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ON BOUNDS AND LIMIT THEOREMS FOR SECONDARY CREEP IN SYMMETRIC PRESSURE VESSELS

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Warren S. Edelstein*

Richard A. Valentin**

*Mathematics Department, Illinois Institute of Technology

**Components Technology Division, ANL

ARGONNE NATIONAL LABORATORY
U of C-AUA-USAEC

BASE TECHNOLOGY

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Summary

The field equations governing creep in spherical and incompressible cylindrical pressure vessels subject to a nondecreasing internal pressure are reduced to a single equation in the effective stress. Using this equation, bounds are obtained for the effective stress and the displacement at any point in the body at any time. Also, in the case where the pressure tends to a limit as $t \to \infty$, limit theorems are obtained which describe the long term behavior of the effective stress and the displacement.

1. Introduction

A priori bounds for creep problems are of great practical importance in engineering applications, since they allow rapid evaluation of preliminary designs and reduce the need for comprehensive and expensive computer solutions. Bounds for creep derived by authors such as Leckie, Martin, and Ponter [4], [6] on the basis of energy considerations are well known. These give estimates for the displacement on the surface of bodies of fairly general geometry subject to creep. While results of this type are quite general, their application to specific cases first requires the solution of an associated boundary value problem. A type of bound not having this limitation is due to Einarsson [2], [3] and does not seem to be as well known. Einarsson's bounds result from direct calculus considerations and may be calculated a priori in terms of known quantities without the need for solving any boundary value problems. While they are thus quite simple to apply, they are limited in their generality; the bounds of [2] and [3] apply only to spheres and cylinders under constant internal pressure.

Our work furnishes a new set of direct calculus bounds for a generalization of the problems considered by Einarsson; however, the formulation is more direct and proceeds using totally different arguments. We deal with the two problems of the hollow sphere and the infinite incompressible hollow cylinder subject to internal pressure p. However, in all of our results we relax the constraint that p be constant in time and allow it to be nondecreasing. An earlier version of our bounds for the case of spherical geometry and constant pressure appears in [1]. Like Einarsson, we include the effect of the elastic as well as the creep strains and reduce the field equations to a single integral equation. Our method of reduction leads directly to an equation for the effective stress, here denoted by σ, rather than to an associated quantity such as the y defined by Einarsson. Thus the bounds which result relate immediately to all of the physical quantities of the problem without the need for further transformation.

For comparison purposes we record the equation central to developing Einarsson's bounds. The cylinder equation, given by (51) of [2], and the sphere equation, (94) of [2], are given respectively by the cases $j = 2$ and $j = 3$ of the equation

$$\frac{\partial y}{\partial t}(x,t) = -1 + \left[1 + x^{-j+j/n}\left\{\frac{jc^j}{c^j - 1}\int_1^c \frac{y(\xi,t)}{\xi^{j+1}}\,d\xi - y(x,t)\right\}\right]^n , \quad (j = 2,3) , \tag{1.1}$$

where we have neglected the effects of strain-hardening included in (51) and (94) of [2]; i.e., we have set $m = 0$. For the cylinder case ($j = 2$), the quantity y is related to the radial stress σ_r through equations (37) and (55) of [2]. In (3.12) of the present paper we show the relation, again for the cylinder, between y and the effective stress. The derivation of (1.1) is given in [2]; however, the analysis leading to error bounds for numerical solutions in the case of secondary creep, which is of major interest to us, is contained in [3]. In section 2 we obtain equations for the sphere (2.28) and the cylinder (2.32). These are then unified in equation (2.36). This latter equation plays the role analogous to (1.1) and is the equation on which the analysis of the paper is based.

Our first results in section 3, which have no counterpart in [2] and [3], state that $\frac{\partial\sigma}{\partial r} < 0$ and $\frac{\partial}{\partial r}(r^j\sigma) > 0$ for all $t > 0$, where $j = 2$ for cylinders and $j = 3$ for spheres. These results, together with some others, are used to obtain bounds (3.8) for $\sigma(r,t)$. They, in turn, may be used to furnish upper and lower bounds for the radial displacement $u(r,t)$ at any point in the body, although only bounds for the displacement $u(b,t)$ at the outer surface [(3.9) and (3.10)] are stated. These bounds reduce to the exact solution at t = 0. However, as indicated in section 5, they lose accuracy as t increases. At the end of section 3, a comparison is made between (3.8), restricted to the special case of constant pressure, and the bounds on the effective stress implied by those of Einarsson for y. It is shown that the two sets of bounds, which were derived by completely different arguments, also give different results.

In section 4, Theorem 4.1, it is proven that if p(t) tends to a constant fast enough, then σ tends to a limit as $t \to \infty$. The main idea in our proof is due to Einarsson and was used by him in [3] to prove that solutions y of an equation (3) of [3], which is a generalization of (1.1), tend to a constant y_∞. This idea is essentially an application of linear methods to a nonlinear problem, in that it involves the introduction of an appropriately chosen inner product and norm, the estimation of the norms of certain quantities, and the application of these norms together with a Sobolev-type inequality to the derivation of a pointwise estimate which implies uniform convergence. We apply this method to the equation (4.23) below, which is a modification of (2.36) involving a power of the effective stress. For $\dot{p} \neq 0$, this equation is not contained in Einarsson's generalized equation (3) in that it does not have Einarsson's property C. However, even in the case $\dot{p} \equiv 0$ we feel that our convergence argument is worth stating in detail, since, by virtue of having specialized, we are able to introduce various simplifications. Equation (4.13) gives an explicit expression for the limit of the effective stress.

