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TKO — A THREE-DIMENSIONAL NEUTRON-DIFFUSION CODE FOR THE IBM-704

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OPERATED FOR THE U. S. ATOMIC ENERGY COMMISSION
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UC-34: Physics and Mathematics
TID-4500 (15th Ed.)

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W. R. Cadwell

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A description is given of a code for the IBM-704 computer which solves the few-group, time-independent, neutron-diffusion equations in x,y,z geometry. The number of lethargy groups may be one, two, three, or four, and the solution is obtained over a rectangular parallelepiped which is symmetric with respect to the plane x = y. A mesh of horizontal and vertical planes is imposed on this parallelepiped; all region interfaces must occur on mesh planes. Input parameters are specified regionwise, and completely variable mesh spacing is permitted. The number of mesh points on and to one side of the plane x = y is limited to 2675 and 4725 on 16,384-word and 32,768-word computers, respectively. Either a zero flux or a zero current boundary condition may be applied at each boundary plane. The code has been constructed to provide great ease of input preparation and simplicity of code operation. The running time of a typical two-group, 2600-point problem is 1.5 hours.

TKO – A THREE-DIMENSIONAL NEUTRON-DIFFUSION CODE FOR THE IBM-704

W. R. Cadwell

I. INTRODUCTION

Group Equations

TKO is a three-dimensional, reactor-design code for the IBM-704 computer. It finds a discrete solution over a rectangular parallelepiped to the few-group, time-independent, neutron-diffusion equations for a heterogeneous reactor. These equations are of the form

$$\left\{-D_i \nabla^2 \varphi_i + \left(\Sigma_i^a + \Sigma_i^R\right) \varphi_i = \frac{\chi_i \psi}{\lambda} + \Sigma_{i-1}^R \varphi_{i-1}\right\}_{i=1}^{k} , \qquad (1)$$

where

$$1 \le k \le 4 ,$$

$$\Sigma_0^R = \Sigma_k^R \equiv 0 , \qquad (2)$$

$$\chi_1 = 1 \text{ if } k = 1; \sum_{i=1}^{k-1} \chi_i = 1 \text{ and } \chi_k = 0 \text{ if } k > 1 , \qquad (3)$$

and

$$\psi = \sum_{i=1}^{k} \left(\nu \Sigma^f\right)_i \varphi_i . \qquad (4)$$

The physical interpretations of the above symbols are

D = the diffusion coefficient
Σ^a = the absorption cross section
Σ^R = the removal cross section
χ_i = the integral of the fission spectrum over the lethargy range represented by group i
ν_i = the average number of neutrons produced by a fission in group i

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Σ^f = the fission cross section

ψ = the fission source

φ = the neutron flux

λ = the eigenvalue

The rectangular parallelepiped is divided into a number of diffusion regions and rod regions with the interfaces between these regions composed of horizontal and vertical planes only. The parameters D_i, Σ_i^a, Σ_i^R, and $(\nu\Sigma^f)_i$ are regionwise constant.

The neutron flux for one or more groups is not defined interior to a rod region but satisfies

$$\frac{D_i}{\varphi_i}\frac{\partial\varphi_i}{\partial n} = -C_i \tag{5}$$

on its boundary, where C_i is a positive constant and the derivative is taken perpendicular to the boundary in the direction of the rod region. For all other groups the region is treated as a diffusion region.

A composition number, which identifies a set of parameters D_i, Σ_i^a, Σ_i^R, and $(\nu\Sigma^f)_i$, is assigned to each diffusion and rod region. The same number may be assigned to several of these regions if the parameters associated with them are identical. There is no limit to the number of diffusion and rod regions that may be present in the rectangular parallelepiped, but no more than 511 composition numbers may be assigned to them.

To solve the differential equations numerically, a mesh of horizontal and vertical planes is imposed on the rectangular parallelepiped. Since the intervals between successive mesh planes need not be constant, this mesh construction is done in such a way that all external boundaries and internal interfaces lie exactly on mesh planes. The discrete problem is then defined by replacing the differential equations with seven-point difference equations at the points of intersection of the mesh planes.*

The planes are numbered x = 0, 1, 2, ..., ss; y = 0, 1, 2, ..., tt; and z = 0, 1, 2, ..., uu. Either a zero flux or a zero current (symmetry) condition may be applied at each external boundary. The zero flux condition is applied at the boundary plane itself, while the symmetry condition is applied at the first interior plane. Therefore, mesh points on the boundary planes are never points of solution. In addition, the code is restricted to solving only problems which are symmetric with respect to the plane x = y. Hence ss = tt; the points of solution are those lying in and on the boundaries of the isosceles-right-triangular prism x = 1, x = y, y = ss-1, z = 1, and z = uu-1; and the number of points of solution is

$$N = \frac{1}{2}(ss-1)(ss)(uu-1) \ . \tag{6}$$

This number is limited to 2675 and to 4725 on 16,384-word and 32,768-word computers, respectively.

In general, the running time of a problem depends upon N and upon the total number of lethargy groups, K. If ϵ, the input convergence parameter, is chosen to be 0.05, the running time in minutes can be approximated by

$$T = \frac{KN}{60} \ .$$

Thus, a two-group, 2600-point problem requires about 1.5 hours for solution.

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*A statement of the matrix problem which results from this difference approximation is given in Ref 1, together with a description of the techniques used in TKO to solve this problem. These techniques are also considered in some detail in Ref 2.

Additional Features and Restrictions

Computer Equipment Required

The code requires either 16,384 or 32,768 words of core storage, one drum unit of 8192 words, an on-line card reader, an on-line printer, and seven tape units (eight if the tape input flux option is used). The output is edited onto tape, and a simulator program must be used to print this tape on-line if an off-line printer is not available.

One-Group Nonhomogeneous Source Problems

The one-group equation solved by the code is

$$-D_1 \nabla^2 \varphi_1 + \left(\Sigma_1^a + \Sigma_1^R \right) \varphi_1 = \frac{\chi_1 \psi}{\lambda} , \quad (7)$$

where $\chi_1 = 1$ and λ is an input parameter. The value of ψ used at each mesh point is the numerical average of the input flux guesses for the compositions in the eight octants about the point, multiplied by the volume-weighted average of the values of $\nu\Sigma^f$ for these compositions. The code performs a series of iterations to find the flux corresponding to this source.

Series of Problems

Under operator control, the converged flux values on tape at the end of one problem may be used to provide an input flux guess for a subsequent problem or set of problems. The flux guess on the input cards is ignored whenever this option is used.

The problem providing a flux guess and the problem using this guess must be similar in the following respects: 1) They must have the same number of groups; 2) they must have the same number of mesh planes in each coordinate direction; and 3) if the second problem has rod regions with interior points, the first problem must also have rod regions with the same interior points. The last restriction is imposed because the flux guess must be zero at the interior points of rod regions.

Input and Output

Input to the code consists of a description of the mesh (the intervals between successive mesh planes and the placement of diffusion and rod regions) together with values of D_i, Σ_i^a, Σ_i^R, $(\nu\Sigma^f)_i$, and a flux guess, φ_i, for each composition.

The output of the code includes a complete edit of the input, a picture of each different x-y plane with all regions and interfaces indicated, the composition volumes, the composition-integrated flux and source, and the pointwise flux and source values. Under operator control, a preliminary input edit, which includes the pictures of the mesh, may be run before beginning a problem.

II. CODE ROUTINES

Restart Routine

A standard restart may be done at any time during the running of a problem, and an emergency restart at any time during the iterations. The title card is printed on-line in each case.

When a standard restart is done, the problem is continued from its most recent restart point. There is a restart point at the beginning of each routine in the code except in the iteration routine which has such a point at the beginning of the calculations in each group and at the beginning of the eigenvalue calculation.

When an emergency iteration restart is done, the latest flux and source tape is checked. If this tape can be read, the problem is continued from the beginning of the current outer iteration, using this tape as input. If this tape cannot be read, a restart is impossible.

Input Routine

The title card is read and printed on-line. If columns 68-72 do not contain "TKO01", the routine stops and the remainder of the input is ignored.

The 10000, 20000, 30000, and 40000 series cards are then read, and each number is checked against the restrictions listed in Section III under the heading, Card Format. In addition, the numbers are tested to determine if they were properly specified for fixed- or floating-point conversion, and the input as a whole is checked for consistency. If an error is found, the 50000 series is not read.

The composition description given in the 40000 series is expanded, using the overlay process described in Section III under the heading, Construction of the Mesh. The routine stops if there is any region of the mesh for which no composition has been specified.

The 50000 series is then read and checked for consistency and floating-point conversion.

The first input error causes the routine to stop after printing a card number on-line. This is either the number of the first card in the input containing an error or a card number the routine expected, but could not find. In the second case, the difficulty may be caused by an error on the preceding card, by the absence of a blank card following the 40000 series, or by the presence of a blank card anywhere else.

Coefficient Routine

The coefficient routine computes the following quantities in group i at each point s, t, u.* The numerical subscripts apply as indicated in Fig. 1.

