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INVERSION OF THE ANGULAR-MOMENTUM EXPANSION OF MESON PHOTOPRODUCTION AMPLITUDES

BERKELEY, CALIFORNIA

UCRL-8858
Physics and Mathematics
TID-4500 (15th Ed.)

UNIVERSITY OF CALIFORNIA

Lawrence Radiation Laboratory
Berkeley, California

Contract No. W-7405-eng-48

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Printed for the U. S. Atomic Energy Commission

Printed in USA. Price 50 cents. Available from the
Office of Technical Services
U. S. Department of Commerce
Washington 25, D.C.

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ABSTRACT

Helicity amplitudes are written for photoproduction of mesons and related to the amplitudes of Chew, Goldberger, Low, and Nambu. The expansions of both types of amplitudes in terms of amplitudes for particular angular-momentum states are then inverted.

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I. INTRODUCTION

The formulation by Jacob and Wick[1] (hereafter referred to as JW) of scattering theory in terms of helicity states results in a simplification of the amplitudes for photoproduction. The advantages gained by employing these amplitudes are: (a) One is able to write directly a rather simple angular-momentum expansion for the various amplitudes; and (b) the complete set of functions $[d_{mn}^{J}(\theta)]$ used in this expansion have simple orthogonality relations that make the inversion of these expansions quite simple. In this note, we explore the relationship between helicity amplitudes and the amplitudes heretofore used in photomeson theory.

II. HELICITY AMPLITUDES

By the use of the procedure and notation of JW, the four helicity states for the photon-nucleon system are described completely by the total helicity (λ) of the system:

$$\lambda = \lambda_p - \lambda_\gamma , \qquad \lambda_p = \pm \frac{1}{2} , \qquad \lambda_\gamma = \pm 1 . \tag{2.1}$$

The nucleon is taken, for convenience, to be traveling in the z direction so that λ_p will be identical to the z component of nucleon spin. Since the π has no spin, the total final helicity (μ) is just the helicity of the final nucleon.

If we assume parity conservation, the helicity amplitudes ($f_{\mu,\lambda}$) obey

$$f_{-\mu,-\lambda}(\theta, \phi) = \eta_g f_{\mu,\lambda}(\theta, \pi - \phi) , \tag{2.2}$$

where

$$\eta_g = \frac{\eta_n \eta_\gamma}{\eta_n \eta_\pi} (-1)^{\mu-\lambda} . \tag{2.3}$$

If we make the usual intrinsic parity assignment $\eta_n = 1$ and $\eta_\gamma = \eta_\pi = -1$, the scattering amplitude is

*This work done under the auspices of the U. S. Atomic Energy Commission.

$$F = \begin{pmatrix} f_{\frac{1}{2},\frac{3}{2}}(\theta,\phi) & f_{\frac{1}{2},\frac{1}{2}}(\theta,\phi) & f_{\frac{1}{2},-\frac{1}{2}}(\theta,\phi) & f_{\frac{1}{2},-\frac{3}{2}}(\theta,\phi) \\ -f_{\frac{1}{2},-\frac{3}{2}}(\theta,\pi-\phi) & -f_{\frac{1}{2},-\frac{1}{2}}(\theta,\pi-\phi) & -f_{\frac{1}{2},\frac{1}{2}}(\theta,\pi-\phi) & -f_{\frac{1}{2},\frac{3}{2}}(\theta,\pi-\phi) \end{pmatrix} \tag{2.4}$$

where

$$f_{\frac{1}{2},\lambda} = \sum_J (J + \frac{1}{2}) A_\lambda^J e^{i(\lambda-\frac{1}{2})\phi} d_{\lambda,\frac{1}{2}}^J(\theta) . \tag{2.5}$$

Now, using the orthogonality relation for $d_{\lambda,\mu}^J$,

$$\int_0^\pi d_{m\mu}^J(\theta)\, d_{m\mu}^{J'}(\theta) \sin\theta\, d\theta = \frac{\delta_{J,J'}}{(J+\frac{1}{2})} , \tag{2.6}$$

one gets

$$A_\lambda^J = \frac{1}{2\pi} \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi\, d_{\lambda,\frac{1}{2}}^J(\theta)\, e^{-i(\lambda-1/2)\phi} f_{\frac{1}{2},\lambda}(\theta,\phi) . \tag{2.7}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{q}{k} \left| f_{\mu,\lambda} \right|^2 .$$

