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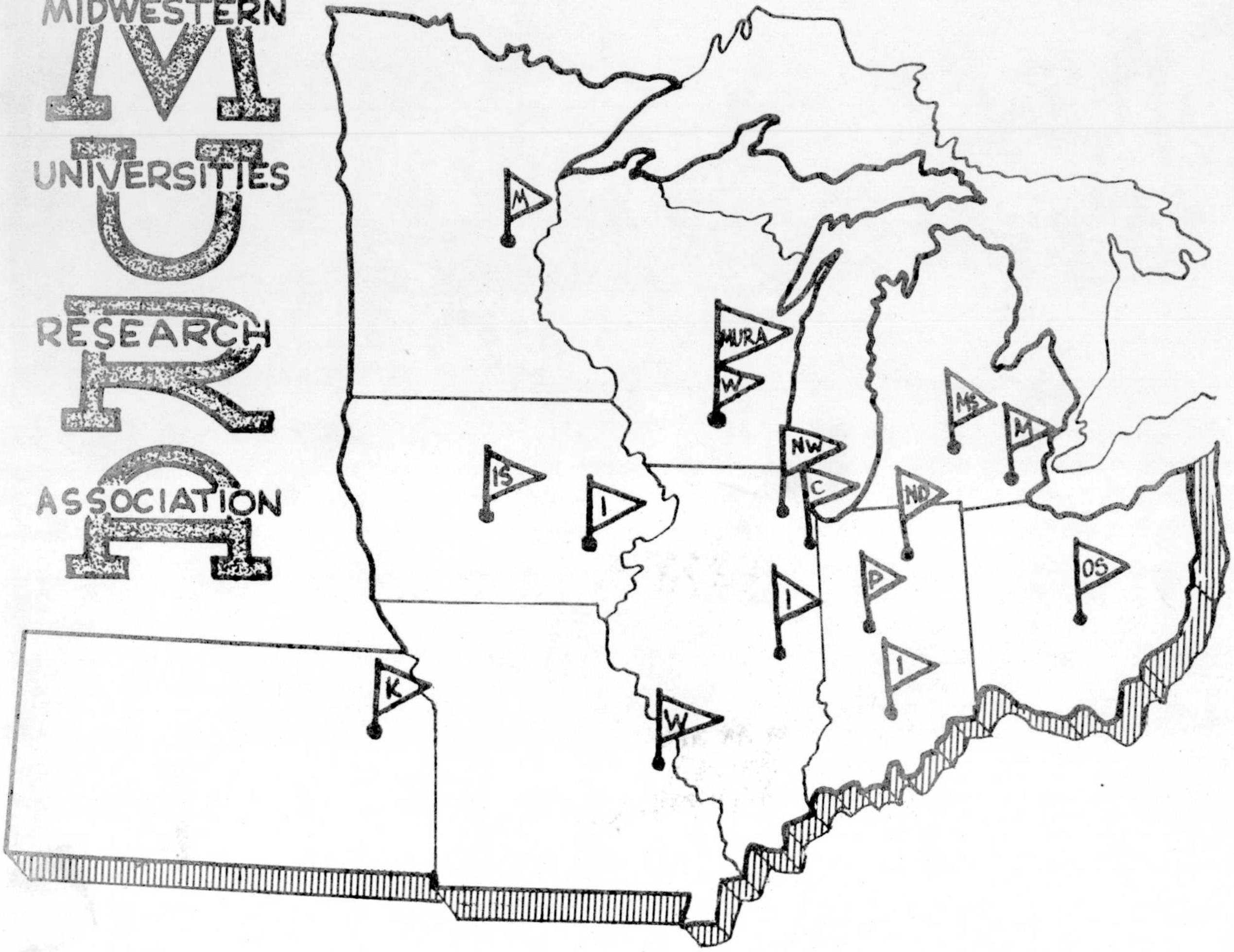

CONCERNING RESONANT BEAM KNOCK-OUT FROM AN A-G SYNCHROTRON

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CONCERNING RESONANT BEAM KNOCK-OUT FROM AN A-G SYNCHROTRON

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ABSTRACT

The efficacy of various types of field-index perturbations (n-bumps) in effecting an instability potentially helpful for beam utilization is examined analytically for an alternating-gradient particle accelerator and the results illustrated by computational examples. The perturbations of interest open up a stop-band, at the frequency ν_x, within which the solution to the (linear) equations for the radial betatron oscillations soon become dominated by a solution of exponentially-increasing amplitude. A field-index perturbation containing circular functions of argument $(\nu_x + r)\theta$ and $(\nu_x - r)\theta$ can open up such a stop-band and a circular function of argument $2\nu_x\theta$ alone can be particularly effective. The azimuthal dependence of the perturbation can also serve to influence the form of the unstable solution, a perturbation which in particular contains a term of argument $(\nu_x \pm r)\theta$ serving to introduce a sine or cosine term of argument $r\theta$ into the solution, and it is suggested that such terms with $r = 1/2$ may be useful in some applications. It is finally pointed out that to achieve some particular features of the solution, such as meeting the condition $dx/d\theta = 0$ at $\theta = 0$, careful engineering attention may be necessary to insure meeting the necessary tolerances for the form of the perturbation. It is suggested that such tolerances, although not discussed explicitly, could be estimated by the methods presented in this report. The analytic work is supplemented by Appendices covering some details of the analytic work, which employs a variational method, and by Appendix I outlining an equivalent approach by conventional perturbation theory.

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CONTENTS

I. INTRODUCTION

The use of the half-integral rsonance, $\nu_x = 1/2$, to effect rapid beam knock-out, or extraction, from a normally constant gradient synchrotron (or betatron) has been previously published[1,2*] and more recently a convenient analytic description of the method has been reported.[3] Although there were some early attempts[4,5,6] to study the applicability of the original method to alternating-gradient accelerators, the use of the analytic approach to guide a broader reinvestigation of resonant knock-out seems timely, especially in view of the great enhancement of utility and versatility which a successful method would provide for alternating-gradient accelerators now nearing completion.[7,8]

In the following sections we attempt to make such an investigation, guided by the analytic approach and with computational tests made throughout the treatment to check the theory with illustrative examples. It is not claimed that the knock-out methods examined here are optimum, or even practicable in all cases, but it is hoped that the discussion will stimulate furthe r consideration and examination of this topic.

It may be recalled that the method used with the constant-gradient synchrotron[1,2,3] employed an azimuthally-dependent perturbation of the field-gradient (n-bump) to drive the operating point into an unstable zone (stop-band), which opened up with a width proportional to the magnitude of the perturbation and within which the solution for the exponentially-increasing betatron oscillations attained its maximum value at <u>one particular azimuth</u> in the machine. In application to an alternaing-gradient accelerator it may

*References are given in Section V.

not be convenient to retain all the features just mentioned, but we shall devote attention in what follows to the dependence of the stop-band width on the strength of the perturbation and on the manner in which the character of the unstable orbits can be controlled.

For the purpose under consideration here the use of an n-bump appears desirable, since the perturbing windings then have very little coupling from the main magnetic field of the accelerator. In analogy to the earlier work,[1, 2, 3] we shall confine our attention in the present report to the effect of various types of n-bumps, although the possible utility of field-bumps may deserve attention at a later time. The use of a half-integral, as distinct from an integral, resonance in the present application does not seem essential and the selection of the particular resonance to be employed may be based on secondary considerations peculiar to the particular accelerator with which the method might be used. With the integral resonance it will be seen, however, that if the unstable orbits show any preference for large amplitudes at some particular azimuth, half of the particles may be driven towards the outer radius at that azimuth and half toward the inner radius of the chamber; in contrast, with a half-integral resonance, the particles would go alternately to large and small radii on successive revolutions. In some cases it may be of importance, from the standpoint of economy or feasibility of the electrical pulsing equipment, to open up the stop-band by an adequate amount with relativity modest perturbations and, in such cases, this consideration may prove to be of dominant weight. In other cases, however, it may be of interest to insure that the beam makes its maximum departure from the equilibrium orbit at one azimuth within the accelerator, to avoid interference by the injector, additional targets, or other structures within the vacuum chamber. As we shall see, it appears that these features can be realized by a suitable form of perturbation, or by a combination of such perturbations.

Basically, then, we shall visualize driving the accelerator to a near-by half-integral or integral resonance and shall direct attention not only to the stop-band width and associated growth rate of unstable oscillations but also to the form of the dominant solution for these oscillations. If the instability associated with a readily-accessible stop-band can be effectively exploited, use of this instability would appear to afford a subtile and economical way of effecting knock-out.

Attention will be directed exclusively to achieving radial instability, it being presumed that axial stability can be maintained. Throughout the report the equations of motion will be taken to be linear, and typically may be regarded as of the Hill form. It is convenient to obtain approximate solutions to problems of this type by means of a variational method[3, 9, 10, 11] and this method will be followed in the body of this report (Sect. II); alternatively, however, the use of harmonic balance or, as demonstrated in Appendix I, standard perturbation methods[12, 13] may be found equally suitable.

In some of the numerical examples, the unperturbed accelerator[7] will be considered to consist of N identical A-G sectors (full sectors) with N = 48 and ν (the number of radial betatron oscillations per circumference) in the range 7 to 7 1/2, while in other examples we take N = 24 and $\nu = 5$. In either case, however, the illustrative material thus will not include the complication of straight-sections, super-periods, or auxiliary lenses, it being felt that nothing significant is lost in the exposition by omitting such elaborations.

II. THEORY

A. The Unperturbed Problem

The differential equation characterizing the radial betatron oscillation may be taken to be of the form[14]

$$d^2x/d\theta^2 + [a + m\,F(\theta)]\,x = 0 \tag{1}$$

in the unperturbed case, where

$$F(\theta) = +1 \quad \text{for} \quad -\pi/2N < [\theta,\ \text{mod.}\ 2\pi/N] < \pi/2N$$

$$F(\theta) = -1 \quad \text{for} \quad \pi/2N < [\theta,\ \text{mod.}\ 2\pi/N] < 3\pi/2N\ .$$

It is seen that $\theta = 0$ then corresponds to the center of a "radially focusing" semi-sector. In what follows we shall usually neglect, for convenience, the constant term "a", thus ignoring the normally-small "centrifugal focusing" for the radial oscillations.

As noted previously,[3, 9, 10, 11] eigenvalues and, with less accuracy, eigenfunctions for periodic solutions of equations such as (1) may be conveniently obtained by a variational method. This method may be expressed in the form of the statement

$$\langle (dx/d\theta)^2 \rangle - a\langle x^2 \rangle - m\langle x^2\,F(\theta)\rangle = \text{min.}, \tag{2}$$

where $\langle\ \rangle$ denotes that the quantity within the symbol is to be averaged over an entire period.[15]

A solution of (2) may be obtained readily by adopting the simple trial function[16]

$$x = A \cos \nu\theta + B_1 \cos (N - \nu)\theta + B_2 \cos (N + \nu)\theta \quad \text{(3a)}$$

or

$$x = A \sin \nu\theta + B_1 \sin (N - \nu)\theta + B_2 \sin (N + \nu)\theta \,. \quad \text{(3b)}$$

By substitution of the trial solutions (3a) or (3b) into (2), setting the partial derivatives of the resultant algebraic expressions separately equal to zero, and solving the resultant simultaneous equations, one obtains

$$B_1 = \pm \frac{2m}{\pi} \frac{1}{(N - \nu)^2 - a} A \quad \text{(4a)}$$

$$B_2 = + \frac{2m}{\pi} \frac{1}{(N + \nu)^2 - a} A \quad \text{(4b)}$$

$$\nu^2 = a + \left(\frac{2m}{\pi}\right)^2 \left[\frac{1}{(N - \nu)^2 - a} + \frac{1}{(N + \nu)^2 - a}\right], \quad \text{(4c)}$$

where the upper and lower signs for B_1 refer respectively to the even (cosine) or odd (sine) solutions (3a) or (3b).

In the case of present interest we take $a = 0$ and we write

$$B_1 = \pm \left(\frac{2m}{\pi}\right) \frac{1}{(N - \nu)^2} A \quad \text{(5a)}$$

$$B_2 = + \left(\frac{2m}{\pi}\right) \frac{1}{(N + \nu)^2} A \quad \text{(5b)}$$

$$\nu^2 = \left(\frac{2m}{\pi}\right)^2 \left[\frac{1}{(N - \nu)^2} + \frac{1}{(N + \nu)^2}\right]$$

$$= \left(\frac{2m}{\pi N}\right)^2 \left[\frac{1}{(1 - \nu/N)^2} + \frac{1}{(1 + \nu/N)^2}\right]$$

$$= 2\left(\frac{2m}{\pi N}\right)^2 \frac{1 + (\nu/N)^2}{\left[1 - (\nu/N)^2\right]^2} \,. \quad \text{(5c)}$$

For this "unperturbed problem," the interval 2π of the accelerator as a whole plays no basic role and the results (4a - c) or (5a - c) need not be considered to be restricted to integral or half-integral values of ν--from another point of view one may reason that the frequencies ν of the oscillation and N of the structure may be regarded as effectively commensurate in some (possibly large) interval and the variational statement then considered as applying in that interval [ref. 11, Appendix II]. It may also be noted that, to the degree of approximation employed here, the function F (θ) in the differential equation (1) could equally well be replaced by its first Fourier component, $\frac{4}{\pi}$ cos N θ.

For $\nu \ll N$, equation (5c) may be written in the simplified approximate form[17]

$$\nu^2 \cong 2\left(\frac{2\,m}{\pi N}\right)^2 . \qquad (6)$$

Although equation (5c) as it stands is a rather accurate relation between ν and m for the parameters of interest here (vide Table I below), it may be presumed that improved accuracy could be obtained by use of a more elaborate trial function, employing, for example, additional circular functions of argument $(2N - \nu)\theta$, $(2N + \nu)\theta$, $(3N - \nu)\theta$, $(3N + \nu)\theta$, etc. If we undertake to improve (5c) in this way, and simplify small correction terms by the aid of (6), it appears that we obtain a more accurate result of the form given below.

$$\nu^2 = 2\left(\frac{2\,m}{\pi N}\right)^2 \left\{ \frac{1 + (\nu/N)^2}{\left[1 - (\nu/N)^2\right]^2} + 0.015 + 0.06\,(\nu/N)^2 \right\} . \qquad (7)$$

We present below in Table I the values of m computed by means of equations (6), (5c), and (7) for a few values of ν in the range 7 to 7 1/2, for N = 48, in comparison to the exact results obtained by a direct matrix computation for the solutions to equation (1). It will be noted that, although the results obtained by use of (7) are definitely superior, the relation given by equation (5c) is fairly accurate. In the following sections we shall endeavor to treat the perturbed accelerator to an order of accuracy comparable with that used in deriving (5c).

TABLE I

Comparison of Exact Values of m with Values Given by Analytic Formulas (N = 48)

ν	m			
	Exact	By (6)	By (5c)	By (7)
7	358.68	$373._2$	361.4_4	358.7_1
7.375	376.53	$393._2$	379.4_6	376.5_9
7.5	382.43	$399._9$	385.4_2	382.5_1
Error		$(4 \text{ to } 5)/10^2$	$< 1/10^2$	$(1 \text{ to } 2)/10^4$

B. Analytic Estimates of Stop-Band Widths, Lapse-Rates and Character of Orbits in an Unstable Zone

1. Method:

When a perturbation is applied to modify the coefficient of x in equation (1), the frequency of the betatron oscillations will be modified and a zone of instability, or stop-band, may be opened up (Fig. 1). If the perturbation is an even function of θ (periodic, with period 2π or a sub-multiple thereof), the periodic eigensolutions which are associated with the edges of such a stop-band are conveniently even or odd functions of θ. By use of suitable even or odd trial functions in a variational procedure similar to that used in Sect. A, the location of the stop-band boundaries may be determined rather well and the form of the eigensolutions estimated.

Within the stop-band, moreover, it appears that the solutions to the differential equation can be rather well expressed in terms of the eigenfunctions associated with the boundary of that particular zone of instability. This fact, which we develop below, permits us to estimate the rate of growth of the unstable oscillations, as well as other features of the orbits which are of interest. We undertake below to develop the general relation which connects the solutions within the stop-band to the associated eigenfunctions, and then proceed to examine the effect of specific types of perturbations.

2. Approximate Character of Solutions within a Stop-Band:

To obtain an approximate description of the solutions within an unstable zone, with an estimate of the characteristic exponent which determines the rate of growth (lapse rate) of the unstable solution, we follow a procedure, based on a suggestion by McLachlan,[18] which we have previously[3, 11] found useful in similar applications. Although the solutions quite generally could be written in terms of an ascending (or descending) exponential factor times a periodic function of θ, where this periodic function could be expanded in terms of a complete set of eigenfunctions, we assume here that it suffices to employ only the characteristic solutions associated with the boundaries of the stop-band of interest. We thus visualize an approximate solution of the form

$$x = e^{\pm \mu \theta} \Big[C\, c\,(\theta) \pm S\, s\,(\theta) \Big] , \tag{8}$$

where c (θ) and s (θ) represent, respectively, the even and odd eigen-solutions at the boundaries of the stop-band. An expression of the form (8) is then substituted into the differential equation of interest, which is of the form

$$d^2x/d\theta^2 + \Big[m\, F\,(\theta) + f\,(\theta) \Big]\, x = 0 , \tag{9}$$

where f (θ) represents the perturbation. Use is made of the fact that c (θ) and s (θ) satisfy (9) for m equal to m_{even} or m_{odd}, respectively. In this way we find that, if (8) were a true solution of (9), the following relation would be satisfied identically in θ:

$$\Big[\mu^2 + (m - m_{even})\, F\,(\theta)\Big]\, C\, c\,(\theta) + 2\mu\, S\, s'\,(\theta)$$
$$\pm \Big[\mu^2 + (m - m_{odd})\; F\,(\theta)\Big]\, S\, s\,(\theta) \pm 2\mu C\, c'\,(\theta) = 0 , \tag{10}$$

where the prime denotes differentiation with respect to θ.

In order to adjust the parameters in (8) so that (10) is satisfied in an approximate sense, we multiply (10) in turn by c (θ) and by s (θ) and integrate, to obtain the conditions represented by the algebraic equations which follow:

$$\Big[\mu^2 \langle c^2 \rangle + (m - m_{even}) \langle c^2\, F \rangle \Big]\, C + 2\mu \langle c\, s' \rangle\, S = 0 \tag{11a}$$

$$2\mu \langle c'\, s \rangle\, C + \Big[\mu^2 \langle s^2 \rangle + (m - m_{odd}) \langle s^2\, F \rangle \Big]\, S = 0. \tag{11b}$$

For the simultaneous homogeneous equations (11a, b) to have a non-trivial solution, the determinant of the coefficients must vanish, thus determining μ and thereby fixing the ratio of S to C.

Since the analysis described here is no more than approximate, we regard it as sufficiently accurate to propose the solution

$$\mu^2 \cong \frac{(m_{even} - m)(m - m_{odd}) \langle c^2 F\rangle \langle s^2 F\rangle}{-4 \langle c' s\rangle \langle c s'\rangle} \qquad (12a)$$

and

$$C/S \cong \sqrt{\frac{\langle c s'\rangle \langle s^2 F\rangle}{\langle c' s\rangle \langle c^2 F\rangle} \frac{m - m_{odd}}{m_{even} - m}} \; . \qquad (12b)$$

For evaluation of the right-hand sides of equations (12a, b) it should suffice to employ only the dominant terms in c (θ) and s (θ), namely just the terms appearing in the unperturbed solutions [(3a, b), with the coefficients (5a, b)]. By use of these unperturbed solutions we find

$$\langle c^2 F\rangle \cong \frac{2}{\pi} (A B_1 + A B_2)_{even}$$

$$\cong \frac{8 m_o}{\pi^2 N^2} A^2_{even} , \qquad (13a)$$

$$\langle s^2 F\rangle \cong \frac{8 m_o}{\pi^2 N^2} A^2_{odd} , \qquad (13b)$$

$$\langle c s'\rangle \cong \frac{\nu}{2} A_{even} A_{odd} , \text{ and} \qquad (13c)$$

$$\langle c' s\rangle \cong -\frac{\nu}{2} A_{even} A_{odd} . \qquad (13d)$$

The results (12a, b) may thus be expressed in the convenient form:

$$\mu \cong \frac{8 m_o}{\pi^2 N^2 \nu} \sqrt{(m_{even} - m)(m - m_{odd})} \qquad (14a)$$

and

$$C/S \cong \sqrt{\frac{m - m_{odd}}{m_{even} - m}} , \qquad (14b)$$

with

$$\mu_{max.} \cong \frac{4\, m_o}{\pi^2 N^2 \nu} \cdot \left| m_{even} - m_{odd} \right| . \tag{14c}$$

The exponent μ may be said to be expressed here in nepers per radian--the corresponding lapse rate in nepers per revolution would be obtained by multiplying by 2π and in decades per revolution by use of the further factor $\log_{10} e$ ($\doteq$ 0.4343).