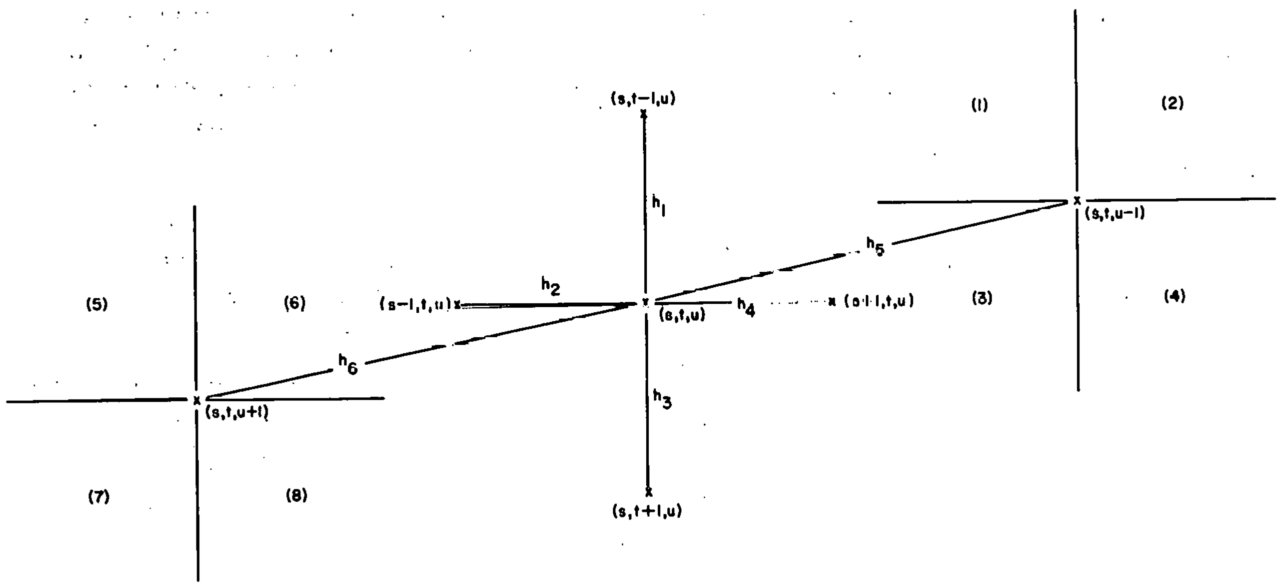

Figure 1

$$C_1'(s,t-1,u) = \frac{D_{i,1}\, h_2h_5 + D_{i,2}\, h_4h_5 + D_{i,5}\, h_2h_6 + D_{i,6}\, h_4h_6}{4h_1} \tag{8}$$

$$C_2'(s-1,t,u) = \frac{D_{i,1}\, h_1h_5 + D_{i,3}\, h_3h_5 + D_{i,5}\, h_1h_6 + D_{i,7}\, h_3h_6}{4h_2} \tag{9}$$

---

* The derivation of the coefficients in x-y geometry is carried out explicitly in Ref 3.

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$$C_3'(s,t+1,u) = \frac{D_{i,3}\,h_2h_5 + D_{i,4}\,h_4h_5 + D_{i,7}\,h_2h_6 + D_{i,8}\,h_4h_6}{4h_3} \qquad (10)$$

$$C_4'(s+1,t,u) = \frac{D_{i,2}\,h_1h_5 + D_{i,4}\,h_3h_5 + D_{i,6}\,h_1h_6 + D_{i,8}\,h_3h_6}{4h_4} \qquad (11)$$

$$C_5'(s,t,u-1) = \frac{D_{i,1}\,h_1h_2 + D_{i,2}\,h_1h_4 + D_{i,3}\,h_2h_3 + D_{i,4}\,h_3h_4}{4h_5} \qquad (12)$$

$$C_6'(s,t,u+1) = \frac{D_{i,5}\,h_1h_2 + D_{i,6}\,h_1h_4 + D_{i,7}\,h_2h_3 + D_{i,8}\,h_3h_4}{4h_6} \qquad (13)$$

$$C_7'(s,t,u) = \frac{(h_1+h_3)\,(h_2+h_4)\,(h_5+h_6)}{8} \qquad (14)$$

$$C_8'(s,t,u) = \frac{h_5}{8}\left(\Sigma^R_{i-1,1}h_1h_2 + \Sigma^R_{i-1,2}h_1h_4 + \Sigma^R_{i-1,3}h_2h_3 + \Sigma^R_{i-1,4}h_3h_4\right) + \frac{h_6}{8}\left(\Sigma^R_{i-1,5}h_1h_2 + \Sigma^R_{i-1,6}h_1h_4 + \Sigma^R_{i-1,7}h_2h_3 + \Sigma^R_{i-1,8}h_3h_4\right) \qquad (15)$$

$$C_0'(s,t,u) = \sum_{j=1}^{6} C_j' + \frac{h_5}{8}\left(\sigma_{i,1}h_1h_2 + \sigma_{i,2}h_1h_4 + \sigma_{i,3}h_2h_3 + \sigma_{i,4}h_3h_4\right) + \frac{h_6}{8}\left(\sigma_{i,5}h_1h_2 + \sigma_{i,6}h_1h_4 + \sigma_{i,7}h_2h_3 + \sigma_{i,8}h_3h_4\right), \qquad (16)$$

where

$$\sigma_{i,\ell} = \Sigma^a_{i,\ell} + \Sigma^R_{i,\ell}$$

$$(\nu\Sigma^f)_{i,s,t,u} = \frac{1}{8C_7'}\Big\{h_5\left[(\nu\Sigma^f)_{i,1}h_1h_2 + (\nu\Sigma^f)_{i,2}h_1h_4 + (\nu\Sigma^f)_{i,3}h_2h_3 + (\nu\Sigma^f)_{i,4}h_3h_4\right] + h_6\left[(\nu\Sigma^f)_{i,5}h_1h_2 + (\nu\Sigma^f)_{i,6}h_1h_4 + (\nu\Sigma^f)_{i,7}h_2h_3 + (\nu\Sigma^f)_{i,8}h_3h_4\right]\Big\} \qquad (17)$$

If one or more of the coefficients C_1', C_2', ..., C_6' refers to a point on a zero flux boundary, this coefficient is set to zero. If the point s, t, u is on a symmetry boundary, the coefficient referring to the point outside the boundary is set to zero and the coefficient referring to the image of this point is doubled. Similarly, if the point s, t, u lies on the diagonal symmetry axis (s = t), C_1' and C_4' are set to zero and C_2' and C_3' are doubled.

If the point s, t, u is on the boundary of a rod region, D_i, Σ^a_i, Σ^R_i, Σ^R_{i-1}, and $(\nu\Sigma^f)_i$ are set to zero in the rod region and C, the logarithmic boundary condition value, is multiplied by the local surface area of the rod interface and the product is added to C_0'. A similar calculation is carried out if the point lies on the boundary of two or more rod regions.

After all boundary conditions have been incorporated, the coefficients C_1', C_2', ..., C_8' are normalized by C_0' to obtain coefficients C_1, C_2, ..., C_8 used in the iteration.

Omega Routine*

The purpose of this routine is to estimate the optimum overrelaxation factors, ω_i, to be used

* The relevant theory underlying this method of estimating overrelaxation factors is the result of work by Dr. R. S. Varga.

in the group iterations. If the matrix of coefficients in group i is designated by M_i and the spectral norm of M_i is designated by $\bar{\mu}(M_i)$, the optimum factors are obtained from the formula

$$\omega_i = \frac{2}{1 + \sqrt{1 - \left[\bar{\mu}\ (M_i)\right]^2}} \quad (18)$$

Given an initial unit vector $\vec{x}^{(0)}$, the iterative scheme used to determine $\bar{\mu}(M_i)$ is defined by

$$y_q^{(n)} = \sum_{r=1}^{q-1} m_{qr}\, x_r^{(n)} + \sum_{r=q+1}^{N} m_{qr}\, x_r^{(n-1)} , \quad (19)$$

$$\pi^{(n)} = \frac{\sum_{q=1}^{N} \left[y_q^{(n)}\right]^2}{\sum_{q=1}^{N} y_q^{(n)}\, x_q^{(n-1)}} , \quad (20)$$

and

$$x_q^{(n)} = \frac{y_q^{(n)}}{\pi^{(n)}} . \quad (21)$$

Ten iterations are usually performed in each group. If $\bar{\mu}(M_i)$ is small, however, the nature of the iterative process is such that underflow may occur before these iterations are completed. In this case, the results of the last valid iteration are used, and the values of $\underline{\omega}_i$ and $\bar{\omega}_i$, defined below, are set to one and two, respectively.

