \alpha_2 p_2(\nu)$, $\alpha_i > 0$, $\alpha_1 + \alpha_2 = 1$ (L. Carter)

Note 2. This is the discrete-continuous marginal version of D30. The discrete-compounded continuous density seems to occur infrequently. As an example, we have included C143.

C13. $q(u) = m\ b^{-m}\ u^{m-1}$; $(0,b)$, $m = k/\ell$, k,ℓ, $k,\ell \in \{1,2,3,\ldots\}$.

R. Set $u = b\cdot\left(\max\ \{r_1,\ldots,r_k\}\right)^\ell$.

J. For $u = bv^\ell$ one has $q\ (u)du = kv^{k-1}\ dv$. Moreover, $\frac{d}{dv}\int_{\max\{r_1,\ldots,r_k\}\leq v} dr_1\cdots dr_k = \frac{d}{dv}(v^k) = kv^{k-1}$. See also C118, Note 2.

C13A. $q(u) = C^{-1}u^{m-1}$; (a,b) m real > 0, $C = (b^m - a^m)/m$, $a \geqslant 0$.

R. Set $u = \left(a^m + (b^m - a^m)\, r_o\right)^{1/m}$

C15. $p(v) = m\beta^m\, v^{-m-1}$; $0 < \beta < v < \infty$, $m = k/\ell$, $k,\ell \in \{1,2,3,\ldots\}$

R. Set $v = \beta/\left(\max\,\{r_1,\ldots,r_k\}\right)^{\ell}$

J. Let $v = 1/u$ and compare C13.

C15A. $p(v) = C^{-1}\, v^{-m-1}$; $0 < \beta < v < \alpha \leqslant \infty$, m real > 0, $C = (\beta^{-m}-\alpha^{-m})/m$.

R. Set $v = 1/\left\{\alpha^{-m} + (\beta^{-m}-\alpha^{-m})\, r_o\right\}^{1/m}$

J. Let $v = 1/u$ and compare C13A.

C17. $p(\nu) = ae^{-a\nu}$; $(0,\infty)$, $a > 0$.

R. Set $\nu = -a^{-1}\,\ell n\; r_o$

J. From $\int_{\nu}^{\infty} p(\nu)\,d\nu = r$.

C22. $q(u) = u^{n-1}\, e^{-u}/\Gamma(n)$; $(0,\infty)$, $n = 1,2,3,\ldots$

R. Set $u = -\,\ell n \prod_1^n r_i$

J. $$\frac{d}{du}\int_{\sum_1^n \nu_i \leqslant u} \prod_1^n e^{-\nu_i}\, d\nu_i = \frac{d}{du}\int_0^u e^{-u}\, A(u)\,du = e^{-u}u^{n-1}/(n-1)!$$

where $A(u) = dV/du$ and $V = \int_{\sum_1^n \nu_i \leqslant u} \prod d\nu_i = u^n/n!$

C25. $p(\nu) = 2\nu^{2n-1}\, e^{-\nu^2}/\Gamma(n)$; $(0,\infty)$, $n = 1,2,3,\ldots$, $2n-1 = 1,3,5,\ldots$

R. Set $\nu = \left(-\ell n \prod_1^n r_i\right)^{\frac{1}{2}}$

J. For $\nu = u^{\frac{1}{2}}$, one has $p(\nu)d\nu = q(u)du$ as in C22.

C26. $p(R) = 2Re^{-R^2}$; $(0,\infty)$

R. Set $R = (-\ell n\ r)^{\frac{1}{2}}$

C27. $p(\nu_1) = 2\ e^{-\nu_1^2}/\sqrt{\pi}$; $(0,\infty)$

R1. Generate r, r' until $S = r^2 + r'^2 \leqslant 1$. For accepted r,r', set

$\nu_1 = \left\{(-\ell n\ r_0)/S\right\}^{\frac{1}{2}} r$, $\nu_2\ \left\{(-\ell n\ r_0)/S\right\}^{\frac{1}{2}} r'$. (Two samples)

J1. Under $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, one finds $\frac{2}{\sqrt{\pi}} e^{-\nu_1} d\nu_1 \times \frac{2}{\sqrt{\pi}} e^{-\nu_2} d\nu_2 =$

$2Re^{-R^2}\ dR \times \frac{2}{\pi}\ d\theta$

R2. Generate r,r' until $S = r^2 + r'^2 \leqslant 1$. Set $\nu_1 = \{(-\ell nS)/S\}^{\frac{1}{2}} r$,

$\nu_2 = \{(-\ell nS)/S\}^{\frac{1}{2}} r'$. (Two samples)

J2. Under $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, where $R \equiv \{-2\ell n\rho\}^{\frac{1}{2}}$, one finds

$$\frac{2}{\sqrt{\pi}} e^{-\nu_1^2} d\nu_1 \times \frac{2}{\sqrt{\pi}} e^{-\nu_2^2} d\nu_2 = \frac{4}{\pi}\ \rho d\rho d\theta.$$

C28. $p_0(\nu_1) = e^{-\nu_1^2}/\sqrt{\pi}$; $(-\infty,\infty)$.

R. Generate r,r' until $S = x^2 + y^2 \leqslant 1$, where $x = 2r-1$, $y = 2r'-1$.

For accepted S, set $\nu_1 = \{(-\ell nS)/S\}^{\frac{1}{2}} x$, $\nu_2 = \{(-\ell nS)/S\}^{\frac{1}{2}} y$. (Two samples)

J. Cf. C27(J2)

C29. $q(u) = 2u^{2n-1}\ e^{-u^2}/\Gamma(n)$; $(0,\infty)$, $n = 1/2, 3/2, 5/2, \ldots$, $2n-1 = 0,2,4,\ldots$

R. Define h by $2n = 2h + 1$ ($h = 0,1,2,\ldots$). Generate r,r' until $S = r^2+r'^2 \leqslant 1$.

For accepted r,r', set $\tau^2 = \{(-\ell nS)/S\}\ r^2$. Set $u = \left\{-\ell n \prod_1^h r_i + \tau^2\right\}^{\frac{1}{2}}$.

J. $$\frac{d}{du} \int_{\left\{\sum_1^{2n} \nu_i^2\right\}^{\frac{1}{2}} \leqslant u} \prod_1^{2n} \left(\frac{2}{\sqrt{\pi}}\ e^{-\nu_i^2}\ d\nu_i\right) = \frac{d}{du} \int_0^u \frac{2^{2n}}{\pi^n}\ e^{-u^2}\ A(u)du$$

$$= 2\, e^{-u^2}\, u^{2n-1}/\Gamma(n), \text{ where } A = dV/du, \text{ and } V = \int_{\left\{\sum_1^{2n} \nu_i^2\right\}^{\frac{1}{2}} \leqslant u} \prod_1^{2n} d\nu_i$$

$$= \pi^n\, u^{2n}/2^{2n}\, n\Gamma(n).$$

Note. $\tau^2 = \{-\ell nS/S\}r'^{-2}$ should be saved for a second transit of C29.

C32. $p(\nu) = \nu^{n-1}\, e^{-\nu}/\Gamma(n)$; $(0,\infty)$, $n = 1/2, 3/2, 5/2, \ldots$

R. Set $\nu = u^2$ where u is obtained from C29. (Avoid squaring the square root!)

J. For $\nu = u^2$, one finds $p(\nu)d\nu = q(u)du$ in C29.

C35. $B(v) = v^{m-1}\,(1-v)^{n-1}/B(m,n)$; $(0,1)$

$b(z) = z^{m-1}/(1+z)^{m+n}\, B(m,n)$; $(0,\infty)$

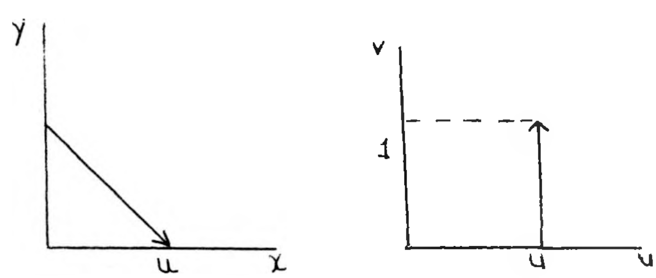
$q(\theta) = 2\sin^{2m-1}\theta\cos^{2n-1}\theta/B(m,n)$; $(0,\pi/2)$

$m,n\varepsilon \{1/2, 1, 3/2, 2, \ldots\}$ in all.

R. Sample $x^{m-1}\, e^{-x}/\Gamma(m)$, $y^{n-1}\, e^{-y}/\Gamma(n)$ for x,y on $(0,\infty)$ by C22 and/or C32. Set $v = x/(x+y)$, $z = v/(1-v) = x/y$, $\theta = \arcsin\sqrt{v}$.

J. The densities are equivalent under the substitutions indicated. For the transformation $x = uv$, $y = u(1-v)$, with inverse $u = x+y$, $v = x/(x+y)$, one finds

$$\Gamma^{-1}(m)x^{m-1}e^{-x}dx\cdot\Gamma^{-1}(n)y^{n-1}e^{-y}dy=\Gamma^{-1}(m+n)u^{m+n-1}e^{-u}du\; B^{-1}(m,n)v^{m-1}(1-v)^{n-1}dv$$



C43. $q_1(t) = \Gamma\left(\frac{N+1}{2}\right) / \sqrt{N\pi}\,\Gamma(N/2)\left(1 + \frac{t^2}{N}\right)^{\frac{N+1}{2}}$; $(-\infty,\infty)$, N= 1,2,3,...

R1. Sample $w^{N/2-1}\, e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C22 or C32; set $x = (w/N)^{\frac{1}{2}}$.

Sample $e^{-y^2}/\sqrt{\pi}$ for y on $(-\infty,\infty)$ by C28. Set $t = y/x$.

J1. See Sampler I.

R2. Sample $b(z) = z^{-\frac{1}{2}}/(1+z)^{\frac{N+1}{2}} B\left(\frac{1}{2}, \frac{N}{2}\right)$ for z on $(0,\infty)$ by C35. Set $t = \sqrt{Nz}$. Change sgn t with probability 1/2.

J2. By symmetry, we may sample density $2q_1(t)$ for t on $(0,\infty)$, with provision for sign change (Cf. C128). But $2q_1(t)\,dt = b(z)dz$ for $t = \sqrt{Nz}$ on $(0,\infty)$. The two rules are essentially identical.

C62. $p(\nu) = \sum_0^\infty a_j \nu^j$; $(0,1)$, $a_j \geq 0$.

R. Define $A_j = a_j/(j+1)$. Set K min $\left\{k; \sum_0^k A_j \geq r_0\right\}$. Set $\nu = r_1^{1/(K+1)} = \exp\left(\frac{1}{K+1} \ln r_1\right)$ or set $\nu = \max\{r_1,\ldots,r_{K+1}\}$.

J. The rule follows from C3, and from C13A or C13.

C63. $q(y) = \int_a^b dx\, f(x,y)$; (c,d), $f(x,y) \geq 0$.

R1. Sample marginal density $p(x) = \int_c^d f(x,y)\,dy$ for x on (a,b). For this x, sample the x-dependent y density $f_x(y) = f(x,y)/p(x)$ for y on (c,d).

J1. Under the rule, the probability of choosing y on $(y,y+dy)$ is $\int_a^b p(x)dx \cdot f_x(y)dy = q(y)dy$.

Note. One may regard C6 (n=2) and C8 (p,q,q_1) as special cases.

R2. Sample $f(\xi,\eta)$ for (ξ,η). Set $y = \eta$.