From the results (14a - c) it is clear that the lapse rate attains a maximum value, near the center of the stop-band, which is directly proportional to the width, $\left| m_{even} - m_{odd} \right|$, of the stop-band. The growing unstable solution, when the exponential factor is factored out, is seen to have its azimuthal dependence in the center of the band represented by an equal admixture of the even and odd eigensolutions which prevail at the edge of the band. To the extent that equation (6) is an adequate approximate expression for ν, the maximum lapse rate given by (14c) may be expressed in the very simple approximate form

$$\mu_{max.} \approx \frac{\nu}{2} \frac{\left| m_{even} - m_{odd} \right|}{m_o} \text{ nepers per radian.} \tag{15}$$

3. Effect of Perturbation

The perturbation applied to n will necessarily be periodic in θ and in a typical case might be of the form $f(\theta) = \epsilon \cos r\theta + \zeta \cos s\theta$, where $f(\theta)$ is the perturbation function appearing in equation (9) and the constants r and s are integers. We shall see that to open up a stop-band effectively at some integral or half-integral betatron frequency ν it is generally necessary to have present in the perturbation both of the terms shown, with $\frac{r+s}{2} = \nu$, since the stop-band width depends on the product $\epsilon \zeta$. Before treating this case, however, we shall consider the case in which a perturbation $f(\theta) = \lambda \cos 2\nu$ is applied, since in this special case the stop-band is found to open up with a width proportional to the first power of the perturbation. In the interests of clarity we then also discuss the special case in which $f(\theta) = \eta \cos \nu\theta$, which is a form of perturbation capable of producing a stop-band about the frequency ν when ν is an integer.

For determination of the stability boundaries and the form of the associated eigenfunctions for the differential equation (9) we make use of the variational statement

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \langle x^2 f(\theta) \rangle = \text{min.} \qquad (16)$$

Throughout the analysis we shall presume that $\nu \ll N$. It moreover will be noted that, to the degree of accuracy employed, $F(\theta)$ could be replaced by the first term of its Fourier expansion $\frac{4}{\pi} \cos N\theta$.

a. The perturbation $\lambda \cos 2\nu\theta$

With $f(\theta) = \lambda \cos 2\nu\theta$, the variational statement (16) which we apply at the stop-band boundaries becomes

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \lambda \langle x^2 \cos 2\nu\theta \rangle = \text{min.} \quad (17)$$

Trial functions of the form (3a, b) which were employed in the unperturbed problem (Sect. II) should now be supplemented by circular functions with arguments $3\nu\theta$, $(N - 3\nu)\theta$, and $(N + 3\nu)\theta$ in order that cross terms in x^2 can contribute to $\langle x^2 \cos 2\nu\theta \rangle$. By insertion of such elaborated trial functions, one even and one odd, into (17), one is led in each case to a set of simultaneous algebraic equations for the coefficients and for the associated values of the parameter m. We list below, in Table II, approximate solutions to these equations, retaining only terms which are first order in λ (and presuming $\nu \ll N$). Details are given in Appendix II.

TABLE II

COEFFICIENTS OF cos hθ or sin hθ IN THE EIGENFUNCTIONS CORRESPONDING TO THE STABILITY BOUNDARIES OF FREQUENCY ν FOR THE EQUATION

$$d^2x/d\theta^2 + [m\,F(\theta) + \lambda \cos 2\nu\theta]\,x = 0.$$

The upper sign is for the even (cosine) eigenfucntion and the lower sign for the odd (sine) eigenfunction.

h	Coefficient of cos hθ or sin hθ
ν	1 [Normalized]
3ν	$\frac{\lambda}{16\nu^2}\frac{1-17(\nu/N)^2}{1-21(\nu/N)^2} \cong \frac{\lambda}{16\nu^2}\left[1+4(\nu/N)^2\right] \cong \frac{\lambda}{16\nu^2}$
$N-3\nu$	$\pm\frac{\lambda}{8\nu^2}\frac{m}{\pi N^2}\frac{(1+3\nu/N)^2\left[1+\left(\frac{3\nu}{N-\nu}\right)^2\right]}{1-21(\nu/N)^2} \cong \pm\frac{\lambda}{8\nu^2}\frac{m}{\pi N^2} \cong \pm\frac{\lambda}{16\sqrt{2}\,\nu N}$
$N-\nu$	$\pm\frac{2m}{\pi N^2}\frac{1}{(1-\nu/N)^2}$
$N+\nu$	$\frac{2m}{\pi N^2}\frac{1}{(1+\nu/N)^2}$
$N+3\nu$	$\frac{\lambda}{8\nu^2}\frac{m}{\pi N^2}\frac{(1-3\nu/N)^2\left[1+\left(\frac{3\nu}{N+\nu}\right)^2\right]}{1-21(\nu/N)^2} \cong \frac{\lambda}{8\nu^2}\frac{m}{\pi N^2} \cong \frac{\lambda}{16\sqrt{2}\,\nu N}$
$\frac{m-m_o}{m_o}$	$\mp\frac{1}{2}\left(\frac{\pi N}{4m_o}\right)^2\lambda \cong \mp\frac{1}{4\nu^2}\lambda$
Relative Width, $\frac{W}{m_o}=\frac{m_{even}-m_{odd}}{m_o}$	$-\left(\frac{\pi N}{4m_o}\right)^2\lambda \cong -\frac{1}{2\nu^2}\lambda$

These results are readily interpretable[19] as the analogues of the corresponding results for the first stop-band of the Mathieu equation if we replace the rapidly-varying term m F (θ) by its smooth-approximation equivalent constant. The effect of the particular perturbation considered here is seen to be powerful, in the sense of being first order in λ. It thus affords the opportunity of readily forming large rates of growth if such should be desired, but it should be noted that the modifications to the orbit are of high frequency.

By use of the relative width, given in the last line of Table II, and by reference to equation (14c), one estimates the lapse-rate in the center of the stop-band to be given by

$$\mu_{\text{max.}} \cong \frac{\lambda}{4\nu} \quad \text{nepers per radian.} \tag{18}$$

b. The perturbation $\eta \cos \nu\theta$

With $f(\theta) = \eta \cos \nu\theta$, the variation statement (16) becomes

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \eta \langle x^2 \cos \nu\theta \rangle = \text{min.} \tag{19}$$

In this case the supplementary terms in the trial functions should include a constant (for the even eigenfunction), and circular functions of argument $2\nu\theta$, $(N - 2\nu)\theta$, $N\theta$, and $(N + 2\nu)\theta$. By use of such supplementary terms the results listed in Table III are obtained (Appendix III).

We note that this perturbation, $f(\theta) = \eta \cos \nu\theta$, is able alone to open up a stop-band at the frequency ν, albeit with a width proportional

TABLE III

COEFFICIENTS OF cos hθ or sin hθ IN THE EIGENFUNCTIONS
CORRESPONDING TO THE STABILITY BOUNDARIES
OF FREQUENCY ν FOR THE EQUATION

$$d^2x/d\theta^2 + [m F(\theta) + \eta \cos \nu\theta] x = 0.$$

h	Coefficient of cos h θ	Coefficient of sin h θ
0	$-\left(\frac{\pi N}{4 m}\right)^2 \left[1 + (\nu/N)^2\right]\eta \cong -\frac{1}{2\nu^2}\left[1 + 4(\nu/N)^2\right]\eta$	- - -
ν	1 [Normalized]	1 [Normalized]
2ν	$\frac{1}{6\nu^2} \frac{1 - 7(\nu/N)^2}{1 - 11(\nu/N)^2} \eta$	$\frac{1}{6\nu^2} \frac{1 - 7(\nu/N)^2}{1 - 11(\nu/N)^2} \eta$
$N - 2\nu$	$\frac{m}{3\pi \nu^2 N^2} \frac{(1 + 2\nu/N)^2 \left[1 + \left(\frac{2\nu}{N - \nu}\right)^2\right]}{1 - 11(\nu/N)^2} \eta$	$-\frac{m}{3\pi \nu^2 N^2} \frac{(1 + 2\nu/N)^2 \left[1 + \left(\frac{2\nu}{N - \nu}\right)^2\right]}{1 - 11(\nu/N)^2} \eta$
$N - \nu$	$\frac{2m}{\pi N^2} \frac{1}{(1 - \nu/N)^2}$	$-\frac{2m}{\pi N^2} \frac{1}{(1 - \nu/N)^2}$
N	$-\frac{\pi}{4m}\eta$	$-4\frac{m\nu}{\pi N^5}\eta \cong -\sqrt{2}\frac{\nu^2}{N^4}\eta$
$N + \nu$	$\frac{2m}{\pi N^2} \frac{1}{(1 + \nu/N)^2}$	$\frac{2m}{\pi N^2} \frac{1}{(1 + \nu/N)^2}$
$N + 2\nu$	$\frac{m}{3\pi \nu^2 N^2} \frac{(1 - 2\nu/N)^2 \left[1 + \left(\frac{2\nu}{N + \nu}\right)^2\right]}{1 - 11(\nu/N)^2} \eta$	$\frac{m}{3\pi \nu^2 N^2} \frac{(1 - 2\nu/N)^2 \left[1 + \left(\frac{2\nu}{N + \nu}\right)^2\right]}{1 - 11(\nu/N)^2} \eta$
$\frac{m - m_o}{m_o}$	$\frac{5}{6}\left(\frac{\pi N}{4 m_o}\right)^4 \eta^2 \cong \frac{5}{12\nu^2}\left(\frac{\pi N}{4 m_o}\right)^2 \eta^2$	$-\frac{1}{12\nu^2}\left(\frac{\pi N}{4 m_o}\right)^2 \eta^2 \cong -\frac{1}{6}\left(\frac{\pi N}{4 m_o}\right)^4 \eta^2$
Relative Width, $\frac{W}{m_o} = \frac{m_{even} - m_{odd}}{m_o}$	$\left(\frac{\pi N}{4 m_o}\right)^4 \eta^2 \cong \frac{1}{2\nu^2}\left(\frac{\pi N}{4 m_o}\right)^2 \eta^2 \approx \frac{1}{4\nu^4}\eta^2$	

to the square of the perturbation. In addition to introducing the term of frequency $2\nu\theta$, and some higher-frequency terms, we note that it introduces a constant term (in the even eigenfunction), with a sign opposite to that of η.

From equation (14c), and reference to the last line of Table III, we estimate the maximum lapse-rate to be

$$\mu_{max.} \cong \left(\frac{\pi N}{8\ m_o}\right)^2 \ \frac{\eta^2}{\nu} \tag{20a}$$

$$\approx \frac{\eta^2}{8\ \nu^3} \quad \text{nepers per radian.} \tag{20b}$$

c. The perturbation $\epsilon \cos(\nu - r)\theta + \zeta \cos(\nu + r)\theta$

With $f(\theta) = \epsilon \cos(\nu - r)\theta + \zeta \cos(\nu + r)\theta$, the variational statement (16) becomes

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \epsilon \langle x^2 \cos(\nu - r)\theta \rangle - \zeta \langle x^2 \cos(\nu + r)\theta \rangle = \text{min}, \tag{21}$$

with r taken different from ν and from zero since the special cases $f(\theta) = \lambda \cos 2\nu\theta$ and $f(\theta) = \eta \cos \nu\theta$ have been considered previously in (a) and (b). The two terms, of frequencies $\nu - r$ and $\nu + r$, are considered together here, since, to open up a stop-band at ν, both terms must be present together.[20] The width of the resultant stop-band, specifically, is proportional to the product $\epsilon\zeta$; if only one such perturbation term is present, however, the results obtained here of course still may be used to give the m-value associated with the oscillation frequency ν and to describe the possible forms of betatron oscillation.

Appropriate supplemental terms in the trial functions are selected so that such terms will give cross products with the terms of the original (unperturbed) functions when forming x^2 such as to contribute to $\langle x^2 \cos(\nu - r)\theta \rangle$ or $\langle x^2 \cos(\nu + r)\theta \rangle$. Such terms are evidently of frequency r, $2\nu - r$, $2\nu + r$, $N - r$, $N + r$, $N - 2\nu - r$, $N - 2\nu + r$,

$N + 2\nu - r$, and $N + 2\nu + r$. By inserting such supplemented trial functions into (21), one is led (Appendix IV) to the results listed in Table IV, in which the upper and lower signs respectively refer to the even and odd eigenfunctions. It is seen that, as just mentioned, the width of the stop-band is proportional to the product $\epsilon \zeta$, thus requiring the presence of both a $\cos(\nu - r)\theta$ and a $\cos(\nu + r)\theta$ term in the perturbation. To obtain a simple, low-frequency term (as $\cos \frac{1}{2}\theta$ or $\cos \theta$) in the solution one would take $r = 1/2$ or $r = 1$, although the width of the stop-band and the consequent lapse-rate may not be as great as with larger values of r.

From equation (14c), and reference to the last line of Table IV, we estimate the maximum lapse-rate to be obtained with this type of perturbation as

$$\mu_{max.} \cong \frac{\epsilon \zeta}{4\nu(\nu^2 - r^2)} = \frac{\epsilon \zeta}{4(\nu - r)\nu(\nu + r)} \quad \text{nepers per radian.} \tag{22}$$

4. Introduction of a Phase-Shift in the Perturbation

From equation (14b), Sect. II B 2, it is evident that, with the types of perturbation considered so far, operation in the interior of a stop-band (where growth can occur) will result in a mixture of even and odd terms in the ascending solution. In practice, however, it may be desirable that this solution have zero slope at $\theta = 0$ (the center of the first radially-focusing semi-sector) so that a maximum orbit displacement can occur at that azimuth. An almost equivalent condition, which is slightly simpler to treat, is that the ascending solution after removal of the exponential factor shall be an even function of θ. We indicate below some examples wherein this condition

TABLE IV

COEFFICIENTS OF cos hθ OR sin hθ IN THE EIGENFUNCTIONS CORRESPONDING TO THE STABILITY BOUNDARIES OF FREQUENCY ν FOR THE EQUATION

$$d^2x/d\theta^2 + \left[mF(\theta) + \epsilon\cos(\nu - r)\theta + \gamma\cos(\nu + r)\theta\right]x = 0.$$

The upper sign is for the even (cosine) eigenfunction and the lower sign for the odd (sine) eigenfunction.

h	Coefficient of cos hθ or sin hθ
r	$-\frac{1}{2(\nu^2 - r^2)}\,\frac{1 + (\nu^2 - 2r^2)/N^2}{1 - (3\nu^2 + 2r^2/N^2}\,(\epsilon \pm \gamma) \cong -\frac{1}{2(\nu^2 - r^2)}\left[1 + 4(\nu/N)^2\right](\epsilon \pm \gamma)$
ν	1 [Normalized]
$2\nu - r$	$\frac{1}{2(3\nu - r)(\nu - r)}\,\frac{1 - (7\nu^2 - 8\nu r + 2r^2)/N^2}{1 - (11\nu^2 - 8\nu r + 2r^2)/N^2}\,\epsilon \cong \frac{1}{2(3\nu - r)(\nu - r)}\left[1 + 4(\nu/N)^2\right]\epsilon$
$2\nu + r$	$\frac{1}{2(3\nu + r)(\nu + r)}\,\frac{1 - (7\nu^2 + 8\nu r + 2r^2)/N^2}{1 - (11\nu^2 + 8\nu r + 2r^2)/N^2}\,\gamma \cong \frac{1}{2(3\nu + r)(\nu + r)}\left[1 + 4(\nu/N)^2\right]\gamma$
$N - 2\nu - r$	$\pm\frac{1}{(3\nu + r)(\nu + r)}\,\frac{m}{\pi N^2}\left[1 + \frac{2\nu + r}{N}\right]^2\left[1 + \left(\frac{2\nu + r}{N - \nu}\right)^2\right]\left[1 - \frac{11\nu^2 + 8\nu r + 2r^2}{N^2}\right]^{-1}\gamma$
$N - 2\nu + r$	$\pm\frac{1}{(3\nu - r)(\nu - r)}\,\frac{m}{\pi N^2}\left[1 + \frac{2\nu - r}{N}\right]^2\left[1 + \left(\frac{2\nu - r}{N - \nu}\right)^2\right]\left[1 - \frac{11\nu^2 - 8\nu r + 2r^2}{N^2}\right]^{-1}\epsilon$
$N - \nu$	$\pm\frac{2m}{\pi N^2}\,\frac{1}{(1 - \nu/N)^2}$
$N - r$	$-\frac{1}{\nu^2 - r^2}\,\frac{m}{\pi N^2}\left(1 + \frac{r}{N}\right)^2\left[1 - \frac{3\nu^2 + 2r^2}{N^2}\right]^{-1}\left\{\pm\left[1 + \left(\frac{r}{N - \nu}\right)^2\right]\epsilon + \left[1 + \left(\frac{r}{N + \nu}\right)^2\right]\gamma\right\}$
$N + r$	$-\frac{1}{\nu^2 - r^2}\,\frac{m}{\pi N^2}\left(1 - \frac{r}{N}\right)^2\left[1 - \frac{3\nu^2 + 2r^2}{N^2}\right]^{-1}\left\{\left[1 + \left(\frac{r}{N + \nu}\right)^2\right]\epsilon \pm \left[1 + \left(\frac{r}{N - \nu}\right)^2\right]\gamma\right\}$
$N + \nu$	$\frac{2m}{\pi N^2}\,\frac{1}{(1 + \nu/N)^2}$
$N + 2\nu - r$	$\frac{1}{(3\nu - r)(\nu - r)}\,\frac{m}{\pi N^2}\left[1 - \frac{2\nu - r}{N}\right]^2\left[1 + \left(\frac{2\nu - r}{N + \nu}\right)^2\right]\left[1 - \frac{11\nu^2 - 8\nu r + 2r^2}{N^2}\right]^{-1}\epsilon$
$N + 2\nu + r$	$\frac{1}{(3\nu + r)(\nu + r)}\,\frac{m}{\pi N^2}\left[1 - \frac{2\nu + r}{N}\right]^2\left[1 + \left(\frac{2\nu + r}{N + \nu}\right)^2\right]\left[1 - \frac{11\nu^2 + 8\nu r + 2r^2}{N^2}\right]^{-1}\gamma$
$\frac{m - m_o}{m_o}$	$\frac{1}{2}\left(\frac{\pi N}{4m_o}\right)^2\left[\frac{\epsilon^2}{(3\nu - r)(\nu + r)} \pm \frac{\epsilon\gamma}{\nu^2 - r^2} + \frac{\gamma^2}{(3\nu + r)(\nu - r)}\right]$
Relative Width, $\frac{W}{m_o} = \frac{m_{even} - m_{odd}}{m_o}$	$\left(\frac{\pi N}{4m_o}\right)^2\frac{\epsilon\gamma}{\nu^2 - r^2}$

is met by introduction of phase-shifts into the applied perturbation and by suitable adjustments of the coefficients.

a. The perturbation $- \xi_1 \left[\sin 60^\circ \cos(7\theta + 52.5^\circ) + \sin 52.5^\circ \cos(8\theta + 60^\circ)\right]$, with $\nu = 7.5$ and $N = 48$

As a specific example we consider an accelerator with $N = 48$, and make use of a stop-band at $\nu = 7\frac{1}{2}$ which results from a perturbation such that the differential equation reads

$$d^2x/d\theta^2 + \left\{ m F(\theta) - \xi_1 \left[\sin 60^\circ \cos(7\theta + 52.5^\circ) + \sin 52.5^\circ \cos(8\theta + 60^\circ)\right]\right\} x = 0 \quad (23a)$$

or

$$d^2x/d\theta^2 + \left\{ m F(\theta) - \xi_1 \left[\sin 60^\circ \cos 7(\theta + 7.5^\circ) + \sin 52.5^\circ \cos 8(\theta + 7.5^\circ)\right]\right\} x = 0. \quad (23b)$$

By setting $\varphi = \theta + 7.5^\circ$ this is seen to assume the standard form for which Table IV applies, since, with $N = 48$, $F(\theta)$ is periodic with a period $2\pi/48$ or 7.5°:

$$d^2x/d\varphi^2 + \left[m F(\theta) - \xi_1 (\sin 60^\circ \cos 7\varphi + \sin 52.5^\circ \cos 8\varphi)\right] x = 0, \quad (24)$$

for which $\epsilon = - \xi_1 \sin 60^\circ$, $\zeta = - \xi_1 \sin 52.5^\circ$, $r = 1/2$.