The following quantities are calculated at the conclusion of the iterations in each group:

1) A lower bound for the estimate of ω, denoted $\underline{\omega}$, which is Eq (18) evaluated with $\left[\bar{\mu}(M)\right]^2 = \underline{\pi}$, where

$$\underline{\pi} = \operatorname*{Min.}_{q} \left[\frac{y_q^{(10)}}{x_q^{(9)}}\right] ; \quad (22)$$

2) An upper bound for the estimate of ω, denoted $\bar{\omega}$, which is Eq (18) evaluated with $\left[\bar{\mu}(M)\right]^2 = \bar{\pi}$, where

$$\bar{\pi} = \operatorname*{Max.}_{q} \left[\frac{y_q^{(10)}}{x_q^{(9)}}\right] ; \quad (23)$$

3) The first estimate of $\left[\bar{\mu}(M)\right]^2$, which is $\pi^{(10)}$;

4) Four additional estimates of $\left[\bar{\mu}(M)\right]^2$ obtained from the general formula

$$\delta_\ell^{(n)} = \frac{\left[\delta_{\ell-1}^{(n-1)} - \delta_{\ell-1}^{(n)}\right]^2}{\left[\delta_{\ell-1}^{(n-1)} - \delta_{\ell-1}^{(n)}\right] + \left[\delta_{\ell-1}^{(n-1)} - \delta_{\ell-1}^{(n-2)}\right]} + \delta_{\ell-1}^{(n)} , \quad \ell = 1, 2, 3, 4, \quad (24)$$

where $\delta_0^{(n)} = \pi^{(n)}$; the four estimates are $\delta_1^{(10)}$, $\delta_2^{(10)}$, $\delta_3^{(10)}$, and $\delta_4^{(10)}$. Since the nature of the convergence of $\delta_\ell^{(n)}$ is not known precisely and since the evaluation of Eq (24) may result in great loss of significance, a check is made to insure that the values $\delta_\ell^{(10)}$, $1 \le \ell \le 4$, are acceptable. The condition for acceptance is $\underline{\pi} < \delta_\ell^{(10)} < \bar{\pi}$. If some $\delta_\ell^{(10)}$ is not acceptable, this and all subsequent values are set to zero. The last nonzero value is then chosen as the final estimate of $[\bar{\mu}(M)]^2$ to be used in Eq (18).

5) An estimate of lower and upper bounds, $\underline{N}$ and $\overline{N}$, on the number of inner iterations that will be required per outer iteration.

The bounds are chosen as the smallest positive integers such that

$$(\omega - 1)^{\underline{N}} \leq \begin{cases} \epsilon \text{ if } k = 1 \\ \text{Max. } (0.05, \epsilon) \text{ if } k > 1 , \end{cases} \tag{25}$$

$$\overline{N}(\omega - 1)^{\overline{N}-1} \leq \begin{cases} \epsilon \text{ if } k = 1 \\ \text{Max. } (0.05, \epsilon) \text{ if } k > 1 . \end{cases} \tag{26}$$

The on-line printout at the end of the calculations in each group consists of the number of iterations performed, $\pi^{(10)}$, $\delta_1^{(10)}$, $\delta_2^{(10)}$, $\delta_3^{(10)}$, $\delta_4^{(10)}$, $\underline{\omega}$, ω, $\overline{\omega}$, $\underline{N}$, and $\overline{N}$.

Flux Expansion Routine

The input flux values contained in the 50000 series are normally expanded to provide a pointwise guess for each group. The value at a point is obtained by taking the numerical average of the values corresponding to the eight compositions surrounding the point.

Under operator control, these input values may be ignored and the flux guess may be taken from the binary output tape of a previous problem. The problem providing the flux guess and the current problem must have the same number of groups and the same number of mesh planes in each coordinate direction. In addition, if the current problem has rod regions with interior points, the problem providing the flux guess must also have rod regions identically placed in the mesh.

Iteration Routine

Outer Iterations

A single outer iteration consists of (1) a set of inner iterations in each group, (2) calculation of the source and eigenvalue, (3) extrapolation and renormalization of the source and, (4) an on-line supervisory printout.

At the beginning of the first outer iteration, the initial source is calculated using the volume-averaged values of $\nu\Sigma^f$ provided by the coefficient routine and the pointwise input flux values. Thus,

$$\psi^{(0)}_{s,t,u} = \sum_{i=1}^{k} (\nu\Sigma^f)_{i,s,t,u} \, \varphi^{(0)}_{i,s,t,u} . \tag{27}$$

The norm of this source vector divided by the input eigenvalue,

$$N^{(0)} = \frac{\sum_{s,t,u} \left[\psi^{(0)}_{s,t,u}\right]}{\lambda^{(0)}} , \tag{28}$$

is also calculated for subsequent renormalization purposes.

Inner Iterations

To start the iterations in group i, the group source,

$$S_{i,s,t,u} = \frac{C_7 \chi_i \psi_{s,t,u}}{\lambda} + C_8 \, \varphi_{i-1,s,t,u} , \tag{29}$$

is calculated, where C_7 and C_8 are the normalized coefficients at the point s, t, u.

Inner iteration n + 1 then consists of solving the equation

$$\varphi^{(n+1)}_{i,s,t,u} = \omega_i \Big\{ C_1 \varphi^{(n+1)}_{i,s,t-1,u} + C_2 \varphi^{(n+1)}_{i,s-1,t,u} + C_3 \varphi^{(n)}_{i,s,t+1,u} + C_4 \varphi^{(n)}_{i,s+1,t,u} + C_5 \varphi^{(n+1)}_{i,s,t,u-1} + C_6 \varphi^{(n)}_{i,s,t,u+1} + S_{i,s,t,u} - \varphi^{(n)}_{i,s,t,u} \Big\} + \varphi^{(n)}_{i,s,t,u} \tag{30}$$

at each point, where ω_i is the overrelaxation factor. During the iteration, the residual

$$R_i^{(n+1)} = \sum_{s,t,u} \left| \varphi^{(n+1)}_{i,s,t,u} - \varphi^{(n)}_{i,s,t,u} \right| \tag{31}$$

is also calculated. Inner iterations are continued in group i until

$$R_i^{(n)} \le \begin{cases} (2-\omega_i) \cdot \epsilon & \text{if } k = 1 \\ 0.05\, R_i^{(1)} & \text{if } k > 1 \end{cases} . \tag{32}$$

In addition, if the group source is non-negative, a negative flux check is applied, and the iterations are continued beyond this point, if necessary, to obtain a non-negative flux.

Source and Eigenvalue Calculations

After the inner iterations in each group have been completed, the resulting flux values are used to compute an improved source approximation,

$$\psi^{(m)}_{s,t,u} = \sum_{i=1}^{k} (\nu\Sigma^f)_{i,s,t,u}\, \varphi^{(m)}_{i,s,t,u} , \tag{33}$$

where m is the outer iteration index. (This and all subsequent calculations are bypassed in a one-group problem.)

A new eigenvalue approximation and corresponding bounds are then obtained from the following formulas:

$$\lambda^{(m)} = \lambda^{(m-1)} \frac{\sum_{s,t,u} \left[\psi^{(m)}_{s,t,u}\right]^2}{\sum_{s,t,u} \psi^{(m-1)}_{s,t,u} \cdot \psi^{(m)}_{s,t,u}} , \tag{34}$$

$$\bar{\lambda}^{(m)} = \lambda^{(m-1)} \underset{s,t,u}{\text{Max.}} \left[\frac{\psi^{(m)}_{s,t,u}}{\psi^{(m-1)}_{s,t,u}} \right] , \tag{35}$$

and

$$\underline{\lambda}^{(m)} = \lambda^{(m-1)} \underset{s,t,u}{\text{Min.}} \left[\frac{\psi^{(m)}_{s,t,u}}{\psi^{(m-1)}_{s,t,u}} \right] . \tag{36}$$

If the inequality

$$\epsilon^{(m)} = \frac{\bar{\lambda}^{(m)} - \underline{\lambda}^{(m)}}{2\lambda^{(m)}} < \epsilon^2 \tag{37}$$

is satisfied, the problem is considered converged, the source extrapolation and renormalization are bypassed, and the problem is edited following the supervisory printout. Otherwise, the source is extrapolated and renormalized, and the supervisory printout is followed by another outer iteration.

Source Extrapolation and Renormalization

If the input value of $\bar{\sigma}$,† the approximation to the homogeneous spectral norm of the outer iteration matrix, is zero, no source extrapolation is done for the first three outer iterations, and a value of $\bar{\sigma}$ is calculated at the end of the fourth iteration using the formula

$$\bar{\sigma} = \frac{1}{k} \sum_{i=1}^{k} \frac{R_i^{(1),(4)}}{R_i^{(1),(3)}} . \tag{38}$$

Here $R_i^{(1),(m)}$ is the initial residual in group i for outer iteration m. If this value of $\bar{\sigma}$ is greater than one, the extrapolation is bypassed and the calculation is repeated at the end of the fifth iteration; if less than one, this value is used to begin the extrapolations as indicated below. (In the case of a nonzero input $\bar{\sigma}$, the source extrapolation is bypassed for the first outer iteration, and the input value is used to begin the extrapolations at the end of the second iteration.)