C63A. $q(y) = \int_a^b dx\, p(x)\, f_x(y)$; (c,d), $p(x)$ density on (a,b), $f_x(y)$ continuous y-density on (c,d) for each value of parameter x on (a,b).

R. Sample $p(x)$ for x. For this x, sample $f_x(y)$ for y.

J. Corollary of C63.

C64. $q(y) = \frac{m}{\sqrt{2\pi b}} \int_0^1 dx \cdot x^{m-3/2}\, e^{-y^2/2bx}$; $(-\infty,\infty)$, $m,b > 0$.

R. Sample mx^{m-1} for x on (0,1) by C13 or C13A. By C28, sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$. Set $y = \nu(2bx)^{\frac{1}{2}}$.

J. Following C63 for the integrand f(x,y), one finds

$p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = mx^{m-1}$ on (0,1) and $f_x(y)dy = f(x,y)dy/p(x)$

$= e^{-y^2/2bx}dy/(2\pi bx)^{\frac{1}{2}} = e^{-\nu^2}d\nu/\sqrt{\pi}$ where $y = \nu(2bx)^{\frac{1}{2}}$.

C65. $$q(y) = \frac{1}{2^{\frac{N-1}{2}}\sqrt{\pi N}\,\Gamma(N/2)} \int_0^{\infty} dx \cdot x^N e^{-x^2/2} \exp -\left(\frac{xy}{\sqrt{N}} - \delta\right)^2/2; \quad (-\infty,\infty),$$

δ arbitrary, N = 1,2,3,...

R. Sample $w^{N/2-1} e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C22 or C32. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $y = \left(\sqrt{2}\cdot\nu+\delta\right)\cdot\sqrt{N/2w}$.

J. Following C63 for the integrand f(x,y), one finds $p(x)dx = dx\int_{-\infty}^{\infty} f(x,y)dy =$

$dx \cdot x^{N-1} e^{-\frac{x^2}{2}}/2^{N/2-1}\,\Gamma(N/2) = w^{N/2-1} e^{-w}\,dw/\Gamma(N/2)$, where $x = \sqrt{2w}$.

Moreover, $f(x,y)dy/p(x) = \frac{x\,dy}{\sqrt{2\pi N}} \exp -\left(\frac{xy}{\sqrt{N}} - \delta\right)^2/2 = e^{-\nu^2}d\nu/\sqrt{\pi}$ for

$y = \left(\sqrt{2}\,\nu+\delta\right)\sqrt{N}/x$.

C66. $$q(y) = \int_0^{\infty} \frac{dx\cdot x^{\frac{n-4}{2}} e^{-x/2H^2}}{\sqrt{2\pi K^2(1-\rho^2)}\,(2H^2)^{\frac{n-1}{2}}\,\Gamma\left(\frac{n-1}{2}\right)} \exp \frac{-\left(y-(\rho Kx/H)\right)^2}{2K^2(1-\rho^2)x}\ ; \quad (-\infty,\infty),$$

n = 5,6,..., H, K > 0, $-1 < \rho < 1$.

R. Sample $z^{\frac{n-3}{2}} e^{-z}/\Gamma\left(\frac{n-1}{2}\right)$ for z on $(0,\infty)$ by C22 or C32. Sample $e^{-u^2}/\sqrt{\pi}$ for u on $(-\infty,\infty)$ by C28. Set $x = 2H^2z$, and $y = \frac{\rho Kx}{H} + u\sqrt{2K^2(1-\rho^2)x}$.

J. The rule follows from C63 upon noting the relations:

$$p(x)\,dx = x^{\frac{n-3}{2}} e^{-x/2H^2} dx/(2H^2)^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)$$

$$= z^{\frac{n-3}{2}} e^{-z} dz/\Gamma\left(\frac{n-1}{2}\right) \text{ for } x = 2H^2 z$$

$$f(x,y)\,dy/p(x) = e^{-u^2} du/\sqrt{\pi} \text{ for } y = \frac{\rho K x}{H} + u\sqrt{2K^2(1-\rho^2)x}\ .$$

C67. $q(y) = (e^{-ay} - e^{-by})/yC$; $(0,\infty)$, $0 < a < b$, $C = \ln(b/a)$

R. Generate r_1, r_2 and set $y = -(\ln r_2)/a\, e^{Cr_1}$

J. For $f(x,y) = e^{-xy}/C$ on $(a,b) \times (0,\infty)$, one has the marginal densities $q(y) = (e^{-ay} - e^{-by})/yC$ on $(0,\infty)$, $p(x) = 1/xC$ on (a,b). Moreover $f(x,y)/p(x) = x\, e^{-xy}$. Following C63, we sample $p(x)$ on (a,b) using $P(x) = \int_a^x p(x)\,dx = C^{-1} \ln(x/a) = r_1$, and thus setting $x = ae^{Cr_1}$ as in C14. For this x, we sample $f(x,y)/p(x)$ for y by C17, setting $y = -(\ln r_2)/x$.

C68. $q(y) = \left\{\exp(-ay^{1/n}) - \exp(-by^{1/n})\right\}/y^{1/n} C$; $(0,\infty)$, $0 < a < b$, $n > 0$, $(n \neq 1)$, $C = \Gamma(n+1)(b^{1-n} - a^{1-n})/(1-n)$.

R. Sample $C^{-1}\Gamma(n+1)x^{-n}$ for x on (a,b), using C13, C13A, C15, or C15A. Sample $z^{n-1} e^{-z}/\Gamma(n)$ for z on $(0,\infty)$ by C22, C32, or R18. Set $y = z^n/x^n$.

J. For $f(x,y) = \exp(-xy^{1/n})/C$ on $(a,b) \times (0,\infty)$, one has, for marginal densities, the given $q(y)$ on $(0,\infty)$, and $p(x) = C^{-1}\Gamma(n+1)x^{-n}$ on (a,b). Moreover, $f(x,y)/p(x) = x^n \exp(-xy^{1/n})/\Gamma(n+1)$. Following C63 we therefore sample $p(x)$ for x on (a,b). For this x, we then sample $f(x,y)/p(x)$ for y on $(0,\infty)$. Since $f(x,y)\,dy/p(x) = z^{n-1}e^{-z}\, dz/\Gamma(n)$ for $y = z^n/x^n$, the rule follows.

C69. $q(y) = \int_a^y dx \cdot f(x,y)$; (a,b), $f(x,y) \geqslant 0$ on region R bounded by $x = a$,

$y = b$, $x = y$, where $a < b$.

R. Define $p(x) = \int_x^b f(x,y)dy$ for each x on (a,b). Sample density $p(x)$ for x on (a,b). For this x, sample density $f_x(y) = f(x,y)/p(x)$, $x < y < b$ for y on (x,b).

J. Under the rule, the probability of choosing y on $(y,y+dy)$ is

$$\int_a^y dx\ p(x) \cdot \frac{f(x,y)}{p(x)}\ dy = q(y)dy.$$

Note. This is an obvious modification of C63 for a density $f(x,y)$ defined on the region R. For $(a,b) = (-\infty,\infty)$, R is the region above the line $y = x$. C8 (s) is a special case of C69.

C70. $s(w) = 1/2\ \big(g(w) + g(-w)\big)$; $(-\infty,\infty)$, $g(w)$ density on $(-\infty,\infty)$.

R. Sample density $g(w)$ for w on $(-\infty,\infty)$. Change sign of w with probability 1/2.

J. The obvious rule may be regarded as an instance of C3.

C71. $s(w) = (\rho/2)\ e^{\rho^2/2} \left\{ \phi(w-\rho)e^{-\rho w} + \phi(-w-\rho)e^{\rho w} \right\}$; $(-\infty,\infty)$, $\rho > 0$,

$$\phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}dx.$$

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $w = \nu\sqrt{2} - \frac{1}{\rho}\ \ell n r$. Change sign of w with probability 1/2.

J. By C70 it suffices to sample density $g(w) = \rho e^{\rho^2/2}\ \phi(w-\rho)\ e^{-\rho w}$ for w on $(-\infty,\infty)$ with provision for changing sgn w. For $w = y+\rho$, we find that $g(w)dw = q(y)dy$, where $q(y) = \int_{-\infty}^{y} dx \cdot \rho e^{-\rho^2/2}\ e^{-x^2/2}\ e^{-\rho y}/\sqrt{2\pi}$. Following C69 with $f(x,y) = \rho\ e^{-\rho^2/2}\ e^{-x^2/2}\ e^{-\rho y}/\sqrt{2\pi}$, we find that:

$$p(x) = \int_x^\infty f(x,y)\,dy = e^{-(x+\rho)^2/2}/\sqrt{2\pi}\ ;\ p(x)\,dx = e^{-\nu^2}d\nu/\sqrt{\pi} \text{ for}$$

$$x = -\rho + \nu\sqrt{2}\ ;\ f(x,y)/p(x) = \rho\, e^{-\rho(y-x)},\ y > x;\ \frac{f(x,y)}{p(x)}\,dy = e^{-v}dv \text{ for}$$

$y = x + \rho^{-1}v$. The rule follows from these relations and C17.

C72. $t(z) = e^{\frac{1}{2}(\sigma/\phi)^2}\left\{\Phi\left(\frac{z-\zeta}{\sigma} - \frac{\sigma}{\phi}\right)e^{-\frac{z-\zeta}{\phi}} + \Phi\left(-\frac{z-\zeta}{\sigma} - \frac{\sigma}{\phi}\right)e^{\frac{z-\zeta}{\phi}}\right\}/2\phi;$

$(-\infty,\infty)$, σ, $\phi > 0$, ζ arbitrary, $\phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}\,dx$.

℞. Sample $s(w)$ for w as in C71, where $\rho = \sigma/\phi$. Set $z = \zeta + \sigma w$.

J. Under the stated substitution, one finds $t(z)dz = s(w)dw$.

C73. $p(x) = \beta/(1+\beta x)\ \ell n\ (1+\beta)$; $(0,1)$, $\beta > -1$.

℞. Set $x = \beta^{-1}\left\{-1 + \exp\left[r\ \ell n\ (1+\beta)\right]\right\}$.

J. The rule results from setting $\int_0^x p(x)dx = r$. (Cf. C1.)

C74. $p(x) = (1+\theta x)\exp\left\{-\left(x+\frac{1}{2}\theta x^2\right)\right\}$; $(0,\infty)$, $\theta > 0$.

℞. Set $x = \theta^{-1}\left\{-1 + (1-2\theta\ \ell n\ r)^{\frac{1}{2}}\right\}$

J. For $\nu = x + \frac{1}{2}\theta x^2$, one has $p(x)dx = e^{-\nu}d\nu$. (Cf. C17)

C75. $q(x) = \left[1 + \theta(1-e^{-x})\right]\exp\left\{-\left[x + \theta(x+e^{-x}-1)\right]\right\}$; $(0,\infty)$, $\theta > 0$.

℞. Set $\nu_0 = -\ell n\ r_0$. Solve equation $\nu_0 = x + \theta(x+e^{-x}-1)$ for x. (See note below).

J. For $\nu = x + \theta(x+e^{-x}-1)$, one has $q(x)dx = e^{-\nu}d\nu$ as in C17.

Note (Newton's method for x). For $f(x) = x + \theta(x+e^{-x}-1) - \nu_0$ on $[0,\infty)$, one has $f(\nu_0) = \theta(\nu_0+e^{-\nu_0}-1) > 0$, $f(0) = -\nu_0 < 0$, $f'(x) = 1 + \theta(1-e^{-x}) \geq 1$, $f''(x) = \theta e^{-x} > 0$. Newton's recursion reads

$$x_{n+1} = x_n - f(x_n)/f'(x_n) = \frac{\nu_o + \theta\{1 - (1+x_n)\exp(-x_n)\}}{1 + \theta\{1 - \exp(-x_n)\}}$$

With initial $x_0 \equiv \nu_0$, the sequence $\{x_n\}$ rapidly converges down to x.