In section 5 we derive an asymptotic expression for u(b,t) as $t \to \infty$ and a new bound for u(b,t) which is suggested by the analysis of section 4 and which appears to be more appropriate for large times than the previously derived bound.

2. Derivation of Equations

Following Einarsson [2] and numerous other authors, we write the strain-stress relations for secondary creep with initial elastic response in the form[1]

$$\varepsilon_{ij} = \varepsilon_{ij}^{(e)} + \varepsilon_{ij}^{(c)} \tag{2.1}$$

where

$$\varepsilon_{ij}^{(e)} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right] \tag{2.2}$$

and

$$\frac{\partial\varepsilon_{ij}^{(c)}}{\partial t} = \frac{3}{2} K \sigma_e^{n-1} s_{ij} \quad . \tag{2.3}$$

Here ε_{ij}, $\varepsilon_{ij}^{(e)}$, and $\varepsilon_{ij}^{(c)}$ denote total strain, elastic strain, and creep strain respectively, while σ_{ij}, s_{ij}, and σ_e stand for stress, stress deviation, and the effective stress. The latter two quantities are defined in terms of the stress by the equations

$$s_{ij} = \sigma_{ij} - \frac{\delta_{ij}}{3}\sigma_{kk} \quad , \tag{2.4}$$

$$\sigma_e^2 = \frac{3}{2} s_{ij} s_{ij} \quad . \tag{2.5}$$

The quantity σ_e is taken to be the positive square root of σ_e^2 (c.f. Odqvist [5], page 20). We assume that Young's modulus E and Poisson's ratio ν satisfy the inequalities

$$E > 0 \quad , \qquad -1 < \nu \leq \frac{1}{2} \tag{2.6}$$

and that creep constants K,n are positive with $n \geq 1$. With the assumption that the creep strains $\varepsilon_{ij}^{(c)}$ are initially zero, the total strain-stress relations take the form

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[1] Subscripts have the range 1, 2, 3, δ_{ij} stands for the Kronecker delta, and summation over repeated indices is implied. We shall also use a superposed dot to denote differentiation with respect to time. Our symbol for Poisson's ratio is different from that used by Einarsson.

$$\varepsilon_{ij} = \frac{1}{E}\,[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}] + \frac{3}{2}\,K\int_0^t \sigma_e^{n-1}\, s_{ij}\,d\tau \quad . \tag{2.7}$$

We shall now sketch the derivation of the equation which governs the effective stress in a hollow spherical shell of inner radius a and outer radius b subject to a nondecreasing internal pressure $p(t) \geq 0$ and zero body force. In spherical coordinates r, θ, ϕ with the displacement components denoted by u_r, u_θ, u_ϕ, the assumption of a spherically symmetric deformation takes the form

$$u_r = u(r,t) \quad , \qquad u_\theta = u_\phi = 0 \quad . \tag{2.8}$$

Subject to (2.8), the infinitesimal strain-displacement relations (see e.g. Sokolnikoff [7], page 184) in spherical coordinates reduce to

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad , \qquad \varepsilon_\theta = \varepsilon_\phi = \frac{u}{r} \quad , \qquad \varepsilon_{r\theta} = \varepsilon_{r\phi} = \varepsilon_{\theta\phi} = 0 \quad . \tag{2.9}$$

It then follows immediately from (2.6), (2.7), and (2.9) that

$$\sigma_{r\theta} = \sigma_{r\phi} = \sigma_{\theta\phi} = 0 \quad , \qquad \sigma_\theta = \sigma_\phi \quad , \tag{2.10}$$

so that by (2.4) and (2.5),

$$\sigma_e = |\sigma_r - \sigma_\theta| \quad ,$$

$$s_\theta = -\frac{1}{2}\, s_r = \frac{1}{3}\,(\sigma_\theta - \sigma_r) \equiv \frac{1}{3}\,\sigma \quad . \tag{2.11}$$

It will later be proven that the quantity σ is, in fact, the effective stress. With the above simplifications, the strain-stress relations (2.7) for ε_r and ε_θ become

$$\varepsilon_r = \frac{1}{E}\,[\sigma_r - 2\nu\sigma_\theta] - \textstyle\sum \tag{2.12}$$

$$\varepsilon_\theta = \frac{1}{E}\,[(1-\nu)\sigma_\theta - \nu\sigma_r] + \frac{1}{2}\textstyle\sum \tag{2.13}$$

where

$$\textstyle\sum(r,t) \equiv K\displaystyle\int_0^t \sigma_e^{n-1}\,\sigma\, d\tau \quad . \tag{2.14}$$

We adjoin to (2.12) and (2.13) the quasistatic stress equation of motion

$$\frac{\partial}{\partial r}\sigma_r - \frac{2}{r}\sigma = 0 \tag{2.15}$$

(assuming zero body force) and the strain equation of compatibility

$$\varepsilon_r = \frac{\partial}{\partial r}(r\,\varepsilon_\theta) \tag{2.16}$$

to obtain four equations for the four unknowns ε_r, ε_θ, σ_r, and σ_θ. This system can be rapidly collapsed into a single equation for σ. In fact, if we substitute the expressions (2.12) and (2.13) for ε_r and ε_θ into (2.16) and utilize (2.11) and (2.15), we find that

$$\frac{\partial}{\partial r}(r^3\sigma) + \frac{E}{2(1-\nu)}\frac{\partial}{\partial r}(r^3\textstyle\sum) = 0 \quad ,$$

from which it immediately follows that

$$\sigma(r,t) + \frac{E}{2(1-\nu)}\textstyle\sum(r,t) = \frac{f(t)}{r^3} \quad . \tag{2.17}$$