By reference to the first line of Table IV we note that the coefficient of the $\cos\frac{1}{2}\theta$ and $\sin\frac{1}{2}\theta$ terms which arise in the perturbed eigenfunctions may be written as of the form

$$- k(\epsilon \pm \zeta)$$

or, in the present example,

$$k\, \xi_1 (\sin 60^\circ \pm \sin 52.5^\circ).$$

The ascending solution prevailing in the interior of the stop-band would then be expected to be of the form [cf. (8)]:

$$x = e^{\mu\varphi}\Big\{C\big[\cos 7.5\varphi + k\,\xi_1\,(\sin 60^\circ + \sin 52.^\circ5)\cos\tfrac{1}{2}\varphi + \cdots\big]$$
$$+ S\big[\sin 7.5\varphi + k\,\xi_1\,(\sin 60^\circ - \sin 52.^\circ5)\sin\tfrac{1}{2}\varphi + \cdots\big]\Big\}. \quad (25)$$

Since, by (14b), the ratio C/S is given at least approximately by $\sqrt{(m - m_{odd}) / (m_{even} - m)}$, we can, by suitable choice of m, arrange to have C/S = tan 33.°75. The solution (25) may then be written

$$x = A_1 e^{\mu\varphi}\Big\{\sin 33.^\circ75\big[\cos 7.5\varphi + k\,\xi_1\,(\sin 60^\circ + \sin 52.^\circ5)\cos\tfrac{1}{2}\varphi + \cdots\big]$$
$$+ \cos 33.^\circ75\big[\sin 7.5\varphi + k\,\xi_1\,(\sin 60^\circ - \sin 52.^\circ5)\sin\tfrac{1}{2}\varphi + \cdots\big]\Big\}$$

$$= A_1 e^{\mu\varphi}\Big\{\cos 56.^\circ25 \cos 7.5\varphi + \sin 56.^\circ25 \sin 7.5\varphi$$
$$+ k\,\xi_1 \sin 67.^\circ5\big[\cos 3.^\circ75 \cos\tfrac{1}{2}\varphi + \sin 3.^\circ75 \sin\tfrac{1}{2}\varphi\big] + \cdots\Big\}$$

$$= A_1 e^{\mu\varphi}\big[\cos 7.5(\varphi - 7.5) + k\,\xi_1 \sin 67.^\circ5 \cos\tfrac{1}{2}(\varphi - 7.5) + \cdots\big]$$

$$= A\, e^{\mu\varphi}\big[\cos 7.5\,\theta + k\,\xi_1 \sin 67.^\circ5 \cos\tfrac{1}{2}\theta + \cdots\big], \quad (26)$$

which is of the form desired. The appropirate value of m would be expected to be roughly

$$m \doteq 0.69\, m_{odd} + 0.31\, m_{even}, \quad (27)$$

which although not centrally located, is comfortably within the zone of instability and [by (14a)] should lead to a lapse-rate estimated as some 92.5 per cent of μ_{max}. In practice, the value of m most suitable for the present purpose might be determined by empirical computation, to achieve a condition such that the pure ascending solution (which soon becomes the dominant solution) is characterized at $\theta = 0$ by zero slope.[21]

b. The perturbation $-\xi_2 \cos(7\theta + 45^\circ)$, with $\nu = 7$ and N = 48

As a second specific example we consider an accelerator in which again N = 48, and make use of a stop-band at $\nu = 7$ which results from a perturbation such that the differential equation is

$$d^2x/d\theta^2 + \left[m\ F\ (\theta) - \xi_2 \cos\ (7\ \theta + 45^o)\right]\ x = 0. \tag{28}$$

By setting $\phi = \theta - 45^o$ this equation becomes of the standard form for which Table III applies, since an interval of 45^o corresponds to a whole number of periods for F (θ):

$$d^2x/d\varphi^2 + \left[m\ F\ (\varphi) - \xi_2 \cos 7\ \phi\right] x = 0. \tag{29}$$

By reference to Table III and equation (14b) one is thus led to expect an ascending solution, at the center of the $\nu = 7$ stop-band, of the form

$$\begin{aligned} x &= A_1\ e^{\mu\varphi}\left\{k\ \xi_2 + (\cos 7\varphi + \sin 7\varphi) + \cdots\right\} \\ &= A_1\ e^{\mu\varphi}\left\{k\ \xi_2 + \left[\cos 7\ (\theta - 45^o) + \sin 7\ (\theta - 45^o)\right] + \cdots\right\} \\ &= A_1\ e^{\mu\varphi}\left\{k\ \xi_2 + \left[\cos\ (7\ \theta + 45^o) + \sin\ (7\ \theta + 45^o)\right] + \cdots\right\} \\ &= \sqrt{2}\ A_1\ e^{\mu\varphi}\left\{k\ \xi_2/\sqrt{2} + \cos 7\ \theta + \cdots\right\} \\ &= A\ \ e^{\mu\theta}\left\{k\ \xi_2\ /\ \sqrt{2} + \cos 7\ \theta + \cdots\right\}, \end{aligned} \tag{30}$$

which may be representative of a useful form for the unstable orbits.

The location of the operating point which would be chosen in this case would be very close to the <u>center</u> of the stop-band [(14b)] and, accordingly, one would expect a lapse-rate virtually equal to the maximum [<u>cf.</u> (14a)].

c. The perturbation $-\ \xi_3\left[\sin 60^o \cos\ (6\ \theta + 45^o) + \sin 45^o \cos\ (8\ \theta + 60^o)\right]$,

with $\nu = 7$ and N = 48

As another example in which $\nu = 7$ and N = 48 we consider the case governed by the differential equation

$$d^2x/d\theta^2\ \left\{m\ F\ (\theta) - \xi_3\left[\sin 60^o \cos\ (6\ \theta + 45^o) + \sin 45^o \cos\ (8\ \theta + 60^o)\right]\right\}\ x = 0. \tag{31}$$

Here, if we set $\varphi = \theta + 7\overset{o}{.}5$, we again obtain an expression of the standard form for which Table IV applies:

$$d^2x/d\varphi^2 + \left[m\,F(\varphi) - \xi_3 (\sin 60^\circ \cos 6\varphi + \sin 45^\circ \cos 8\varphi)\right] x = 0. \quad (32)$$

By reference to equation (8) and Table IV, we see that the ascending solution prevailing in the interior of the stop-band would be expected to have the form:

$$x = e^{\mu\varphi}\Big\{C\left[\cos 7\varphi + k\,\xi_3 (\sin 60^\circ + \sin 45^\circ) \cos\varphi + \cdots\right] + S\left[\sin 7\varphi + k\,\xi_3 (\sin 60^\circ - \sin 45^\circ) \sin\varphi + \cdots\right]\Big\}. \quad (33)$$

A choice of m such that $C/S = \tan 37.^\circ 5$ permits one then to write the solution (33) as

$$x = A_1 e^{\mu\varphi}\Big\{\sin 37.^\circ 5 \left[\cos 7\varphi + k\,\xi_3 (\sin 60^\circ + \sin 45^\circ) \cos\varphi + \cdots\right] + \cos 37.^\circ 5\left[\sin 7\varphi + k\,\xi_3 (\sin 60^\circ - \sin 45^\circ) \sin\varphi + \cdots\right]\Big\}$$

$$= A_1 e^{\mu\varphi}\Big\{\cos 52.^\circ 5 \cos 7\varphi + \sin 52.^\circ 5 \sin 7\varphi + k\,\xi_3 \sin 75^\circ \left[\cos 7.^\circ 5 \cos\varphi + \sin 7.^\circ 5 \sin\varphi\right] + \cdots\Big\}$$

$$= A_1 e^{\mu\varphi}\left[\cos 7(\varphi - 7.^\circ 5) + k\,\xi_3 \sin 75^\circ \cos(\varphi - 7.^\circ 5) + \cdots\right]$$

$$= A\,e^{\mu\theta}\left[\cos 7\theta + k\,\xi_3 \sin 75^\circ \cos\theta + \cdots\right], \quad (34)$$

which is of the form sought. The appropriate value of m to be selected would be [by (14b)] roughly

$$m \doteq 0.63\, m_{odd} + 0.37\, m_{even}, \quad (35)$$

for which [by (14a)] the resultant lapse-rate would be expected to be about 96.6 per cent of μ_{max}. In practice, of course, the value of m most suitable for the present purpose might be best determined by empirical computation.[21]

d. The perturbation $-\xi_4 \cos 4\theta - \xi_5 \cos(10\theta + 90^\circ)$, with $\nu = 5$ and $N = 24$

An effective and useful perturbation might employ one term of the form $\cos(2\nu\theta + \delta_1)$ to open up a stop-band and a second term of the form $\cos\left[(\nu - 1)\theta + \delta_2\right]$ to provide a term $\cos\theta$ in the solution. To illustrate this possibility we consider a case in which N = 24, the stop-band of interest corresponds to $\nu = 5$, and the differential equation is

$$d^2x/d\theta^2 + \left[m\,F(\theta) - \xi_4 \cos 4\theta - \xi_5 \cos(10\theta + 90^\circ)\right] x = 0. \tag{36}$$

The last term, of frequency 10 (= 2ν) in equation (35) will, as noted in Table II, serve to open up a stop-band at $\nu = 5$ with a width directly proportional to the strength of the perturbation (ξ_5).

To determine the expected character of the solutions, in regard to their azimuthal dependence, we put $\varphi = \theta + 45^\circ$ and equation (35) becomes

$$d^2x/d\varphi^2 + \left[m\,F(\varphi) + \xi_4 \cos 4\varphi - \xi_5 \cos 10\varphi\right] x = 0, \tag{37}$$

since, with N = 24, the periodicity of F insures that $F(\theta) = F(\varphi)$. By reference to the first line of Table IV we thus see that in the center of the stop-band the ascending solution would be expected to be of the following form, through the lower frequency terms:

$$\begin{aligned} x &= A_1 e^{\mu\varphi}\left[\cos 5\varphi + \sin 5\varphi - 0.025\,\xi_4(\cos\varphi + \sin\varphi) + \cdots\right] \\ &= -\sqrt{2}\,A_1 e^{\mu\varphi}\left[\cos 5\theta + 0.025\,\xi_4 \cos\theta + \cdots\right] \\ &= A\, e^{\mu\theta}\left[\cos 5\theta + 0.025\,\xi_4 \cos\theta + \cdots\right]; \end{aligned} \tag{38}$$

This may be a useful form for the ascending unstable orbit, with the lapse-rate controlled by the coefficient ξ_5 in (36).

We turn now to some computational tests, intended to check and illustrate the analytic work of this section.

III. COMPUTATIONS

A. Method

Computations were performed with the MURA IBM 704 computer to check and illustrate the general character of the knock-out phenomena described by the analysis presented in this report. The examples may not be ideal for illustrative purposes, since the azimuthal dependence of the solutions may be felt to be not markedly changed by the perturbation in these examples--on the other hand it may be expected that the theory should perform fairly well, in a quantitative sense, with perturbations as small as those used here and that such perturbations, moreover, should not be excessively difficult to realize technically.

In performing the computations, solution of the differential equation for x was accomplished by use of the DUCK-BUMP program,[22, 23] in which integration is by a Runge-Kutta method, in fixed-point, and the square-wave function F (θ) could be generated by use of suitable "bumps" in the ψ channel of the program. Fourier analysis of the resulting orbits was implemented by the FORANAL program.[24] In one example the particle motion was also studied by successive matrix multiplications, in which case use was made of the MESSY-MESSY program.[25]

The initial DUCK-BUMP computations were made for $\nu = 7\frac{1}{2}$, with a perturbation ϵ (cos 7 θ + cos 8 θ) and ϵ given the values $\epsilon = 6$ or $\epsilon = 6 / \sqrt{3} = 2\sqrt{3}$. A few side checks were made in addition, however, (i) to verify that the expected betatron frequency was correctly given by the program when no perturbation was present, and (ii) to compare the results for $\epsilon = -6$ with those found for $\epsilon = +6$.

In the following sections we report the results of computations made, in turn, for examples in which $\nu = 7\frac{1}{2}$, $\nu = 7$, and $\nu = 5$. In the first case the perturbation employs cos 7 θ and cos 8 θ, so that circular functions of argument $\frac{1}{2}\theta$ may be expected to arise in the solution. In the second case this same type of perturbation is also studied, so that both a constant term and circular functions of argument θ may be expected to arise; in addition, a perturbation in which $\cos 7\theta$ is present alone is also studied. In the case $\nu = 5$, the perturbation employs $\cos 10\theta$ and $\cos 4\theta$ terms, the first to open up the stop-band and the second to control the solution so that circular functions of argument θ appear. In each case the location of the stability boundaries and resultant stop-band width are studied, the Fourier composition of the eigenfunctions examined, the lapse-rate within the stop-band determined, and the utility of phase-shifts illustrated.

B. Computations in which $\nu = 7\frac{1}{2}$

[N = 48]

1. Stability Boundaries for a Perturbation $\epsilon\,(\cos 7\theta + \cos 8\theta)$:

With $\nu = 7\frac{1}{2}$, N = 48, and $f(\theta) = \epsilon\,(\cos 7\theta + \cos 8\theta)$, a number of DUCK-BUMP computations was made to obtain results suitable for comparison with the analytic theory summarized in Table IV. Runs made to determine the values of m associated with the stability boundaries for $\epsilon\ (= \zeta) = 6$ led to

$$m_{even} = 384.426,$$

$$m_{odd} = 381.999,$$

$$\text{Width} = m_{even} - m_{odd} = 2.427;$$

this width is seen, from the summary presented in Table V below, to be in reasonably good agreement with that expected from the analytic theory.

2. Eigenfunctions for a Perturbation $\epsilon\,(\cos 7\theta + \cos 8\theta)$:

The x-values for the eigensolutions associated with the boundaries of the $\nu = 7\frac{1}{2}$ stop-band were entered into the FORANAL program (192 points per problem, corresponding to $\Delta\theta = 4\pi$) to obtain the Fourier coefficients for which analytic estimates were presented in Table IV. To insure that terms which involve the perturbation to first order could be distinguished from higher-order terms, additional runs of this same sort were made for ϵ (= ζ) = $6 / \sqrt{3} = 2\sqrt{3}$. It appeared from this work that the terms expected to show zero- or first-order dependence on the strength of the perturbation did, in fact, rather accurately show the expected difference, while all other coefficients appeared to be definitely of higher order.

Before introducing a quantitative comparison of theory with the computational results, it should be mentioned that the FORANAL program,[24] as applied here to half-integral eigensolutions, provides coefficients only through those of harmonic order $h = 47\frac{1}{2}$. It must moreover be noted that the Fourier coefficients printed by the FORANAL program are necessarily influenced, because of the discrete nature of the input data, by higher-order coefficients for the true function.[26] In effect, cosine coefficients of order h as printed should be interpreted as having been supplemented by the sum of other cosine coefficients of order $96\,M \pm h$ (M denoting an integer); similarly the sine coefficient of order h is to be regarded as supplemented by other sine coefficients of order $96\,M + h$ and decreased by those of order $96\,M - h$. A reasonable comparison might thus best be made between a computed cosine coefficient and the sum of the analytic values for h and $96 - h$; likewise a comparison might be made between a computed sine coefficient and the difference of the analytic values for h and $96 - h$.

Table V summarizes this computational work and compares the results with the theoretical expressions listed in Table IV. For the odd eigenfunction

TABLE V

COSINE COEFFICIENTS, C_h, AND SINE COEFFICIENTS, S_h, OF ORDER h IN THE FOURIER EXPANSION OF EVEN AND ODD EIGENFUNCTIONS FOR THE EQUATION

$$d^2x/d\theta^2 + \left[mF(\theta) + \epsilon\,(\cos 7\theta + \cos 8\theta)\right] x = 0, \text{ with } \nu = 7\ 1/2 \text{ and } N = 48.$$

[The digitally computed coefficients are based on $\epsilon = 2\sqrt{3}$; the stop-band widths, on $\epsilon = 6$]

[$m_o = 382.429$]

FOR EVEN EIGENFUNCTION			FOR ODD EIGENFUNCTION		
	Value			Value	
Coefficient	From Analytic Result (Table IV)	From Digital Computations	Coefficient	From Analytic Result (Table IV)	From Digital Computations
$h = r = 1/2$					
$\frac{1}{\epsilon} C_{1/2}$	-0.020	-0.020	$\frac{1}{\epsilon} S_{1/2}$	0	- - -
$h = \nu = 7.5$					
$C_{7.5}$	1 [Normalized]	1 [Normalized]	$S_{7.5}$	1 [Normalized]	1 [Normalized]
$h = 2\nu - r = 14.5$					
$\frac{1}{\epsilon} C_{14.5}$	0.0037	0.0037	$\frac{1}{\epsilon} S_{14.5}$	0.0037	0.0037
$h = 2\nu + r = 15.5$					
$\frac{1}{\epsilon} C_{15.5}$	0.0031	0.0031	$\frac{1}{\epsilon} S_{15.5}$	0.0031	0.0031
$h = N - 2\nu - r = 32.5$					
$\frac{1}{\epsilon} C_{32.5}$	0.00080		$\frac{1}{\epsilon} S_{32.5}$	-0.00080	
$h = N + 2\nu + r = 63.5$					
$\frac{1}{\epsilon} C_{63.5}$	0.00020		$\frac{1}{\epsilon} S_{63.5}$	0.00020	
$\frac{C_{32.5} + C_{63.5}}{\epsilon}$	0.00100	0.00096	$\frac{S_{32.5} - S_{63.5}}{\epsilon}$	-0.00100	-0.00096
$h = N - 2\nu + r = 33.5$					
$\frac{1}{\epsilon} C_{33.5}$	0.00088		$\frac{1}{\epsilon} S_{33.5}$	-0.00088	
$h = N + 2\nu - r = 62.5$					
$\frac{1}{\epsilon} C_{62.5}$	0.00024		$\frac{1}{\epsilon} S_{62.5}$	0.00024	
$\frac{C_{33.5} + C_{62.5}}{\epsilon}$	0.00112	0.00109	$\frac{S_{33.5} - S_{62.5}}{\epsilon}$	-0.00112	-0.00108

TABLE V
(continued)

FOR EVEN EIGENFUNCTION			FOR ODD EIGENFUNCTION		
Coefficient	Value		Coefficient	Value	
	From Analytic Result (Table IV)	From Digital Computations		From Analytic Result (Table IV)	From Digital Computations
$h = N - \nu = 40.5$					
$C_{40.5}$	0.148		$S_{40.5}$	-0.148	
$h = N + \nu = 55.5$					
$C_{55.5}$	0.079		$S_{55.5}$	0.079	
$C_{40.5} + C_{55.5}$	0.227	0.221	$S_{40.5} - S_{55.5}$	-0.227	-0.220
$h = N - r = 47.5$					
$\frac{1}{\epsilon} C_{47.5}$	-0.0021		$\frac{1}{\epsilon} S_{47.5}$	~ 0	- - -
$h = N + r = 48.5$					
$\frac{1}{\epsilon} C_{48.5}$	-0.0020		$\frac{1}{\epsilon} S_{48.5}$	~ 0	- - -
$\frac{C_{47.5} + C_{48.5}}{\epsilon}$	-0.0041	-0.0039	$\frac{S_{47.5} - S_{48.5}}{\epsilon}$	~ 0	- - -
$\frac{1}{\epsilon^2} \frac{m - m_o}{m_o}$	0.000145	0.000145	$\frac{1}{\epsilon^2} \frac{m - m_o}{m_o}$	-0.000029	-0.000031
$\frac{1}{\epsilon^2} \cdot$ (Relative Width) $\equiv \frac{1}{\epsilon^2} \frac{m_{even} - m_{odd}}{m_o}$				Analytic: 0.000174	Observed: 0.000176

obtained with $\epsilon = \zeta = 6$, the Fourier coefficient associated with $\sin \frac{1}{2} \theta$ is, as expected, quite small ($\cong -5.6 \times 10^{-4}$ in comparison to the unit coefficient taken for $\sin 7.5\,\theta$). For the even eigenfunction, the coefficients corresponding to $h = 6.5$ and $h = 8.5$, although varying as ϵ^2, were found to become virtually as large in absolute value as the coefficient corresponding to $h = 14.5$ ($\pm$ 0.021 vs. 0.023), for ϵ $(= \zeta) = 6$.

3. Computations Near the Center of the $\nu = 7\frac{1}{2}$ Stop-Band, for a Perturbation 6 (cos 7 θ + cos 8 θ):

With $\epsilon = \zeta = 6$ and no phase shift present in the perturbation, computations made with m = 383.213--i.e., near the center of the $\nu = 7\frac{1}{2}$ stop-band--indicated that the ascending solution could be obtained by use of initial conditions such that $p_o/x_o = 0.032_{035}$. The lapse-rate was found to be given by

0.1632 nepers/revolution, or 0.0709 decades/revolution,

corresponding to an increase by a factor 1.1773 each revolution. This computational result for the lapse-rate may be compared with the value

0.137 nepers/revolution, or 0.059 decades/revolution,

implied by the value $\mu = 0.0218$ nepers/radian given by the analytic result (14a) when using the observed values for m at the boundaries.

When the exponential increase was divided out from the solution, the remaining periodic azimuthal dependence was found to be a sum of even and odd functions which were respectively very close in form to the eigenfunctions, c (θ) and s (θ), found at the zone boundaries in the previous computations. For the value of m employed (m = 383.213), the ratio C/S = 0.994, or virtually unity, was suggested by forming the ratio of the coefficients of cos 7.5 θ and sin 7.5 θ in the Fourier analysis of the computational results. In terms of equation (14b), this ratio is consistent with the fact that m was taken virtually in the center of the stop-band.