C76. $p(x) = \exp(-x-e^{-x})$; $(-\infty,\infty)$

R. Set $x = -\ln(-\ln r)$

J. For $x = -\ln \nu$, one has $p(x)dx = e^{-\nu}(-d\nu)$ on $(0,\infty)$ as in C17.

C77. $q(z) = \theta^{-1} e^{-(z-\zeta)/\theta} \exp\left[-e^{-(z-\zeta)/\theta}\right]$; $(-\infty,\infty)$, $\theta > 0$, ζ arbitrary.

R. Set $z = \zeta - \theta \ln(-\ln r)$.

J. For $z = \zeta + \theta x$, one has $q(z)dz = p(x)dx$ as in C76.

C78. $p(t) = \frac{\rho^\phi}{\sigma\Gamma(\phi)} e^{-\phi t/\sigma} \exp(-\rho e^{-t/\sigma})$; $(-\infty,\infty)$, $\rho,\sigma,\phi > 0$

R. Sample $w^{\phi-1} e^{-w}/\Gamma(\phi)$ for w on $(0,\infty)$ by C22, C32, or R18. Set $t = -\sigma \ln(w/\rho)$.

J. For the given transformation, one has $p(t)dt = w^{\phi-1} e^{-w}(-dw)/\Gamma(\phi)$.

C79. $p(x) = C\left(1 + \frac{x}{a}\right)^{ab} e^{-bx}$; $(-a,\infty)$, $a,b > 0$, $C = (ab)^{ab}/a\, e^{ab}\, \Gamma(ab)$.

R. Define $n = ab + 1$. Sample $z^{n-1} e^{-z}/\Gamma(n)$ for z on $(0,\infty)$ by C22, C32, or R18. Set $x = (z-ab)/b$.

J. Under the latter substitution, one has $p(x)dx = z^{n-1} e^{-z}dz/\Gamma(n)$ on $(0,\infty)$.

C80. $p(x) = b^n/\Gamma(n)\, x^{n+1}\, e^{b/x}$; $(0,\infty)$, $b > 0$, $n > 0$.

R. Sample $z^{n-1} e^{-z}/\Gamma(z)$ for z on $(0,\infty)$ by C22, C32, or R18. Set $x = b/z$.

J. For, $p(x)dx = z^{n-1} e^{-z}(-dz)/\Gamma(n)$.

C81. $p(x) = (2\pi)^{-\frac{1}{2}} \lambda^{-1}\delta \left\{1 + \left(\frac{x-\xi}{\lambda}\right)^2\right\}^{-\frac{1}{2}} \exp -\frac{1}{2}\left\{\gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right)\right\}^2$;

$(-\infty,\infty)$, ξ, γ arbitrary, λ, $\delta > 0$.

R. Sample $e^{-y^2}/\sqrt{\pi}$ for y on $(-\infty,\infty)$ by C28. Set $x = \xi + \lambda \sinh \frac{y\sqrt{2} - \gamma}{\delta}$

J. One finds $p(x)dx = e^{-y^2}dy/\sqrt{\pi}$.

C82. $p(x) = (2\pi)^{-\frac{1}{2}} \lambda^{-1} \delta \left(\frac{x-\xi}{\lambda}\right)^{-1} \left(1 - \frac{x-\xi}{\lambda}\right)^{-1} \exp - \frac{1}{2}\left\{\gamma + \delta \ln \frac{x-\xi}{\xi+\lambda-x}\right\}^2$;

$(\xi,\xi+\lambda)$, ξ, γ arbitrary, $\lambda,\delta > 0$.

R. Sample $e^{-y^2}/\sqrt{\pi}$ for y on $(-\infty,\infty)$ by C28. Set $x = \xi + \left\{\lambda/(1+E)\right\}$, where $E = \exp\left\{-\delta^{-1}(y\sqrt{2} - \gamma)\right\}$.

J. For the x,y substitution, one finds $p(x)dx = e^{-y^2}dy/\sqrt{\pi}$:

C83. $q(u) = \exp(-\ln^2 u)/e^{\frac{1}{4}}\sqrt{\pi}$; $(0,\infty)$.

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $u = \exp(\nu+\frac{1}{2})$. (Cashwell)

J. For the latter transformation, one finds $q(u)du = e^{-\nu^2}d\nu/\sqrt{\pi}$.

C84. $p(x) = \frac{(x-\theta)^{-1}}{\sqrt{2\pi b}} \exp\left\{-\left[\ln(x-\theta)-\zeta\right]^2/2b\right\}$; (θ,∞), $b > 0$, θ,ζ arbitrary.

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $x = \theta + \exp\left\{\zeta + \nu\sqrt{2b}\right\}$.

J. For the given transformation, one has $p(x)dx = e^{-\nu^2}/\sqrt{\pi}$. (See log-normal, C53).

C85. $p(y_1,y_2) = \frac{1}{2\pi\,\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp -\left\{\left(\frac{y_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{y_1-\mu_1}{\sigma_1}\right)\left(\frac{y_2-\mu_2}{\sigma_2}\right) + \left(\frac{y_2-\mu_2}{\sigma_2}\right)^2\right\}\Big/2(1-\rho^2)$;

y_1, y_2 on $(-\infty,\infty)$, σ_1, $\sigma_2 > 0$, μ_1, μ_2 arbitrary, $-1 < \rho < 1$.

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for both ν_1, ν_2 on $(-\infty,\infty)$ by C28.
Set $y_1 = \mu_1 + \sqrt{2}\,\sigma_1\left(\sqrt{R}\,\nu_1 + \rho\nu_2\right)$, $y_2 = \mu_2 + \sqrt{2}\,\sigma_2\,\nu_2$, where $R \equiv 1 - \rho^2$.

J. Under the indicated transformation, one has $p(y_1, y_2)\, dy_1\, dy_2$

$$= \left(e^{-\nu_1^2}\, d\nu_1/\sqrt{\pi}\right)\left(e^{-\nu_2^2}\, d\nu_2/\sqrt{\pi}\right).$$

C85A. $p(y_1,\ldots, y_n) = C^{-1} \exp -\sum_{ij} y_i\, a_{ij}\, y_j$; y_i on $(-\infty,\infty)$, $A = [a_{ij}]$ positive definite.

R. Construct a matrix S such that $S^T AS = I$. (See note.) Sample $e^{-\nu^2}/\sqrt{\pi}$ n times for $z_1, \ldots, z_n$ by C28. Define y_i by the linear transformation

$$\begin{vmatrix} y_1 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{vmatrix} = S \begin{vmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{vmatrix} .$$

J. In (column!) vector notation, we have Y = SZ, and hence

$Z^T Z = Z^T IZ = Z^T S^T ASZ = Y^T AY$, i.e., $\sum z_i^2 = \sum y_i a_{ij} y_j$.

Therefore, under the transformation Y = SZ, with Jacobian |det S|, we see that $p(y_1 \ldots y_n)\, dy_1 \ldots dy_n = \prod_1^n \left(e^{-z_i^2} dz_i / \sqrt{\pi} \right)$. Note that $C = \pi^{n/2}$ |det S| necessarily.

Note. The matrix S may be obtained from the Gram-Schmidt process. Without going into its machinery, we remark here that it is a definite algorithm for constructing from any n linearly independent vectors an equivalent set which are orthonormal with respect to a given inner product. If we define in E^n the inner product $(X,Y) = X^T AY$ then the Gram-Schmidt algorithm, applied to the "1-spot" vectors δ_i, produces a set $w_1, \ldots, w_n$ which are orthonormal relative to (X,Y). Hence we may define S by $\left[w_1, \ldots, w_n \right] = \left[\delta_1, \ldots, \delta_n \right] S$.

For then $\delta_{ij} = (w_i, w_j) = \left(\sum_k \delta_k\, s_{ki}, \sum_\ell \delta_\ell\, s_{\ell j} \right)$

$= \sum_{k,\ell} s_{ki}\, (\delta_k, \delta_\ell)\, s_{\ell j} = \sum_{k,\ell} s_{ki}\, a_{k\ell}\, s_{\ell j}$, or, in matrix form,

$I = S^T AS$, as required. Note that S is simply the matrix whose columns are the column vectors w_j.

C86. $h(x) = g(x) + g(-x)$; $(0,\infty)$, $g(y)$ density on $(-\infty,\infty)$.

R. Sample density $g(y)$ for y on $(-\infty,\infty)$. Set $x = |y|$.

J. $h(x)$ is the symmetric sum density $\frac{1}{2}\left(g(x) + g(-x)\right)$ restricted to $(0,\infty)$, and doubled. See C70.

C87. $s(x) = (2\pi\sigma^2)^{-\frac{1}{2}} \cosh(\xi x/\sigma^2) \exp -\left\{(x^2+\xi^2)/2\sigma^2\right\}$; $(-\infty,\infty)$, ξ arbitrary, $\sigma > 0$.

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $x = \xi + \nu\sqrt{2\sigma^2}$. Change sign of x with probability 1/2.

J. One notes that $s(x) = \frac{1}{2}\left(g(x) + g(-x)\right)$ where $g(x) = e^{-(x-\xi)^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$ and uses C70.

C88. $h(x) = \sqrt{2/\pi\sigma^2} \cosh(\xi x/\sigma^2) \exp\left\{-(x^2+\xi^2)/2\sigma^2\right\}$; $(0,\infty)$

R. Sample $e^{-\nu^2}/\sqrt{\pi}$ for ν on $(-\infty,\infty)$ by C28. Set $x = |\xi+\nu\sqrt{2\sigma^2}|$

J. See C86, C87.

C89. $p(x) = a/(e^x+b+e^{-x})$; $(-\infty,\infty)$, $-2 < b < 2$,

$a = B/\left\{\pi/2 - \arctan(b/2B)\right\}$, $B = (1-b^2/4)^{\frac{1}{2}}$.

R. Set $x = \ln\left\{-\frac{b}{2} + B\left[\frac{(b/2B)+\tan(Br/a)}{1-(b/2B)\tan(Br/a)}\right]\right\}$

J. For $x = \ln y$, one finds $p(x)dx = a\,dy/\left[\left(y + \frac{b}{2}\right)^2 + B^2\right] \equiv q(y)dy$ on $(0,\infty)$, with $\int_0^y q(y)dy = \frac{a}{B}\left\{\arctan\frac{y+\frac{b}{2}}{B} - \arctan\frac{\left(\frac{b}{2}\right)}{B}\right\} = r$ yielding the rule.

C90. $p(x) = (2/\pi)/(e^x+e^{-x})$; $(-\infty,\infty)$

R1. Set $x = \ln\tan\left(\frac{\pi}{2}r\right)$.

R2. Generate r,r' until $S \equiv r^2 + r'^2 \leqslant 1$. For accepted r,r', set $x = \ln(r'/r)$.

J. Special case of C89 $(b = 0)$.

Note the equivalent form $p(x) = \frac{1}{\pi}\operatorname{sech} x$.

C91. $p(x) = 1/(e^x+2+e^{-x})$; $(-\infty,\infty)$.