Before dealing with the unknown function f(t), we first record the boundary conditions

$$\sigma_r(a,t) = -p \quad , \qquad \sigma_r(b,t) = 0 \quad . \tag{2.18}$$

Together with (2.15), they imply that

$$\int_a^b \sigma(\xi,t)\,\frac{d\xi}{\xi} = \frac{p}{2} \quad . \tag{2.19}$$

This result suggests that the natural way to eliminate f(t) is to multiply both sides of equation (2.17) by r^{-1}, integrate with respect to the space variable, and then apply (2.19). Substituting the resulting expression for f(t) back into (2.17) and making use of (2.14), we obtain

$$\sigma(r,t) = \frac{\beta_s p(t)}{2r^3} + \mu_s\left(\frac{\beta_s}{r^3}\int_a^b\int_0^t \sigma_e^{n-1}\sigma\,d\tau\,\frac{d\xi}{\xi} - \int_0^t \sigma_e^{n-1}\sigma\,d\tau\right) \quad . \tag{2.20}$$

Here

$$\mu_s \equiv \frac{EK}{2(1-\nu)} \quad , \tag{2.21}$$

$$\beta_s^{-1} \equiv \int_a^b \frac{d\xi}{\xi^4} \quad . \tag{2.22}$$

We assume from now on the existence of a differentiable solution σ of (2.20). It is physically plausible that $\sigma \geq 0$, so that $\sigma = \sigma_e$. Using (2.20), we can prove this conjecture easily, provided we make the basic assumption that

$$p(0) > 0 \quad , \quad \dot{p}(t) \geq 0 \quad (t \geq 0) \quad . \tag{2.23}$$

Our proof depends on the following elementary fact which we record as a lemma for future reference.

Lemma 2.1. *Let Q and G be continuous functions of t on [0,t] (T > 0), and suppose that F is* C^1 *and satisfies the equation*

$$\dot{F} + QF = G \tag{2.24}$$

on [0,T]. Suppose $G \geq 0$ *(resp.* $G \leq 0$*) on [0,T] and* $F(0) \geq 0$ *(resp.* $F(0) \leq 0$*). Then* $F \geq 0$ *(resp.* $F \leq 0$*) for all t in [0,T]. Furthermore, if, in addition, either* $F(0) > 0$ *or* $G > 0$ *on (0,T] (resp.* $F(0) < 0$ *or* $G < 0$ *on [0,T]), then* $F > 0$ *on (0,T] (resp.* $F < 0$ *on (0,T]).*

The lemma follows from the representation

$$F(t)\exp\left[\int_0^t Q(\tau)d\tau\right] = F(0) + \int_0^t G(\tau)\exp\left[\int_0^\tau Q(\lambda)d\lambda\right] d\tau \quad . \tag{2.25}$$

Suppose now that there exists $t > 0$ and $a \leq r \leq b$ such that $\sigma(r,t) < 0$. Let

$$\mathcal{S} = \{t > 0: \text{ there exists } a \leq r \leq b \text{ such that } \sigma(r,t) < 0\} \quad .$$

Then $t_1 \equiv \text{g.}\ell\text{.b. } \mathcal{S}$ exists and $t_1 > 0$ by (2.20), (2.23), and the assumed continuity of σ. Due to the Bolzano-Weierstrass Theorem, there must exist $a \leq r_1 \leq b$ such that $\sigma(r_1,t_1) \leq 0$. However, differentiating (2.20) with respect to t, we find that

$$\dot{\sigma} + \mu_s \sigma_e^{n-1} \sigma = \frac{\beta_s}{r^3} \left[\frac{\dot{p}}{2} + \mu_s \int_a^b \sigma_e^{n-1} \sigma \frac{d\xi}{\xi} \right] , \tag{2.26}$$

$$\sigma(r,0) = \frac{\beta_s p(0)}{2r^3} . \tag{2.27}$$

Due to the continuity of σ and the definition of t_1, we must have $\sigma \geq 0$ in $[a,b] \times [0,t_1]$. Therefore (2.23), (2.26), (2.27) and Lemma 2.1 imply that $\sigma(r_1,t_1) > 0$ on $[a,b]$, which is a contradiction. Since we now know that $\sigma \geq 0$ on $[a,b] \times [0,\infty)$ we may apply Lemma 2.1 to (2.26) and (2.27) again to see that $\sigma > 0$ on $[a,b] \times [0,\infty)$.