Given the value of $\bar{\sigma}$, an index ℓ is set to zero and

$$p = \cosh^{-1}\left(\frac{2}{\bar{\sigma}} - 1\right) \tag{39}$$

is calculated. The extrapolation factors for this iteration are then

$$\alpha^{(m)} = \frac{2}{2 - \bar{\sigma}}, \quad \beta^{(m)} = 0 . \tag{40}$$

At the end of any subsequent outer iteration, the index ℓ is increased by one, and the extrapolation factors are

$$\alpha^{(m+\ell)} = \frac{4}{\bar{\sigma}} \frac{\cosh \ell p}{\cosh (\ell + 1) p} \tag{41}$$

and

$$\beta^{(m+\ell)} = \frac{\cosh (\ell - 1) p}{\cosh (\ell + 1) p} . \tag{42}$$

With these factors the source extrapolation at the end of iteration m + ℓ is given (Ref 5) by

$$\overset{**}{\psi}{}^{(m+\ell)}_{s,t,u} = \overset{**}{\psi}{}^{(m+\ell-1)}_{s,t,u} + \alpha^{(m+\ell)} \left[\psi^{(m+\ell)}_{s,t,u} - \overset{**}{\psi}{}^{(m+\ell-1)}_{s,t,u}\right]$$
$$+ \beta^{(m+\ell)} \left[\overset{**}{\psi}{}^{(m+\ell-1)}_{s,t,u} - \overset{**}{\psi}{}^{(m+\ell-2)}_{s,t,u}\right] . \tag{43}$$

The norm of this source vector divided by the current eigenvalue,

$$N^{(m+\ell)} = \frac{\sum_{s,t,u} \left|\overset{*}{\psi}{}^{(m+\ell)}_{s,t,u}\right|}{\lambda^{(m+\ell)}} , \tag{44}$$

is calculated, and the source is renormalized by

$$\overset{**}{\psi}{}^{(m+\ell)}_{s,t,u} = \frac{N^{(0)}}{N^{(m+\ell)}} \overset{*}{\psi}{}^{(m+\ell)}_{s,t,u} . \tag{45}$$

Supervisory Printout

At the end of each outer iteration, the initial and final residuals and number of inner iterations performed are printed for each group. A negative sign preceding the inner iteration count indicates

† For the definition of $\bar{\sigma}$ and for methods of computing estimates of $\bar{\sigma}$, see Ref 4.

that extra iterations were required because of a negative flux check. In addition, the printout includes the values of

$$m, \bar{\lambda}^{(m)}, \lambda^{(m)}, \underline{\lambda}^{(m)}, \epsilon^{(m)}, \alpha^{(m)}, \beta^{(m)}, \text{ and } \frac{N^{(0)}}{N^{(m)}} .$$

The value of $\bar{\sigma}$ used to start the source extrapolations is also printed at the end of the second or fourth iteration.

If the problem has converged, the number of the tape unit containing the current flux and source values is printed before the edits are begun. Under operator control, the problem may be forced to edit at the end of any outer iteration. In this case, "Edit Forced" is printed in addition to the tape number.

Input Edit Routine

All of numbers in the 10000, 20000, 30000, and 50000 input series are edited into the first file of the output tape. This editing is normally done at the end of a problem, in which case the edited values of lambda and sigma are the final values calculated by the code. Under operator control, this routine may be entered immediately following the input routine to provide a preliminary input edit. In this case the values of lambda and sigma are those contained on the input cards. In either case the picture routine is always entered following this routine.

Picture Routine

The expanded composition description prepared in the input routine is edited into the second file of the output tape. The editing results in a picture of each different x-y plane with the composition reflected about the diagonal. The compositions are identified by number, and a dotted line is placed along each interface and on the external boundary. The mesh spacing is not indicated.

Normally, this routine is immediately followed by the average routine. If a preliminary input edit is being done, however, there is a stop at the end of this routine. The operator may remove the problem from the computer at this time and print the preliminary edit. A restart continues the normal sequence of calculations, beginning with the coefficient routine.

Average Routine

The composition-integrated volumes and composition-integrated and averaged flux and source values are edited into the third file of the output tape.

The integrated flux for group i, composition c, is given by

$$\overline{\varphi}_{i,c} = \sum \frac{h_s h_t h_u}{8} \Big(\varphi_{i,s-1,t-1,u-1} + \varphi_{i,s-1,t,u-1} + \varphi_{i,s,t-1,u-1} + \varphi_{i,s,t,u-1} + \varphi_{i,s-1,t-1,u} + \varphi_{i,s-1,t,u} + \varphi_{i,s,t-1,u} + \varphi_{i,s,t,u} \Big) , \qquad (46)$$

where h_s, h_t, and h_u are the intervals associated with the mesh cube* indicated. The sum is taken over all mesh cubes of composition c, ignoring mesh cubes outside symmetry planes. The integrated source for composition c is

$$\overline{S}_c = \sum_{i=1}^{k} (\nu\Sigma^f)_{i,c} \overline{\varphi}_{i,c} . \qquad (47)$$

* The term "mesh cube" is used in place of the more accurate "mesh rectangular parallelepiped."

Edit Routine

The final routine of the code edits into the fourth file of the output tape those flux and source vectors called for in columns 62-66 of the title card.

The pointwise values are edited in such a way that a picture of each x-y plane is formed, the values being reflected about the diagonal. Each page is labeled with the problem title, group number, plane number, final eigenvalue, page number, and row and column identification. A maximum of 12 columns and 27 rows are printed per page. The editing is done in fixed-point with three digits to the right of the decimal point and up to four digits to the left.

The source value edited at a point is given by

$$S_{s,t,u} = \sum_{i=1}^{k} \varphi_{i,s,t,u} \frac{\sum_{\ell=1}^{8} (\nu\Sigma^f)_{i,\ell}}{\sum_{\ell=1}^{8} n_\ell}, \tag{48}$$

where ℓ refers to the octant about the point s, t, u; $n_\ell = 1$ if octant ℓ contains a fuel composition $[(\nu\Sigma^f)_i \neq 0$ for some i]; $n_\ell = 0$ otherwise. If the point is interior to a nonfuel composition, the source is set to zero.

III. INPUT PREPARATION

Title Card

A title card must precede the input deck of each problem. This card is used to identify on-line and off-line output and to provide edit control information.

Columns 1-60 of the card are for problem identification. Any combination of alphabetic and numeric information may be used, and any of these columns may be left blank.

Column 61 must be blank.

Columns 62-66 are used to control the flux and source edits. The columns refer to φ_1, φ_2, φ_3, φ_4, and S, respectively. An "N" in a given column causes the edit of the corresponding flux or source to be bypassed. A column is not tested if there is no corresponding flux; for example, columns 64 and 65 are irrelevant in a two-group problem.

Column 67 must be blank.

Columns 68-72 must contain "TKO01".

Data Cards

All of the input to this code is in fixed-point decimal form with each data card punched according to a definite pattern. Columns 1-7 and 11 are blank, columns 8-10 contain "DEC", and columns 12-16 contain the card number followed by a comma in column 17. The input itself begins in column 18 and may extend through column 72. Successive numbers are separated by commas, but no comma is allowed following the last number on a card. The first blank column indicates the end of the information on a card, and any number punched beyond this blank column is not used. Signs may be used but only minus signs are necessary.

The card numbering system divides the input deck into five series, as follows:

10000 series: Miscellaneous parameters and control information
20000 series: Mesh intervals in the x coordinate direction
30000 series: Mesh intervals in the z coordinate direction
40000 series: Composition description
50000 series: Material- and group-dependent parameters.

The input deck, preceded by the title card, must be arranged in order of increasing card number with one blank card following the 40000 series.

Since the input is in fixed-point form, it is necessary to indicate which of the numbers are to be converted to floating-point for use in numerical calculations. This is accomplished by including a decimal point in those numbers which are to be converted. (It should be noted that the number zero need never be accompanied by a decimal point.) In particular, the numbers on card 10002, the mesh intervals on the 20000 and 30000 series cards, and all numbers on the 50000 series cards must contain decimal points; the numbers on card 10001, the plane numbers on the 20000 and 30000 series cards, and all numbers on the 40000 series cards must not contain decimal points. To make this distinction more evident, all numbers which must have decimal points are designated by upper-case or Greek letters in this section under the heading, Card Format; all numbers which must not have decimal points are designated by lower-case letters.

The only real limitation on the range of input numbers is that contained in the IBM-704 itself. Extreme convergence difficulties and even underflow-overflow situations may arise, however, if values are assigned to input quantities which are well outside the normal range of neutron-diffusion parameters.

The number of digits that may be provided in input numbers is completely arbitrary, but no more than eight significant digits can be used by the computer. The only limitation on the amount of input which may be used is that the total of all numbers on the 10000, 20000, 30000, and 40000 series cards, including card numbers, must not exceed 13,350.

Construction of the Mesh

To solve the differential equations numerically, a mesh of horizontal and vertical planes must be imposed on the rectangular parallelepiped. The mesh planes are numbered x = 0, 1, 2, . . . , ss; y = 0, 1, 2, . . . , ss; and z = 0, 1, 2, . . . , uu. The point (0, 0) must lie in the upper left corner of an x-y plane, but plane z = 0 may occur at either the top or bottom of the parallelepiped.

In the first step of the mesh construction, the outer boundaries of the mesh, ss and uu, are specified, together with the conditions to be applied at the boundary planes. Because of the diagonal symmetry, only the boundary conditions for planes x = 0, x = ss, z = 0, and z = uu are required. A zero flux boundary condition is applied at the boundary plane, while a symmetry boundary condition is applied at the first interior plane.

The second step in the mesh construction is the specification of the intervals between successive mesh planes. Only the intervals in the x and z coordinate directions are required because of the diagonal symmetry. A change in the interval at the first or last interior plane is not permitted if this plane is a symmetry boundary.

The final step consists of a description of the material composition of the mesh. For purposes of composition description, the z boundary and interface planes, designated 0, u_1, u_2, . . . , uu, essentially divide the three-dimensional mesh into a set of two dimensional meshes. The composition between planes z = 0 and z = u_1 is not z dependent, and this region may be considered a single x-y plane of composition. The region between planes z = u_1 and z = u_2 may be considered a second composition plane, and so forth. A z interface is not permitted at the first or last interior mesh plane if this plane is a symmetry boundary.