R. Set $x = \ell n \left| r/(1-r) \right|$.

J. For $x = \ell n\, y$, one finds $p(x)dx = dy/(y+1)^2 = q(y)dy$ on $(0,\infty)$, with

$$\int_0^y q(y)dy = y/(y+1) = r \text{ yielding the rule.}$$

Note the equivalent forms $p(x) = 1\Big/\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2 = \frac{1}{4}\,\mathrm{sech}^2(x/2)$

$= e^x/(e^x+1)^2 = e^{-x}/(1+e^{-x})^2$.

C92. $p(x) = a/(e^x+b+e^{-x})$; $(-\infty,\infty)$, $b > 2$, $a = C/\ell n\left(\frac{b}{2} + C\right)$, where $C = \left(\frac{b^2}{4} - 1\right)^{\frac{1}{2}}$.

R. Define $s = \frac{b}{2} + C$, $d = \frac{b}{2} - C$. Set $x = \ell n \left|(E-1)/(s-dE)\right|$, where $E = \exp(2r\, \ell n s)$.

J. For $x = \ell n\, y$, one finds $p(x)dx = ady \Big/ \left[\left(y + \frac{b}{2}\right)^2 - C^2\right] \equiv q(y)dy$ on $(0,\infty)$,

with $\int_0^y q(y)dy = \frac{a}{2C}\, \ell n \frac{s(y+d)}{d(y+s)} = r$ yielding the rule. (Note that $sd = 1$).

C93. $q(y) = n/\left\{\lambda + \cosh \alpha\, (y-y_0)\right\}$; $(-\infty,\infty)$, $\lambda > -1$.

R. Define $b = 2\lambda > -2$. Sample density $a/(e^x + b + e^{-x})$ by C89, 90, 91, or 92. Set $y = y_0 + (x/\alpha)$.

J. For the given substitution, one obtains $q(y)dy = h(x)dx$ for an $h(x)$ in the cited references. Note that $n = \alpha\, a/2$.

C94. $r(t) = n/t\left\{\frac{1}{2}\left(\frac{t}{t_0}\right)^{\alpha} + \lambda + \frac{1}{2}\left(\frac{t}{t_0}\right)^{-\alpha}\right\}$; $(0,\infty)$, α, $t_0 > 0$, $\lambda > -1$.

R. Define $b = 2\lambda > -2$. Sample $a/(e^x + b + e^{-x})$ by C89, 90, 91, or 92. Set $t = t_0 e^{x/\alpha}$.

J. For the given substitution, one obtains $r(t)dt = h(x)dx$ for an $h(x)$ in the cited references. Note that $n = \alpha\, a/2$.

C95. $q(x) = m\, \beta^m\, e^x/(\beta+e^x)^{m+1}$; $(-\infty,\infty)$, β, $m > 0$.

R. Sample $m\, \beta^m\, v^{-m-1}$ for v on (β,∞) by C15 or C15A. Set $x = \ell n\, (v-\beta)$.

J. For $x = \ell n(v-\beta)$, one has $q(x)dx = m\ \beta^m\ v^{-m-1}\ dv = p(v)dv$ on (β,∞).

For arbitrary $m > 0$, one sets $\int_v^\infty p(v)dv = r$ to obtain $v = \beta\ r^{-1/m}$

$= \beta \exp\left\{-\frac{1}{m}\ \ell n\ r\right\}$. For $\beta = m = 1$, Cf. C91.

C96. $r(y) = m\ \beta^{-1}\ e^{-y}/(1+\beta^{-1}\ e^{-y})^{m+1}$; $(-\infty,\infty)$, β, $m > 0$.

R. Sample $m\ \beta^m\ v^{-m-1}$ for v on (β,∞) by C15 or C15A. Set $y = -\ \ell n(v-\beta)$.

J. With $y = -x$, one has $r(y)dy = m\ \beta^m\ e^{-y}dy/(\beta+e^{-y})^{m+1}$

$= m\ \beta^m\ e^x(-dx)/(\beta+e^x)^{m+1}$ as in C95.

C97. $q(x) = (x-a)^{m-1}\ (b-x)^{n-1}/(b-a)^{m+n-1}\ B(m,n)$; (a,b), $a < b$, $m,n > 0$.

R. Sample $B(v) = v^{m-1}(1-v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C35 or R19.

Set $x = a + (b-a)v$.

J. For the x,v substitution, one has $q(x)dx = B(v)dv$.

C98. $p(x) = \Gamma(Q)(b-a)^{Q-R-1}\ (x-b)^R/\Gamma(Q-R-1)\ \Gamma(R+1)(x-a)^Q$; (b,∞), $b > a$,

$Q > R + 1 > 0$.

R. Define $m = R + 1 > 0$, $n = Q-R-1 > 0$. Sample $b(z) = z^{m-1}/(1+z)^{m+n}\ B(m,n)$ for z on $(0,\infty)$ by C35 or R19. Set $x = b + (b-a)z$.

J. For the x,z substitution, one has $p(x)dx = b(z)dz$.

C99. $p(E) = 2(M/N)^{M/2}\ E^{M-1}/B(M/2,N/2)\left(1 + \frac{M}{N}\ E^2\right)^{\frac{M+N}{2}}$; $(0,\infty)$, M, $N\ \varepsilon\{1,2,3...\}$.

R. Define $m = M/2$, $n = N/2$. Sample $\nu_1^{n-1}\ e^{-\nu_1}/\Gamma(n)$, $\nu_2^{m-1}\ e^{-\nu_2}/\Gamma(m)$ for ν_1, ν_2 on $(0,\infty)$ by C22 and/or C32. Set $E = (N\nu_2/M\nu_1)^{\frac{1}{2}}$.

J. For $E = F^{\frac{1}{2}}$, one finds $p(E)dE = q(F)dF$ for Snedecor's F, as in C45.

C100. $p(x) = 1/B(m,m)(e^{x/2}+e^{-x/2})^{2m}$; $(-\infty,\infty)$, $m > 0$.

R. Sample $b(z) = z^{m-1}/B(m,m)(1+z)^{2m}$ for z on $(0,\infty)$ by C35 or R19. Set $x = \ell n\ z$.

J. For $x = \ell n\ z$, one has $p(x)dx = b(z)dz$ on $(0,\infty)$.

Note the equivalent form $\mathrm{sech}^{2m}(x/2)/4^m\ B(m,m)$.

$\underline{\text{C101.}\quad p(x) = \rho^m e^{-mx/\sigma}/\sigma B(m,n)(1+\rho e^{-x/\sigma})^{m+n}}$; $(-\infty,\infty)$, $\rho,\sigma > 0$, $m,n > 0$.

℞. Sample $b(z)$ for z on $(0,\infty)$ by C35 or R19. Set $x = -\sigma\, \ell n(z/\rho)$.

J. For the x,z substitution, one has $p(x)dx = z^{m-1}(-dz)/(1+z)^{m+n} B(m,n)$ on $(0,\infty)$.

Note 1. See C120 for the case $m = N - k + 1$, $n = k$, $\rho = \sigma = 1$.

Note 2. For $m = n = 1 = \rho = \sigma$, $p(x)$ reduces to C91.

$\underline{\text{C102.}\quad e(x) = \rho^m e^{-mx/\sigma}(1-\rho e^{-x/\sigma})^{n-1}/\sigma B(m,n)}$; $(\sigma\, \ell n\, \rho, \infty)$, $\rho,\sigma > 0$, $m,n > 0$.

℞. Sample $B(v) = v^{m-1}(1-v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C35 or R19. Set $x = -\sigma\, \ell n(v/\rho)$.

J. For the x,v substitution, one has $e(x)dx = B(v)(-dv)$ on $(0,1)$.

Note 1. See C123 for case $m = N - k + 1$, $n = k$, $\rho = \sigma = 1$.

Note 2. For $m = n = 1 = \rho = \sigma$, $e(x) = e^{-x}$ on $(0,\infty)$.

$\underline{\text{C103.}\quad p(x) = \Gamma(n+\tfrac{1}{2})\left(1 - \frac{x^2}{a^2}\right)^{n-1}/a\sqrt{\pi}\,\Gamma(n)}$; $(-a,a)$, $a > 0$, $n > 0$.

℞. Sample $v^{-\frac{1}{2}}(1-v)^{n-1}/B(\tfrac{1}{2},n)$ for v on $(0,1)$ by C35 or R19. Set $x = av^{\frac{1}{2}}$. Change sgn x with probability $\tfrac{1}{2}$.

J. By symmetry, we may sample $2\,p(x)$ for x on $(0,a)$, with provision for sign. (Cf. C128, Note). For $x = av^{\frac{1}{2}}$ we find $2\,p(x)dx$ $= v^{-\frac{1}{2}}(1-v)^{n-1}/B(\tfrac{1}{2},n) = B(v)$ on $(0,1)$.

$\underline{\text{C104.}\quad b(z) = (\sin m\pi/\pi)\, z^{m-1}/(1+z)}$; $(0,\infty)$, $0 < m < 1$.

℞. Sample $b(z) = z^{m-1}/B(m,1-m)(1+z)$ for z on $(0,\infty)$ by C35 or R19.

Note that $B(m,1-m) = \Gamma(m)\,\Gamma(1-m) = \pi/\sin m\pi$.

$\underline{\text{C105.}\quad p(x) = C^{-1}\, x(x-a)^{m-1}\,(b-x)^{n-1}}$; (a,b), $a < b$, $m,n > 0$.

$C = (b-a)^{m+n-1}\,\Gamma(m)\,\Gamma(n)(na+mb)/\Gamma(m+n+1)$.

℞. If $r_o \leqslant A_1 \equiv (m+n)a/na+mb$, sample $B(v) = v^{m-1}(1-v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C35 or R19. If $r_o > A_1$, sample $B(v) = v^m(1-v)^{1-n}/B(m+1,n)$ for v on $(0,1)$ by C35 or R19. Set $x = a + (b-a)v$.

J. For $x = a + (b-a)v$, one finds $p(x)dx = \left(a_1(v) + a_2(v)\right)dv$ where $a_1(v) = C^{-1}\,(b-a)^{m+n-1}\,av^{m-1}(1-v)^{n-1}$, $a_2(v) = C^{-1}\,(b-a)^{m+n}\,v^m(1-v)^{n-1}$ and $A_1 = \int_0^1 a_1(v)dv = (m+n)a/(na+mb)$. The rule then follows from C3.

C106. $p(x) = x^{m-1}\,(1-x)^{n-1}/(x+a)^{m+n}\,C$; $(0,1)$ $a > 0$, $m,n > 0$,

$C = B(m,n)/a^n(1+a)^m$.

R. Sample $b(z) = z^{m-1}/(1+z)^{m+n}\,B(m,n)$ for z on $(0,\infty)$ by C35 or R19. Set $x = az/(1+a+az)$.

J. Under the x,z substitution, one has $p(x)dx = b(z)dz$ on $(0,\infty)$.

Note. The x,z substitution is the iterate of $x = y/(1+y)$ and $y = az/(1+a)$.

C107. $p(x) = (a+x)^{m-1}\,(a-x)^{n-1}/C$; $(-a,a)$, $m,n > 0$, $C = (2a)^{m+n-1}B(m,n)$.

R. Sample $B(v)$ for v on $(0,1)$ by C35 or R19. Set $x = a(2v-1)$.

J. For, $p(x)dx = B(v)dv$.

C108. $p(x) = F(x) + x^{-2}F(x^{-1})$; $(0,1)$, $F(y)$ density on $(0,\infty)$.