Equation (2.20) now becomes

$$\sigma(r,t) = \frac{\beta_s p(t)}{2r^3} + \mu_s \left(\frac{\beta_s}{r^3} \int_a^b \int_0^t \sigma^n(\xi,\tau) d\tau \frac{d\xi}{\xi} - \int_0^t \sigma^n(r,\tau) d\tau \right) . \tag{2.28}$$

It follows from (2.9), (2.13), (2.14), (2.15), and (2.18) that the radial displacement $u(r,t)$ is expressed in terms of the effective stress σ by the equation

$$u(r,t) = \frac{r}{E} \left[(1 - \nu)\sigma(r,t) - (1 - 2\nu)p(t) + 2(1 - 2\nu) \int_a^r \sigma(\xi,t) \frac{d\xi}{\xi} \right]$$

$$+ \frac{Kr}{2} \int_0^t \sigma^n(r,\tau) d\tau . \tag{2.29}$$

A quantity of special interest is $u(b,t)$, the displacement history of the outer surface of the cylinder. Due to (2.19), it takes the form

$$u(b,t) = \frac{(1 - \nu)b}{E} \left[\sigma(b,t) + \mu_s \int_0^t \sigma^n(b,\tau) d\tau \right] . \tag{2.30}$$

Let us consider now the case of a hollow incompressible cylindrical pressure vessel of inner radius a and outer radius b subject to an internal pressure $p(t)$ which satisfies the restrictions (2.23). We again denote the radial displacement by $u(r,t)$ and the effective stress by $\sigma(r,t)$. In this case

$$\sigma = \frac{\sqrt{3}}{2} (\sigma_\theta - \sigma_r) \quad . \tag{2.31}$$

The cylinder equations analogous to (2.28) and (2.29) are

$$\sigma(r,t) = \frac{\sqrt{3}\ \beta_c p(t)}{2r^2} + \mu_c \left(\frac{\beta_c}{r^2} \int_a^b \int_0^t \sigma^n(\xi,\tau)d\tau \frac{d\xi}{\xi} - \int_0^t \sigma^n(r,\tau)d\tau \right) \quad , \tag{2.32}$$

$$u(r,t) = \frac{\sqrt{3}\ r}{2E} \left[\sigma(r,t) + \mu_c \int_0^t \sigma^n(r,\tau)d\tau \right] \quad . \tag{2.33}$$

Here,

$$\beta_c^{-1} = \int_a^b \frac{d\xi}{\xi^3} \quad , \tag{2.34}$$

$$\mu_c = EK \quad . \tag{2.35}$$

The difference between the derivation of the incompressible cylindrical and the spherical equations is not sufficiently great to justify the inclusion of the former in this paper. Both (2.28) and (2.32) are included in the general equation

$$\sigma(r,t) = \frac{\beta P(t)}{r^j} + \mu \left(\frac{\beta}{r^j} \int_a^b \int_0^t \sigma^n(\xi,\tau)d\tau \frac{d\xi}{\xi} - \int_0^t \sigma^n(r,\tau)d\tau \right) \tag{2.36}$$

where $j = 2$ or 3 and

$$P(0) > 0 \quad , \qquad \dot{P} \geq 0 \quad , \qquad \mu > 0 \quad , \tag{2.37}$$

$$\beta^{-1} \equiv \int_a^b \frac{d\xi}{\xi^{j+1}} \quad . \tag{2.38}$$

The derivation of (2.28) and (2.32) included the proof that $\sigma > 0$ on $[a,b]$. With assumptions (2.37), this can also be proven directly from (2.36) and (2.38).

Equations (2.30) and (2.33) for the case $r = b$ both have the form

$$u(b,t) = \kappa b \left[\sigma(b,t) + \mu \int_0^t \sigma^n(b,\tau)d\tau \right] \quad , \tag{2.39}$$

where κ is a material constant. Notice that by (2.36) and (2.39),

$$\dot{u}(b,t) = \kappa b[\dot{\sigma}(b,t) + \mu\sigma^n(b,t)]$$

$$= \frac{\kappa\beta}{b^{j-1}} \left[\dot{P}(t) + \mu \int_a^b \sigma^n(\xi,t) \frac{d\xi}{\xi} \right] .$$

This representation of $\dot{u}$, together with (2.37) and the fact that $\sigma > 0$ implies that $u(b,t)$ is a nondecreasing function of time.

3. Monotone Properties and Bounds for the Effective Stress

It follows from (2.36) after some differentiations that

$$\frac{\partial\dot{\sigma}}{\partial r} + n\mu\sigma^{n-1}\frac{\partial\sigma}{\partial r} = -j\ r^{-j-1}\left[\beta\dot{P} + \mu\beta\int_a^b \sigma^n(\xi,t)\,\frac{d\xi}{\xi}\right],$$

$$\frac{\partial\sigma}{\partial r}(r,0) = -j\beta P(0)r^{-j-1} .$$

Applying Lemma 2.1 to these two equations together with (2.37), we see that

$$\frac{\partial\sigma}{\partial r} < 0 \qquad \text{on} \qquad [a,b] \times [0,\infty) . \tag{3.1}$$

Also, if we multiply (2.36) by r^j and differentiate with respect to r we see that

$$\frac{\partial}{\partial r}[r^j\sigma(r,t)] = -\mu\int_0^t \frac{\partial}{\partial r}[r^j\sigma^n(r,\tau)]d\tau . \tag{3.2}$$

From this equation it follows that

$$\frac{\partial}{\partial r}[r^j\dot{\sigma}] + \mu\sigma^{n-1}\frac{\partial}{\partial r}(r^j\sigma) = -\mu(n-1)r^j\sigma^{n-1}\frac{\partial\sigma}{\partial r} ,$$

$$\frac{\partial}{\partial r}[r^j\sigma(r,0)] = 0 .$$

This pair of equations, together with (3.1) and Lemma 2.1, imply that for $t > 0$,

$$\frac{\partial}{\partial r}(r^j\sigma) > 0 \tag{3.3}$$

in the case of nonlinear creep, i.e., $n > 1$. In the linear viscoelastic case $n = 1$, they can be solved using (2.25) to obtain

$$\sigma(r,t) = g(t)r^{-j} \tag{3.4}$$

for some function g. For the rest of this paper we shall restrict ourselves to the nonlinear case, so that (3.3) holds. It then follows from (3.2) that

$$\frac{\partial}{\partial r}\left[\int_0^t r^j \sigma^n d\tau\right] < 0 \tag{3.5}$$

for $t > 0$.