A two-dimensional overlay process is used to describe these composition planes. A particular plane is described by successively laying rectangular blocks of specified composition over the x-y mesh. Any block of composition may be laid over all or parts of blocks specified previously. For each mesh rectangle in the plane, the last specification which includes this rectangle determines its composition. The mesh rectangles to the right of the symmetry diagonal may be described in full, described inaccurately, or not described at all as dictated by convenience, since an automatic reflection about this diagonal is provided. An interface at the first or last interior mesh line in the composition plane is not permitted if this mesh line is a symmetry boundary. After the first composition plane has been completely described, any succeeding composition plane may be described by overlaying changes on the previous composition plane.

Card Format

CARD NUMBER	DESCRIPTION
10001	k, n, ss, uu, a, b, c, d

10001: k, n, ss, uu, a, b, c, d

k: The number of groups ($1 \le k \le 4$).

n: The largest composition number for which input is provided in the 50000 series ($1 \le n \le 511$). Input must be provided in each group for all compositions c = 1, 2, 3, . . . , n.

ss: The last mesh planes in the x and y coordinate directions ($3 \le ss \le 28$). These are either zero flux boundaries or one interval beyond symmetry boundaries.

uu: The last mesh plane in the z coordinate direction ($3 \le uu \le 28$). This is either a zero flux boundary or one interval beyond a symmetry boundary. The number of points of solution is given by the product

$$\frac{1}{2}(ss - 1)(ss)(uu - 1) \quad ,$$

and must not exceed 2675 and 4725 on 16,384-word and 32,768-word computers, respectively.

a: If a = 0, planes x = 0 and y = 0 are zero flux boundaries. If a = 1, planes x = 1 and y = 1 are symmetry boundaries.

b: If b = 0, planes x = ss and y = ss are zero flux boundaries. If b = 1, planes x = ss-1 and y = ss-1 are symmetry boundaries.

c: If c = 0, plane z = 0 is a zero flux boundary. If c = 1, plane z = 1 is a symmetry boundary.

d: If d = 0, plane z = uu is a zero flux boundary. If d = 1, plane z = uu-1 is a symmetry boundary.

10002: λ_0, ϵ, χ_1, χ_2, χ_3, $\bar{\upsilon}_0$

λ_0: Initial approximation to the eigenvalue.

ϵ: Convergence parameter used to terminate the iterations. In a one-group problem, the criterion is

$$R^{(n)} \le (2 - \omega)\epsilon \quad ,$$

where $R^{(n)}$ is the flux residual after n inner iterations. In a two-, three-, or four-group problem, the criterion is

$$\frac{\bar{\lambda}^{(m)} - \underline{\lambda}^{(m)}}{2\lambda^{(m)}} \le \epsilon^2 \quad ,$$

where m is the outer iteration index.

CARD NUMBER	DESCRIPTION

χ_i: The integrals of the fission spectrum. Three values must always be provided.

$$(\chi_1 = 1 \text{ if } k = 1;\ 0.95 < \sum_{i=1}^{k-1} \chi_i < 1.05 \text{ and } \chi_k \equiv 0 \text{ if } k > 1.)$$

$\bar{\sigma}_0$: Approximation to the homogeneous spectral norm of the iteration matrix. This should be zero unless a good approximation is available from a previous problem.

20001
20002
⋮

$H, s_1, H, s_2, \ldots$

The sequence of mesh intervals in the x coordinate direction. The first value is the mesh interval between planes $x = 0$ and $x = s_1$, the second is the interval between planes $x = s_1$ and $x = s_2$, and so forth. There may be any number of sets (H, s_i) per card and any number of cards in this series, but no set may overlap two cards. The sequence s_i must be strictly increasing, and the last value must equal ss on card 10001. If $a = 1$ on card 10001, no s_i may equal 1. If $b = 1$ on card 10001, no s_i may equal ss-1.

30001
30002
⋮

$H, u_1, H, u_2, \ldots$

The sequence of mesh intervals in the z coordinate direction. The first value is the mesh interval between planes $z = 0$ and $z = u_1$, the second is the interval between planes $z = u_1$ and $z = u_2$, and so forth. There may be any number of sets (H, u_i) per card and any number of cards in this series, but no set may overlap two cards. The sequence u_i must be strictly increasing, and the last value must equal uu on card 10001. If $c = 1$ on card 10001, no u_i may equal 1. If $d = 1$ on card 10001, no u_i may equal uu-1.

40001
40002
⋮

$c, s_0, s_1, t_0, t_1, \ldots$

Description of the composition between planes $z = 0$ and $z = u_1$.

c: Number of the composition whose boundaries are described by the following four words $(1 \leq c \leq n)$.

s_0: Left-hand boundary of c.

s_1: Right-hand boundary of c $(00 \leq s_0 < s_1 \leq ss)$.

t_0: Upper boundary of c.

t_1: Lower boundary of c $(00 \leq t_0 < t_1 \leq ss)$.

There may be any number of sets (c, s_0, s_1, t_0, t_1) per card and any number of cards in this sub-series, but no set may overlap two cards. If $a = 1$ on card 10001, no s_0, s_1, t_0, or t_1 may equal 1. If $b = 1$ on card 10001, no s_0, s_1, t_0, or t_1 may equal ss-1.

$4\ \underline{u_1}\ 01$
$4\ \underline{u_1}\ 02$
⋮

As above, giving the changes in the composition between planes $z = 0$ and $z = u_1$ necessary to generate the composition between planes $z = u_1$ and $z = u_2$.

CARD NUMBER	DESCRIPTION
4 $\underline{u_2}$ 01 4 $\underline{u_2}$ 02 ⋮ ⋮	As above, giving the changes in the composition between planes $z = u_1$ and $z = u_2$ necessary to generate the composition between planes $z = u_2$ and $z = u_3$. The sequence u_i must be strictly increasing and is independent of the sequence in the 30000 series. If c = 1 on card 10001, no u_i may equal 1. If d = 1 on card 10001, no u_i may equal uu-1.
4 $\underline{uu}$ 01	A control card signaling the end of the composition description. This card must contain only the card number and must be followed by a blank card. The comma following the card number is optional.
51001	D_1, Σ_1^a, Σ_1^R, $(\nu\Sigma^f)_1$, φ_1 The values of the above parameters for group 1, composition 1, where φ is the input flux guess. If D = 0, Σ^R, $\nu\Sigma^f$, and φ must be zero and Σ^a must be replaced by C. In this case the logarithmic derivative boundary condition is applied at the boundaries of this composition in this group.
51002 ⋮ 51 $\underline{n}$	As above for group 1, compositions 2 through n.
52001 ⋮ 52 $\underline{n}$ ⋮	As above for group 2, compositions 1 through n.
5$\underline{k}$001 ⋮ 5$\underline{k\ n}$	D_k, Σ_k^a, $(\nu\Sigma^f)_k$, φ_k As above for the last group with $\Sigma_k^R \equiv 0$ omitted. If k = 1, this format, rather than the one above for group 1, is to be used.

IV. OPERATING INSTRUCTIONS

Card Reader

Use the 72-72 card reader board.

On-line Printer

Use the SHARE-2 board and 120-column paper.

Off-line Printer

Use 120-column paper with the carriage control switch set to PROGRAM. If an off-line printer is not available, a simulator program must be used.

Tapes

At the beginning of a problem, mount the instruction tape on logical unit 1 and mount blanks on units 2-7. If an input flux guess from tape is to be used, mount this tape on unit 8 unless a preliminary input edit is being done. In this case the flux tape is not mounted until the problem is restarted.

All tapes except the instruction tape are rewound by the code at the beginning of each problem and each restart, and all tapes including the instruction tape are rewound at the end of each problem.

Sense Switches

1) Up: Normal.
   Down: Computer will stop at 2174 at end of current iteration. Problem may be removed for later restart or operator may START to force an edit of this iteration.

2) Up: Normal.
   Down: Restart.

3) Up: Normal.
   Down: Binary output of previous problem, mounted on tape unit 8, used as input flux guess.

4) Up: Normal.
   Down: Preliminary input edit.

5) Up: Normal.
   Down: Emergency iteration restart (in conjunction with sense switch 2).

6) Not used.

Starting Procedure

Mount the necessary tapes, ready the input deck followed by three blank cards in the card reader, ready the printer with the SHARE-2 board, set the necessary sense switches and tape selector switches, CLEAR, and LOAD TAPE.

Standard Restart Procedure

A restart may be done at any time during the running of a problem. Without changing any tapes, rewind tape 1, depress sense switch 2, CLEAR, and LOAD TAPE. START when the computer stops at 0151. (Sense switch 2 may be released at this time.)

An attempt to restart a problem which has not reached the first restart point in the code results in a stop at 0143. Ready the input deck in the card reader and START to begin the problem again.

If a problem is restarted with sense switch 2 not depressed, the computer stops with a select on the card reader. CLEAR, press the start key on the card reader to clear the select, rewind tape 1, depress sense switch 2, and LOAD TAPE to restart again.

Emergency Iteration Restart

An emergency restart may be done at any time during the iterations of a problem. It should never be done unless a standard restart has been attempted and has failed. In no case will an emergency restart succeed if the difficulty is the result of an error on tape 2 or tape 4.