4. Computations with a Phase Shift Present in the Perturbation:

Motivated by the development outlined in Sect. II B 4a, a computational investigation was made for the perturbation[27]

$$f(\theta) = -7.231\,622\,101 \left[\sin 60^\circ \cos(7\theta + 52.^\circ5) + \sin 52.^\circ5 \cos(8\theta + 60^\circ)\right]$$

$$= -6.262\,768\,45 \cos(7\theta + 52.^\circ5) - 5.737\,231\,55 \cos(8\theta + 60^\circ) .$$

For this perturbation, the boundaries of the $\nu = 7\frac{1}{2}$ stop-band were found to be located at

$$m_{even} = 384.427 \qquad \text{and} \qquad m_{odd} = 382.001 ,$$

corresponding to a width $m_{even} - m_{odd} = 2.426$. This width may be compared with the value expected from the analytic theory, namely 2.38.

It was then determined from the computations that selection of the value $m = 382.91_{75}$ would lead, for this perturbation, to the ascending unstable solution being characterized by $dx/d\theta = 0$ at $\theta = 0$, as desired. A run made under these conditions exhibited a lapse-rate

$$0.158\dot{2} \text{ nepers/revolution} \qquad (\mu = 0.0252 \text{ nepers/radian})$$

or $\qquad 0.0687$ decades/revolution,

corresponding to growth by a factor 1.171_{35} per revolution. With the exponential factor divided out from the solution, the azimuthal dependence with this strength perturbation was found to be such that the coefficients of $\cos\frac{1}{2}\theta$ and $\cos 7.5\,\theta$ terms were roughly in the ratio 0.069. These results are illustrated in Fig. 2.

As we have seen previously [cf. (22)], one expects the rate of exponential growth to be proportional to the product $\epsilon\,\zeta$. By comparison of the parameters in the present case with thoseemployed in the tests reported in sub-section2 above ($\epsilon = \zeta = 6$) it is therefore to be expected that the lapse-rates would be similar in these two cases; specifically, we are not operating quite in the center of the stop-band and may expect the present lapse-rate to be 0.97 ($\epsilon \xi /36$) times that found previously, so that we indeed may expect 0.158 nepers/revolution on this basis. From the observed values of m at the stability boundaries, equation (14a) would give $\mu = 0.0211$ nepers/radian, or 0.133 nepers/revolution, directly.

By reference to equation (26), and taking k = 0.010 [from Table V, in which results for $\epsilon = \zeta$ are summarized], we would expect the ratio of the $\cos \frac{1}{2} \theta$ to $\cos 7.5\ \theta$ coefficients to be given by

$$k\, \xi_1 \sin 67^\circ\!.5 = 0.010 \times 7.231\ 622\ 101 \times 0.9238\ 7953 = 0.067.$$

From these results we infer that the development of Sect. II B 4a is at least semi-quantitatively valid.

Additional computations for a perturbation of essentially this same form were performed by matrix multiplication, using the MESSY-MESSY program, and are reported later (Sect. E and Fig. 3).

C. Computations in which $\nu = 7$

[N = 48]

1. Stability Boundaries for a Perturbation $\eta\,(\cos 7\,\theta + \cos 8\,\theta)$:

With $\nu = 7$, N = 48, and $f(\theta) = \eta\,(\cos 7\,\theta + \cos 8\,\theta)$, a number of DUCK-BUMP computations was made to obtain results suitable for comparison with the analytic theory summarized in Tables III and IV. It will be recognized that for $\nu = 7$ the term $\eta \cos 8\,\theta$ in the perturbation $f(\theta)$ is expected to affect the location of the stop-band boundaries, but not contribute to the width of the resonance; the term $\eta \cos 7\,\theta$ is expected to lead to the appearance of a constant term in the Fourier expansion of the even eigenfunction, while the $\eta \cos 8\,\theta$ perturbation would engender a $\cos \theta$ term in the solution.

Runs made to determine the stability boundaries for η (= ζ) = 6 led to

$$m_{even} = 360.349,$$

$$m_{odd} = 358.943,$$

$$\text{Width} = m_{even} - m_{odd} = 1.406;$$

this width (and the location of the individual boundaries) is seen, from the summary presented in Table VI below, to be in reasonably good agreement with the analytic theory. With the magnitude of the perturbation reduced by a factor $1/\sqrt{3}$, so that η (= ζ) $=6/\sqrt{3} = 2\sqrt{3}$, it was found that the boundaries were given by

$$m_{even} = 359.234,$$

$$m_{odd} = 358.765,$$

$$\text{Width} = m_{even} - m_{odd} = 0.469;$$

the width is thus seen to be proportional to the square of the perturbation, as expected.

2. Eigenfunctions for a Perturbation η (cos 7 θ + cos 8 θ):

The x-values for the eigensolutions associated with the boundaries of the ν = 7 stop-band were entered into the FORANAL program (192 points in an interval $\Delta\theta = 2\pi$) to obtain the Fourier coefficients desired for comparison with the analytic theory. Output from the FORANAL runs then gave values for Fourier coefficients ostensibly through the 96th order. Only for the coefficients listed was the dependence on the strength of the perturbation found to be of order no higher than the first, save for such higher Fourier coefficients as that with h = 89 (the cosine or sine coefficients amounting to $\pm$ 0.0037 in this case, relative to the component

TABLE VI

COSINE COEFFICIENTS, C_h, AND SINE COEFFICIENTS, S_h, OR ORDER h IN THE FOURIER EXPANSION OF EVEN AND ODD EIGENFUNCTIONS FOR $d^2x/d\theta^2 + [mF(\theta) + \eta(\cos 7\theta + \cos 8\theta)]x = 0$, with $\nu = 7$ and N = 48.

$[m_o = 358.676]$

FOR EVEN EIGENFUNCTION			FOR ODD EIGENFUNCTION		
Coefficient	Value: From Analytic Result (Tables III & IV)	Value: From Digital Computation	Coefficient	Value: From Analytic Result (Tables III & IV)	Value: From Digital Computation
h = 0 $\frac{1}{\eta} C_o$	-0.0113	-0.0110	—	- - - -	- - - -
h = r = 1 $\frac{1}{\eta} C_1$	-0.0114	-0.0112	$\frac{1}{\eta} S_1$	0.0114	0.0113
h = ν = 7 C_7	1 [Normalized]	1 [Normalized]	S_7	1 [Normalized]	1 [Normalized]
h = 2ν = 14 $\frac{1}{\eta} C_{14}$	0.0037_8	0.00381	$\frac{1}{\eta} S_{14}$	0.0037_8	0.00380
h = 2ν + r = 15 $\frac{1}{\eta} C_{15}$	0.0031_7	0.00315	$\frac{1}{\eta} S_{15}$	0.0031_7	0.00319
h=N-2ν-r=33 $\frac{1}{\eta} C_{33}$	0.00074	0.00072	$\frac{1}{\eta} S_{33}$	-0.00074	-0.00073
h=N-2ν = 34 $\frac{1}{\eta} C_{34}$	0.00082	0.00081	$\frac{1}{\eta} S_{34}$	-0.00082	-0.00081
h = N-ν = 41 C_{41}	0.1358	0.1338	S_{41}	-0.1358	-0.1336
h = N - r = 47 $\frac{1}{\eta} C_{47}$	-0.00115	-0.00111	$\frac{1}{\eta} S_{47}$	-0.00115	-0.00112
h = N = 48 $\frac{1}{\eta} C_{48}$	-0.0021_9	-0.00206	$\frac{1}{\eta} S_{48}$	-0.000013	-0.000013_3
h = N + r = 49 $\frac{1}{\eta} C_{49}$	-0.00106	-0.00100	$\frac{1}{\eta} S_{49}$	0.00106	0.00101
h = N + ν = 55 C_{55}	0.0755	0.0721	S_{55}	0.0755	0.0720
h = N + 2ν=62 $\frac{1}{\eta} C_{62}$	0.00024	0.00022	$\frac{1}{\eta} S_{62}$	0.00024	0.00022
h=N+2ν+r=63 1 C_{63}	0.00019	0.00018	$\frac{1}{\eta} S_{63}$	0.00019	0.00018
$\frac{1}{\eta^2} \frac{m - m_o}{m_o}$	0.000143 or 0.000136	0.000130	$\frac{1}{\eta^2} \frac{m - m_o}{m_o}$	0.000 023	0.000 021
$\frac{1}{\eta^2}$ (Relative Width) $\equiv \frac{1}{\eta^2} \frac{m_{even} - m_{odd}}{m_o}$				Analytic: 0.000 120 or 0.000 113	Observed: 0.000 109

with h = 7, independent of η) which corresponds to h = 2 N - ν and may be omitted from our summary since such terms were ignored in the analysis.

The comparison of the computational and analytic results is given in Table VI. The theoretical values are obtained by substitution into the formulas listed in Tables III and IV, with $\eta = \zeta$ and r = 1, so that (for example) one expects the stability boundaries in this case to be given by the sum of the analytic expressions shown in these Tables.

3. Computations Near the Center of the ν = 7 Stop-Band, for a Perturbation 6 (cos 7 θ + cos 8 θ):

With $\eta = \zeta = 6$ and no phase shift present in the perturbation, computations made with m = 359.646--i.e., at the center of the ν = 7 stop-band--indicated that the ascending solution could be obtained by use of initial conditions such that $p_o/x_o = 0.34_{455}$. The lapse-rate was found to be given by

0.0934 nepers/revolution, or 0.0406 decades/revolution,

corresponding to an increase by a factor 1.0979 each revolution. This computational result for the lapse-rate may be compared with the value

0.080 nepers/revolution, or 0.035 decades/revolution,

implied by the value μ = 0.013 nepers/radian given by the analytic result (14a, c) when using the observed values for m at the boundaries.

When the exponential increase was divided out from the solution, the remaining periodic azimuthal dependence was found to be a sum of even and odd functions which were respectively very close in form to the eigenfunctions, c (θ) and s (θ), found at the zone boundaries in the previous computations. The ratio C/S = 0.996, or virtually unity, was suggested by forming the ratio of the coefficients of $\cos 7\theta$ and $\sin 7\theta$ in the Fourier analysis of the computational results. In terms of equation (14b), a ratio near unity is in accord with the fact that m was selected to lie at the center of the stop-band.

4. Computations with a Phase Shift Present in the Perturbation:

a. The perturbation $-8 \cos (7\theta + 45^{\circ})$

To follow up the development outlined in Sect. II B 4b, a computational investigation was made of the perturbation

$$f(\theta) = -8 \cos (7\theta + 45^{\circ}) .$$

For this perturbation, the boundaries of the $\nu = 7$ stop-band were found to occur at

$$m_{even} = 360.830 \qquad \text{and} \qquad m_{odd} = 358.219,$$

corresponding to a width $m_{even} - m_{odd} = 2.61_1$. This observed width is in fair agreement with the values 2.80, 2.59, and 2.39 suggested by the analytic formulas presented in the last line of Table III for $\eta = -8$.

For the ascending solution to be correctly launched within the $\nu = 7$ stop-band at $\theta = 0$ with $dx/d\theta = 0$, it was then found that one should take $m = 359.653$. A run made under these conditions showed a lapse-rate

0.1721 nepers/revolution ($\mu = 0.0274$ nepers/radian)

or 0.0747 decades/revolution,

corresponding to growth by a factor 1.188 each revolution. With the exponential factored out from the solution, the ratio of the constant term to the coefficient of the $\cos 7\theta$ term was 0.065_4. These results are illustrated in Fig. 4.

The expected lapse-rate, computed from the location of the operating point with respect to the observed stability boundaries, is found, from (14a), to be given by

$$\mu = 0.0234 \text{ nepers/radian, or } 0.147 \text{ nepers/revolution.}$$

By reference to (30) and Table VI, the expected ratio of the constant term to the coefficient of cos 7 θ would be 0.062 or 0.064. Again there appears to be reasonably good, semi-quantitative agreement between the expected values for these quantities and the values found computationally.

b. The perturbation $-7.628\,093\,8847\left[\sin 60^{\circ} \cos(6\theta + 45^{\circ}) + \sin 45^{\circ} \cos(8\theta + 60^{\circ})\right]$

As an example of the development presented in Sect. II B 4c, a series of computations was made with a perturbation given by[28]

$$f(\theta) = -7.628\,093\,8847\left[\sin 60^{\circ} \cos(6\theta + 45^{\circ}) + \sin 45^{\circ} \cos(8\theta + 60^{\circ})\right]$$

$$= -6.606\,123\,086_5 \cos(6\theta + 45^{\circ}) - 5.393\,876\,913_5 \cos(8\theta + 60^{\circ}) .$$

For this perturbation the boundaries of the $\nu = 7$ stop-band were found to lie at

$$m_{even} = 361.122 \qquad \text{and} \qquad m_{odd} = 358.155,$$

corresponding to a width 2.96_7. This width compares well with that expected on the basis of the analytic theory [from the last line of Table IV and ref. 28], namely 2.94. For the ascending unstable solution within the stop-band to be characterized by $dx/d\theta = 0$ at $\theta = 0$, it was then found that one should select $m = 359.494$. With this value of m, a lapse-rate

$$0.1958 \text{ nepers/revolution} \qquad (\mu = 0.03116 \text{ nepers/radian})$$

$$\text{or} \qquad 0.0850_4 \text{ decades/revolution,}$$

corresponding to growth by a factor 1.2163 each revolution, was obtained.

The azimuthal dependence of this solution, with the exponential factor divided out, was found to be characterized by a ratio of the cos θ and cos 7 θ coefficients given roughly by 0.091_8. These results are illustrated in Fig. 5.

The resonance width for the present type of perturbation is expected [Tables III and IV] to be about $\frac{2\nu^2}{\nu^2 - r^2} \frac{\epsilon \cdot \zeta}{\eta^2}$, or $\frac{98}{48} \frac{\epsilon \cdot \zeta}{\eta^2}$, times as great as for a perturbation η cos 7 θ; hence, to compare with the present results with those found in sub-section 3, we may (taking account of the location of the operating points within their respective stop-bands) multiply the lapse-rate in the latter case by $0.995 \frac{98}{48} \frac{\epsilon \cdot \zeta}{36}$ to obtain an expected 0.20_1 nepers/revolution for the present problem. More directly, from the observed values of m at the stability boundaries, equation (14a) would predict

$$\mu = 0.027 \text{ nepers/radian, or } 0.17 \text{ nepers/revolution.}$$

By use of the theoretical solution (34), and by reference to Table VI to obtain $k \cong 0.0113$, the theoretically-expected ratio of the cos θ and cos 7 θ coefficients is

$$k \, \xi_3 \sin 75^\circ = 0.0113 \times 7.628\ 093\ 8847 \times 0.9659\ 2583 = 0.083.$$

Again we infer from these results that the theoretical development is valid, at least in a semi-quantitative sense.

D. Computations in which $\nu = 5$

[N = 24]

Computations in which a cos 10 θ term was present in the perturbation were made to illustrate the effectiveness of such a term in opening up the $\nu = 5$ stop-band. In some of the computations a cos 4 θ term was also introduced in the perturbation, in order to engender in the solution cosine or sine terms of argument θ, and phase shifts were also introduced. In all this work N = 24, and $m_o = 123.735_3$ for $\nu = 5$.

1. Computations Without Phase Shifts in the Perturbation:

a. The perturbation - cos 10 θ

The $\nu = 5$ stability boundaries for the perturbation

$$f(\theta) = - \cos 10\,\theta$$

were found to occur at

$$m_{even} = 125.004_8 \quad \text{and} \quad m_{odd} = 122.4396,$$

corresponding to a width $m_{even} - m_{odd} = 2.565_2$. This observed width is in fair agreement with the values 2.87 or 2.47 obtained from use of the formulas listed at the bottom of Table II ($\lambda = -1$); since the width involves λ to the first power, a substantial width is obtained with a relatively modest perturbation.

The Fourier coefficients for the eigenfunctions corresponding to these stability boundaries were determined with the DUCKNALL program[24] and the results for the prominant coefficients, through the 29^{th}, are listed in Table VII. For the example with which we are concerned here, ν is not sufficiently small in comparison to N that the results given in Table II are trustworthy--accordingly we include in Table VII numerical estimates obtained by solving equations (II, 4 d' - f') of Appendix II explicitly. The cosine and sine coefficients of order $h = N + 3\nu = 39$ were found to be -0.00026 and -0.00024, respectively, in good agreement with the value -0.00025 calculated from (II, 4 d' - f'); these results are not included in

Table VII, however, because the magnitudes are so small, and terms of order $2N \pm \nu$, $2N \pm 3\nu$, $3N \pm \nu$, etc. are likewise omitted. The distinction between m_{even} and m_{odd} was not made in solving equations (II, 4 d' - f'), since this distinction would make only a second-order effect, but was considered in calculating the large coefficients of order $N \pm \nu$.

TABLE VII

COSINE COEFFICIENTS, C_h, AND SINE COEFFICIENTS, S_h, OF ORDER h IN THE FOURIER EXPANSION OF EVEN AND ODD EIGENFUNCTIONS FOR THE EQUATION

$$d^2x/d\theta^2 + [m F(\theta) - \cos 10\theta]\, x = 0, \text{ with } \nu = 5 \text{ and } N = 24$$
$$[m_o = 123.7353]$$

FOR EVEN EIGENFUNCTION			FOR ODD EIGENFUNCTION		
	Value			Value	
Coefficient	Calc. from (5a), (5b) and (II, 4 d' - f')	From Digital Computation	Coefficient	Calc. from (5a), (5b) and (II, 4 d' - f')	From Digital Computation
$h = \nu = 5$ C_5	1 [Normalized]	1 [Normalized]	S_5	1 [Normalized]	1 [Normalized]
$h = N - 3\nu = 9$ C_9	-0.005_4	-0.0063	S_9	$+0.005_4$	+0.0060
$h = 3\nu = 15$ C_{15}	-0.004_2	-0.0046	S_{15}	-0.004_2	-0.0044
$h = N - \nu = 19$ C_{19}	0.220	0.222	S_{19}	-0.216	$-0.21\dot{8}$
$h = N + \nu = 29$ C_{29}	0.095	0.094	S_{29}	0.093	0.092

Information concerning the ascending solution within the $\nu = 5$ stop-band is presented later, in sub-section 2a below, for a perturbation of this same character and strength, save for the introduction of a phase shift.

b. The perturbation $- \cos 10\,\theta + 7.5 \cos 4\,\theta$

The $\nu = 5$ stability boundaries for the perturbation

$$f(\theta) = - \cos 10\,\theta + 7.5 \cos 4\,\theta$$

were found to occur at

$$m_{even} = 125.99_{26} \quad \text{and} \quad m_{odd} = 123.382_{9},$$

corresponding to a width $m_{even} - m_{odd} = 2.60_{97}$. This width, as expected, is little different from that found previously for the case in which the term $- \cos 10\,\theta$ was present alone in the perturbation, although the stop-band as a whole is of course displaced towards larger m-values by an amount ($0.9\dot{7}$) which is in close agreement with the shift predicted by the ϵ^2 term of the analytic formula listed in the next to the last line of Table IV. Without attempting to account at all quantitatively for the slightly greater width in the present case, we may point out that the parameters of the present example are somewhat special in that $3\,\nu = N - 2\,\nu + r$ and $N - 3\,\nu = 2\,\nu - r$; as a result, certain terms in our trial functions can receive contributions from <u>both</u> terms of the perturbation and, without recourse to higher-order effects, one can recognize that the width may receive "cross-term" ($\lambda\epsilon$) contributions.

The chief value which may be attached to the $\cos 4\,\theta$ term of the perturbation is its effectiveness in introducing $\cos\theta$ or $\sin\theta$ into the solutions of the differential equation. The Fourier coefficients of interest in this case were again obtained by the DUCKNALL program and are listed in Table VIII. The calculated values which are listed for comparison are those of Table VII, supplemented by the contributions suggested by Table IV ($\epsilon = 7.5$).

TABLE VIII

COSINE COEFFICIENTS, C_h, AND SINE COEFFICIENTS, S_h, OF ORDER h IN THE FOURIER EXPANSION OF EVEN AND ODD EIGENFUNCTIONS FOR THE EQUATION

$$d^2x/d\theta^2 + \left[mF(\theta) - \cos 10\,\theta + 7.5 \cos 4\,\theta\right] x = 0, \text{ with } \nu = 5 \text{ and } N = 24.$$

$$\left[m_o = 123.7353\right]$$

FOR EVEN EIGENFUNCTION			FOR ODD EIGENFUNCTION		
Coefficient	Value: Calc. from Tables IV & VII	Value: From Digital Computation	Coefficient	Value: Calc. from Tables IV & VII	Value: From Digital Computation
$\nu = r = 1$					
C_1	-0.190	-0.186	S_1	-0.190	-0.200
$h = \nu = 5$					
C_5	1 [Normalized]	1 [Normalized]	S_5	1 [Normalized]	1 [Normalized]
$h = 2\nu - r = N - 3\nu = 9$					
C_9	0.080	0.086	S_9	0.091	0.091
$h = 3\nu = N - 2\nu + r = 15$					
C_{15}	0.029	0.033	S_{15}	-0.038	-0.038
$h = N - \nu = 19$					
C_{19}	0.220	0.224	S_{19}	-0.216	-0.219
$h = N - r = 23$					
C_{23}	-0.027	-0.027	S_{23}	+0.027	+0.028
$h = N + r = 25$					
C_{25}	-0.023	-0.023	S_{25}	-0.023	-0.024
$h = N + \nu = 29$					
C_{29}	0.095	0.095	S_{29}	0.093	0.093

Information concerning the ascending solution within the $\nu = 5$ stop-band is presented later, in sub-section 2b below, for a perturbation of this same type but with phase-shifts present.