We can now obtain bounds for $\sigma(r,t)$ for all $t > 0$. By virtue of (3.5) and (2.36),

$$\sigma(r,t) \leq \frac{\beta P(t)}{r^j} + \mu\left(\frac{a^j}{r^j}\int_0^t \sigma^n(a,\tau)d\tau - \int_0^t \sigma^n(r,\tau)d\tau\right) .$$

Therefore

$$\sigma(a,t) \leq \frac{\beta P(t)}{a^j} . \tag{3.6}$$

A very similar argument yields the lower bound

$$\sigma(b,t) \geq \frac{\beta P(t)}{b^j} . \tag{3.7}$$

Inequalities (3.6) and (3.7) together with (3.1) imply that for any $a \leq r \leq b$ and $t > 0$,

$$\frac{\beta P(t)}{b^j} \leq \sigma(r,t) \leq \frac{\beta P(t)}{a^j} . \tag{3.8}$$

By (2.36) and (2.39), $u(b,t)$ has the representation

$$u(b,t) = \kappa b\left[\frac{\beta P(t)}{b^j} + \frac{\mu\beta}{b^j}\int_a^b \int_0^t \xi^j \sigma^n(\xi,\tau)d\tau \frac{d\xi}{\xi^{j+1}}\right] .$$

This equation, together with (3.5) and (3.8), implies the following upper and lower bounds on the displacement of the outer surface of the vessel:

$$u(b,t) \leq \kappa b\left[\frac{\beta P(t)}{b^j} + \frac{\mu\beta^n}{b^j a^{(n-1)j}}\int_0^t [P(\tau)]^n d\tau\right] , \tag{3.9}$$

$$u(b,t) \geq \kappa b\left[\frac{\beta P(t)}{b^j} + \frac{\mu\beta^n}{b^{nj}}\int_0^t [P(\tau)]^n d\tau\right] . \tag{3.10}$$

Notice that these bounds converge to the exact solution as t tends to 0.

In the case where P is independent of t, Einarsson [2], [3] also states a priori bounds on the quantity $y(x,t)$, $x = r/a$, which imply bounds for the effective stress[2] σ_e. For the incompressible cylinder,

$$1 \leq y \leq c^{2-\frac{2}{n}} \tag{3.11}$$

for all time and all $1 \leq x \leq c = b/a$. It follows from (34), (37), and (55) of [2] that, in the absence of strain hardening, σ_e and y are related by the equation

$$\sigma_e = \frac{\sqrt{3}}{n}\left(c^{\frac{2}{n}} - 1\right)^{-1} p\left[\left(\frac{c}{x}\right)^{\frac{2}{n}} + \frac{c^{\frac{2}{n}}}{x^2}\left\{\frac{2}{1 - c^{-2}}\int_1^c y(\xi,\tau)\,\frac{d\xi}{\xi^3} - y(x,\tau)\right\}\right]. \tag{3.12}$$

Here, p stands for the internal pressure, and τ is a dimensionless quantity proportional to t which is defined by (39) and (40) of [2]. Taken together, (3.11) and (3.12) imply the following bounds on σ_e:

$$\frac{\sigma_e}{p} \leq \frac{\sqrt{3}}{n}\left(c^{\frac{2}{n}} - 1\right)^{-1}\left[\left(\frac{c}{x}\right)^{\frac{2}{n}} + \left(\frac{c}{x}\right)^2\left(1 - c^{\frac{2}{n}-2}\right)\right], \tag{3.13}$$

$$\frac{\sigma_e}{p} \geq \frac{\sqrt{3}}{n}\left(c^{\frac{2}{n}} - 1\right)^{-1}\left[\left(\frac{c}{x}\right)^{\frac{2}{n}} + \frac{c^{\frac{2}{n}}}{x^2}\left(1 - c^{2-\frac{2}{n}}\right)\right]. \tag{3.14}$$

Notice that at the inner boundary of the cylinder, where $x = 1$, the lower bound in (3.14) becomes negative for $c > 2^{\frac{n}{2n-2}}$. On the other hand, at the at the outer boundary $x = c$, (3.8), with $P = \frac{\sqrt{3}}{2}p$ for the cylindrical case, implies

$$\frac{\sigma_e}{p} \leq \frac{\sqrt{3}}{1 - c^{-2}}, \tag{3.15}$$

whereas Einarsson's upper bound (3.13) becomes

$$\frac{\sigma_e}{p} \leq \frac{\sqrt{3}\left(2 - c^{\frac{2}{n}-2}\right)}{n\left(c^{\frac{2}{n}} - 1\right)}. \tag{3.16}$$

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[2] In our discussion of Einarsson's results, we again use σ_e to denote effective stress in order to avoid confusion with Einarsson's notation $\sigma \equiv p^{-1}\sigma_r$.