R. Sample $F(y)$ for y on $(0,\infty)$. If $y \leqslant 1$, set $x = y$. If $y > 1$, set $x = 1/y$.

J. Under the rule, the probability of an x on $(x,x+dx) \subset (0,1)$ is $F(x)\,dx + F(y)(-dy)$ where $y = 1/x$. But $F(y)(-dy) = F(1/x)dx/x^2$.

Note. The rule is a disguised version of C3.

C109. $p(x) = (x^{m-1}+x^{n-1})/(1+x)^{m+n}\,B(m,n)$; $(0,1)$, $m,n > 0$.

R. Sample $b(z)$ for z on $(0,\infty)$ by C35 or R19. If $z \leqslant 1$, set $x = z$. If $z > 1$, set $x = 1/z$.

J. For $b(z) = z^{m-1}/(1+z)^{m+n}\,B(m,n)$, one has $x^{-2}b(x^{-1}) = x^{n-1}/(1+x)^{m+n}\,B(m,n)$. The rule then follows from C108.

C110. $p(\zeta) = \Gamma(m)\,c^{2m-1}/\sqrt{\pi}\,\Gamma(m-\frac{1}{2})\left[c^2 - (\zeta-\zeta_0)^2\right]^m$; $(-\infty,\infty)$, $c > 0$, ζ_0 arbitrary,

$m = 1, 3/2, 2, 5/2, \ldots$

℞. Define $N = 2m-1 = 1,2,3,\ldots$ Sample $w^{\frac{N}{2}-1} e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C22 or C32; set $x = (w/N)^{\frac{1}{2}}$. Sample $e^{-y^2}/\sqrt{\pi}$ for y on $(-\infty,\infty)$ by C28. Set $\zeta = \zeta_0 + \left(\frac{c}{\sqrt{N}} \cdot \frac{y}{x}\right)$.

J. With N as defined, and $\zeta = \zeta_0 + ct/\sqrt{N}$, one finds $p(\zeta)d\zeta = q_1(t)dt$ as in C43 (Student's t).

C111. $q(t) = 1/\pi\lambda\left[1 + \left(\frac{t-\theta}{\lambda}\right)^2\right]$; $(-\infty,\infty)$, $\lambda > 0$, θ arbitrary.

℞1. Set $t = \theta + \lambda \tan \frac{\pi}{2}(2r-1)$.

J1. From $Q(t) \equiv \int_{-\infty}^{t} q(t)dt = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{t-\theta}{\lambda} = r$, the rule follows (C1).

℞2. Generate r, r' until $S = x^2 + y^2 \leqslant 1$, where $x = r$, $y = 2r' - 1$. For accepted x,y, set $t = \theta + \lambda(y/x)$.

J2. (x,y) is chosen, uniformly in area, in quadrants I, IV of the unit circle. Hence $y/x = \tan\theta$ where θ is uniform on $(-\pi/2, \pi/2)$, as required in J1.

C112. $s(t) = \left[1 + \left(\frac{t^2+\theta^2}{\lambda^2}\right)\right] \Big/ \pi\lambda \left[1 + 2\left(\frac{t^2+\theta^2}{\lambda^2}\right) + \left(\frac{t^2-\theta^2}{\lambda^2}\right)^2\right]$; $(-\infty,\infty)$, $\lambda > 0$, θ arbitrary.

℞. Sample q(t) for t on $(-\infty,\infty)$ as in C111. Change sign of t with probability $\frac{1}{2}$.

J. One notes that $s(t) = \frac{1}{2}\left\{q(t) + q(-t)\right\}$ and uses C70.

C113. $h(t) = 2\left[1 + \left(\frac{t^2+\theta^2}{\lambda^2}\right)\right] \Big/ \pi\lambda \left[1 + 2\left(\frac{t^2+\theta^2}{\lambda^2}\right) + \left(\frac{t^2-\theta^2}{\lambda^2}\right)^2\right]$; $(0,\infty)$, $\lambda > 0$, θ arbitrary.

℞. Sample $q(t_1)$ for t_1 on $(-\infty,\infty)$ as in C111. Set $t = |t_1|$.

J. See C86.

C114. $p(x) = 1/2m\lambda B(m,n)\left\{1 + \left|\frac{x-\theta}{\lambda}\right|^{1/m}\right\}^{m+n}$; $(-\infty,\infty)$, $\lambda > 0$, θ arbitrary, $m,n > 0$.

℞. Sample $b(z) = z^{m-1}/(1+z)^{m+n} B(m,n)$ for z on $(0,\infty)$ by C35 or R19. Set $y = z^m$. Change sgn y with probability $\frac{1}{2}$. For final y, set $x = \theta + \lambda y$.

J. For $x = \theta + \lambda y$, one has $p(x)dx = dy/2m\ B(m,n)\left(1 + |y|^{1/m}\right)^{m+n}$ for y on $(-\infty,\infty)$. By symmetry, we may sample density $q(y) = 1/m\ B(m,n)\left(1+y^{1/m}\right)^{m+n}$ for y on $(0,\infty)$, changing sgn y with probability 1/2. But for $y = z^m$, one has $q(y)dy = z^{m-1}\ dz/(1+z)^{m+n}\ B(m,n) = b(z)dz$ on $(0,\infty)$.

C115. $q(x) = k\ C_k^N \left[1 - P(x)\right]^{N-k} P^{k-1}(x)\ p(x)$; (a,b), $p(x)$ density on (a,b), $P(x) = \int_a^x p(x)dx$, $k = 1,2,\ldots,N$.

℞. Sample $p(x)$ independently N times for $x_1,\ldots,x_N$. Order the x_i as $x_1' \leq x_2' \leq \cdots \leq x_N'$. Set $x = x_k'$.

J. The probability distribution function for x_k' is $Q(x) \equiv \Pr(x_k' \leq x) = C_k^N P^k(x)\left[1 - P(x)\right]^{N-k} + C_{k+1}^N P^{k+1}(x)\left[1 - P(x)\right]^{N-k-1} + \cdots + C_{N-1}^N P^{N-1}(x)\left[1 - P(x)\right] + P^N(x)$. Hence the density function for x_k' is $q(x) = \frac{d}{dx} Q(x) = C_k^N\ kP^{k-1}(x)\ p(x)\left[1 - P(x)\right]^{N-k}$.

Note. The rule is feasible for moderate N, and may compare favorably with more direct methods when available.

C116. $q(x) = N\left[1 - P(x)\right]^{N-1} p(x)$, $N\ P^{N-1}(x)\ p(x)$, or $\left((2M+1)!/(M!)^2\right)\left[\left(1-P(x)\right)P(x)\right]^M p(x)$; (a,b), $p(x)$ density on (a,b), $P(x) \equiv \int_a^x p(x)dx$, $M,N \in \{1,2,3,\ldots\}$.

℞. Sample $p(x)$ for $x_1,\ldots,x_N$; set $x = \min\{x_i\}$, $x = \max\{x_i\}$ for first two densities. In last case ($N = 2M+1$ odd), order the x_i as $x_1' \leq \cdots \leq x_{2M+1}'$. Set $x = x_{M+1}$.

J. Cases $k = 1$, $k = N$, and $k = M+1$ for $N = 2M+1$ in C115.

$\underline{\text{C117.}\quad q(x) = k\, C_k^N\, (x-a)^{k-1}(b-x)^{N-k}/(b-a)^N;}$ (a,b), $k = 1,\ldots,N$.

R1. Generate $r_1,\ldots,r_N$. Order as $r_1' \leq \cdots \leq r_N'$. Set $x = a + (b-a)r_k'$.

J1. For the uniform density $p(x) = 1/(b-a)$ on (a,b), one has

$$P(x) = \int_a^x p(x)dx = (x-a)/(b-a),\ 1 - P(x) = (b-x)/(b-a).$$ The rule

follows from C115.

R2. Define $m = k$, $n = N-k+1$; sample $(x-a)^{m-1}\,(b-x)^{n-1}/(b-a)^{m+n-1}\,B(m,n)$ for x on (a,b) as in C97.

$\underline{\text{C118.}\quad q(x) = k\, C_k^N\, x^{k-1}\, (1-x)^{N-k};}$ $(0,1)$, $k = 1,\ldots,N$.

R1. Generate $r_1,\ldots,r_N$. Order as $r_1' \leq \cdots \leq r_N'$. Set $x = r_k'$.

J1. Special case of C117.

R2. Define $m = k$, $n = N-k+1$. Sample $B(x) = x^{m-1}(1-x)^{n-1}/B(m,n)$ as in C35.

J2. For m,n as defined, note that $q(x) = B(x)$.

Note 1. The method of R1. provides a useful test for "random number" generators.

Note 2. For $k = N$, the rule R1 samples $q(x) = N\,x^{N-1}$ for x on $(0,1)$ by setting $x = \max\{r_1,\ldots,r_N\}$. The direct method (C1) would set $x = r^{1/N}$. Cf. C13, 13A.

$\underline{\text{C119.}\quad q(x) = k\, C_k^N \left|1 - \exp(-e^{-x})\right|^{N-k} \exp(-ke^{-x})\; e^{-x};}$ $(-\infty,\infty)$, $k = 1,\ldots,N$.

R. Generate $r_1,\ldots,r_N$. Order as $r_1' \leq \cdots \leq r_N'$. Set $x = -\ell n\,(-\ell n\, r_k')$

J. For the $p(x)$ in C76, one has $P(x) = \int_{-\infty}^x p(x)dx = \exp(-e^{-x})$. The rule follows from C115 and C76.

Note. For $k = N$, $q(x)dx = N\exp(-Ne^{-x})\; e^{-x}dx = N\, e^{-Ny}(-dy) = e^{-z}(-dz)$, and one may set $x = -\ell n \left\{\frac{1}{N}(-\ell n\; r)\right\}$. Cf. C17.

C120. $q(x) = k\, C_k^N\, e^{-(N-k+1)x}/(1+e^{-x})^{N+1}$; $(-\infty,\infty)$, $k = 1,\ldots,N$.

R1. Generate $r_1,\ldots, r_N$. Order as $r_1' \leqslant \cdots \leqslant r_N'$. Set $x = \ln [r_k'/(1-r_k')]$.

J1. For the p(x) of C91, one has $P(x) = \int_{-\infty}^{x} p(x)\,dx = 1/(1+e^{-x})$. The rule follows from C115 and C91.

R2. Define $m = N-k+1$, $n = k$, $\rho = \sigma = 1$. Sample $e^{-mx}/B(m,n)(1+e^{-x})^{m+n}$ for x as in C101.

C121. $\tilde{q}(t) = k\, C_k^N \left[\frac{1}{2} - \frac{1}{\pi} \arctan\left(\frac{t-\theta}{\lambda}\right)\right]^{N-k} \left[\frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{t-\theta}{\lambda}\right)\right]^{k-1} \left\{\pi\lambda\left[1 + \left(\frac{t-\theta}{\lambda}\right)^2\right]\right\}^{-1}$;

$(-\infty,\infty)$, $\lambda > 0$, θ arbitrary, $k = 1,\ldots,N$.

R. Sample q(t) as in C111 for $t_1,\ldots,t_N$. Order as $t_1' \leqslant \cdots \leqslant t_N'$. Set $t = t_k'$.

J. The rule is clear from C111 and C115.

C122. $q(x) = k\, C_k^N\, b\, x^{b-1} (1-e^{-x^b})^{k-1}\, e^{-(N-k+1)x^b}$; $(0,\infty)$, $b > 0$.

R. Generate $r_1,\ldots,r_N$. Order as $r_1' \geqslant r_2' \geqslant \cdots \geqslant r_N'$ (Sic!)

Set $x = \exp\left\{b^{-1} \ln(-\ln r_k')\right\}$.