For an emergency restart, follow the standard restart procedure with sense switch 5 depressed in addition to sense switch 2. If a restart is impossible because the last complete flux tape cannot be read, the computer stops at 0324 after printing the number of the unit in error. If this tape can be read, the number of the alternate flux tape is printed and the computer stops at 0312. The operator may change the reel on this alternate unit or may change to a different unit before continuing. (Sense switch 5 may be released at this time.)

An attempted emergency restart when a problem is not being iterated results in a stop at 0234. START for a standard restart.

Removing a Problem

To remove a problem from the computer on short notice, rewind, remove, and label tapes 2-6. Tape 7 must also be saved if it is not in rewound position. If an input flux guess from tape 8 is being used and the first iteration has not been completed, tape 8 must be saved. To restart the problem, remount these tapes, mount a blank on unit 7 if this tape was not saved, and follow the standard restart procedure.

If more time is available for removing the problem, depress sense switch 1. This causes the computer to stop at 2174 at the end of the current iteration. (Sense switch 1 must not be released until the computer stops.) Remove and label tapes 2-6 (tape 7 is blank and tape 8 has been used). To restart, remount these tapes, mount a blank on unit 7, and follow the standard restart procedure.

Instead of removing the problem when the computer stops at 2174, the operator may START to force an edit of the current iteration. At the end of the problem, remove and label tapes 2-6 and print tape 7. If further iterations are desired, remount tapes 2-6, mount a blank on unit 7, and follow the standard restart procedure.

Preliminary Input Edit

The first two files, containing the input edit and pictures of the mesh, are normally written on tape 7 at the end of a problem. However, if sense switch 4 is depressed at the beginning of the problem, these files are written immediately after the input is read. At the end of the picture edit, all tapes are rewound and the computer stops at 1667. (Sense switch 4 must not be released until the computer stops.) Remove and label tapes 2 and 3 and print tape 7. This allows the problem originator to examine these edits before proceeding with the problem. To restart, remount tapes 2 and 3, mount blanks on units 4-7, and follow the standard restart procedure.

If an input flux guess from tape 8 is to be used, do not depress sense switch 3 and mount this flux tape until the restart which follows the preliminary input edit.

Decimal Output

The title card is used to identify the first page of on-line output and each page of off-line output. This title is also printed on-line at every restart. Additional on-line printing consists of the results of the omega calculation, condensed results of each iteration, and the number of the tape unit containing the binary output at the end of the problem.

All output for off-line printing is written on tape 7. Four files are generated during the running of a problem. The first file contains an edit of the input; the second contains a series of pictures of the mesh; the third contains the composition-averaged flux and source; and the fourth contains a series of picture edits of the flux and source values.

If a preliminary input edit is done, tape 7 contains only the first and second of these files. At the end of the problem, only the third and fourth files are present.

Decimal output of several problems cannot be accumulated on tape 7, since this tape is rewound at the end of each problem.

Binary Input and Output

Near the end of two-, three-, and four-group problems, the number of the tape unit containing the latest flux values in binary form is printed on-line. (The binary output of a one-group problem is always on unit 6.) This tape may be saved and used to provide a pointwise input flux guess for a subsequent problem. The two problems must be similar in that they must have the same number of groups and the same number of mesh planes in each coordinate direction, and any rod regions with interior points must be similarly placed in the mesh.

To make use of this tape input option, mount the binary flux tape on unit 8 and depress sense switch 3 at the beginning of the problem. (If a preliminary input edit is being done, do not mount tape 8 or depress sense switch 3 until the restart following this edit.) At the end of the first iteration, tape 8 may be removed and sense switch 3 released. The information on tape 8 is not destroyed in this process. Hence, the binary output of one problem may be used to provide an input flux guess for a series of subsequent problems.

Instruction Tape Preparation

To write an instruction tape, ready a blank tape 1, ready the TKO-1 deck (WBTK1000 - WBTK1707) in the card reader, CLEAR, and LOAD CARDS. After the tape is written, it is rewound and the computer stops at 0343. The instruction tape contains one file of 28 records.

Program Stops

The following stops are in octal. Words in parentheses indicate the routine in which the stop occurs. Unless otherwise indicated, a standard restart should be done at each of these stops.

0006: Error loading binary tape loader from tape 1. LOAD TAPE to try again.

0077: Error loading record of instructions from tape 1. START to try again.

0143: Restart attempted before reaching first restart point. Ready input deck in card reader and START to begin problem again.

0151: Normal restart stop: START.

0234: Emergency iteration restart attempted while not in iteration routine. START to do a standard restart.

0245: Error determining memory assignments. (Iteration)

0312: Latest flux tape can be read and emergency iteration restart is possible. Number of alternate flux tape has been printed. This tape or unit may be changed before continuing.

0324: Latest flux tape cannot be read and emergency iteration restart is impossible. Number of tape in error has been printed.

0355: Tape 2 error. (Flux Expansion)

0371: Tape 8 error. (Flux Expansion)

0377: Tape 5 error. (Flux Expansion)

0446: Tape 3 error. (Flux Expansion)

0565: Error positioning tape 1. (Restart)

0566: Tape 2 error. (Restart)

0567: Error positioning tape 7. (Restart)

0675: Computer error. (Coefficient)

1222, 1223: Error positioning tape 1. (Input)

1317: Input error. Columns 68-72 of title card do not contain "TKO01", or card number of card containing error has been printed.

1320: Entire mesh not filled with composition. Problem cannot be run.

1320: Tape 2 error. (Input Edit)

1321: Tape 2 error. (Input)

1322: Tape 3 error. (Input)

1323: Card reader EOF error. (Input)

1343: Computer error. (Omega)

1346: Tape 3 error. (Input Edit)

1565: Square root error. (Iteration)

1570: Log error. (Iteration)

1621: Hyperbolic cosine error. (Iteration)

1625: Tape 4 error. (Omega)

1625: Computer error. (Input Edit)

1634: Drum error. (Omega)

1667: Preliminary input edit completed.

1706: Tape EOF error. (Omega)

1707: Tape 2 error. (Omega)

2004: Input card does not contain "DEC" in columns 8-10.

2006: Tape 2 error. (Picture)

2007: Tape 3 error. (Picture)

2172: Overflow or underflow. (Coefficient)

2173: Check sums of first pass do not agree with those of second pass. (Coefficient)

2174: Incorrect number of records written on tape. (Coefficient)

2174: Iteration during which sense switch 1 depressed is now completed. Remove problem for later restart or START to force an edit of this iteration.

2175: Tape 2 error. (Coefficient)

2176: Tape 3 error. (Coefficient)

2177: Tape 4 error. (Coefficient)

2201, 2203: Tape error. Number of tape in AC. (Coefficient)

2263: Computer error. (Coefficient)

2322: Input number out of range.

2335: Problem completed.

2350: Tape 2 error. (Edit)

2351: Tape 3 error. (Edit)

2352: Tape 5 error. (Edit)

2353: Tape 6 error. (Edit)

2414: Computer error. (Omega)

2517: Tape 2 error. (Average)

2520: Tape 3 error. (Average)

2521: Tape 5 error. (Average)

2522: Tape 6 error. (Average)

2603: Illegal punch on input card.

3627: Tape 2 error. (Iteration)

3630: Tape 4 error. (Iteration)

3631: Tape 5 error. (Iteration)

3632: Tape 6 error. (Iteration)

3633: Drum error. (Iteration)

3634: Overflow or underflow during initial group calculation. (Iteration)

3635: Overflow or underflow during current inner iteration. (Iteration)

3636: Overflow or underflow during source calculation. (Iteration)

3637: Overflow or underflow during source extrapolation. (Iteration)

3640: Overflow or underflow during source renormalization. (Iteration)

```
SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143                         TK001
        DEC 10001,2,6,10,5,1,0,1,0
        DEC 10002,1.,.05,1.,0,0,0
        DEC 20001,1.0,2,1.5,3,2.0,4,2.5,8,3.0,10
        DEC 30001,3.5,2,4.0,4,4.5,5
        DEC 40001,1,0,10,0,10.3,0,4,0.8.3,0.6.0,6.4.0.3,0.6,4.0.4.0,5
        DEC 40002,5,0,3,0,4,6.0,2,0.2
        DEC 40201,2,0,4,0,8,2,0,6,0,6,5,0,3.0.4.6.0.2.0,2
        DEC 40401,1,0,10,0,10
        DEC 40501

        DEC 51001,1.1,.0021,.031,.0000,100.
        DEC 51002,1.2,.0022,.032,.0042,200.
        DEC 51003,1.3,.0023,.033,.0043,300.
        DEC 51004,1.4,.0024,.034,.0044,400.
        DEC 51005,1.5,.0025,.035,.0045,500.
        DEC 51006,1.6,.0026,.036,.0046,600.
        DEC 52001,.1,.051..000,10.
        DEC 52002,.2,.052,.062,20.
        DEC 52003,.3,.053,.063,30.
        DEC 52004,.4,.054,.064,40.
        DEC 52005,.5,.055,.065,50.
        DEC 52006,.6,.056,.066,60.
```