2. Computations with Phase Shifts Introduced into the Perturbation:

a. The pertrubation sin 10 θ

The perturbation

$$f(\theta) = \sin 10\,\theta$$

considered here is seen, by the substitution $\theta = \varphi - 45^{\circ}$, to be $-\cos 10\,\varphi$; the perturbation is thus, in essence, exactly that considered in sub-section 1a above and must lead to the same stability boundaries (eigenvalues of m). With N = 24, the ascending exponential solution within the $\nu = 5$ stop-band was found to be characterized by $dx/d\theta = 0$ at $\theta = 0$ when $m = 123.75_{69}$.

Under these conditions the lapse-rate was found computationally to be

0.3775 nepers/revolution ($\mu = 0.06008$ nepers/radian)

or 0.1639 decades/revolution,

corresponding to growth by a factor 1.4586 per revolution. With the exponential growth factored out, Fourier analysis of the periodic azimuthal dependence indicated a character substantially the same as that for the even eigenfunction (sub-section 1a), the dominant terms being those which appear in the even solution for the unperturbed problem:

$$\cos 5\,\theta + 0.220 \cos 19\,\theta + 0.093 \cos 29\,\theta\ .$$

Using the observed values of m at the stability boundaries (sub-section 1a), the lapse-rate suggested by equation (14a) is

$\mu = 0.045$ nepers/radian, or 0.28 nepers/revolution,

which is about 3/4 of the amount found computationally. The absence of any strong Fourier coefficients other than those noted is to be expected from the theoretical analysis (Sect. II B 3a).

b. The perturbation $\sin 10\,\theta - 7.5 \cos 4\,\theta$

As noted in Sect. II B 4d, the perturbation

$$f(\theta) = \sin 10\,\theta - 7.5 \cos 4\,\theta$$

is seen by the substitution $\theta = \phi - 45^{\circ}$ to be $-\cos 10\,\varphi + 7.5 \cos 4\,\varphi$, which is of the form considered in sub-section 2b above. With N = 24, the ascending solution within the $\nu = 5$ stop-band was found to be characterized by $dx/d\theta = 0$ at $\theta = 0$ when $m = 124.759_{44}$.

Under these conditions the lapse-rate was found computationally to be

0.3980 nepers/revolution (μ = 0.06334 nepers/radian)

or 0.1728 decades/revolution

corresponding to growth by a factor 1.489 per revolution. With the exponential increase factored from the solution, the chief Fourier coefficients for the periodic azimuthal dependence are as listed in Table IX and are seen to be substantially the same as those shown in Table VIII for the even eigenfunction when no phase shift is present, save for a change of sign for those coefficients which depend on the strength of the perturbation. These results are illustrated in Fig. 6.

Since, as noted from sub-sections 1a, b, the width of the stop-band in the present case is only very slightly greater than for the perturbation $\sin 10\,\theta$ alone, it is not surprising that the lapse-rate found here is not much different from that reported for the computations of the preceding sub-section (2a). The effect of the additional term $-7.5 \cos 4\,\theta$ which has been added to the perturbation is to introduce an appreciable $\cos \theta$ term into the solution. From equation (38) we expect, with $\xi_4 = 7.5$, the coefficient of $\cos\theta$ in the solution to be $0.025 \times 7.5 = 0.19$ relative to the $\cos 5\theta$ term. In summary it is clear that the relatively modest perturbation $\sin 10\theta$ has engendered an instability characterized by a marked rate of growth and that the additional perturbation $-7.5 \cos 4\theta$ has introduced a noticeable fundamental azimuthal dependence of a type which favors the attainment of large displacements at $\theta = 0$.

TABLE IX

COSINE COEFFICIENTS OF ORDER h IN THE EVEN ASCENDING SOLUTION FOR THE EQUATION

$$d^2x/d\theta^2 + \left[m F(\theta) + \sin 10\theta - 7.5 \cos 4\theta\right] x = 0,$$

WITH $\nu = 5$, $N = 24$, AND FOURIER ANALYSIS MADE AFTER DIVISION BY $\exp(\mu\theta)$.

C_h	Value of Cosine Coefficient from Computational Solution
C_1	0.191
C_5	1 [Normalized]
C_9	-0.085
C_{15}	-0.032
C_{19}	0.221
C_{23}	0.026
C_{25}	0.022
C_{29}	0.093

E. Matrix Computations for a Piecewise-Constant Perturbation, with $\nu = 7\frac{1}{2}$ and N = 48

1. Motivation:

It will be readily appreciated that in practice it could be considerably more convenient to provide a perturbation (n-bump) which is piecewise-constant, rather than a continuously-varying function of azimuth. Specifically in many cases it might prove most convenient to employ a perturbation for which the change in focusing index, n, is constant over each individual semi-sector. Detailed examination of such an arrangement would most naturally be carried out by the standard matrix methods[14] and for computational work the MESSY-MESSY program[25] is helpful. With a large number of sectors in the accelerator one would expect, however, the results to differ in no essential way from those obtained by study of differential equations of the type considered heretofore.

2. The Stop-Band for $\nu = 7\frac{1}{2}$ with Piecewise-Constant Perturbation:

In analogy to the computational example of Sect. III B 4, for which the analytic theory of Sect. II B 4a was intended to apply, the response to a similar piecewise-constant perturbation was studied by matrix multiplications, aided by the MESSY-MESSY program. The quantity designated by δ in the description of this program,[25] representing the square root of the magnitude of the focusing constant, was given values in accordance with the expression

$$\delta = \sqrt{\left| m\,F(\theta) - 6.262\,768\,45 \cos(7\,\theta + 52.^{\circ}5) - 5.737\,231\,55 \cos(8\,\theta + 60^{\circ})\right|}, \quad (39)$$

specific individual values then being obtained by substitution into (39) of the values of θ corresponding to the mid-point of the individual semi-sectors of a structure for which $N = 48$ and for which $\theta = 0$ corresponds to the center line of one of the radially-focusing semi-sectors. Appropriate matrix multiplications (with $t = \pi/48$ for a full semi-sector, or $t = \pi/96$ for one-half of a semi-sector) were then made to obtain the matrix representing an entire revolution, from $\theta = 0$ to $\theta = 2\pi$.

By interpolation of the results of three such runs, made with m successively given the values 383.3, 382.9, and 382.5, it appeared[21] that the pure ascending exponential solution would be correctly launched with $(dx/d\theta)_o = 0$ if $m = 382.983$, and these same orientation runs suggested that a lapse-rate of

0.162 nepers per revolution

would then be expected. A MESSY-MESSY run made under these conditions, strictly with $m = 382.982\,67$, indicated a lapse-rate

$$2\pi\mu \equiv \text{Cosh}^{-1}\frac{\text{Trace}}{2} = 0.160 \text{ nepers per revolution;}$$

it will be noted that this value is close to the result 0.158 nepers per revolution reported in Sect. III B 4 for DUCK-ANSWER computations pertaining to a similar (but continuous) perturbation.

From the matrix element "A" of the successive cumulative product matrices printed out at quarter-sector intervals, in the course of the last-named MESSY-MESSY run, one in effect has the coordinates of a representative pure ascending solution, with $(dx/d\theta)_o = 0$. A Fourier analysis of these data, taken at half-sector intervals and with the exponential increase factored out, led to the major cosine coefficients for the periodic azimuthal dependence (period 4π) listed in Table X, wherein, for comparison, we also include the corresponding results for the DUCK-ANSWER run described in Sect. III B 4. It is noted that in either case the coefficient of $\cos\frac{1}{2}\theta$ is about 7 per cent of the dominant $\cos 7.5\,\theta$ coefficient. These results are illustrated in Fig. 3.

TABLE X

COSINE COEFFICIENTS, OF ORDER h, FOR THE ASCENDING SOLUTION WITHIN THE $\nu = 7\frac{1}{2}$ STOP-BAND, WITH N = 48 .

Results for the solution, with the exponential factor removed, as obtained by the MESSY-MESSY program for the perturbation implied by Eqn. (39).

C_h	Value of Cosine Coefficient	
	for MESSY-MESSY solution	for DUCK-ANSWER solution†
$C_{1/2}$	0.072	0.069
$C_{7.5}$	1 [Normalized]	1 [Normalized]
$C_{40.5} + C_{55.5}$ *	0.221	0.219

*See note, Sect. III B 2, on the interpretation of FORANAL output data.[26]

†For purposes of comparison with analysis of MESSY-MESSY computation.

IV. SUMMARY AND DISCUSSION

It appears from the foregoing work that a perturbation which changes the field index of a particle accelerator (n-bump) can create a stop-band within which, as a result of the perturbation, a substantial rate of growth of oscillation amplitude will occur. It moreover appears that the form of the ascending solution, which is the solution that soon dominates the motion, can be controlled to some extent by the nature of the azimuthal distribution of the perturbation. Specifically, a perturbation whose spatial dependence contains a circular function of argument $2\nu\theta$ will open up the stop-band in first order, in that the width of the stop-band is directly proportional to the strength of the perturbation, whereas otherwise a combination of functions with arguments $(\nu \pm r)\theta$ is required and such a combination produces a stop-band width proportional to the square of the perturbation. Terms with argument $(\nu + r)\theta$ or $(\nu - r)\theta$ may be useful, nonetheless, in introducing into the solution (orbit) a spatial dependence which includes circular functions of argument $r\theta$.

The use of such instabilities may be effective in achieving rather economically a rapid knock-out of a beam onto an internal target. It may also be useful for extraction of the beam, either by moving it rapidly into a "peeler" structure or possibly by directing the beam through the fringe field of the magnet itself. In the case that the fringe field is involved, specific computations including the non-linear features of the motion in such regions would be desirable, but in any extraction method the introduction of a term such as $\cos \frac{1}{2}\theta$ into the growing solution would seem to afford an azimuthal dependence which would be helpful. It may be noted that, as is the case for some of the examples considered in this report, the exponential growth of the solution may be sufficiently great in one revolution as to outweigh the azimuthal variation introduced by the $\cos \frac{1}{2}\theta$ term (and by the other terms of some importance which are associated with this term in the solution)--such a situation should be avoidable, however, if that be desirable, by proper adjustment of the relative magnitudes of the coefficients of those terms in the perturbation which create the stop-band and those which primarily influence the form of the solution. Such "tailoring" of the perturbation may be possible, of course, only at the expense of heavier currents (or ampere turns) in some portions of the perturbation windings, but it is hoped that the analytic discussion contained in this report will provide a helpful perspective concerning the rôle played by the various terms under consideration.

In the body of the present report, the location and width of the stop-bands have been specified, for convenience, in terms of the parameter "m"; in practice it may be more economical to "tune" the accelerator to the desired resonance by use of the constant "a" (representing an azimuthally constant addition to the field index), and actually a combination of changes in "a" and "m" may be desirable to keep the axial motion free of resonance effects while exploiting the desired radial resonance. Such possible perturbation arrangements, however, should be readily investigated in any specific case by methods paralleling those presented here.

As a final caution it should be noted that, in order to achieve ascending exponential solutions for which (in our illustrations) $dx/d\theta = 0$ at $\theta = 0$, a very definite perturbation schedule should be followed. Thus one might best undertake to perturb the accelerator to the vertex of the stop-band and then proceed up a definite central curve in its interior. Departures from the intended central curve will introduce phase shifts into the solution, such that the orbit maxima are shifted or, otherwise expressed, $dx/d\theta$ will differ from zero at the azimuth $\theta = 0$. Such control of the perturbation currents may, then, require fairly careful programming and electronic engineering, and the tolerances necessary to achieve desired limits of performance could be estimated for a specific case by application of methods similar to those discussed in this report. Such planning of the design, and the requisite engineering effort may, however, be well warranted if the obtainable performance is felt materially to enhance the versatility and usefulness of the accelerator.

V. REFERENCES AND NOTES

1. C. L. Hammer and A. J. Bureau, Rev. Sci. Instr. 26, 594 (1955).

2. C. L. Hammer and A. J. Bureau, Rev. Sci. Instr. 26, 598 (1955). The method described in references 1 and 2, in which use is made of the half-integral resonance $\nu_r = 1/2$, bears a certain resemblance to the use of an integral resonance for regenerative beam extraction from a cyclotron, as described by J. L. Tuck and L. C. Teng [Phys. Rev. 81, 305 (1951)] and analyzed by K. J. LeCouteur [Proc. Phys. Soc. (London) B64, 1073 (1951)].

3. C. L. Hammer and L. J. Laslett, "Electron Beam Control in a Conventional Synchrotron," Second International Conference on the Peaceful Uses of Atomic Energy (Geneva, Switzerland, 1958), Paper A/CONF.15/P/726.

4. Loren Robert McMurray, "A Possible Method for Beam Extraction from an Alternating Gradient Synchrotron," MS Thesis, Iowa State College, Ames, Iowa (1955).

5. Loren R. McMurray and Daniel J. Zaffarano, "A Possible Method for Beam Extraction from an Alternating Gradient Synchrotron," Ames Laboratory Report ISC-647, Iowa State College, Ames, Iowa (November 28, 1955).

6. G. H. Meisters, "Beam Extraction from an Alternating Gradient Synchrotron," Ames Laboratory Report AL-87, Iowa State College, Ames, Iowa (1955).

7. As one example of an accelerator for which an effective knock-out method might be useful, one might mention that described in Technical Report No. 60 of the Laboratory for Nuclear Science, Massachusetts Institute of Technology (Cambridge Design Study Group, M. Stanley Livingston, Director), "Design Study for a 15-Bev Accelerator" (June 30, 1953).

8. We are indebted to Dr. George A. Kolstad for stimulating remarks concerning the potential utility of an effective knock-out method for alternating-gradient accelerators (October 8, 1958).

9. L. Jackson Laslett, "Approximation of Eigenvalues, and Eigenfunctions, by Variational Methods," MURA Notes (1 February, 1955).

10. L. Jackson Laslett, "Character of Particle Motion in the Mark V FFAG Accelerator," MURA Report LJL (MURA)-5, Appendix III (30 July, 1955).

11. L. Jackson Laslett and A. M. Sessler, "Concerning Coupling Resonances in the Spirally-Ridged FFAG Accelerator," MURA-263 (May 6, 1957).

12. Leonard I. Schiff, "Quantum Mechanics," Ed. 2 (McGraw-Hill Book Company, New York, 1955), Chapt. VII.

13. See a series of reports by G. Parzen, for example MURA-200. We are indebted to Dr. Parzen for a number of stimulating discussions during the course of the present work.

14. E. D. Courant and H. S. Snyder, Annals of Physics <u>3</u>, 1 (1958).

15. Strictly the variational statement might best be regarded as an isoperimetric problem,[9] with "$-a$" playing the role of a Lagrange Multiplier.

16. It may be pointed out that a general solution of the form

$$R\left[\cos\nu\theta + \beta\cos(N-\nu)\theta + \gamma\cos(N+\nu)\theta\right] + S\left[\sin\nu\theta - \beta\sin(N-\nu)\theta + \gamma\sin(N+\nu)\theta\right]$$

can be written

$$\frac{R-iS}{2}\exp(i\nu\theta)\left[1+\beta(\cos N\theta - i\sin N\theta)+\gamma(\cos N\theta + i\sin N\theta)\right]$$

$$+\frac{R+iS}{2}\exp(-i\nu\theta)\left[1+\beta(\cos N\theta + i\sin N\theta)+\gamma(\cos N\theta - i\sin N\theta)\right],$$

which is clearly of the Floquet Form [E. T. Whittaker and G. N. Watson, "Modern Analysis" (Cambridge University Press, 1927), sect. 19.4].

17. This result is identical to that obtained by applying the "smooth approximation" to the equation $d^2x/d\theta^2 + (4m/\pi)(\cos N\theta)x = 0$. [K. R. Symon, MURA Report KRS(MURA)-1 (July 1, 1954); K. R. Symon, <u>et al.</u>, Phys. Rev. <u>103</u>, 1837 (1956), Appendix A.]

18. N. W. McLachlan, "Theory and Application of Mathieu Functions" (Clarendon Press, Oxford, 1947), sects. 4.90 - 4.91.

19. The basic differential equation governing the radial betatron motion when the perturbation $f(\theta) = \lambda\cos 2\nu\theta$ is present, namely

$$d^2x/d\theta^2 + \left[mF(\theta) + \lambda\cos 2\nu\theta\right]x = 0,$$

may, by use of the smooth approximation,[17] be replaced by

$$d^2x/d\theta^2 + \left[2\left(\frac{2m}{\pi N}\right)^2 + \lambda\cos 2\nu\theta\right]x = 0.$$

Since, to this degree of approximation, $\nu^2 = 2\left(\frac{2m_o}{\pi N}\right)^2$,

this becomes

$$d^2x/d\theta^2 + \left[\nu^2 + \left(\frac{4m_o}{\pi N}\right)^2\frac{m-m_o}{m_o} + \lambda\cos 2\nu\theta\right]x = 0.$$

In the standard Mathieu Form, obtained by the substitution $2\nu\theta = 2\tau$, the last equation may be written

$$d^2x/d\tau^2 + \left[1 + \left(\frac{4m_o}{\pi\nu N}\right)^2\frac{m-m_o}{m_o} + \frac{\lambda}{\nu^2}\cos 2\tau\right]x = 0.$$

For the first resonance of this equation, which corresponds to the situation of interest here, it is well known that the stop-band opens up linearly with λ in the manner

$$\left(\frac{4m_o}{\pi\nu N}\right)^2\frac{m-m_o}{m_o} = \mp\frac{1}{2}\frac{\lambda}{\nu^2} \qquad \text{or} \qquad \frac{m-m_o}{m_o} = \mp\frac{1}{2}\left(\frac{\pi N}{4m_o}\right)^2\lambda.$$

The eigenfunctions, moreover, may be approximated by

$$A_{even}\left[\cos\tau + \frac{\lambda}{16\nu^2}\cos 3\tau\right] = A_{even}\left[\cos\nu\theta + \frac{\lambda}{16\nu^2}\cos 3\nu\theta\right]$$

and

$$A_{odd}\left[\sin\tau + \frac{\lambda}{16\nu^2}\sin 3\tau\right] = A_{odd}\left[\sin\nu\theta + \frac{\lambda}{16\nu^2}\sin 3\nu\theta\right].$$

[cf. Ref. 11, Appendix I; or Whittaker and Watson (ref. 16), Ex. 1, Sect. 19.3.] These results are thus seen to be consistent with the corresponding results listed in Table II.

20. We wish to record that the dependence of stop-band width on the product $\epsilon\zeta$ in cases of the type discussed here was pointed out to us by Dr. Parzen prior to completion of the analytic work described in the present report.