J. For the Weibull density $p(x) = b\, x^{b-1}\, e^{-x^b}$ on $(0,\infty)$ (C52) one has

$P(x) = \int_0^x p(x)\,dx = 1 - e^{-x^b}$. The rule follows from C115.

Note that $x = x(r)$ is decreasing.

C123. $q(x) = k\, C_k^N\, e^{-(N-k+1)x} (1-e^{-x})^{k-1}$; $(0,\infty)$, $k = 1,\ldots,N$.

R1. Generate $r_1,\ldots,r_N$. Order as $r_1' \geqslant r_2' \geqslant \cdots \geqslant r_N'$. Set $x = -\ln r_k'$.

J1. For $p(x) = e^{-x}$, one has $P(x) = \int_0^x p(x)\,dx = 1 - e^{-x}$, and the rule follows from C115. Note $x = x(r)$ is decreasing.

R2. Define $m = N-k+1$, $n = k$, $\rho = \sigma = 1$. Sample $e(x)$ for x on $(0,\infty)$ as in C102.

C124. $q(x) = k\, C_k^N \left[1 - \Gamma_x(n)/\Gamma(n)\right]^{N-k} \left[\Gamma_x(n)/\Gamma(n)\right]^{k-1} x^{n-1}\, e^{-x}/\Gamma(n)$;

$(0,\infty)$, $n > 0$, $k = 1,\ldots,N$, $\Gamma_x(n) = \int_0^x x^{n-1}\, e^{-x}\, dx$.

R. Sample $p(x) = x^{n-1}\, e^{-x}/\Gamma(n)$ N times for $x_1,\ldots,x_N$ by either C22, C32, or R18. Order as $x_1' \leq \cdots \leq x_N'$. Set $x = x_k'$.

J. Since $p(x)$ has distribution $P(x) = \int_0^x \frac{1}{\Gamma(n)} x^{n-1}\, e^{-x}\, dx = \Gamma_x(n)$, the rule follows from C115.

C125. $q(x) = k\, C_k^N \left(\frac{m}{x}\right)\left(\frac{\beta}{x}\right)^{m(N-k+1)} \left(1 - \left(\frac{\beta}{x}\right)^m\right)^{k-1}$; (β,∞), $m,\beta > 0$.

R. Sample the density $p(x) = m\beta^m\, x^{-m-1}$ for $x_1,\ldots,x_N$ on (β,∞) by C15 or C15A. Order as $x_1' \leq \cdots \leq x_N'$. Set $x = x_k'$.

J. Since $P(x) = \int_\beta^x p(x)dx = 1 - \left(\frac{\beta}{x}\right)^m$, the rule follows from C115.

C126. $t(x) = \begin{cases} 4(x-a)/(c-a)^2; & (a,b) \\ 4(c-x)/(c-a)^2; & (b,c) \end{cases}$; $a < c$, $b = (a+c)/2$

R. Set $x = a + \frac{1}{2}(c-a)(r_1+r_2)$.

J. Under the transformation $x = a + \frac{1}{2}(c-a)u$, one has $t(x)dx = s(u)du$ as in C55.

C127. $t(x) = 1 - |x|$; $(-1,1)$

R. Set $x = r_1 - r_2$.

J. Special case of C126 with $a = -1$, $b = 0$, $c = 1$.

C128. $q(x) = \begin{cases} a_1(x); & (a,b) \\ a_2(x); & (b,c) \end{cases}$

R. Define $A_1 = \int_a^b a_1(x)dx$, $A_2 = \int_b^c a_2(x)dx$, $A_1 + A_2 = 1$. If $r_0 \leqslant A_1$ sample density $a_1(x)/A_1$ for x on (a,b). If $r_0 > A_1$, sample $a_2(x)/A_2$ for x on (b,c).

J. Special case of C3.

Note. For a symmetric density $q(x) = q(-x)$ on (-c,c), one samples the density $2q(x)$ for x on (0,c) and changes sgn x with probability 1/2. This obvious rule may be regarded as a special case.

C129. $t(x) = \begin{cases} h(x-a)/(b-a); & (a,b) \\ h(c-x)/(c-b); & (b,c),\ a < b < c,\ h = 2/(c-a) \end{cases}$

R. Define $A_1 = (b-a)/(c-a)$. For $r_0 \leqslant A_1$, set $x = a + (b-a) \cdot \max\{r_1, r_2\}$. For $r_0 > A_1$, set $x = c - (c-b) \max\{r_1, r_2\}$.

J. The rule is a consequence of C128 and C13 (b = 1, m = 2) Cf. R15.

C130. $q(x) = \begin{cases} A\ ae^{-ax} & ; [0,\infty) \\ B\ be^{bx} & ; (-\infty,0],\ A,B,a,\ b > 0,\ A + B = 1. \end{cases}$

R. For $r_0 \leqslant A$, set $x = -a^{-1} \ln r_1$. If $r_0 > A_1$, set $x = b^{-1} \ln r_1$.

J. Special case of C128.

C131. $q(x) = a_i(x); [x_i, x_{i+1}),\ i = 0,1,2,\ldots$

R. Define $A_i = \int_{x_i}^{x_{i+1}} a_i(x)$, where $\sum_0^\infty A_i = 1$. Set $K = \min\left\{k; \sum_0^k A_i \geqslant r\right\}$.

Sample density $a_K(x)/A_K$ for x on $[x_K, x_{K+1})$.

J. Modification of C3.

C132. $q(x) = \begin{cases} a_0(x) = px/a^2; \ (0,a) \\ a_i(x) = pq^{i-1}\{(1+ip)a-px\}/a^2; \ \big(ia,(i+1)a\big), \ i = 1,2,\ldots \end{cases}$

$a,p,q > 0, \ p + q = 1.$

R. Define $A_0 = p/2$. If $r_0 \leqslant A_0$ set $x = a\sqrt{r_1}$. (Cf. C13). If $r_0 > A_0$, set $K = \min\left\{k; \ 1 + q + \cdots + q^{k-1} \geqslant 2r_0/p(1+q)\right\}$, $K \geqslant 1$.
Set $x = p^{-1}a\left\{1 + Kp - \left[1 - (1+q)pr_1\right]^{\frac{1}{2}}\right\}$.

J. Following C131, one finds $A_0 = p/2$, $A_i = pq^{i-1}(1+q)/2$, $i \geqslant 1$,
$a_0(x)/A_0 = 2x/a^2$, $a_i(x)/A_i = 2\left\{(1+ip)a-px\right\}/a^2(1+q)$.

C133. $p(t) = \left(\dfrac{\lambda}{2\pi t^3}\right)^{\frac{1}{2}} e^{-\lambda(t-\mu)^2/2\mu^2 t}$; $(0,\infty)$, $\lambda,\mu > 0$.

R. Define $\phi = \lambda/2\mu$, and sample $q(x)$ for x as in R16. Set $t = \mu x$.

J. For $t = \mu x$, $\phi = \lambda/2\mu$, one has $p(t)dt = q(x)dx$ in R16.

Note the Brownian motion form $p(t) = \dfrac{d\ e^{-(d-vt)^2/2\beta t}}{\sqrt{2\pi\beta t^3}}$

C134. $f(x,y) = C^{-1}\, x^{m-1}\, y^{n-1}\, F(x+y)$; $x,y > 0$, $x+y \leqslant a \leqslant \infty$, a fixed, $m,n > 0$,

$C = A\cdot B(m,n)$, $A \equiv \int_0^a u^{m+n-1}\, F(u)du$.

R. Sample density $A^{-1}\, u^{m+n-1}\, F(u)$ for u on $(0,a)$. Sample $v^{m-1}(1-v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C35 or R19. Set $x = uv$, $y = u(1-v)$,

J. Since $|\partial(x,y)/\partial(u,v)| = u$, one has $f(x,y)dx\, dy =$
$A^{-1}\, u^{m+n-1}\, F(u)du \cdot B^{-1}(m,n)\, v^{m-1}(1-v)^{n-1}dv$ on $(0,a) \times (0,1)$.

Note. For $F(u) = e^{-u}$, $a = \infty$, we obtain the relation

$$\Gamma^{-1}(m)x^{m-1}e^{-x}dx\ \Gamma^{-1}(n)y^{n-1}e^{-y}dy = \Gamma^{-1}(m+n)u^{m+n-1}e^{-u}du B^{-1}(m,n)v^{m-1}(1-v)^{n-1}dv$$

which gives a simple basis for C35, R18, R19.

C135. $f(x,y) = x^{m-1} y^{n-1}/(1-x-y)^n C$; $x,y > 0$, $x + y < 1$, $m > 0$, $0 < n < 1$,

$C = \Gamma(n)\,\Gamma(1-n)/m = \pi/m \sin n\pi$.

R. Sample $u^{m+n-1}(1-u)^{(1-n)-1}/B(m+n,1-n)$ and $v^{m-1}(1-v)^{n-1}/B(m,n)$ for u,v on (0,1) by C35 or R19. Set $x = uv$, $y = u(1-v)$.

J. Case $a = 1$, $F(u) = (1-u)^{-n}$ of C134.

C136. $q(y) = \left\{(1+ay)e^{-ay} - (1+by)e^{-by}\right\}/y^2(b-a)$; $(0,\infty)$, $0 < a < b$.

R. Set $y = (-\ln r_2)/\left\{a + (b-a)r_1\right\}$.

J. For $p(x) = 1/(b-a)$ on (a,b) and $f_x(y) = xe^{-xy}$ on $(a,b) \times (0,\infty)$, one has $\int_a^b p(x)\, f_x(y)dx = q(y)$. See C63A.

C137. $q(y) = \left(e^{-ay^2} - e^{-by^2}\right)/y^2 C$; $(0,\infty)$, $0 < a < b$, $C = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right)$.

R. Sample $z^{-\frac{1}{2}} e^{-z}/\sqrt{\pi}$ for z on $(0,\infty)$ by C32. Set $y = \sqrt{z}/\left\{\sqrt{a} + \left(\sqrt{b} - \sqrt{a}\right)r\right\}$.

J. For $f(x,y) = C^{-1}e^{-xy^2}$ on $(a,b) \times (0,\infty)$, one has for marginal densities the q(y) above, and $p(x) = x^{-\frac{1}{2}}/2\left(\sqrt{b} - \sqrt{a}\right)$. The latter is sampled by setting $x = \left\{\sqrt{a} + \left(\sqrt{b} - \sqrt{a}\right)r\right\}^2$ (C13A). For this x, one has $f(x,y)dy/p(x) = 2x^{\frac{1}{2}} e^{-xy^2}dx/\sqrt{\pi} = z^{-\frac{1}{2}} e^{-z}dz/\sqrt{\pi}$ on $(0,\infty)$, for $y = \sqrt{z}/\sqrt{x}$ (C63).

C138. $q(y) = e^{-y}\int_0^y x^{n-1}dx/(y-x)^n C$; $(0,\infty)$, $0 < n < 1$, $C = \Gamma(n)\Gamma(1-n) = \pi/\sin n\pi$.

R. Sample $x^{n-1} e^{-x}/\Gamma(n)$ and $z^{-n} e^{-z}/\Gamma(1-n)$ for x and z on $(0,\infty)$ by C32 or R18. Set $y = x+z$.