869 023

869 024

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001

GROUP	IT. NO.	PI	SHK1	SHK2	SHK3	SHK4	WMIN	OMEGA	WMAX	NMIN	NMAX
1	10	0.93929456	0.94141793	0.94250779	0.94199800	0.94192892	1.42565065	1.61163029	1.72986846	7	13
2	10	0.84152818	0.84233275	0.83990558	0.	0.	1.21897873	1.42845103	1.50023520	4	7

GROUP	INITIAL RES.	NO. INNER IT.	FINAL RES.
1	7096.6651	13	265.2531
2	720.8395	6	22.2825

OUTER IT.	MAX. LAMBDA	LAMBDA	MIN. LAMBDA	EPSILON SQ.
1	.839090	.561176	.245103	.529234

ALPHA	BETA	RENORM.
.000000	.000000	1.009672

GROUP	INITIAL RES.	NO. INNER IT.	FINAL RES.
1	209.5590	9	6.6149
2	57.6757	8	1.8824

OUTER IT.	MAX. LAMBDA	LAMBDA	MIN. LAMBDA	EPSILON SQ.
2	.585473	.518251	.458828	.122185

ALPHA	BETA	RENORM.
.000000	.000000	.986788

GROUP	INITIAL RES.	NO. INNER IT.	FINAL RES.
1	32.5421	13	1.4608
2	38.5807	7	1.7269

OUTER IT.	MAX. LAMBDA	LAMBDA	MIN. LAMBDA	EPSILON SQ.
3	.550304	.535482	.524417	.024172

ALPHA	BETA	RENORM.
.000000	.000000	.996524

GROUP	INITIAL RES.	NO. INNER IT.	FINAL RES.
1	7.2289	13	.2898
2	8.4447	6	.4142

OUTER IT.	MAX. LAMBDA	LAMBDA	MIN. LAMBDA	EPSILON SQ.
4	.542705	.539469	.538041	.004323

ALPHA	BETA	RENORM.	SIGMA
1.123919	.000000	.998387	.220512

GROUP	INITIAL RES.	NO. INNER IT.	FINAL RES.
1	1.5156	12	.0627
2	1.3597	6	.0486

OUTER IT.	MAX. LAMBDA	LAMBDA	MIN. LAMBDA	EPSILON SQ.
5	.540635	.540001	.539722	.000846

ALPHA	BETA	RENORM.
.000000	.000000	1.000000

BINARY OUTPUT ON TAPE 6

869 025

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 PAGE 1

2 GROUPS 6 COMPS SS=10 UU= 5 B.C.=1,0,1,0

LAMBDA	EPSILON	CHI 1	CHI 2	CHI 3	SIGMA
.540001	.050000	1.000000	.000000	.000000	.220512

MESH	XX	MESH	XX	MESH	XX	MESH	XX	MESH	XX
1.0000	2	1.5000	3	2.0000	4	2.5000	8	3.0000	10

MESH	ZZ	MESH	ZZ	MESH	ZZ
3.5000	2	4.0000	4	4.5000	5

COMP	D 1	SIG A 1	SIG R 1	N SIG F 1	D 2	SIG A 2	N SIG F 2
1	1.100000	.002100	.031000	.000000	.100000	.051000	.000000
2	1.200000	.002200	.032000	.004200	.200000	.052000	.062000
3	1.300000	.002300	.033000	.004300	.300000	.053000	.063000
4	1.400000	.002400	.034000	.004400	.400000	.054000	.064000
5	1.500000	.002500	.035000	.004500	.500000	.055000	.065000
6	1.600000	.002600	.036000	.004600	.600000	.056000	.066000

```
COLUMN                                      1
    0   1   2   3   4   5   6   7   8   9   0
ROW
  0 .........................................
    .006 006.005 005.004 004.003 003.001 001.
  1 .       .       .       .       .       .
    .006 006.005 005.004 004.003 003.001 001.
  2 . . . . .       .       .       .       .
    .005 005 005 005.004 004.003 003.001 001.
  3 .           . . .   . . .       .       .
    .005 005 005.004 004.003 003 003.001 001.
  4 . . . . . . .   . . .   . . . . .       .
    .004 004 004 004.003 003.001 001 001 001.
  5 .           . . .       .               .
    .004 004 004.003 003 003.001 001 001 001.
  6 . . . . . . .   . . . . .               .
    .003 003 003 003.001 001 001 001 001 001.
  7 .               .                       .
    .003 003 003 003.001 001 001 001 001 001.
  8 . . . . . . . . .                       .
    .001 001 001 001 001 001 001 001 001 001.
  9 .                                       .
    .001 001 001 001 001 001 001 001 001 001.
 10 .........................................
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869 027

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 PLANE 2 PAGE 3

```
COLUMN                                        1
  0    1    2    3    4    5    6    7    8    9    0
ROW
 0 ...............................................
   .006 006.005 005.002 002 002 002.001 001.
 1 .         .         .                   .         .
   .006 006.005 005.002 002 002 002.001 001.
 2 . . . . .         .                   .         .
   .005 005 005 005.002 002 002 002.001 001.
 3 .              . . .                   .         .
   .005 005 005.002 002 002 002 002.001 001.
 4 . . . . . . .                . . . . .         .
   .002 002 002 002 002 002.001 001 001 001.
 5 .                            .                   .
   .002 002 002 002 002 002.001 001 001 001.
 6 .                   . . . . .                   .
   .002 002 002 002.001 001 001 001 001 001.
 7 .                   .                             .
   .002 002 002 002.001 001 001 001 001 001.
 8 . . . . . . . . .                             .
   .001 001 001 001 001 001 001 001 001 001.
 9 .                                                 .
   .001 001 001 001 001 001 001 001 001 001.
10 ...............................................
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8C9 028

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 PLANE 4 PAGE 4

ROW \ COLUMN	0	1	2	3	4	5	6	7	8	9
0	001	001	001	001	001	001	001	001	001	001
1	001	001	001	001	001	001	001	001	001	001
2	001	001	001	001	001	001	001	001	001	001
3	001	001	001	001	001	001	001	001	001	001
4	001	001	001	001	001	001	001	001	001	001
5	001	001	001	001	001	001	001	001	001	001
6	001	001	001	001	001	001	001	001	001	001
7	001	001	001	001	001	001	001	001	001	001
8	001	001	001	001	001	001	001	001	001	001
9	001	001	001	001	001	001	001	001	001	001

```
                                        COMPOSITION-INTEGRATED VOLUME
  1       4.516863   3 952000   3 280000   3 136500   3 175375   2 115000
                                    COMPOSITION-INTEGRATED FLUX - GROUP 1
  1       5 292097   5 301810   5 106859   4 810222   4 984075   3 672337
                                     COMPOSITION-AVERAGED FLUX - GROUP 1
  1       1 565134   2 317027   2 381538   2 593569   2 561126   2 584641
                                    COMPOSITION-INTEGRATED FLUX - GROUP 2
  1       5.189639   5 176727   4 628786   4 448289   4 550879   3 373803
                                     COMPOSITION-AVERAGED FLUX - GROUP 2
  1       1 366516   2 185637   2 224565   2 323417   2 314115   2 325046
                                        COMPOSITION-INTEGRATED SOURCE
  1         000000   4 122247   3 842084   3 322555   3 402355   2 277637
                                         COMPOSITION-AVERAGED SOURCE
  1         000000   1 128410   1 157887   1 236304   1 229425   1 241424
```

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TKOO1 GROUP 1 PLANE 1 LAMBDA .540001 PAGE 6

	1	2	3	4	5	6	7	8	9
1	74.980	74.740	73.354	69.590	61.238	49.999	36.509	22.674	8.899
2	74.740	74.485	73.089	69.298	60.927	49.636	36.162	22.435	8.812
3	73.354	73.089	71.698	67.758	59.316	47.715	34.318	21.173	8.366
4	69.590	69.298	67.758	63.573	54.787	42.487	29.164	17.760	7.260
5	61.238	60.927	59.316	54.787	45.671	33.585	21.316	12.860	5.542
6	49.999	49.636	47.715	42.487	33.585	23.941	15.124	9.130	4.017
7	36.509	36.162	34.318	29.164	21.316	15.124	10.004	6.164	2.758
8	22.674	22.435	21.173	17.760	12.860	9.130	6.164	3.864	1.752
9	8.899	8.812	8.366	7.260	5.542	4.017	2.758	1.752	.803

869 032

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143   TK001 GROUP 1 PLANE 2 LAMBDA .540001 PAGE 7

	1	2	3	4	5	6	7	8	9
1	70.726	70.515	69.297	66.014	58.049	47.113	34.283	21.055	8.277
2	70.515	70.292	69.067	65.740	57.739	46.760	33.950	20.832	8.197
3	69.297	69.067	67.879	64.297	56.113	44.892	32.181	19.655	7.783
4	66.014	65.740	64.297	60.200	51.668	39.809	27.215	16.483	6.756
5	58.049	57.739	56.113	51.668	42.944	31.336	19.883	11.982	5.168
6	47.113	46.760	44.892	39.809	31.336	22.254	14.098	8.518	3.751
7	34.283	33.950	32.181	27.215	19.883	14.098	9.335	5.758	2.578
8	21.055	20.832	19.655	16.483	11.982	8.518	5.758	3.612	1.639
9	8.277	8.197	7.783	6.756	5.168	3.751	2.578	1.639	.752