III. CONNECTION BETWEEN AMPLITUDES

The amplitudes of Chew, Goldberger, Low, and Nambu[2] (hereafter referred to as CGLN) are defined by

$$\mathcal{F} = i\,\vec{\sigma}\cdot\vec{\epsilon}\,\mathcal{F}_1 + \frac{\vec{\sigma}\cdot\vec{q}\,\vec{\sigma}\cdot(\vec{k}\times\vec{\epsilon})}{q\,k}\mathcal{F}_2 + \frac{i\,\vec{\sigma}\cdot\vec{k}\,\vec{q}\cdot\vec{\epsilon}}{q\,k}\mathcal{F}_3 + \frac{i\,\vec{\sigma}\cdot\vec{q}\,\vec{q}\cdot\vec{\epsilon}}{q^2}\mathcal{F}_4, \tag{3.1}$$

where the differential cross section is then defined by

$$\frac{d\sigma}{d\Omega} = \frac{q}{k}\left|\chi_f\,\mathcal{F}\,\chi_i\right|^2 \tag{3.2}$$

The χ's are the usual Pauli spinors representing the z component of nucleon spin.

The helicity amplitudes differ from those of CGLN in that (a) they refer to circularly polarized photons, and (b) the final nucleon spin is quantized along the direction of motion rather than the z-axis.

For a definite helicity of the photon, we can take the linear combinations of the spin-up and spin-down Pauli spinors that correspond to the helicity states. When these linear combinations are used as the final state, the resulting amplitudes are just the helicity amplitudes. The linear combinations required are obtained by rotating the z axis to lie along the direction of motion. A particular photon helicity, F, can be represented as

$$F_{\mu,\lambda_p}^{\lambda_\gamma} = \chi_\mu\, F^{\lambda_\gamma}\,\chi_{\lambda_p}, \tag{3.3}$$

where the χ's are the usual two-component Pauli spinors. Also we can write

$$F_{\mu,\lambda_p}^{\lambda_\gamma} = \chi_\mu\, R^{-1}\,\mathcal{F}^{\lambda_\gamma}\,\chi_{\lambda_p}. \tag{3.4}$$

and thus we have

$$F^{\lambda_\gamma} = R^{-1}\,\mathcal{F}^{\lambda_\gamma}. \tag{3.5}$$

If for convenience in comparison we take $\phi = 0$, then we have

$$R^{-1} = e^{i\frac{\theta}{2}\sigma_y}. \tag{3.6}$$

The $\lambda_\gamma = -1$ state of a photon traveling in the negative z direction is represented by

$$\vec{\epsilon}^{\,-} = \frac{\vec{\epsilon}_x + i\,\vec{\epsilon}_y}{\sqrt{2}} \tag{3.7}$$

The amplitude of CGLN for this case becomes:

$$\mathcal{F}^{-} = \frac{i}{\sqrt{2}} \Big\{ (\mathcal{F}_1 + \mathcal{F}_2)(\sigma_x + i\,\sigma_y) + \sin\theta\,(\mathcal{F}_3 - \mathcal{F}_4)\sigma_z$$

$$-2\cos\frac{\theta}{2}\, e^{-i\theta\frac{\sigma_y}{2}} \left[\mathcal{F}_2(\sigma_x + i\,\sigma_y) - \sin\theta\;\mathcal{F}_4\sigma_z\right] \Big\}. \tag{3.8}$$