The latter is obviously a much better bound than the former for large c. This is not necessarily true for all values of c and n. For example, in the case where n = 6, we have

	c = 1.1	c = 1.2	c = 1.3
$\dfrac{\sqrt{3}}{1 - c^{-2}}$	9.9799	5.6685	4.2423
$\dfrac{\sqrt{3}\left(2 - c^{-\frac{5}{3}}\right)}{6\left(c^{\frac{1}{3}} - 1\right)}$	10.2563	5.8144	4.2774

4. A Limit Theorem for the Effective Stress

In order to establish a limit theorem for $\sigma(r,t)$ as $t \to \infty$, we introduce the inner product[3]

$$(v,w) \equiv \beta \int_a^b v(\xi)w(\xi) \frac{d\xi}{\xi^{j+1}} \tag{4.1}$$

and the corresponding norm

$$\|v\|^2 \equiv \beta \int_a^b v^2(\xi) \frac{d\xi}{\xi^{j+1}} \quad . \tag{4.2}$$

Notice that for the linear functional $\ell(v)$ defined by

$$\ell(v) \equiv \beta \int_a^b v(\xi) \frac{d\xi}{\xi^{j+1}} \tag{4.3}$$

we have for any integrable function v and constant function c

$$\ell(v) = (v,1) \quad , \qquad \ell(vw) = (v,w) \tag{4.4}$$

$$\ell(c) = c \quad , \tag{4.5}$$

$$(c, v-\ell(v)) = 0 \quad . \tag{4.6}$$

For the norm (4.2) one has the following Sobolev-type inequality:

Lemma 4.1. *If* v *is any function* C^1 *in* [a,b] *and* ζ *is any positive constant, then*

$$v^2(r) \le \left[\frac{b^{j+1}}{\beta\zeta} + 1\right] \|v\|^2 + \frac{b^{j+1}\zeta}{\beta} \left\|\frac{dv}{dr}\right\|^2 \tag{4.7}$$

for all r *in* [a,b].

We shall also need the following Lemma.

Lemma 4.2. *Suppose that* u(t) *satisfies the inequality*

$$\dot{u}(t) + Qu(t) \le F(t) \tag{4.8}$$

for all $t > 0$, *where* Q *is a positive constant, and*

$$|F(t)| \le A\, t^{-\alpha-1} \quad , \qquad \alpha > 0 \tag{4.9}$$

[3] Compare [3] equation (40).

<u>for all</u> $t \geq t_1$. <u>Then</u>

$$u(t) \leq \left[u(0) + \int_0^{t_2} F(\tau)e^{Q\tau}d\tau\right]e^{-Qt} + A\,t^{-\alpha} \tag{4.10}$$

<u>for all</u> t <u>such that</u>

$$t \geq t_2 \equiv \max\{t_1, Q^{-1}(\alpha + 1)\} \quad . \tag{4.11}$$

In what follows, we shall write $f(t) = O(g(t))$ if any only if there exist constants t_1 and $A > 0$ such that

$$|f(t)| \leq A\,g(t)$$

for all $t \geq t_1$. We shall also write $f(r,t) = O(g(t))$ <u>uniformly in</u> $[a,b]$ provided the constants A and t_1 can be chosen to be independent of r.

<u>Theorem</u> 4.1. <u>Suppose that for some constant</u> $\alpha > 2$,

$$\dot{P}(t) = O(t^{-\alpha}) \quad , \tag{4.12}$$

$$\lim_{t\to\infty} P(t) \equiv P(\infty) \quad .$$

<u>Then</u>

$$\lim_{t\to\infty} \sigma(r,t) = \frac{j}{n}\,P(\infty)\left[a^{-\frac{j}{n}} - b^{-\frac{j}{n}}\right]^{-1} r^{-\frac{j}{n}} \tag{4.13}$$

<u>uniformly in</u> $[a,b]$.

<u>Proof</u>. Equation (2.36), when differentiated with respect to time, becomes

$$\dot{\sigma}(r,t) = \frac{\beta\,\dot{P}(t)}{r^j} + \mu\left(\frac{\beta}{r^j}\int_a^b \sigma^n(\xi,t)\,\frac{d\xi}{\xi} - \sigma^n(r,t)\right) \quad . \tag{4.14}$$

We define

$$w(r,t) \equiv r^j\sigma^n(r,t)$$
$$z(t) \equiv \ell w(t) \tag{4.15}$$

where the linear functional ℓ is given by (4.3). With this notation, (4.14) takes the form

$$\dot{\sigma}(r,t) = \frac{\beta\,\dot{P}(t)}{r^j} + \frac{\mu}{r^j}\,[z(t) - w(r,t)] \quad . \tag{4.16}$$

In order to show that $\sigma(r,t)$ tends to a limit $\sigma_\infty(r)$ uniformly in $[a,b]$ as $t \to \infty$, it suffices to prove that

$$\dot{\sigma}(r,t) = O[t^{-\alpha+1}] \tag{4.17}$$

uniformly in $[a,b]$. If we then take the limit of (4.14) as $t \to \infty$ using (4.12), (4.17) and the uniform convergence of σ^n on $[a,b]$, we see that

$$\sigma_\infty^n(r) = C^n r^{-j} \tag{4.18}$$

for some constant $C > 0$. In order to evaluate C, we multiply both sides of equation (2.36) by r^{-1} and integrate with respect to r from a to b. This yields the identity[4]

$$\int_a^b \sigma(r,t) \frac{dr}{r} = P(t) \quad . \tag{4.19}$$

Since this identity must hold in the limit as $t \to \infty$, we may use it together with (4.18) to evaluate C. In this way we arrive at (4.13).