21. If the matrix which serves to carry the vector $\begin{pmatrix} x_{m-1} \\ p_{m-1} \end{pmatrix}$ of a particle through one revolution is written in the form [cf. ref. 14]

$$\begin{pmatrix} \pm\cosh\mu + \alpha\sinh\mu & \beta\sinh\mu \\ \gamma\sinh\mu & \pm\cosh\mu - \alpha\sinh\mu \end{pmatrix},$$

where $\gamma \equiv \frac{1-\alpha^2}{\beta}$, the general solution after n revolutions, for a particle starting with initial values x_o, p_o is

$$x_n = \frac{(\pm)^{n+1}}{2}\left\{\left[(\alpha \pm 1)\, x_o + \beta p_o\right] e^{n\mu} - \left[(\alpha \mp 1)\, x_o + \beta p_o\right] e^{-n\mu}\right\}$$

$$p_n = \frac{(\pm)^{n+1}}{2}\left\{-\left[\frac{\alpha^2 - 1}{\beta}\, x_o + (\alpha \mp 1)\, p_o\right] e^{n\mu} + \left[\frac{\alpha^2 - 1}{\beta}\, x_o + (\alpha \pm 1) p_o\right] e^{-n\mu}\right\}.$$

Here the upper and lower signs correspond, respectively, to operation in an integral or half-integral stop-band. For a solution launched with $p_o = 0$ to correspond entirely to a rising exponential, it is therefore necessary that $\alpha = \pm 1$, the product of the main-diagonal matrix elements must then be unity, and one of the two coefficients β or γ must vanish. It is, in fact, evident that it is γ which must vanish in this case, as clearly can be established directly from the observation that a "pure" exponential solution, if correctly launched with $p_o = 0$, would continue to give $p = 0$ after one revolution. The sign of α $(= \pm 1)$ can serve as noted here, to distinguish between the cases in which $p_o = 0$ will lead to an ascending or decreasing exponential:

Character of Exponential Solution Arising when $p_o = 0$ and $\alpha = \pm 1$

α \ Resonance	Integral	Half-Integral
+1	Rising	Falling
− 1	Falling	Rising

The matrix element γ Sinh μ can, of course, be determined directly by a one-revolution computation for an orbit launched with $p_o = 0$, since it is then given simply by p_1 / x_o. The parameters of the differential equation (e.g., m) can be adjusted to make this matrix element vanish. [We consider that the condition $(dx/d\theta)_o = 0$ for the rising exponential, which is the condition adopted here, is not only simpler to apply but probably preferable to the condition $\left((d/d\theta)(e^{-\mu\theta} x)\right)_o = 0.$]

22. J. N. Snyder, DUCK-ANSWER (I.B.M. Program 75), MURA-237, Int. (1957). In the use of this program for examples in which N = 48, $\tau = 168\,\theta$ with 16 steps of the variable τ in each interval $\Delta\tau = \pi$. Thus 5376 steps were made per revolution, or 112 within each full sector. The square-wave function $F(\theta)$ of unit amplitude (+1) was generated in the Ψ-channel of the program by use of the "bump" feature,[23] and introduced into the ρ- or x-equation by setting $\mu_2 = 1$. Accordingly one set $S_{14} = -(1/2)(m/28224)$. A perturbation $\epsilon \cos 7\theta + \eta \cos 8\theta$ can be represented by setting

$$B_1 = -\epsilon/282240 \quad \text{with } N_1 = 96$$

and

$$B_2 = -\eta/282240 \quad \text{with } N_2 = 126.$$

Similarly, a perturbation $\epsilon \cos 6\theta + \zeta \cos 8\theta$ can be obtained by

$$B_1 = -\epsilon/282240 \quad \text{with } N_1 = 112$$

and

$$B_2 = -\zeta/282240 \quad \text{with } N_2 = 126.$$

For examples in which N = 24, 192 Runge-Kutta steps were employed in an interval $\Delta\tau = \pi$, corresponding to 1920 steps per revolution or 80 steps per full sector. Ψ was again obtained by use of the "bump" feature, but with values ± 20. In this case $S_{14} = -m/1000$ and a perturbation $\epsilon \cos 4\theta + \zeta \cos 6\theta + \lambda \cos 10\theta$ was representable by

$$A_1 = -\lambda/250;$$

$$B_1 = -\epsilon/250, \quad \text{with } N_1 = 5; \text{ and}$$

$$C_1 = -\zeta/250, \quad \text{with } N_2 = 5.$$

Phase shifts can be introduced into the arguments of the circular functions, when desired, by use of the parameters α_1, β_1, and γ_1.

23. J. N. Snyder, Invariant Duck-Bumps (I.B.M. Program 77), MURA-238, Int. (1957). With the Runge-Kutta interval chosen as noted in reference 22 for N = 48 sectors/rev., we set $N_B = 112$, $\eta = 28$, $\eta' = 84$, $2^{-5}\Delta\Psi = -.03125$, $2^{-5}\Delta\Psi' = .03125$, and launch the Ψ solution with $\Psi_o 10^{-2} = .01$ to generate the desired unit square wave. In the examples pertaining to N = 24 sectors/rev., we set $N_B = 80$, $\eta = 20$, $\eta' = 60$, $2^{-5}\Delta\Psi = -.625$, $2^{-5}\Delta\Psi' = .625$, and $\Psi_o 10^{-2} = .2$ to generate the desired square wave ($\Psi = \pm 20$).

24. J. N. Snyder, FORANAL (I.B.M. Program 52), MURA-228, Int. (1957). With a large number of input data the limitation of the original program to output coefficients of order not to exceed 24 was not considered to be a necessary or desirable limitation. We are indebted to J. McNall of the

MURA Computer Division, for relaxing this limitation and, more recently, for incorporating the Fourier-analysis methods of FORANAL directly into the DUCK-ANSWER program [John McNall, DUCKNALL (I. B. M. Program 219), MURA-438, Int. (1958)].

25. Elizabeth Zographos, MESSY-MESSY (I. B. M. Program 78), MURA-239, Int. (1957). Matrices of "Type 1" and "Type 2" respectively describe the passage of a particle through a focusing or defocusing section. Matrices of "Type 3" may also be used, if desired, to represent either a lens or a straight section. We are indebted to Mrs. Zographos Chapman for preparing an overwrite to the MESSY-MESSY program so that the elements of the successive cumulative matrix products can be printed. A test of the program in its present application was made by a routine run for an unperturbed A-G structure for which $\nu = 7.375$.

26. L. Jackson Laslett, MURA-435, Int. (1958).

27. For this perturbation, in which the coefficient ξ_1 of equations (23a, b) has the value 7.231 622 101,

$$\epsilon = -7.231\ 622\ 101 \sin 60^{\circ} = -6.262\ 768\ 45,$$
$$\zeta = -7.231\ 622\ 101 \sin 52^{\circ}.5 = -5.737\ 231\ 55, \text{ and}$$
$$\epsilon + \zeta = -12.$$

28. For this perturbation, in which the coefficient ξ_3 of equation (31) has the value 7.628 093 8847,

$$\epsilon = -7.628\ 093\ 8847 \sin 60^{\circ} = -6.606\ 123\ 086_5,$$
$$\zeta = -7.628\ 093\ 8847 \sin 45^{\circ} = -5.393\ 876\ 913_5, \text{ and}$$
$$\epsilon + \zeta = -12.$$

APPENDIX I. PERTURBATION METHOD

The analytical problem of solving the Hill equation

$$\frac{d^2}{d\theta^2} \psi_\ell (\theta) + \left(\frac{4 m_\ell}{\pi} \cos N \theta + \sum_m \xi_m \cos m \theta\right) \psi_\ell (\theta) = 0 \qquad \text{(I-1)}$$

for the eigenfunctions ψ_ℓ can also be accomplished for small ξ_m through the use of stationary state perturbation theory.[12] As the unperturbed equation, one considers the Mathieu equation

$$H_o \phi_{\nu,\ell} = \frac{d^2 \phi_{\nu,\ell}}{d \theta^2} + \frac{4 m_\ell}{\pi} \cos N \theta \, \phi_{\nu,\ell} = -a_\nu \phi_{\nu,\ell} \qquad \text{(I-2)}$$

where for certain a_ν and m_ℓ, the $\phi_{\nu,\ell}$ are periodic. Comparison of this equation to equation (1) with $\xi_m = 0$ shows that

$$\phi_{\ell,\ell} = \psi_\ell \qquad \text{(I-3)}$$

if m_ℓ is chosen so that $a_\ell = 0$. The eigenfunctions $\phi_{\nu,\ell}$ for ν and ℓ either integer or 1/2 integer are given to a good approximation, for $\nu < N$, by

$$\phi_{\nu,\ell} = A_{\nu,\ell} \begin{Bmatrix} \cos \nu \theta \\ \sin \nu \theta \end{Bmatrix} + B_{\nu,\ell} \begin{Bmatrix} \cos (N - \nu) \theta \\ \sin (N - \nu) \theta \end{Bmatrix} + C_{\nu,\ell} \begin{Bmatrix} \cos (N + \nu) \theta \\ \sin (N + \nu) \theta \end{Bmatrix} \qquad \text{(I-4)}$$

where the notation means either the even or the odd functions are to be used and where

$$B_{\nu,\ell} = \pm \frac{2 m_\ell}{\pi} \frac{A_{\nu,\ell}}{(N - \nu)^2 - a_\nu} ; \qquad \begin{cases} + \text{ for even solutions;} \\ - \text{ for odd solutions.} \end{cases}$$

$$C_{\nu,\ell} = \frac{2 m_\ell}{\pi} \frac{A_{\nu,\ell}}{(N + \nu)^2 - a_\nu} ;$$

$$\nu^2 = a_\nu + \left(\frac{2 m_\ell}{\pi}\right)^2 \left\{\frac{1}{(N - \nu)^2 - a_\nu} + \frac{1}{(N - \nu)^2 - a_\nu}\right\}^* \qquad \text{(I-5)}$$

*These equations are the same as eqns. 3, 4 and 5 of the main text. They are repeated here only in an attempt to draw more closely the parallel to the familiar quantum mechanical problem through the use of similar notation.

Setting $\nu = \ell$ and $a_\ell = 0$ gives the condition

$$2\left(\frac{2\, m_\ell}{\pi}\right)^2 = \frac{\ell^2 (N^2 - \ell^2)^2}{N^2 + \ell^2},$$

$$\cong \ell^2 (1 - 3\, \ell^2/N^2) . \qquad \text{(I-6)}$$

A good approximation for a_ν is given by

$$a_\nu = \frac{\nu^2 - 2\left(\frac{2\, m_\ell}{N\, \pi}\right)^2 (1 + 3\, \nu^2/N^2)}{1 + 2\left(\frac{2\, m_\ell}{\pi\, N}\right)^2 \Big/ N^2} \cong \frac{\nu^2 - \ell^2}{1 + 4\, \ell^2/N^2} ; \quad \nu \ll N. \qquad \text{(I-7)}$$

Assuming the perturbation term to be

$$H_1 = \sum_{m=1}^{2\ell} \xi_m \cos m\, \theta \qquad \text{(I-8)}$$

so that both first and second order effects will occur and applying stationary state perturbation theory one obtains the solution of equation (1) to be to first order

$$\psi_\ell(\theta) = \phi_{\ell,\ell}(\theta) + \sum_{\substack{\nu \\ \nu \neq \ell}} \left[(H_1)_{\ell\nu} / a_\nu \right] \phi_{\nu,\ell}(\theta) , \qquad \text{(I-9)}$$

where

$$(H_1)_{\ell\nu} = \sum_{m=0}^{2\ell} \int d\,\theta\; \phi_{\nu,\ell}(\theta)\; \xi_m \cos m\,\theta\; \phi_{\ell,\ell}(\theta) . \qquad \text{(I-10)}$$

The change in the eigenvalue a_ℓ to second order is given by

$$(-\Delta a_\ell) = (H_1)_{\ell\ell} + \sum_{\substack{\nu \\ \nu \neq \ell}} \left[(H_1)_{\ell,\nu} \right]^2 / a_\nu . \qquad \text{(I-11)}$$

For $\ell \ll N$, the only matrix elements of importance are, to terms in $1/N^3$,

(a) $(H_1)_{\ell,\ell+m} = \xi_m/2$;

(b) $(H_1)_{\ell|\ell-m|} = \xi_m/2 + \xi_{2\ell-m}/2$; $m \neq \ell$

(c) $(H_1)_{\ell,0} = \xi_\ell/\sqrt{2}$, (I-12) for the even functions and

$$\text{(a)}\quad (H_1)_{\ell,\ell+m} = \xi_m/2\,;$$

$$\text{(b)}\quad (H_1)_{\ell,|\ell-m|} = (\xi_m/2 - \xi_{2\ell-m}/2)\,(\ell-m)/|\ell-m|;\ \ell \neq m$$

$$\text{(c)}\quad (H_1)_{\ell,0} = 0\,; \qquad \text{(I-13)}$$

for the odd functions.

The negative differential of equation (7) gives approximately the change in the eigenvalue m_ℓ equivalent to the change Δa_ℓ as

$$\Delta m_\ell / m_\ell = \frac{(\Delta a_\ell/4)\,(N\pi/2m_\ell)^2\,(1 + [4/N^2][2m_\ell/N\pi]^2)}{1 + (4/N^2)\,(\ell^2 - [2m_\ell/N\pi]^2)}$$

$$\cong (\Delta a_\ell/2)\,(1 + 3\ell^2/N^2)/\ell^2\,. \qquad \text{(I-14)}$$

Changing m_ℓ by this amount approximately adjusts Δa_ℓ to zero in the final differential equation so that ψ_ℓ is the solution of equation (1) with m_ℓ replaced by $(m_\ell + \Delta m_\ell)$ instead of being the solution to the equation

$$\frac{d^2}{d\theta^2}\psi_\ell + (\Delta a_\ell + \frac{4}{\pi} m_\ell \cos N\theta + \sum_m \xi_m \cos m\theta)\,\psi_\ell = 0\,.$$

A splitting of the m_ℓ level (removal of the parity degeneracy), that is, an opening of the m_ℓ stop-band, will occur to first order only when $m = 0$ ($\xi_o = 0$), as can be seen from equations (12b) and (13b) because of the presence of the $\cos 2\ell\theta$ term in the perturbation. The magnitude of the splitting as obtained from equations (11) and (14) is

$$[(\Delta m_\ell)_{even} - (\Delta m)_{odd}\,/m_\ell = -\frac{[\xi_{2\ell}/4][N\pi/2m_\ell]^2\,[1 + (4/N^2)(2m_\ell/N\pi)^2]}{1 + [4/N^2][\ell^2 - (2m_\ell/N\pi)^2]}$$

$$\cong -(\xi_{2\ell}/2)\,(1 + 3\ell^2/N^2)/\ell^2\,, \qquad \text{(I-15)}$$

where the subscript "even" refers to the matrix elements using the even $\phi_{\nu,\ell}$ and the subscript "odd" refers to the matrix elements using the odd

$\phi_{\nu,\ell}$. This result is identical to the one obtained using the variational method if terms of order $1/N^2$ are dropped, as can be seen from Table II.

An opening of the m_ℓ stop-band will occur to second order only when terms of the type, $\xi_\ell \cos \ell\theta$ or $[\xi_m \cos m\theta + \xi_{2\ell-m} \cos(2\ell - m)\theta]$, are present in the perturbation. The magnitude of this splitting is

$$\left[(\Delta m_\ell)_{even} - (\Delta m_\ell)_{odd}\right]/m_\ell = -\left\{\xi_\ell^2/2a_o + \sum_{m-\ell}^{K}\left[\xi_m\,\xi_{2\ell-m}\right]/a_{|\ell-m|}\right\}\left\{\frac{[1/4][N\pi/2m_\ell]^2\left[1 + (4/N^2)(2m_\ell/N\pi)^2\right]}{1 + [4/N^2]\left[\ell^2 - (2m_\ell/N\pi)^2\right]}\right\}$$

$$\cong +\left\{\xi_\ell^2/2\ell^2 + \sum_{m=1}^{K}\left[\xi_m\,\xi_{2\ell-m}\right]/\left[\ell^2 - (\ell-m)^2\right]\right\}\left\{\left[1 + 7\ell^2/N^2\right]/2\ell^2\right\}, \quad \text{(I-16)}$$

where $K = \ell - 1$ if ℓ is integer or

$K = \ell - \frac{1}{2}$ if ℓ is $\frac{1}{2}$ odd integer.

This result is to be compared to those shown in Tables III and IV, again for the case where terms of order $1/N^2$ are ignored.

For the application discussed in this paper, that of finding a perturbation that can be used for the resonant extraction of a beam from a conventional A.G. synchrotron, it would appear from equations (4) and (15) that the maximum effect is obtained for a perturbation of the type

$$H_1 = \xi_{2\ell} \cos 2\ell\theta + \xi_{\ell-1} \cos(\ell - s)\theta + \xi_{\ell+1} \cos(\ell + s)\theta \quad \text{(I-17)}$$

where

$s = 1$ if ℓ is integer

$s = \frac{1}{2}$ if ℓ is $\frac{1}{2}$ odd integer,

since the first term opens the stop-band linearly in $\xi_{2\ell}$ and the second and third terms give rise to a $\cos s\theta$ term in the solution $\psi_\ell(\theta)$. It is interesting to note that for the special case of $\ell = \frac{1}{2}$, a single perturbation term $\xi_1 \cos\theta$ accomplishes both effects.

APPENDIX II. VARIATIONAL SOLUTION FOR THE PERTURBATION $f(\theta) = \lambda \cos 2\nu\theta$

We employ the variational statement (17) of Sect. II 3 Ba,

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \lambda \langle x^2 \cos 2\nu\theta \rangle = \text{min.}, \qquad \text{(II-1)}$$

and employ as trial functions

$$x \doteq A \cos \nu\theta + B \cos (N - \nu)\theta + C \cos (N + \nu)\theta + D \cos 3\nu\theta + E \cos (N - 3\nu)\theta + F \cos (N + 3\nu)\theta \qquad \text{(II-2a)}$$

or a similar expression containing sine functions. (II-2b).

The variational statement (II-1) then becomes

$$\frac{\nu^2 A^2}{2} + \frac{(N-\nu)^2 B^2}{2} + \frac{(N+\nu)^2 C^2}{2} + \frac{(3\nu)^2 D^2}{2} + \frac{(N-3\nu)^2 E^2}{2} + \frac{(N+3\nu)^2}{2} F^2$$

$$- \frac{2m}{\pi} \left[\pm AB + AC \pm DE + DF \right]$$

$$\mp \lambda \frac{A^2}{4} \qquad - \frac{\lambda}{2} \left[AD + BC + BE + CF \right] = \text{min.}, \qquad \text{(II-3)}$$

where the upper and lower signs respectively refer to the even or odd trial functions. Differentiation with respect to the parameters A through E, in turn, leads to the simultaneous algebraic equations

$$\left(\nu^2 \mp \frac{\lambda}{2}\right) A \mp \frac{2m}{\pi} B - \frac{2m}{\pi} C - \frac{\lambda}{2} D = 0 \qquad \text{(II-4a)}$$

$$\mp \frac{2m}{\pi} A + (N-\nu)^2 B - \frac{\lambda}{2} C - \frac{\lambda}{2} E = 0 \qquad \text{(II-4b)}$$

$$- \frac{2m}{\pi} A - \frac{\lambda}{2} B + (N+\nu)^2 C - \frac{\lambda}{2} F = 0 \qquad \text{(II-4c)}$$

$$- \frac{\lambda}{2} A + (3\nu)^2 D \mp \frac{2m}{\pi} E - \frac{2m}{\pi} F = 0 \qquad \text{(II-4d)}$$

$$-\frac{\lambda}{2} B \qquad \mp \frac{2m}{\pi} D \qquad + (N - 3\nu)^2 E \qquad = 0 \qquad \text{(II-4e)}$$

$$-\frac{\lambda}{2} C \qquad - \frac{2m}{\pi} D \qquad + (N + 3\nu)^2 F \qquad = 0 \qquad \text{(II-4f)}$$

The coefficients D, E, and F will be of order λ and, to this order, it will suffice to use the unperturbed values of B/A and C/A in solving (II-4d-f):

$$(3\nu)^2 D \mp \frac{2m}{\pi} E - \frac{2m}{\pi} F = A \qquad \text{(II-4d')}$$

$$\mp \frac{2m}{\pi} D + (N - 3\nu)^2 E = \pm \frac{2m}{\pi} \frac{A}{(N-\nu)^2} \qquad \text{(II-4e')}$$

$$-\frac{2m}{\pi} D + (N + 3\nu)^2 F = \frac{2m}{\pi} \frac{A}{(N+\nu)^2} \qquad \text{(II-4f')}$$

The determinant of the coefficients is

$$9\nu^2 (N^2 - 9\nu^2)^2 - \left(\frac{2m}{\pi}\right)^2 \left[(N+3\nu)^2 + (N-3\nu)^2 \right],$$ of which the last term is the smaller.

This last term may be simplified, by use of equation (5c), to become approximately

$$-\nu^2 N^4 \frac{1 + 9(\nu/N)^2}{1 + 3(\nu/N)^2}, \quad \text{or} \quad -\nu^2 N^4 \left[1 + 6(\nu/N)^2\right].$$

Hence the determinant may be expressed in the form

$$9\nu^2 N^4 - 162\nu^4 N^2 - \nu^2 N^4 - 6\nu^4 N^2$$

$$= 8\nu^2 N^4 - 168\nu^4 N^2 = 8\nu^2 N^4 \left[1 - 21(\nu/N)^2\right]. \qquad \text{(II-5)}$$

In solving equations (II-4d'-f') by determinants, one obtains

Numerator for D: $$\frac{\lambda}{2} \left\{ (N^2 - 9\nu^2)^2 + \left(\frac{2m}{\pi}\right)^2 \left[\left(\frac{N-3\nu}{N+\nu}\right)^2 + \left(\frac{N+3\nu}{N-\nu}\right)^2 \right] \right\} A$$

$$\cong \frac{\lambda}{2} \left[N^4 - 18\nu^2 N^2 + \nu^2 N^2\right] A = \frac{\lambda}{2} N^4 \left[1 - 17(\nu/N)^2\right];$$

$$\text{Numerator for E:} \pm \frac{\lambda}{2} \left\{ \frac{2m}{\pi} (3\nu)^2 \frac{(N+3\nu)^2}{(N-\nu)^2} + \left(\frac{2m}{\pi}\right)^3 \left[\frac{1}{(N+\nu)^2} - \frac{1}{(N-\nu)^2}\right] + \frac{2m}{\pi} (N+3\nu)^2 \right\} A$$

$$\cong \pm \lambda \frac{m}{\pi} (N+3\nu)^2 \left[1 + \left(\frac{3\nu}{N-\nu}\right)^2\right] A = \pm \lambda \frac{m N^2}{\pi} (1+3\nu/N)^2 \left[1+\left(\frac{3\nu}{N-\nu}\right)^2\right] A;$$

$$\text{Numerator for F:} \frac{\lambda}{2} \left\{ \frac{2m}{\pi} (3\nu)^2 \frac{(N-3\nu)^2}{(N+\nu)^2} + \left(\frac{2m}{\pi}\right)^3 \left[\frac{1}{(N-\nu)^2} - \frac{1}{(N+\nu)^2}\right] + \frac{2m}{\pi} (N-3\nu)^2 \right\} A$$

$$\cong \lambda \frac{m}{\pi} (N-3\nu)^2 \left[1 + \left(\frac{3\nu}{N+\nu}\right)^2\right] A = \lambda \frac{m N^2}{\pi} (1-3\nu/N)^2 \left[1+\left(\frac{3\nu}{N+\nu}\right)^2\right] A.$$

Hence we write

$$D \cong \frac{\lambda}{16\nu^2} \frac{1-17(\nu/N)^2}{1-21(\nu/N)^2} A; \quad E \cong \pm \frac{\lambda}{8} \frac{m}{\pi \nu^2 N^2} \frac{(1+3\nu/N)^2 \left[1+\left(\frac{3\nu}{N-\nu}\right)^2\right]}{1-21(\nu/N)^2} A;$$

$$F \cong \frac{\lambda}{8} \frac{m}{\pi \nu^2 N^2} \frac{(1-3\nu/N)^2 \left[1+\left(\frac{3\nu}{N+\nu}\right)^2\right]}{1-21(\nu/N)^2} A. \quad \text{(II-6a-c)}$$

The solutions for B and C, as given by (II-4b, c), will be modified to first order in λ by the presence of the terms $-(\lambda/2)$ C and $-(\lambda/2)$ B, respectively. A more marked first-order modification of the values for B and C arises implicitly, however, from the first-order change of m, which then affects the terms $\mp (2m/\pi)$ A and $-(2m/\pi)$ A in (II-4b, c). We believe it permissible to disregard here the first-mentioned effect.