Substituting this in Eq. (3.5), we obtain

$$R^{-1}\mathcal{F}^{-} = \frac{i}{\sqrt{2}} \begin{pmatrix} \cos\frac{\theta}{2}\sin\theta\,(\mathcal{F}_3 + \mathcal{F}_4) & 2(\mathcal{F}_1 - \mathcal{F}_2)\cos\frac{\theta}{2} - \sin\theta\,\sin\frac{\theta}{2}(\mathcal{F}_3 - \mathcal{F}_4) \\ -\sin\frac{\theta}{2}\,\sin\theta\,(\mathcal{F}_3 - \mathcal{F}_4) & -2(\mathcal{F}_1 + \mathcal{F}_2)\sin\frac{\theta}{2} - \sin\theta\,\cos\frac{\theta}{2}(\mathcal{F}_3 + \mathcal{F}_4) \end{pmatrix}$$

(3.9)

$$= F^{-} = \begin{pmatrix} f_{\frac{1}{2},\frac{3}{2}} & f_{\frac{1}{2},\frac{1}{2}} \\ -f_{\frac{1}{2},-\frac{3}{2}} & f_{\frac{1}{2},-\frac{1}{2}} \end{pmatrix}$$

This yields

$$\frac{i}{\sqrt{2}}\mathcal{F}_1 = -\frac{f_{\frac{1}{2},\frac{3}{2}} + f_{\frac{1}{2},-\frac{1}{2}}}{4\sin\frac{\theta}{2}} + \frac{f_{\frac{1}{2},\frac{1}{2}} + f_{\frac{1}{2},-\frac{3}{2}}}{4\cos\frac{\theta}{2}}, \qquad (3.10a)$$

$$\frac{i}{\sqrt{2}}\mathcal{F}_2 = -\frac{f_{\frac{1}{2},\frac{3}{2}} + f_{\frac{1}{2},-\frac{1}{2}}}{4\sin\frac{\theta}{2}} - \frac{f_{\frac{1}{2},\frac{1}{2}} + f_{\frac{1}{2},-\frac{3}{2}}}{4\cos\frac{\theta}{2}}, \qquad (3.10b)$$

$$\frac{i}{\sqrt{2}}\mathcal{F}_3 = \frac{1}{2\sin\theta}\left\{\frac{f_{\frac{1}{2},\frac{3}{2}}}{\cos\frac{\theta}{2}} + \frac{f_{\frac{1}{2},-\frac{3}{2}}}{\sin\frac{\theta}{2}}\right\}, \qquad (3.10c)$$

and

$$\frac{i}{\sqrt{2}}\mathcal{F}_4 = \frac{1}{2\sin\theta}\left\{\frac{f_{\frac{1}{2},\frac{3}{2}}}{\cos\frac{\theta}{2}} - \frac{f_{\frac{1}{2},-\frac{3}{2}}}{\sin\frac{\theta}{2}}\right\}. \qquad (3.10d)$$

By expanding the f's, one obtains an expansion for the $\mathcal{F}$'s in terms of first and second derivatives of Legendre ploynomials which can then be compared to the expansion of the $\mathcal{F}$'s given by CGLN. This yields the relations:

$$M_{\ell+} = \frac{-i\sqrt{2}}{4(\ell+1)}\left\{A_{\frac{1}{2}}^{\ell+1/2} - A_{-\frac{1}{2}}^{\ell+1/2} + \sqrt{\frac{\ell+2}{\ell}}\left(A_{\frac{3}{2}}^{\ell+1/2} - A_{-\frac{3}{2}}^{\ell+1/2}\right)\right\} \qquad (3.11a)$$

$$M_{(\ell+1)-} = -\frac{i\sqrt{2}}{4(\ell+1)}\left\{-A_{\frac{1}{2}}^{\ell+1/2} - A_{-\frac{1}{2}}^{\ell+1/2} + \sqrt{\frac{\ell}{\ell+2}}\left(A_{\frac{3}{2}}^{\ell+1/2} + A_{-\frac{3}{2}}^{\ell+1/2}\right)\right\} \qquad (3.11b)$$

$$E_{\ell+} = -\frac{i\sqrt{2}}{4(\ell+1)}\left\{A_{\frac{1}{2}}^{\ell+1/2} - A_{-\frac{1}{2}}^{\ell+1/2} - \sqrt{\frac{\ell}{\ell+2}}\left(A_{\frac{3}{2}}^{\ell+1/2} - A_{-\frac{3}{2}}^{\ell+1/2}\right)\right\} \tag{3.11c}$$