Inequality (4.17) will follow from (4.12) and (4.16) provided we can show that

$$|z(t) - w(r,t)| = O(t^{-\alpha+1}) \tag{4.20}$$

uniformly in $[a,b]$. This estimate will follow from the Sobolev inequality Lemma 4.1 once we demonstrate that

$$||z - w||^2 (t) = O(t^{-2\alpha+1}) \quad , \tag{4.21}$$

$$\left|\left|\frac{\partial w}{\partial r}\right|\right|^2 (t) = O(t^{-2\alpha+ 2}) \quad . \tag{4.22}$$

The derivation of (4.21), (4.22) uses the differential equation

$$\frac{\partial w}{\partial t} = n\sigma^{n-1}(r,t) \ (\beta\dot{P}(t) + \mu[z(t) - w(r,t)]) \tag{4.23}$$

which is equivalent for $\sigma > 0$ to (4.16). Using (4.23) together with (4.6) we see that

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[4] Compare (2.19) for the special case of the sphere.

$$\frac{d}{dt}\left(\|w - z\|^2\right) = 2(\dot{w} - \dot{z},\ w - z)$$
$$= 2(\dot{w},\ w - z)$$
$$= 2(n\sigma^{n-1}[\beta\dot{P} + \mu(z - w)],\ w - z) \quad .$$

An application of the arithmetic-geometric mean inequality with parameter[5]

$$0 < \theta_1 < \frac{2\mu}{\beta}$$

to the right-hand side of this equation then yields the differential inequality

$$\frac{d}{dt}\left(\|w - z\|^2\right) + Q_1\|w - z\|^2 \le C_1\dot{P}^2(t) \tag{4.24}$$

where

$$Q_1 = n(2\mu - \beta\theta_1)\ \inf\ \sigma^{n-1} \quad , \tag{4.25}$$

$$C_1 = \frac{n\beta}{\theta_1}\ \sup\ \sigma^{n-1} \quad . \tag{4.26}$$

The infimum and supremum are taken over all (r,t) in $[a,b] \times [0,\infty]$. That Q_1 and C_1 are finite and positive follows from (2.37) and (3.8). Inequality (4.24) together with (4.12) and Lemma 4.2 imply (4.21).

In order to establish (4.22), we use (4.23) again to see that

$$\frac{d}{dt}\left(\left\|\frac{\partial w}{\partial r}\right\|^2\right) = 2\left(\frac{\partial \dot{w}}{\partial r}\ ,\ \frac{\partial w}{\partial r}\right)$$
$$= 2\left(n(n - 1)\sigma^{n-2}\ \frac{\partial \sigma}{\partial r}\ [\beta\dot{P} + \mu(z - w)] - n\mu\sigma^{n-1}\ \frac{\partial w}{\partial r}\ ,\ \frac{\partial w}{\partial r}\right) \quad .$$

Since by (3.1) and (3.3),

$$\left|\frac{\partial \sigma}{\partial r}\right| < \frac{1}{r}\ \sigma \quad ,$$

it follows that

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[5]This inequality states that for a and b real and $\theta > 0$,

$$2ab \le \frac{a^2}{\theta} + \theta b^2 \quad .$$

$$\frac{d}{dt}\left(\left\|\frac{\partial w}{\partial r}\right\|^2\right) \le 2\beta \int_a^b \sigma^{n-1}\left\{\frac{n(n-1)j}{a}\,|\beta\dot{P} + \mu(z-w)|\,\left|\frac{\partial w}{\partial r}\right| - n\mu\left(\frac{\partial w}{\partial r}\right)^2\right\}\frac{d\xi}{\xi^{j+1}} .$$

Again applying the arithmetic-geometric mean inequality, this time with parameter

$$0 < \theta_2 < \frac{2\mu a}{j(n-1)} ,$$

we obtain the differential inequality

$$\frac{d}{dt}\left(\left\|\frac{\partial w}{\partial r}\right\|^2\right) + Q_2\left\|\frac{\partial w}{\partial r}\right\|^2 \le C_2\left(\dot{P}^2 + \|z-w\|^2\right) \tag{4.27}$$

where

$$Q_2 = n\left(2\mu - \frac{[n-1]j\theta_2}{a}\right)\inf \sigma^{n-1} ,$$

$$C_2 = \frac{n(n-1)j}{a\theta_2}(\beta^2 + \mu^2)\sup \sigma^{n-1} .$$

Inequality (4.22) now follows from (4.12), (4.21), (4.27) and Lemma 4.2. This completes the proof.

5. Further Study of $u(b,t)$

Theorem 5.1. Let $\dot{P}(t) = 0(t^{-\alpha})$ where $\alpha > 2.5$. Then

$$u(b,t) \sim \kappa b^{1-j} \left[\beta P(\infty) + \mu \int_0^\infty [z(\tau) - z(\infty)] d\tau \right] + \kappa\mu b^{1-j} z(\infty) t \tag{5.1}$$

as $t \to \infty$.