J. For the density $f(x,y) = x^{n-1}e^{-y}/(y-x)^n C$ on the region bounded by $x = 0$, $y = \infty$, $y = x$, one has for marginal and x - dependent densities: q(y) as given, $p(x) = x^{n-1} e^{-x}/\Gamma(n)$, and $f(x,y)dy/p(x) = (y-x)^{-n} e^{-(y-x)}dy/\Gamma(1-n) = z^{-n} e^{-z}dz/\Gamma(1-n)$ for $y = x+z$. The rule follows from C69.

C139. $q(y) = \int_y^b dx\ f(x,y)$; (a,b), $f(x,y) \geq 0$ on region R bounded by $x = b$, $y = a$, $y = x$.

R. Define $p(x) = \int_a^x f(x,y)\,dy$ for each x on (a,b). Sample p(x) for x on (a,b). For this x, sample f(x,y)/p(x) for y on (a,x).

J. Obvious variant of C69.

C140. $q(y) = \int_y^b dx\ t(x)/t_1$; (0,b), t(x) density on (0,b), first moment $t_1 = \int_0^b x\ t(x)\,dx$.

R. Sample $p(x) = x\ t(x)/t_1$ for x on (0,b). Set $y = r_0 x$.

J. Corollary of C139.

Note: To sample the "tail-end" density q(y) of t(x), it suffices to be able to sample its "first moment" density p(x).

C141. $q(y) = (B/n)\ e^{-By} \sum_0^{n-1} (By)^i/i!$; $(0,\infty)$, $B > 0$, $n = 1,2,...$

R. Set $y = -(r_0/B)\ \ln \prod_1^{n+1} r_i$.

J. Application of C140 to exponential density $t(x) = B^n\ x^{n-1}\ e^{-Bx}/(n-1)!$, $(0,\infty)$, with first moment $t_1 = n/B$, first moment density given by $p(x)dx = x\ t(x)dx/t_1 = B^{n+1}\ x^n\ e^{-Bx}dx/n! = u^n\ e^{-u}du/n!$ for $x = u/B$. The rule follows from C140 and C22. The tail-end density of t(x) is $\int_y^\infty dx\ t(x)/t_1 = \int_y^\infty dx\ B^{n+1}\ x^{n-1}\ e^{-Bx}/n! = q(y)$ of C141, as may be seen from the formula F3: $\int_0^y x^{n-1}\ e^{-Bx}\ dx = (n-1)!\ B^{-n} \left\{1 - e^{-By} \sum_0^{n-1} (By)^i/i!\right\}$

C141A. $q(y) = \int_y^\infty dx\ B^{n+1}\ x^{n-1}\ e^{-Bx}/\Gamma(n+1)$; $(0,\infty)$, $B,n > 0$.

R. Sample $u^n\ e^{-u}/\Gamma(n+1)$ for u on $(0,\infty)$ by C22, C32, or R18. Set $y = r_0\ u/B$.

J. See C141 (J) for the special case n = 1,2,3,...

C142. $q(y) = (m+1)(b^m - y^m)/mb^{m+1}$; $(0,b)$, $m,b > 0$.

R. Sample $(m+1)x^m/b^{m+1}$ for x on $(0,b)$ by C13 or C13A. Set $y = r_0 x$.

J. Application of C140 to density $t(x) = mx^{m-1}/b^m$, with $t_1 = mb/(m+1)$,

$q(y) = \int_y^b dx\ t(x)/t_1$, as above, $p(x) = x\ t(x)/t_1 = (m+1)x^m/b^{m+1}$.

Note. Direct sampling (C1) leads to the equation

$y^{m+1} - (m+1)b^m y + mb^{m+1} r = 0$.

C143. $q(y) = \lambda e^{-y}/(1-\lambda e^{-y})L(\lambda)$; $(0,\infty)$, $0 < \lambda < 1$, $L(\lambda) = -\ \ell n(1-\lambda)$.

R1. Set $y = -\ \ell n\ \lambda^{-1}\left\{1 - \exp\left[-r\ L(\lambda)\right]\right\}$.

J1. This results from $r = \int_y^\infty q(y)\,dy$ (C1).

R2. Sample $p(j) = \lambda^j/jL(\lambda)$ for j on $\{1,2,\ldots\}$ by D17. Set $y = -\ j^{-1}\ \ell n\ r_0$.

J2. One may realize q(y) as $\sum_{j=1}^{\infty} p(j)\ f_j(y)$ where $p(j) = \lambda^j/jL(\lambda)$ and

$f_j(y) = j\ e^{-jy}$.

R-INDEX

Rejection Techniques

R7.	$f(x)\ h(x)$	Density $\times$ bounded function
R14.	$\sum_1^J \alpha_j\ f_j(x)\ h_j(x)$	Sum of products, Butcher
R15.	$\begin{cases} 2(x-a)/(b-a)(c-a) \\ 2(b-x)/(b-a)(b-c) \end{cases}$	General triangular
R16.	$x^{-3/2}\exp\left\{-\phi(x-1)^2/x\right\}$	Wald (JK2/138)
R17.	$(1-R^2)^{\frac{T}{2}-\frac{1}{2}}/(1+\rho^2-2\rho R)^{\frac{T}{2}}$	Leipnik, circular correlation (JK3/240)
R18.	$x^{m-1}\ e^{-x}$, m real > 0	General Γ-type. (Jöhnk)(JK3/39)
R19.	$v^{m-1}\ (1-v)^{n-1}$ $z^{m-1}/(1+z)^{m+n}$ $\sin^{2m-1}\theta\cos^{2n-1}\theta$, m,n real > 0	General B-type. (Jöhnk)(JK3/39)
R20.	$z^{2m-1}\ e^{-z^2}$, m real > 0	General Gauss-type
R21.	$\sin^2 x/x^2$	Quasi-periodic

Rejection Techniques

Note: In all cases, the process is to be iterated until the condition is satisfied.

R7. $p(x) = A^{-1}\ f(x)\ h(x)$; (a,b), f(x) density on (a,b), $0 \leqslant h(x) \leqslant 1$.

R. Sample f(x) for x on (a,b). Accept x if next $r \leqslant h(x)$.

J. On any trial, the probability of accepting an x on (x,x+dx) is $f(x)dx \cdot h(x)$, the total probability of acceptance (efficiency) being $\int_a^b f(x)dx \cdot h(x) = A$. Hence the relative probability of an accepted x lying on (x,x+dx) is $f(x)dx \cdot h(x)/A = p(x)dx$.

Note. Analysis of the assignments to (x,x+dx) according to the required number of trials shows that the total probability of such an assignment is $f(x)dx \cdot h(x) \left\{1 + (1-A) + (1-A)^2 + \cdots\right\} = f(x)dx \cdot h(x)/A = p(x)dx$. The total probability of assignment on the ν-th trial is $(1-A)^{\nu-1}A$, with sum $A + (1-A)A + (1-A)^2A + \cdots = 1$. The expected number of trials for assignment is $\sum_1^\infty \nu(1-A)^{\nu-1}A = 1/A$, the inverse efficiency.

R14. $p(x) = \sum_{j=1}^{J} \alpha_j\, f_j(x)\, h_j(x)$; (a,b), $\alpha_j > 0$, $f_j(x)$ density on (a,b),

$0 \leqslant h_j(x) \leqslant 1$, all j.

R1. Define $A_j = \int_a^b \alpha_j\, f_j(x)\, h_j(x)dx$ · Fix $K = \min\left\{k; \sum_{j=1}^{k} A_j \geqslant r_0\right\}$. Then:

1. Sample density $f_K(x)$ for x on (a,b).

2. Accept x if next $r \leqslant h_K(x)$. Otherwise, return to (1).

J1. The rule follows from C3, and R7, which samples the K-th density $\alpha_K\, f_K(x)\, h_K(x)/A_K$ by rejection technique.

R2. Define $\sigma = \sum_1^J \alpha_j$. Then:

1. Generate next two random numbers r,r′.

2. Set $K = \min\left\{k; \sum_{j=1}^{k} \alpha_j \geqslant \sigma r\right\}$.

3. Sample density $f_K(x)$ for x on (a,b).

4. Accept x if $r' \leqslant h_K(x)$. Otherwise return to (1).

J2. On any trial, the probability of accepting an x on (x,x+dx) is $\sum_1^J (\alpha_j/\sigma)\, f_j(x)dx \cdot h_j(x)$, the total probability of acceptance being the integral, $1/\sigma$. Hence the relative probability of an accepted x lying on

$(x, x + dx)$ is $\sum_1^J \alpha_j f_j(x)dx\, h_j(x) = p(x)dx$. By the same analysis as in R7 (Note), one finds the expected number of trials for assignment to be σ. Here we may remark that

$$1 = \int_a^b \sum_1^J \alpha_j f_j(x)dx\, h(x) < \sum_1^J \alpha_j \int_a^b f_j(x)dx = \sigma$$

Note 1. In R1, the expected number of trials for assignment is α_j/A_j for the j-th term ($A_j = \int_a^b \alpha_j f_j(x) h_j(x)dx < \alpha_j$) so the average expected number of trials is $\sum_1^J A_j (\alpha_j/A_j) = \sigma$

Note 2. Any density $p(x) = \sum b_j(x) c_j(x)$, with $b_j(x) \geqslant 0$, $B_j = \int_a^b b_j(x)dx$, $0 \leqslant c_j(x) \leqslant c_j$, may be written in the form $p(x) = \sum (B_j c_j)\{b_j(x)/B_j\}\{c_j(x)/c_j\}$ of R14.

R15. $t(x) = \begin{cases} 2(x-a)/(b-a)(c-a); & (a,b) \\ 2(c-x)/(c-b)(c-a); & (b,c),\ a < b < c. \end{cases}$

R. Accept $x = a + (c-a)r_1$ if (1) $x \leqslant b$ and $r_2 < (x-a)/(b-a)$, or if (2) $x > b$ and $r_2 < (c-x)/(c-b)$.

J. The rule may be regarded as a special case of R7 if we write $t(x) = 2\left(\frac{1}{c-a}\right) \cdot \left(t(x) \cdot \frac{c-a}{2}\right)$. Cf. C129.

R16. $q(x) = \sqrt{\frac{\phi}{\pi}}\, x^{-3/2} e^{-\phi(x-1)^2/x}$; $(0,\infty)$, $\phi > 0$.

R. Sample $e^{-z^2}/\sqrt{\pi}$ for z on $(-\infty,\infty)$ by C28. Accept z if next $r < \frac{1}{2}\left(1 - \frac{z}{\sqrt{z^2+4\phi}}\right)$. For accepted z, set $x = \left\{(z^2+4\phi)^{\frac{1}{2}} + z\right\} / \left\{(z^2+4\phi)^{\frac{1}{2}} - z\right\}$

J. For $z = \sqrt{\phi}\,(x-1)/\sqrt{x}$ (increasing), one has $q(x)dx = \frac{2}{\sqrt{\pi}} (x+1)^{-1} e^{-z^2} dz$ on $(-\infty,\infty)$. But from $z^2 = \phi(x-1)^2/x$ and $z^2 + 4\phi = \phi(x+1)^2/x$ follows $(x+1)^{-1} \equiv \frac{1}{2}\left(1 - \frac{x-1}{x+1}\right) = \frac{1}{2}\left(1 - \frac{z}{\sqrt{z^2+4\phi}}\right)$, since sgn z = sgn (x-1).

Hence, $q(x)dx = A^{-1} f(z)\, h(z)dz$ on $(-\infty,\infty)$ where $A^{-1} = 2$, $f(z) = e^{-z^2}/\sqrt{\pi}$ density on $(-\infty,\infty)$, and $h(z) = \frac{1}{2}\left(1 - \frac{z}{\sqrt{z^2+4\phi}}\right)$, $0 < h(z) \leqslant 1$, $-\infty < z < \infty$.