$$E_{(\ell+1)-} = -\frac{i\sqrt{2}}{4(\ell+1)}\left\{A_{\frac{1}{2}}^{\ell+1/2} + A_{-\frac{1}{2}}^{\ell+1/2} + \sqrt{\frac{\ell+2}{\ell}}\left(A_{\frac{3}{2}}^{\ell+1/2} + A_{-\frac{3}{2}}^{\ell+1/2}\right)\right\}, \tag{3.11d}$$

where $\ell = J - \frac{1}{2}$.

IV. INVERSION OF AMPLITUDES

Equation (3.10) and Eq. (2.7) can be combined to express A_λ^ℓ in terms of the $\mathcal{F}$'s, yielding

$$A_{\pm\frac{3}{2}}^{\ell+1/2} = -\frac{i}{\sqrt{2}}\sqrt{\ell(\ell+2)}\int_{-1}^{1} dx\,\frac{(\mathcal{F}_3 \pm \mathcal{F}_4)}{2}\left\{\frac{p_\ell(x) - p_{\ell+2}(x)}{2\ell+3} \pm \frac{p_{\ell-1}(x) - p_{\ell+1}(x)}{2\ell+1}\right\}, \tag{4.1}$$

and

$$A_{\pm\frac{1}{2}}^{\ell+1/2} = \frac{i}{\sqrt{2}}\int_{-1}^{1} dx\left\{(\mathcal{F}_1 \mp \mathcal{F}_2) (+\frac{x \mp 1}{2}(\mathcal{F}_3 \mp \mathcal{F}_4)\right\}(p_{\ell+1}(x) \pm p_\ell(x)) .$$

Finally, these formulae can be combined with Eq. (3.11) to project the E's and M's out of the $\mathcal{F}$'s:

$$M_{\ell+} = \frac{1}{2(\ell+1)}\int_{-1}^{1} dx\left[\mathcal{F}_1\, p_\ell(x) - \mathcal{F}_2 p_{\ell+1}(x) - \mathcal{F}_3\frac{p_{\ell-1}(x) - p_{\ell+1}(x)}{2\ell+1}\right]$$

$$\ell > 0 \tag{4.2a}$$

$$E_{\ell +} = \frac{1}{2(\ell + 1)} \int_{-1}^{1} dx \left[\mathcal{F}_1 \, p_\ell(x) - \mathcal{F}_2 \, p_{\ell+1}(x) + \mathcal{F}_3 \quad \ell \quad \frac{(p_{\ell-1}(x) - p_{\ell+1}(x))}{2\ell + 1} + \mathcal{F}_4 (\ell + 1) \, \frac{p_\ell(x) - p_{\ell+2}(x)}{2\ell + 3} \right] . \tag{4.2b}$$

$$M_{\ell -} = \frac{1}{2\ell} \int_{-1}^{1} dx \left[- \mathcal{F}_1 \, p_\ell(x) + \mathcal{F}_2 \, p_{\ell-1}(x) + \mathcal{F}_3 \, \frac{(p_{\ell-1}(x) - p_{\ell+1}(x))}{2\ell + 1} \right]$$

$$\ell > 0 \tag{4.2c}$$

$$E_{\ell -} = \frac{1}{2\ell} \int_{-1}^{1} dx \left[\mathcal{F}_1 \, p_\ell(x) - \mathcal{F}_2 \, p_{\ell-1}(x) - \mathcal{F}_3 (\ell+1) \, \frac{(p_{\ell-1}(x) - p_{\ell+1}(x))}{2\ell + 1} - \ell \, \mathcal{F}_4 \, \frac{p_{\ell-2}(x) - p_\ell}{2\ell - 1} \right] .$$

$$\ell > 1 \tag{4.2d}$$

V. CONCLUSION

The above formulas give the recipe for projecting multipole amplitudes out of the CGLN amplitudes. Another use for these expressions is in deriving partial-wave dispersion relations for photoproduction. This work is now in progress.

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