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 GROUP 1 PLANE 3 LAMBDA .540001 PAGE 8

	1	2	3	4	5	6	7	8	9
1	54.835	54.663	53.674	51.041	44.435	35.662	25.825	15.689	6.211
2	54.663	54.481	53.489	50.811	44.178	35.388	25.571	15.524	6.152
3	53.674	53.489	52.538	49.602	42.830	33.942	24.223	14.655	5.846
4	51.041	50.811	49.602	46.143	39.271	29.999	20.432	12.328	5.088
5	44.435	44.178	42.830	39.271	32.501	23.542	14.987	9.046	3.917
6	35.662	35.388	33.942	29.999	23.542	16.695	10.655	6.467	2.858
7	25.825	25.571	24.223	20.432	14.987	10.655	7.091	4.393	1.974
8	15.689	15.524	14.655	12.328	9.046	6.467	4.393	2.767	1.259
9	6.211	6.152	5.846	5.088	3.917	2.858	1.974	1.259	.579

8869 034

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 GROUP 1 PLANE 4 LAMBDA .540001 PAGE 9

	1	2	3	4	5	6	7	8	9
1	29.414	29.242	28.396	26.220	22.101	17.496	12.629	7.781	3.227
2	29.242	29.075	28.235	25.057	21.962	17.363	12.512	7.706	3.198
3	28.396	28.235	27.389	25.186	21.234	16.663	11.895	7.312	3.050
4	26.220	26.057	25.186	23.062	19.381	14.806	10.221	6.288	2.686
5	22.101	21.962	21.234	19.381	16.009	11.763	7.766	4.784	2.110
6	17.496	17.363	16.653	14.806	11.763	8.525	5.638	3.490	1.563
7	12.629	12.512	11.895	10.221	7.766	5.638	3.829	2.405	1.091
8	7.781	7.706	7.312	6.288	4.784	3.490	2.405	1.531	.702
9	3.227	3.198	3.050	2.696	2.110	1.563	1.091	.702	.324

	1	2	3	4	5	6	7	8	9
1	40.221	40.118	39.484	37.779	33.539	28.033	20.976	15.042	5.809
2	40.118	40.005	39.368	37.645	33.412	27.881	20.828	14.905	5.755
3	39.484	39.368	38.769	36.937	32.775	27.093	20.077	14.183	5.477
4	37.779	37.645	36.937	34.989	30.739	25.383	18.339	12.212	4.776
5	33.539	33.412	32.775	30.739	26.261	21.324	13.674	8.404	3.604
6	28.033	27.881	27.093	25.383	21.324	16.283	9.873	5.926	2.601
7	20.976	20.828	20.077	18.339	13.674	9.873	6.492	3.990	1.784
8	15.042	14.905	14.183	12.212	8.404	5.926	3.990	2.498	1.132
9	5.809	5.755	5.477	4.776	3.604	2.601	1.784	1.132	.519

869 035

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 GROUP 2 PLANE 2 LAMBDA .540001 PAGE 11

	1	2	3	4	5	6	7	8	9
1	38.148	38.067	37.565	36.232	32.189	26.596	19.773	13.840	5.387
2	38.067	37.977	37.478	36.162	32.050	26.440	19.626	13.710	5.337
3	37.565	37.478	37.056	35.527	31.333	25.634	18.870	13.026	5.077
4	36.282	36.162	35.527	33.509	29.206	23.853	17.072	11.163	4.426
5	32.189	32.050	31.333	29.206	24.803	19.823	12.733	7.804	3.355
6	26.596	26.440	25.634	23.853	19.823	14.935	9.174	5.522	2.428
7	19.773	19.626	18.870	17.072	12.733	9.174	6.050	3.725	1.667
8	13.840	13.710	13.026	11.163	7.804	5.522	3.725	2.335	1.059
9	5.387	5.337	5.077	4.426	3.355	2.428	1.667	1.059	.486

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 GROUP 2 PLANE 3 LAMBDA .540001 PAGE 12

	1	2	3	4	5	6	7	8	9
1	30.357	30.282	29.842	28.760	24.914	20.114	14.858	10.103	4.016
2	30.282	30.201	29.766	28.646	24.782	19.988	14.741	10.006	3.979
3	29.842	29.766	29.402	28.036	24.090	19.353	14.138	9.501	3.787
4	28.760	28.646	28.036	25.977	22.256	17.899	12.687	8.150	3.310
5	24.914	24.782	24.090	22.256	18.756	14.714	9.555	5.859	2.536
6	20.114	19.988	19.353	17.899	14.714	10.961	6.895	4.183	1.848
7	14.858	14.741	14.138	12.687	9.555	6.895	4.586	2.839	1.275
8	10.103	10.006	9.501	8.150	5.859	4.183	2.839	1.788	.814
9	4.016	3.979	3.787	3.310	2.536	1.848	1.275	.814	.374

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 GROUP 2 PLANE 4 LAMBDA .540001 PAGE 13

	1	2	3	4	5	6	7	8	9
1	19.998	19.867	19.238	17.659	14.193	11.160	8.118	5.242	2.112
2	19.867	19.740	19.123	17.521	14.100	11.082	8.049	5.192	2.094
3	19.238	19.123	18.521	16.751	13.620	10.678	7.686	4.929	1.997
4	17.659	17.521	16.751	14.912	12.416	9.668	6.727	4.216	1.756
5	14.193	14.100	13.620	12.416	10.304	7.790	5.058	3.118	1.369
6	11.160	11.082	10.678	9.668	7.790	5.704	3.675	2.263	1.011
7	8.118	8.049	7.686	6.727	5.058	3.675	2.482	1.556	.705
8	5.242	5.192	4.929	4.216	3.118	2.263	1.556	.990	.454
9	2.112	2.094	1.997	1.756	1.369	1.011	.705	.454	.210

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 SOURCE PLANE 1 LAMBDA .540001 PAGE 14

	1	2	3	4	5	6	7	8	9
1	2.999	2.968	2.897	2.746	2.416	1.998	1.478	1.045	.000
2	2.968	2.947	2.888	2.736	2.406	1.986	1.468	1.035	.000
3	2.897	2.888	2.831	2.673	2.349	1.920	1.412	.985	.000
4	2.746	2.736	2.673	2.509	2.181	1.782	1.281	.846	.000
5	2.416	2.406	2.349	2.181	1.851	1.488	.000	.000	.000
6	1.998	1.986	1.920	1.782	1.488	1.129	.000	.000	.000
7	1.478	1.468	1.412	1.281	.000	.000	.000	.000	.000
8	1.045	1.035	.985	.846	.000	.000	.000	.000	.000
9	.000	.000	.000	.000	.000	.000	.000	.000	.000

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 SOURCE PLANE 2 LAMBDA 540001 PAGE 15

	1	2	3	4	5	6	7	8	9
1	2.843	2.814	2.754	2.613	2.278	1.870	1.382	.955	.000
2	2.814	2.796	2.747	2.604	2.267	1.859	1.371	.945	.000
3	2.754	2.747	2.692	2.536	2.211	1.797	1.316	.898	.000
4	2.613	2.604	2.536	2.365	2.049	1.660	1.183	.768	.000
5	2.278	2.267	2.211	2.049	1.733	1.372	.000	.000	.000
6	1.870	1.859	1.797	1.660	1.372	1.028	.000	.000	.000
7	1.382	1.371	1.316	1.183	.000	.000	.000	.000	.000
8	.955	.945	.898	.768	.000	.000	.000	.000	.000
9	.000	.000	.000	.000	.000	.000	.000	.000	.000

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 SOURCE PLANE 3 LAMBDA .540001 PAGE 16

	1	2	3	4	5	6	7	8	9
1	2.256	2.232	2.181	2.048	1.731	1.397	1.030	.692	.000
2	2.232	2.217	2.175	2.040	1.722	1.388	1.021	.686	.000
3	2.181	2.175	2.122	1.971	1.673	1.342	.978	.651	.000
4	2.048	2.040	1.971	1.804	1.545	1.236	.872	.557	.000
5	1.731	1.722	1.673	1.545	1.299	1.011	.000	.000	.000
6	1.397	1.388	1.342	1.236	1.011	.750	.000	.000	.000
7	1.030	1.021	.978	.872	.000	.000	.000	.000	.000
8	.692	.686	.651	.557	.000	.000	.000	.000	.000
9	.000	.000	.000	.000	.000	.000	.000	.000	.000

SAMPLE PROBLEM FOR INCLUSION IN WAPD-TM-143 TK001 SOURCE PLANE 4 LAMBDA .540001 PAGE 17

	1	2	3	4	5	6	7	8	9
1	1.455	1.434	1.378	1.235	.973	.765	.556	.358	.000
2	1.434	1.420	1.370	1.226	.966	.760	.552	.354	.000
3	1.378	1.370	1.311	1.159	.934	.732	.526	.336	.000
4	1.235	1.226	1.159	1.021	.851	.662	.460	.288	.000
5	.973	.966	.934	.851	.706	.532	.000	.000	.000
6	.765	.760	.732	.662	.532	.389	.000	.000	.000
7	.556	.552	.526	.460	.000	.000	.000	.000	.000
8	.358	.354	.336	.288	.000	.000	.000	.000	.000
9	.000	.000	.000	.000	.000	.000	.000	.000	.000

ACKNOWLEDGMENT

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