The locations of the stop-band boundaries may now be determined, through use of (II-4a). Since D is of order λ, the term $-(\lambda/2)$ D will be disregarded. Moreover the explicit correction terms of order λ arising in B and C will affect the location of the boundaries by an amount which is of order

$(\nu/N)^2$ less than the main effect, and hence (as stated above) will also be disregarded. With m_o denoting that value of m which in the unperturbed case gives the same integral or half-integral value of ν as is to be employed with the perturbation, equation (II-4a) becomes, with the simplifications noted above,

$$\mp \frac{\lambda}{2} - 2\left(\frac{2\,m}{\pi N}\right)^2 = -2\left(\frac{2\,m_o}{\pi N}\right)^2 ,$$

or, treating $m - m_o$ as small,

$$\left(\frac{4\,m_o}{\pi N}\right)^2 \frac{m - m_o}{m_o} = \mp \frac{\lambda}{2}$$

$$\frac{m - m_o}{m_o} = \mp \frac{1}{2}\left(\frac{\pi N}{4\,m_o}\right)^2 \lambda . \qquad \text{(II-7)}$$

These results are those listed in Table II.

APPENDIX III. VARIATIONAL SOLUTION FOR THE PERTURBATION $f(\theta) = \eta \cos \nu\theta$

We employ the variational statement (19) of Sect. II 3 B b,

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \eta \langle x^2 \cos \nu\theta \rangle = \text{min.}, \qquad \text{(III-1)}$$

and employ the trial functions

$$x \doteq A \cos \nu\theta + B \cos (N - \nu)\,\theta + C \cos (N + \nu)\,\theta + R + S \cos 2\nu\theta$$

$$+ D \cos (N - 2\nu)\,\theta + E \cos N\theta + F \cos (N + 2\nu)\,\theta \qquad \text{(III-2a)}$$

or a similar expression involving sine functions [with R absent]. (III-2b)

The variational statement (III-1) then becomes

$$\frac{\nu^2 A^2}{2} + \frac{(N - \nu)^2 B^2}{2} + \frac{(N + \nu)^2}{2} + \frac{(2\nu)^2 S^2}{2} + \frac{(N - 2\nu)^2 D^2}{2} + \frac{N^2 E^2}{2} + \frac{(N + 2\nu)^2 F^2}{2}$$

$$- \frac{2\,m}{\pi}\left[\pm AB + AC + 2\,RE \pm DS + FS\right]$$

$$- \frac{\eta}{2}\left[2\,AR + AS + BD + BE + CE + CF\right] = \text{min.}, \qquad \text{(III-3)}$$

with the upper and lower signs referring respectively to the even and odd eigenfunctions and with the terms in R omitted in the latter case. The simultaneous algebraic equations which result are

$$\nu^2 A \mp \frac{2m}{\pi} B - \frac{2m}{\pi} C - \eta R - \frac{\eta}{2} S = 0 \qquad \text{(III-4a)}$$

$$\mp \frac{2m}{\pi} A + (N-\nu)^2 B - \frac{\eta}{2} E - \frac{\eta}{2} D = 0 \qquad \text{(III-4b)}$$

$$-\frac{2m}{\pi} A + (N+\nu)^2 C - \frac{\eta}{2} E - \frac{\eta}{2} F = 0 \qquad \text{(III-4c)}$$

$$-\frac{\eta}{2} B - \frac{\eta}{2} C + N^2 E - \frac{4m}{\pi} R = 0 \qquad \text{(III-4d)}$$

$$-\eta A - \frac{4m}{\pi} E = 0 \qquad \text{(III-4e)}$$

$$-\frac{\eta}{2} A + (2\nu)^2 S \mp \frac{2m}{\pi} D - \frac{2m}{\pi} F = 0 \qquad \text{(III-4f)}$$

$$-\frac{\eta}{2} B \mp \frac{2m}{\pi} S + (N-2\nu)^2 D = 0 \qquad \text{(III-4g)}$$

$$-\frac{\eta}{2} C - \frac{2m}{\pi} S + (N+2\nu)^2 F = 0, \qquad \text{(III-4h)}$$

with the term $-\frac{4m}{\pi} R$ in (III-4d) and the entire equation (III-4e) omitted when seeking the coefficients of the odd eigenfunction.

For the even eigenfunction, equation (III-4e) immediately gives $E = -\frac{\eta}{4}\frac{\pi}{m} A$ and, ignoring quantities of order η^2, equation (III-4d) then gives

$$R = -\frac{\pi}{4m}\left[\frac{\eta}{2}(B+C) - N^2 E\right] - \left(\frac{\pi}{4m}\right)^2 \left[\nu^2 + N^2\right] \eta A = -\left(\frac{\pi N}{4m}\right)^2 \left[1 + (\nu/N)^2\right] \eta A.$$

For the odd eigenfunction, on the other hand, R does not enter and equation (III-4d) gives

$$E = \frac{\eta}{2N^2}(B+C) \cong -4\frac{m\,\nu}{\pi N^5}\eta A, \text{ by taking } B \cong -\frac{2m}{\pi N^2}\left[1 + 2\nu/N\right] \text{ and } C \cong \frac{2m}{\pi N^2}\left[1 - 2\nu/N\right].$$

Through first order in η, the solutions for B and C will be the same function of m as in the unperturbed case, as can be seen by inspection of equations (III-4b-c).

The last three equations, (III-4f-h), are quite analogous to the equations (II-4d-f) of Appendix II and may be solved in a similar manner. Thus the determinant of the coefficients, which enters in the denominator of the solutions, is approximately $4\nu^2 N^4 - 32\nu^4 N^2 - \nu^2 N^4 - \nu^4 N^2 = 3\nu^2 N^4 - 33\nu^4 N^2 = 3\nu^2 N^4\left[1 - 11(\nu/N)^2\right]$.

Then one finds

$$S = \frac{1}{6\nu^2}\,\frac{1 - 7(\nu/N)^2}{1 - 11(\nu/N)^2}\,\eta A; \qquad D = \pm\frac{m}{3\pi\nu^2 N^2}\,\frac{(1 + 2\nu/N)^2\left[1 + \left(\frac{2\nu}{N-\nu}\right)^2\right]}{1 - 11(\nu/N)^2}\,\eta A;$$

$$F = \frac{m}{3\pi\nu^2 N^2}\,\frac{(1 - 2\nu/N)^2\left[1 + \left(\frac{2\nu}{N+\nu}\right)^2\right]}{1 - 11(\nu/N)^2}.$$

Finally, by use of (III-4a), in conjunction with $B \cong \pm\frac{2m}{\pi N^2} A$ and $C \cong \frac{2m}{\pi N^2} A$, one obtains

$$-2\left(\frac{2m}{\pi N}\right)^2 A - \eta\left(R + \frac{S}{2}\right) = -2\left(\frac{2m_o}{\pi N}\right)^2 A,$$

or

$$\left(\frac{4m_o}{\pi N}\right)^2 \frac{m - m_o}{m} = -\eta\left(R + \frac{S}{2}\right)/A.$$

The shift of the stability boundary corresponding to the even eigensolution is then obtained by substitution of the appropriate expressions found above for R and S, rewriting $\frac{1}{2} S/A \cong \frac{1}{12\nu^2}\eta$ in the approximate form $\frac{1}{6}\left(\frac{\pi N}{4m_o}\right)^2 \eta$ to simplify combination with R/A. Likewise the shift for the odd eigenfunction is obtained by ignoring R and substituting the appropriate expression for $\frac{1}{2}$ S/A $\left(\cong \frac{\eta}{12\nu^2}\right)$.

The procedure outlined in this Appendix leads to the results listed in Table III. It may be of interest to note that the net width of the resonance arises from the term R in the even eigensolution, since S does not change sign in passing from the even to the odd solution.

APPENDIX IV. VARIATIONAL SOLUTION FOR THE PERTURBATION $f(\theta) = \epsilon \cos(\nu - r)\theta + \zeta \cos(\nu + r)\theta$

We employ the variational statement (21) of Sect. II 3 B c,

$$\langle (dx/d\theta)^2 \rangle - m \langle x^2 F(\theta) \rangle - \epsilon \langle x^2 \cos(\nu - r)\theta \rangle - \zeta \langle x^2 \cos(\nu + r)\theta \rangle = \text{min.}, \quad \text{(IV-1)}$$

and employ the trial functions

$$\begin{aligned} x \doteq\ & A\cos\nu\theta + B\cos(N-\nu)\theta + C\cos(N+\nu)\theta \\ & + D\cos(2\nu - r)\theta + E\cos(N - 2\nu + r)\theta + F\cos(N + 2\nu - r)\theta \\ & + G\cos(2\nu + r)\theta + H\cos(N - 2\nu - r)\theta + I\cos(N + 2\nu + r)\theta \\ & + J\cos r\theta + K\cos(N - r)\theta + L\cos(N + r)\theta \end{aligned} \quad \text{(IV-2a)}$$

or a similar expression involving sine functions. (IV-2b)

The variational statement, (IV-1), then becomes

$$\begin{aligned} & \frac{\nu^2 A^2}{2} + \frac{(N-\nu)^2 B^2}{2} + \frac{(N+\nu)^2 C^2}{2} \\ & + \frac{(2\nu - r)^2 D^2}{2} + \frac{(N - 2\nu + r)^2 E^2}{2} + \frac{(N + 2\nu - r)^2 F^2}{2} \\ & + \frac{(2\nu + r)^2 G^2}{2} + \frac{(N - 2\nu - r)^2 H^2}{2} + \frac{(N + 2\nu + r)^2 I^2}{2} \\ & + \frac{r^2 J^2}{2} + \frac{(N - r)^2 K^2}{2} + \frac{(N + r)^2 L^2}{2} \\ & - \frac{2m}{\pi}\left[\pm AB + AC \pm DE + DF \pm GH + GI \pm JK + JL\right] \\ & - \frac{\epsilon}{2}\left[AD + AJ + BE + BK + CF + CL\right] \\ & - \frac{\zeta}{2}\left[AG \pm AJ + BH + BL + CI + CK\right] = \text{min.}, \end{aligned} \quad \text{(IV-3)}$$

where the upper and lower signs respectively refer to the even and odd eigenfunctions. Differentiation with respect to the coefficients A, ... L, in turn, leads to the simultaneous algebraic equations

$$\nu^2 A \mp \frac{2m}{\pi} B - \frac{2m}{\pi} C - \frac{\epsilon}{2} D - \frac{\zeta}{2} G - \left[\frac{\epsilon}{2} \pm \frac{\zeta}{2}\right] J = 0 \qquad \text{(IV-4a)}$$

$$\mp \frac{2m}{\pi} A + (N-\nu)^2 B - \frac{\epsilon}{2} E - \frac{\zeta}{2} H - \frac{\epsilon}{2} K - \frac{\zeta}{2} L = 0 \qquad \text{(IV-4b)}$$

$$-\frac{2m}{\pi} A + (N+\nu)^2 C - \frac{\epsilon}{2} F - \frac{\zeta}{2} I - \frac{\zeta}{2} K - \frac{\epsilon}{2} L = 0 \qquad \text{(IV-4c)}$$

$$-\frac{\epsilon}{2} A + (2\nu - r)^2 D \mp \frac{2m}{\pi} E - \frac{2m}{\pi} F = 0 \qquad \text{(IV-4d)}$$

$$-\frac{\epsilon}{2} B \mp \frac{2m}{\pi} D + (N-2\nu+r)^2 E = 0 \qquad \text{(IV-4e)}$$

$$-\frac{\epsilon}{2} C - \frac{2m}{\pi} D + (N+2\nu-r)^2 F = 0 \qquad \text{(IV-4f)}$$

$$-\frac{\zeta}{2} A + (2\nu+r)^2 G \mp \frac{2m}{\pi} H - \frac{2m}{\pi} I = 0 \qquad \text{(IV-4g)}$$

$$-\frac{\zeta}{2} B \mp \frac{2m}{\pi} G \quad (N-2\nu-r)^2 H = 0 \qquad \text{(IV-4h)}$$

$$-\frac{\zeta}{2} C - \frac{2m}{\pi} G \quad (N+2\nu+r)^2 I = 0 \qquad \text{(IV-4i)}$$

$$-\left[\frac{\epsilon}{2} \pm \frac{\zeta}{2}\right] A + r^2 J \mp \frac{2m}{\pi} K - \frac{2m}{\pi} L = 0 \qquad \text{(IV-4j)}$$

$$-\frac{\epsilon}{2} B - \frac{\zeta}{2} C \mp \frac{2m}{\pi} J + (N-r)^2 K = 0 \qquad \text{(IV-4k)}$$

$$-\frac{\zeta}{2} B - \frac{\epsilon}{2} C - \frac{2m}{\pi} J + (N+r)^2 L = 0 \qquad \text{(IV-4l)}$$

The coefficients D, ... L will be of the order of the perturbation (ϵ and/or ζ), while, as in Appendix III, the solutions for B and C will be the same function of m as in the unperturbed case (ignoring effects of second order). Equations (IV-4d-ℓ) may thus be solved, in groups of three, for the coefficients D, ... L in the same way as in the previous cases. The results are those listed in Table IV.

With the coefficients so determined, equation (IV-4a) then serves to give the location of the stability boundaries. Although the shift, $m - m_o$, is second order in the perturbation, we again ignore in this computation possible second order terms in B and C because of the presumption that $\nu \ll N$. Accordingly

$$-2\left(\frac{2m}{\pi N}\right)^2 A - \frac{\epsilon}{2} D - \frac{\zeta}{2} G - \left(\frac{\epsilon}{2} \pm \frac{\zeta}{2}\right) J = -2\left(\frac{2 m_o}{\pi N}\right)^2 \qquad \text{(IV-4a')}$$

$$\left(\frac{4 m_o}{\pi N}\right)^2 \frac{m - m_o}{m_o} = -\frac{1}{2}\left[\epsilon D + \zeta G + (\epsilon \pm \zeta) J\right] / A, \qquad \text{(IV-5)}$$

leading directly to

$$\frac{m - m_o}{m_o} = \frac{1}{2}\left(\frac{\pi N}{4 m_o}\right)^2 \left[\frac{\epsilon^2}{(3\nu - r)(\nu + r)} \pm \frac{\epsilon \zeta}{\nu^2 - r^2} + \frac{\zeta^2}{(3\nu + r)(\nu - r)}\right], \qquad \text{(IV-6)}$$

as entered in Table IV. It may be of interest to note that the splitting of the m-values, to produce a stop-band, arises from the term of frequency r, whose coefficient (J) is proportional to ($\epsilon \pm \zeta$). The width of the resultant stop-band is proportional to the product $\epsilon \zeta$, thus requiring the presence of both a $\cos(\nu - r)\theta$ and a $\cos(\nu + r)\theta$ term in the applied perturbation. If only one such term is present, however, the results obtained here of course still may be used to give the m-values associated with the oscillation frequency ν.

CAPTIONS FOR FIGURES

Fig. 1. Stability Diagram for

$$d^2x/d\theta^2 + \left[mF(\theta) + \epsilon\,(\cos 7\theta + \cos 8\theta) \right] x = 0$$

$$N = 48.$$

Fig. 2. $f(\theta) = -6.262\,768\,45 \cos(7\theta + 52^{o}.5)$

$-5.737\,231\,55 \cos(8\theta + 60^{o})$, $\nu = 7\text{-}1/2$, $N = 48$.

Fig. 3. $f(\theta)$ given piecewise-constant values in accord with

$f(\theta) = -6.262\,768\,45 \cos(7\theta + 52^{o}.5)$

$-5.737\,231\,55 \cos(8\theta + 60^{o})$, $\nu = 7\text{-}1/2$, $N = 48$.

Fig. 4. $f(\theta) = -8 \cos(7\theta + 45^{o})$, $\nu = 7$, $N = 48$.

Fig. 5. $f(\theta) = -6.606\,123\,086_5 \cos(6\theta + 45^{o})$

$-5.393\,876\,913_5 \cos(8\theta + 60^{o})$, $\nu = 7$, $N = 48$.

Fig. 6. $f(\theta) = \sin 10\theta - 7.5 \cos 4\theta$, $\nu = 5$, $N = 24$.

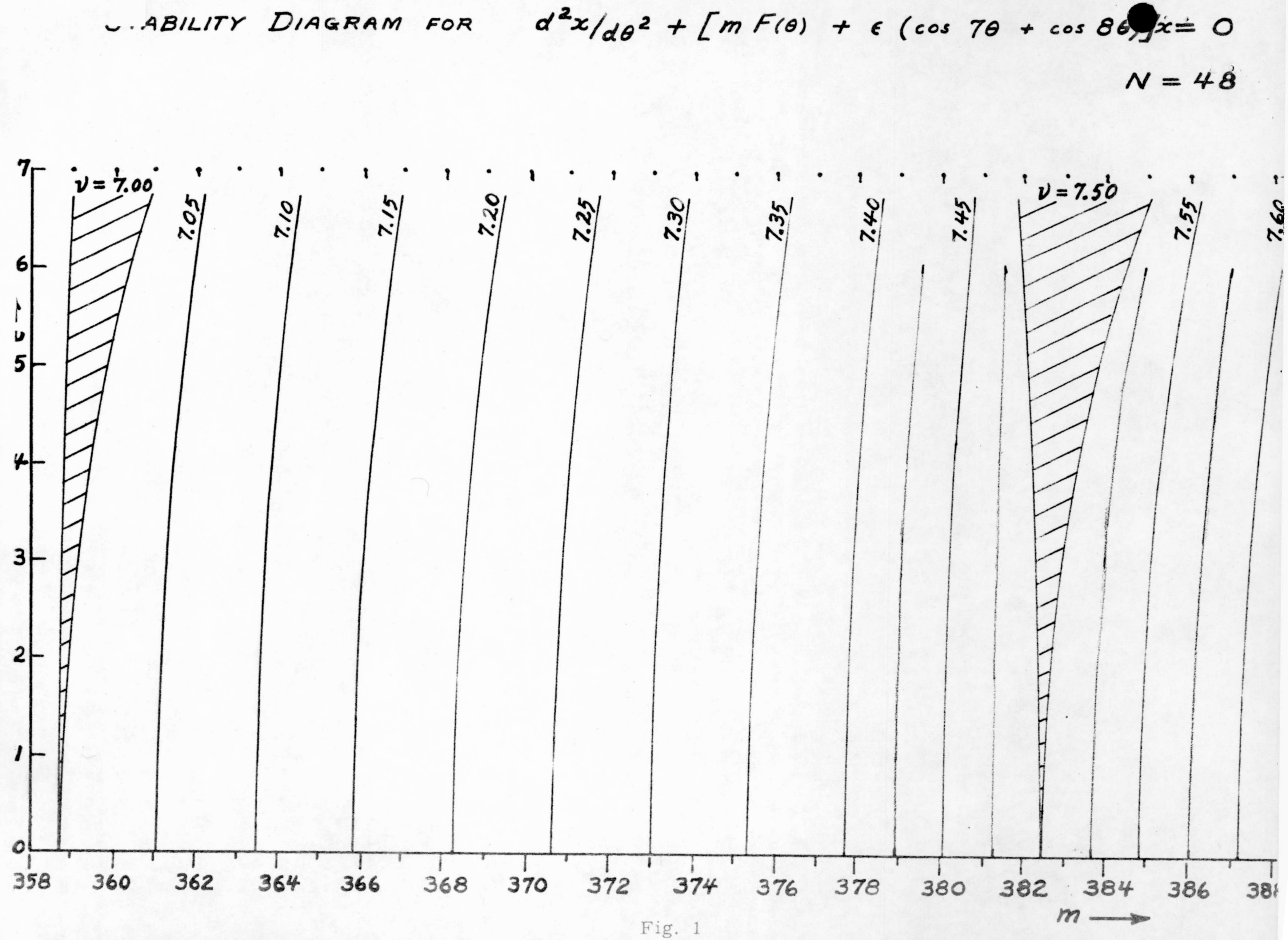
.ABILITY DIAGRAM FOR $d^2x/d\theta^2 + [m F(\theta) + \epsilon (\cos 7\theta + \cos 8\theta)] x = 0$
N = 48
ν = 7.00
7.05
7.10
7.15
7.20
7.25
7.30
7.35
7.40
7.45
ν = 7.50
7.55
7.60
0
1
2
3
4
5
6
7
358
360
362
364
366
368
370
372
374
376
378
380
382
384
386
m