Here $z(\infty)$, the limit of $z(t)$ as $t \to \infty$, is given by

$$z(\infty) = \left(\frac{jP(\infty)}{n}\right)^n \left[a^{-\frac{j}{n}} - b^{-\frac{j}{n}} \right]^{-n} . \tag{5.2}$$

This theorem shows that, if the convergence of P to $P(\infty)$ is sufficiently rapid, then $u(b,t)$ tends asymptotically in the t,u plane to a straight line with slope

$$v_2 = \kappa\mu b^{1-j} \left(\frac{jP(\infty)}{n}\right)^n a^j \left[1 - c^{-\frac{j}{n}} \right]^{-n} . \tag{5.3}$$

Let us consider the important special case in which P is constant. In this case, the upper bound (3.9) on $u(b,t)$ becomes

$$u(b,t) \le \kappa b^{1-j} \beta P + v_1 t \tag{5.4}$$

where

$$v_1 = \kappa\mu b^{1-j} (jP)^n a^j [1 - c^{-j}]^{-n} . \tag{5.5}$$

For $n = 1$, $v_1 = v_2$. However, for $n > 1$ and $1 < c < \infty$, $v_1 > v_2$. In order to see this, we define

$$f_1(x) = 1 - x^n ,$$
$$f_2(x) = n(1-x) . \tag{5.6}$$

Then

$$v_i = \kappa\mu b^{1-j} (jP)^n a^j \left[f_i \left(c^{-\frac{j}{n}} \right) \right]^{-n} \qquad (i = 1,2) , \tag{5.7}$$

and the assertion follows from (5.6), (5.7), and the fact that $f_1(x) < f_2(x)$ for $0 < x < 1$. This shows that (3.9) is a bad estimate for large t and motivates us to complement it by another bound which has asymptotic slope v_2.

Proof of Theorem 5.1. Since, by (2.36), (2.39), and (4.15),

$$u(b,t) = \kappa b^{1-j}\left[\beta P(t) + \mu\int_0^t [z(\tau) - z(\infty)]d\tau\right] + \kappa\mu b^{1-j} z(\infty)t \quad , \tag{5.8}$$

it is clearly sufficient to show that

$$\left|\int_0^\infty [z(\tau) - z(\infty)]d\tau\right| < \infty \quad . \tag{5.9}$$

However,

$$\int_0^t [z(\tau) - z(\infty)]d\tau = -\int_0^t \lambda\dot{z}(\lambda)d\lambda - t\int_t^\infty \dot{z}(\lambda)d\lambda \quad ,$$

so that

$$\left|\int_0^t [z(\tau) - z(\infty)]d\tau\right| \le \int_0^t \lambda|\dot{z}(\lambda)|d\lambda + t\int_t^\infty |\dot{z}(\lambda)|d\lambda \tag{5.10}$$

$$\le \int_0^\infty \lambda|\dot{z}(\lambda)|d\lambda \quad . \tag{5.11}$$

In order to estimate the order of $\dot{z}$, we apply the operator ℓ to both sides of (4.23), thus obtaining the equation

$$\dot{z} = n(\sigma^{n-1}, \beta P + \mu[z - w]) \quad .$$

Applying first the Schwarz and then the Minkowski inequality to the right-hand side of this equation and then appealing to (4.21), we find that

$$|\dot{z}| \le n\|\sigma^{n-1}\| (\beta\dot{P} + \mu\|z - w\|) \tag{5.12}$$

$$\le O(t^{-\alpha}) + O\left(t^{-\alpha+\frac{1}{2}}\right) = O\left(t^{-\alpha+\frac{1}{2}}\right) \quad . \tag{5.13}$$

The proof of the theorem follows from (5.9), (5.11), and (5.13).

The analysis for Theorem 5.1, together with (4.24), can be used to construct a new bound for u(b,t) which has the desired asymptotic behavior. Such a bound can be based on the inequality

$$u(b,t) \leq \kappa b^{1-j} \left[\beta P(t) + \mu \int_0^t \tau |\dot{z}(\tau)| d\tau + \mu t \int_t^\infty |\dot{z}(\tau)| d\tau + \mu z(\infty) t \right] , \quad (5.14)$$

which follows from (5.8) and (5.10), together with the bound

$$|\dot{z}(t)| \leq n \, \|\sigma^{n-1}\| \left(\beta \dot{P}(t) + \mu \left[\|w - z\|^2 (0) + C_1 \int_0^t \dot{P}^2(\tau) e^{Q_1 \tau} d\tau \right]^{\frac{1}{2}} \cdot \exp\left[- \frac{Q_1 t}{2} \right] \right) , \quad (5.15)$$

which is obtained from (4.24) and (5.12). An analogous lower bound for $u(b,t)$ is also readily derived. Bounds for the quantities $\|\sigma^{n-1}\|$, C_1, and Q_1 can be based on either (3.8) or (3.13) and (3.14), depending on the values chosen for n and c. For the case in which the pressure history P is constant in time, (5.14) and (5.15) imply that

$$u(b,t) \leq \kappa b^{1-j} \left[\beta P + \frac{4\mu\zeta_1}{Q_1^2} \left(1 - \exp\left[- \frac{Q_1 t}{2} \right] \right) + \mu z(\infty) t \right] , \quad (5.16)$$

where

$$\zeta_1 = n\mu (\sup \sigma^{n-1}) \, \|w - z\| \, (0)$$

and Q_1 is given by (4.25).

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