The rule follows from R7 (Efficiency 1/2).

Note 1. Naturally the acceptance condition should be coded to avoid square roots and unnecessary computations. We omit the details.

Note 2. For testing purposes, we include the following evaluation of the Wald distribution function $Q(x) = \int_0^x q(\xi)d\xi$ in terms of the well-tabulated normal function $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\zeta^2/2} d\zeta$; $(-\infty,\infty)$. For convenience, we work with $G(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^z e^{-\zeta^2} d\zeta = \Phi(\sqrt{2}\, z)$. From (J) above, with $z = \sqrt{\phi}\,(x-1)/\sqrt{x}$, we have $Q(x) = \int_{-\infty}^z \left(1 - \frac{\zeta}{\sqrt{\zeta^2+4\phi}}\right) e^{-\zeta^2} d\zeta/\sqrt{\pi} = G(z) - I(z)$, where $I(z) = \int_{-\infty}^z \frac{\zeta}{\sqrt{\zeta^2+4\phi}}\, e^{-\zeta^2} d\zeta/\sqrt{\pi}$. Since the latter integrand is odd, we know $I(z) = I(-z)$. For fixed z on $(-\infty,0)$, let $\eta = -\sqrt{\zeta^2+4\phi}$ on $-\infty < \eta < -\sqrt{z^2+4\phi}$. Then $I(z) = -e^{4\phi}\, G\left(-(z^2+4\phi)^{\frac{1}{2}}\right) = I(-z)$ and hence $Q(x) = G(z) + e^{4\phi}\, G\left(-(z^2+4\phi)^{\frac{1}{2}}\right)$ for <u>all</u> x on $(0,\infty)$. In terms of x, therefore, $Q(x) = G\left(\sqrt{\frac{\phi}{x}}\,(x-1)\right) + e^{4\phi}\, G\left(-\sqrt{\frac{\phi}{x}}\,(x+1)\right) = \Phi\left(\sqrt{\frac{2\phi}{x}}\,(x-1)\right) + e^{4\phi}\, \Phi\left(-\sqrt{\frac{2\phi}{x}}\,(x+1)\right)$ (Cf. JK2/141).

R17. <u>$q(R) = (1-R^2)^{\frac{T}{2} - \frac{1}{2}}/(1+\rho^2-2\rho R)^{\frac{T}{2}}\, B\left(\frac{1}{2}, \frac{T}{2} + \frac{1}{2}\right)$</u>; $(-1,1)$, $0 < \rho < 1$, $T = 1,2,3,\ldots$

R1. Define $n = (T+1)/2$. Then:

1. Sample $B(v) = v^{n-1}(1-v)^{n-1}/B(n,n)$ for v on (0,1) by C35.
2. Accept v if next $r < (1-\rho)^T \Big/ \left[(1+\rho)^2 - 4\rho v\right]^{T/2} \equiv h(v)$. Otherwise return to (1).

3. For accepted v, set $R = 2v-1$. Efficiency $A = (1-\rho)^T$.

J1. For $R = 2v-1$, one finds $q(R)dR = A^{-1} B(v)\, h(v)$, $0 < h(v) \leqslant 1$, $0 < v < 1$, for the above A, B(v), h(v). The rule follows from R7. Note the identity $2^T\, \Gamma\left(\frac{T}{2} + \frac{1}{2}\right) \Gamma\left(\frac{T}{2} + 1\right) = \Gamma\left(\frac{1}{2}\right)\Gamma(T+1)$, (F4).

℞2. Define $m = (T+1)/2$, $n = 1/2$. Then:

1. Sample $b(z) = z^{m-1}/(1+z)^{m+n}\, B(m,n)$ for z on $(0,\infty)$ by C35.

2. Accept z if next $r < (1+\rho)^T / \left[(1+\rho)^2 + (1-\rho)^2 z\right]^{T/2} \equiv h(z)$. Otherwise return to (1).

3. For accepted z, set $R = (z-1)/(z+1)$. Efficiency $A = \left[(1+\rho)/2\right]^T$.

J2. For $R = (z-1)/(z+1)$, one finds $q(R)dR = A^{-1} b(z)\, h(z)$, $0 < h(z) \leqslant 1$, $0 \leqslant z < \infty$, for the cited A, b(z), h(z). The rule follows from R7.

Note. The method is only feasible for small T. For $\rho \leqslant 1/3$, use ℞1; for $\rho > 1/3$, use ℞2. The efficiencies are then both minimal at $\rho = 1/3$, with $A = \left(\frac{2}{3}\right)^T$.

R18. $p(x) = x^{m-1} e^{-x}/\Gamma(m)$; $(0,\infty)$, arbitrary $m > 0$, $m \notin \{1/2, 1, 3/2, 2, \ldots\}$.

℞1. Let $m = H + R$, where $H \in \{0, 1/2, 1, 3/2, 2 \ldots\}$ and $0 < R < 1/2$. Define $n = 1/2 - R$. Set $s = \exp \frac{1}{m} \ln r_1$, $t = \exp \frac{1}{n} \ln r_2$, and iterate until $s + t < 1$. For accepted s,t, set $v = s/(s+t)$. Sample $u^{H-\frac{1}{2}} e^{-u}/\Gamma\left(H + \frac{1}{2}\right)$ for u on $(0,\infty)$ by C22 or C32. Set $x = uv$. (Jöhnk)

J1. The rule results from the following remarks:

A. Under the transformation $x = uv$, $y = u(1-v)$, with Jacobian $-u$, one finds that $\Gamma^{-1}(m)\, x^{m-1} e^{-x} dx \cdot \Gamma^{-1}(n)\, y^{n-1} e^{-y} dy =$
$\Gamma^{-1}(m+n)\, u^{m+n-1} e^{-u} du \cdot B^{-1}(m,n)\, v^{m-1}(1-v)^{n-1} dv$ on $(0,\infty) \times (0,1)$.

Hence one may sample the latter two densities and set $x = uv$. The first is possible since $m + n = H + \frac{1}{2} \in \{1/2, 1, 3/2, 2, \ldots\}$.

B. For the density $f(s,t) = ms^{m-1}\, nt^{n-1}$ on $(0,1) \times (0,1)$, we find the probability of acceptance

$$E = P(s+t<1) = \int_0^1 ms^{m-1}\, ds \int_0^{1-s} nt^{n-1} dt = \Gamma(m+1)\ \Gamma(n+1)/\Gamma(m+n+1)$$

and hence the conditional density function

$g(s,t) = E^{-1} f(s,t) = E^{-1}\ ms^{m-1}\ nt^{n-1}$ for $s,t > 0$, $s + t < 1$. Under $g(s,t)$, the density for the value of the function $v = s/(s+t)$ is found to be $q(v) \equiv v^{m-1}(1-v)^{n-1}/B(m,n)$, on $(0,1)$. Hence we may sample $g(s,t)$ for s,t by rejection technique, and set $v = s/(s+t)$ for accepted s,t, as in the rule. The density $q(v)$ may be verified from (i) or (ii):

(i) $$\frac{d}{dv}\int_{s/(s+t)<v} g(s,t)\, ds\ dt = E^{-1}\ \frac{d}{dv}\int_0^v ms^{m-1} ds \int_{s(1-v)/v}^{1-s} nt^{n-1} dt = q(v)$$

(ii) Under transformation $s = S$, $t = S(1-v)/v$, one finds $g(s,t)ds\ dt$ $= nS^{m+n-1}(1-v)^{n-1}\ dS\ dv/v^{n+1}\ B(m,n+1) \equiv h(S,v)dS\ dv$ for $0 < S < 1$, $S < v < 1$, with marginal density $\int_0^v h(S,v)dS = q(v)$. Hence the density of v under $g(s,t)$ is $q(v)$, where $v = s/(s+t)$.

R2. Define H, R, n as in R1. Sample $x^{R-1}\ e^{-x}/\Gamma(R)$ for x as in R1. (i.e., with $H = 0$, $m = R$). Sample $\xi^{H-1}\ e^{-\xi}/\Gamma(H)$ for ξ by C22 or C32. Take $\xi + x$ as final x.

J2. The density of $X = \xi + x$ under $\Gamma^{-1}(H)\ \xi^{H-1}\ e^{-\xi} \times \Gamma^{-1}(R)\ x^{R-1}\ e^{-x}$ is $\Gamma^{-1}(H+R)\ X^{H+R-1}\ e^{-X}$.

Note. The probability E of acceptance in R1 becomes small for large m, but is high for $0 < m = R < 1/2$. Thus R2 is indicated for large m.

R19. $B(v) = v^{m-1}(1-v)^{n-1}/B(m,n)$; $(0,1)$

$b(z) = z^{m-1}/(1+z)^{m+n}\ B(m,n)$; $(0,\infty)$

$q(\theta) = 2\sin^{2m-1}\theta\ \cos^{2n-1}\theta/B(m,n)$; $(0,\pi/2)$, arbitrary $m,n > 0$, not both in $\{1/2, 1, 3/2, 2, ...\}$.

R. Sample $x^{m-1}\ e^{-x}/\Gamma(m)$ for x on $(0,\infty)$ by C22, C32, or R18. Similarly, sample $y^{n-1}\ e^{-y}/\Gamma(n)$ for y (with suitable change of notation). Set $v = x/(x+y)$, $z = x/y$, $\theta = \arcsin\sqrt{v}$.

J. See C35 (J).

R20. $t(z) = 2\Gamma^{-1}(m)\ z^{2m-1}\ e^{-z^2}$; $(0,\infty)$, arbitrary $m > 0$, $m \notin \{1/2, 1, 3/2, 2, \ldots\}$.

℞. Sample p(x) for x on $(0,\infty)$ as in R18. Set $z = x^{1/2}$.

J. $t(z)dz = p(x)dx$

R21. $p(x) = \frac{1}{\pi} \sin^2 x/x^2$; $(-\infty,\infty)$

℞. Define $A_1 = \int_0^1 2p(x)dx$ ($\simeq .57$), $A_2 = 1 - A_1$. If $r_1 \leqslant A_1$, accept $x = r$ if $\sin^2 x/x^2 < r'$. If $r_1 > A_1$, accept $x = 1/r$ if $\sin^2 x < r'$. Change sign of accepted x with probability 1/2.

J. By symmetry, we may sample 2p(x) for x on $(0,\infty)$, with provision for change of sign. Regard 2p(x) as a composite of its values on (0,1) and on $(1,\infty)$. Following C128, we define $A_1 = \int_0^1 \frac{2}{\pi} \sin^2 x\ dx/x^2$, $A_2 = 1-A_1$, and sample the densities $a_1 = 2p(x)/A_1$, $a_2 = 2p(x)/A_2$ with probabilities A_1, A_2. For the first we have $a_1(x)dx = \frac{2}{\pi A_1}\left(\frac{\sin^2 x}{x^2}\right)\left(\frac{dx}{1}\right)$ with $\sin^2 x/x^2 \leqslant 1$. For the second, $a_2(x)dx = \frac{2}{\pi A_2}(\sin^2 x)(dx/x^2)$ with $\sin^2 x \leqslant 1$. Each of these is sampled by the rejection technique of R7. The efficiencies in the two cases are $\pi A_1/2 \simeq .89$, $\pi A_2/2 \simeq .67$.

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Note. References are indicated by initials B, EJ, JK 1, 2, 3 followed by a page number. JK 1, 2, 3 refer to volumes V.1, V.2, V.3, as listed.

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