Fig. 1

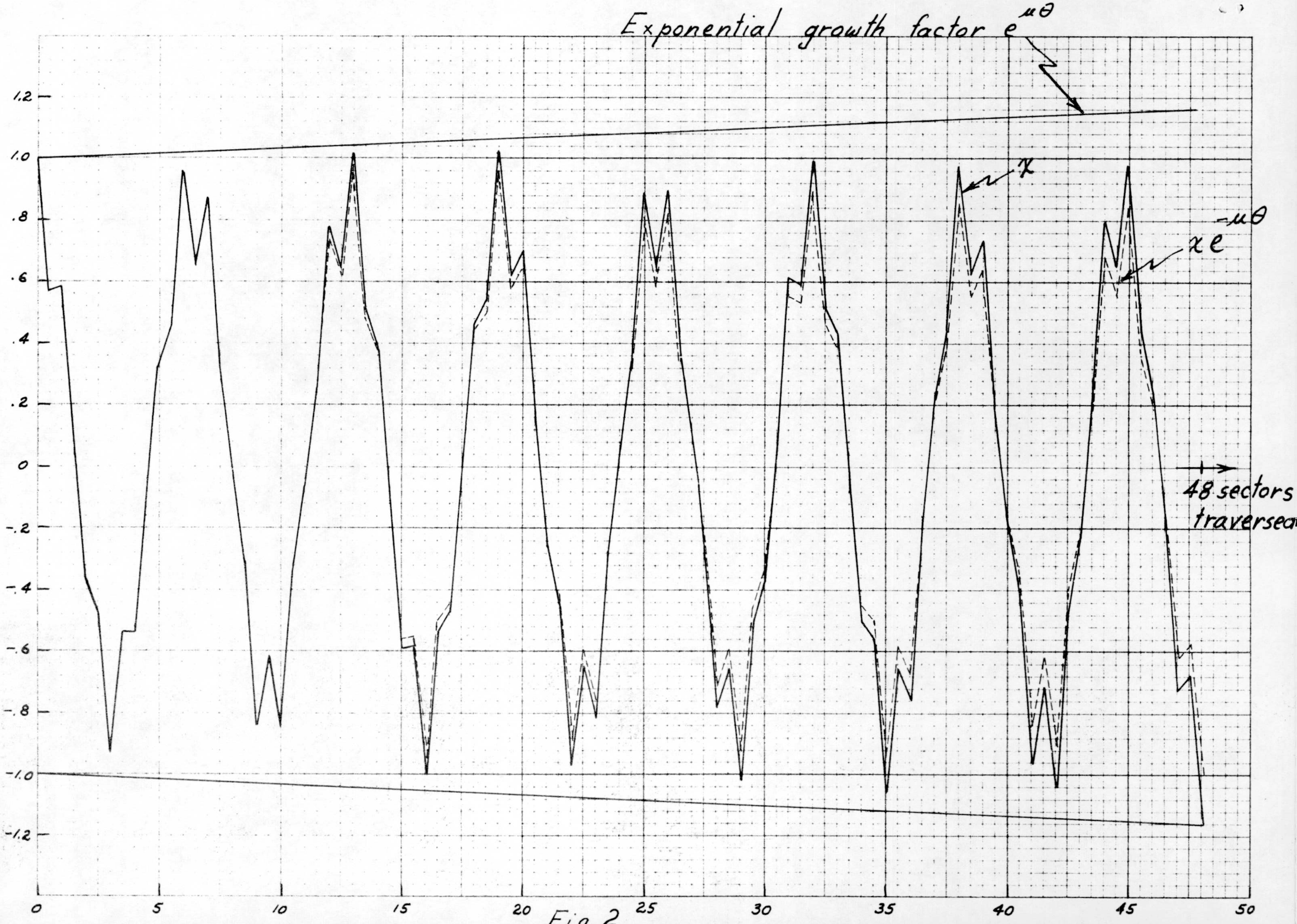
Exponential growth factor $e^{\mu\theta}$
x
$xe^{-\mu\theta}$
48 sectors traversed
1.2
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
-1.2
0
5
10
15
20
25
30
35
40
45
50

Fig. 2

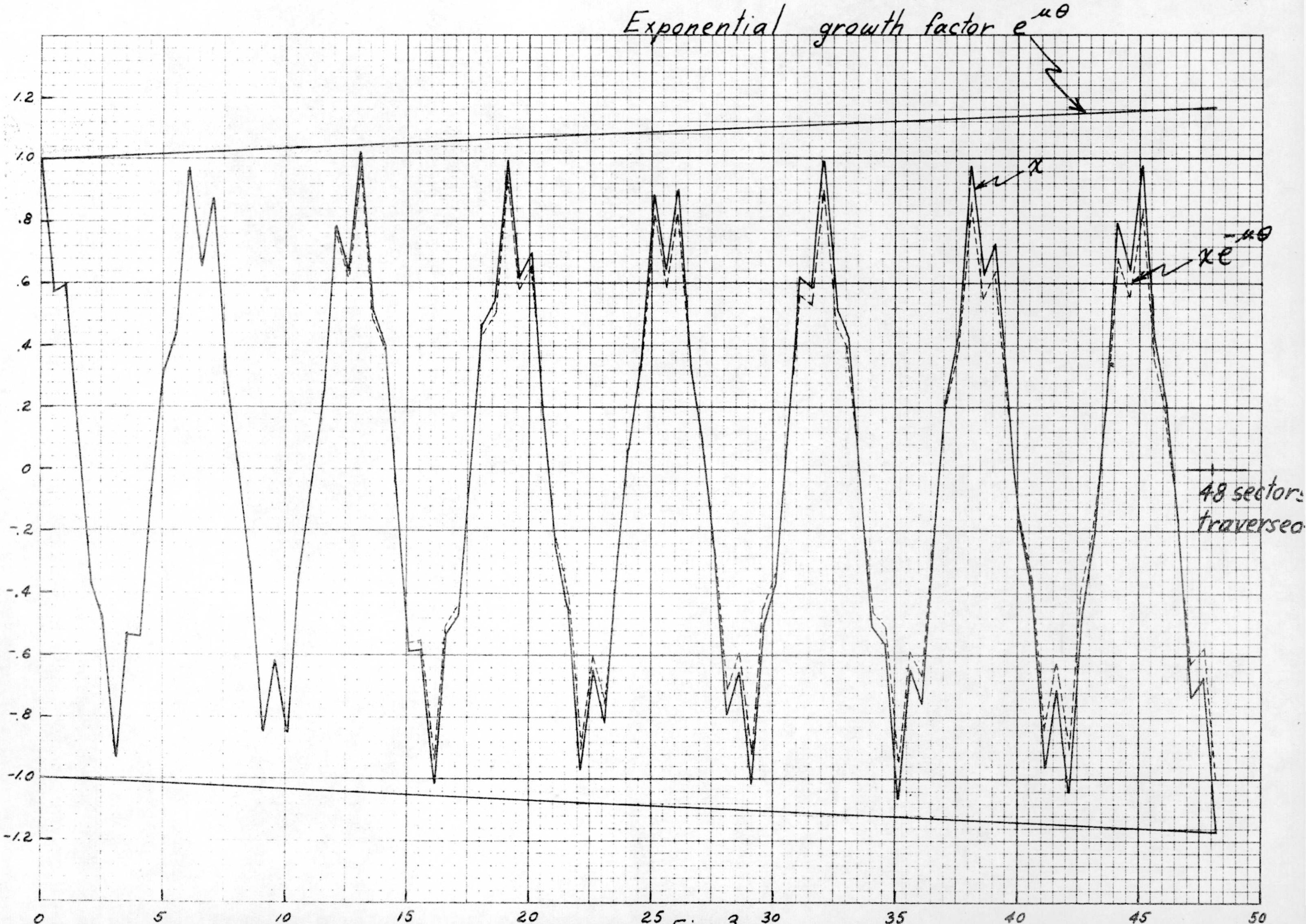
Exponential growth factor $e^{\mu\theta}$
x
$xe^{-\mu\theta}$
48 sectors traversed
1.2
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
-1.2
0
5
10
15
20
25
30
35
40
45
50

Fig. 3

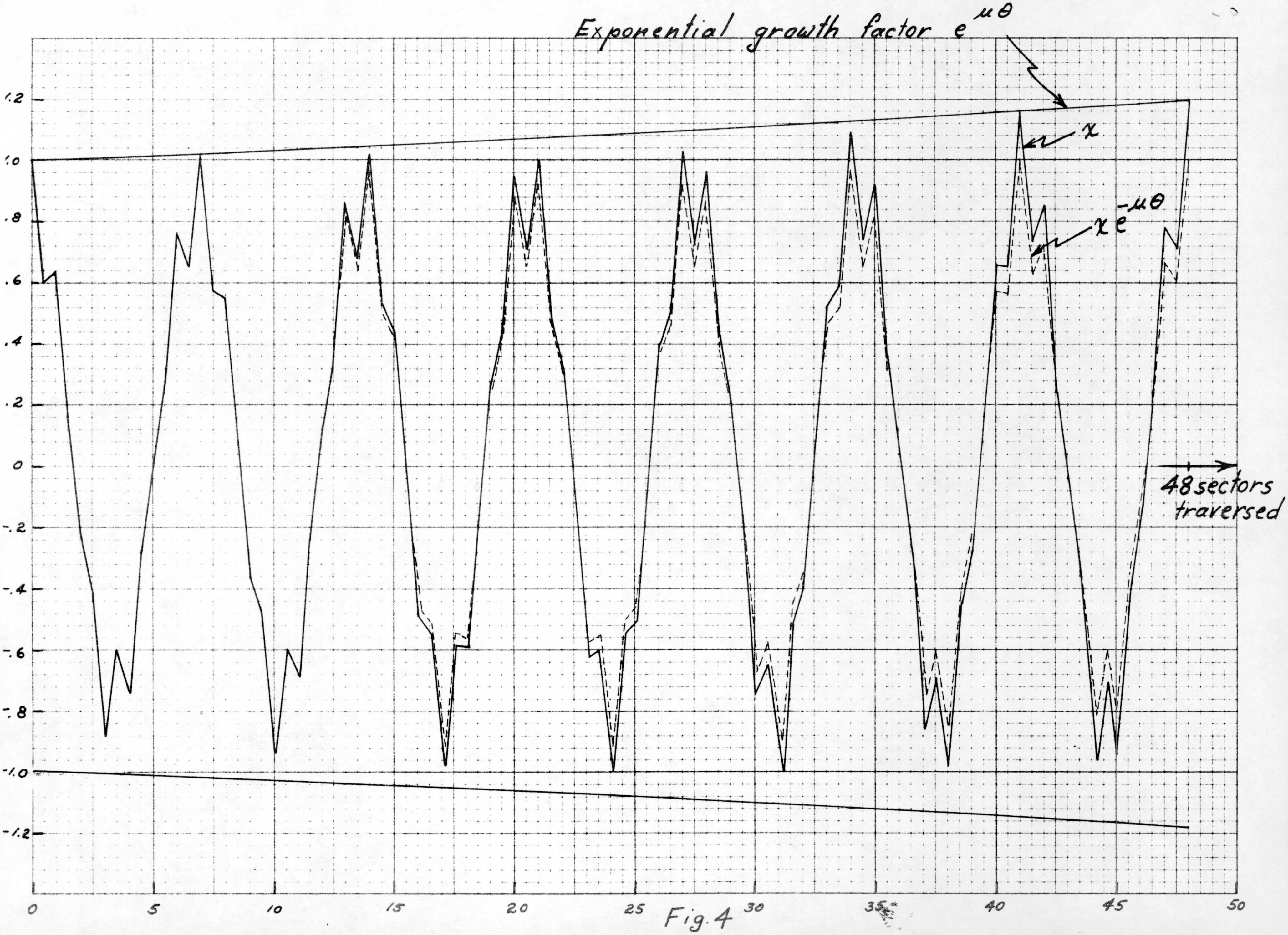
Exponential growth factor $e^{\mu\theta}$
χ
$\chi e^{-\mu\theta}$
48 sectors traversed
1.2
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
-1.2
0
5
10
15
20
25
30
35
40
45
50

Fig. 4

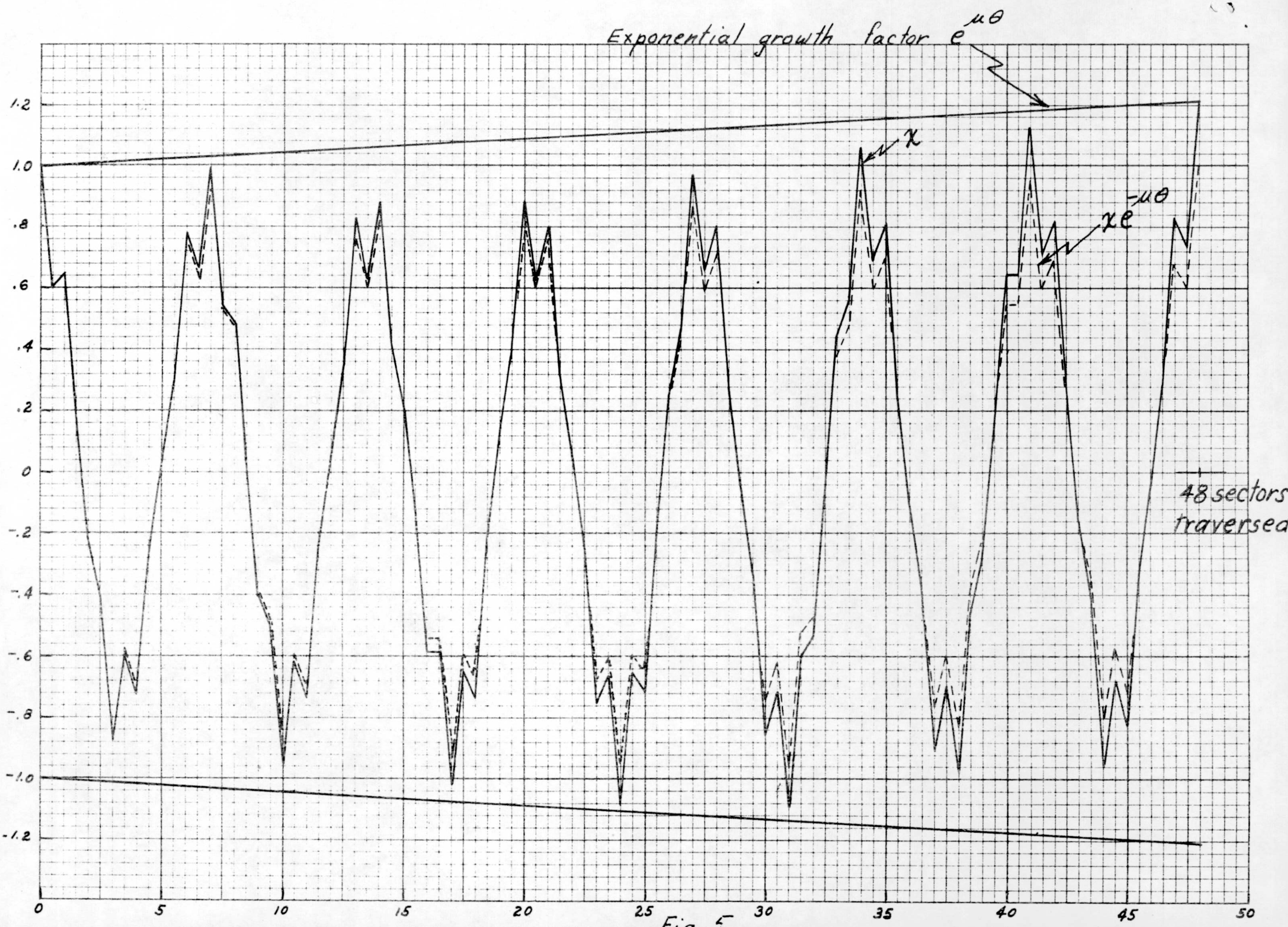
Exponential growth factor $e^{\mu\theta}$
χ
$\chi e^{-\mu\theta}$
48 sectors traversed
1.2
1.0
.8
.6
.4
.2
0
-.2
-.4
-.6
-.8
-1.0
-1.2
0
5
10
15
20
25
30
35
40
45
50

Fig. 5

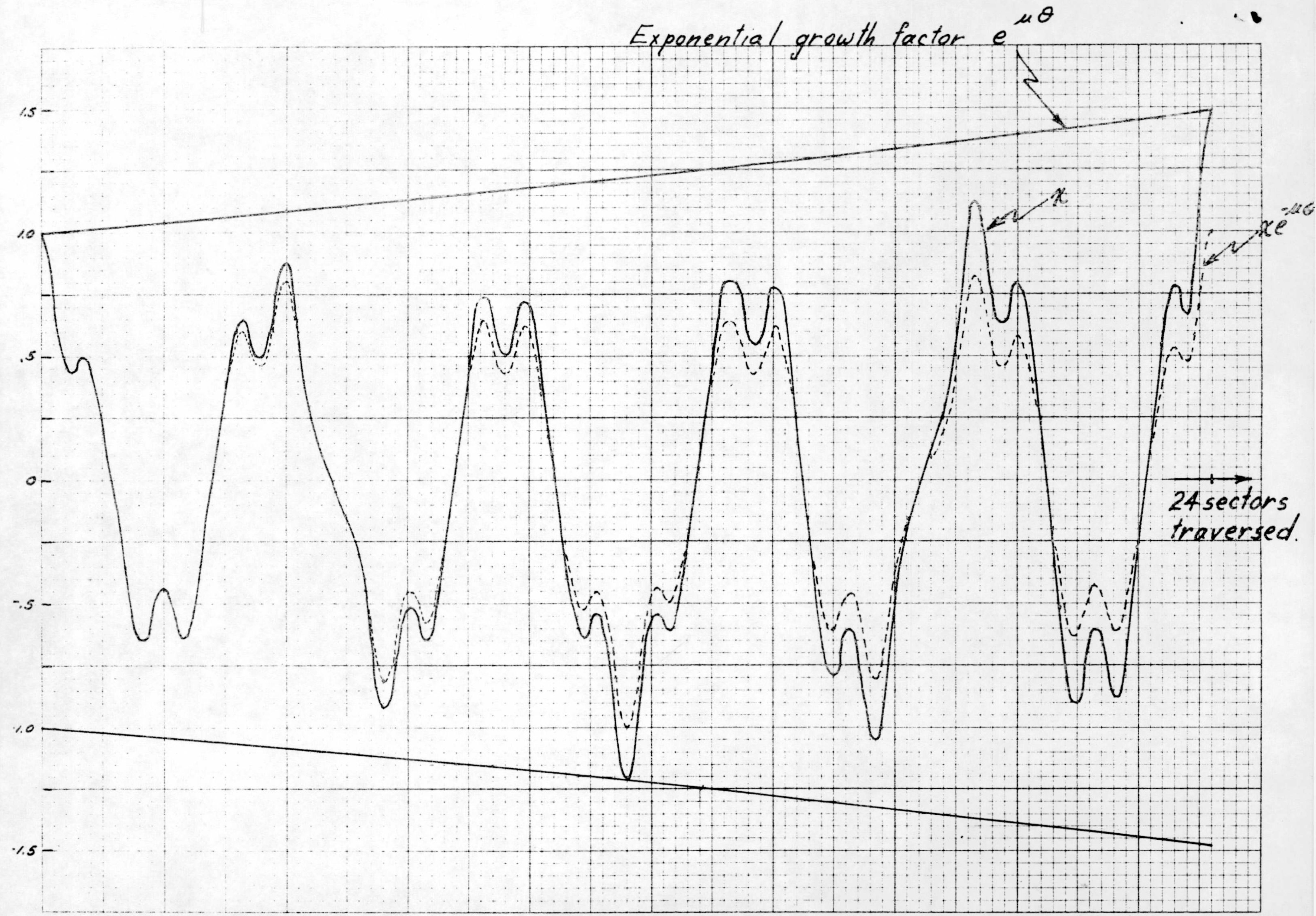
Exponential growth factor $e^{\mu\theta}$
x
$xe^{-\mu\theta}$
24 sectors traversed.
1.5
1.0
.5
0
-.5
-1.0
-1.5
0
5
10
15
20
25

Fig. 6