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KNUDSEN FLOW THROUGH A CHANNEL WITH ROUGH WALLS

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A B S T R A C T

A calculation is made of the Knudsen Flow through a short channel formed by two parallel walls which are roughened by a "roof top" structure on a fine scale. The molecular scattering is assumed to obey the cosine law with respect to the local roof normal. The effect of roughness is found to decrease the flow with respect to a smooth wall channel with cosine law scattering.

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INTRODUCTION

In a preceding report[1], the walls bounding a given flow system have been treated as smooth mathematical surfaces. In this report a simple model of a rough wall is considered.

DESCRIPTION OF THE MODEL

We consider the flow between two infinite parallel plates whose spacing is h. The length of the channel thus formed is L. From a macroscopic standpoint a section of the flow channel would appear as in figure 1.

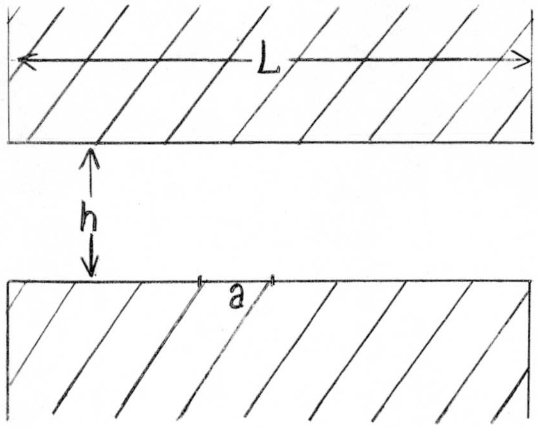

Figure 1
Section of Flow System

However, from a microscopic point of view a segment such as that marked "a" in figure 1 is assumed to be structured as in figure 2. The wall is

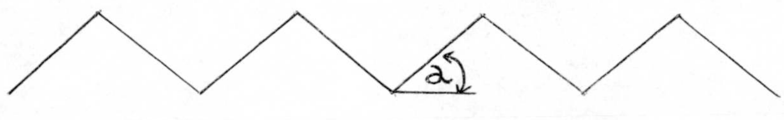

Figure 2
Microscopic View of Channel Wall

[1]DeMarcus, W. C., The Problem of Knudsen Flow, Union Carbide Nuclear Company, Oak Ridge Gaseous Diffusion Plant, September 5, 1956 - September 30, 1957, (K-1302, Parts I - VI).

assumed to be composed of planes all inclined at the same angle α (or its supplement) with respect to the axis of the channel and all of the same size. To complete the model as far as Knudsen Flow is concerned, it is assumed that gas molecules leave the wall distributed in probability according to the cosine law with respect to the local normal. For mathematical reasons the following additional assumption has had to be imposed; namely, that a molecule striking a particular slope undergoes surface diffusion on the same slope only (with zero probability of crossing over to an adjacent slope) to a sufficient extent that it leaves the slope from all distances along it in the direction of the channel with the same probability. (While the writer feels that this last assumption does not play a crucial role in determining the Knudsen Flow, this is not proved; and, hence, the imposition of the additional assumption is logically necessary).

THEORY

As usual, we consider the case where only the end $x = 0$ is exposed to a nonzero gas pressure and normalize the problem so that one molecule per second enters unit area of the channel end at $x = 0$. We then define

$N_1(x)$ = probability that the first collision of an entering molecule takes place at distances greater than x into the flow system.

$\Psi(x)$ = rate density at which molecules after entering a unit area at $x = 0$ strike the slopes whose normal makes an acute angle with the direction of flow ($\Psi(x)dx$ is then a rate).

$\phi(x)$ = rate per unit length at which molecules entering a unit area at $x = 0$ strike the opposite slopes.

$\Psi_1(x)$ = rate density of first collisions on the forward looking slopes.

$\phi_1(x)$ = rate density of first collisions on the backward looking slopes.

$P^{\Psi}(x;L)$ = probability of leaving exit at x = L without intervening collision if emitted from a sloping element at x whose normal makes an acute angle with the direction of flow.

$P^{\phi}(x;L)$ = same probability as preceding except that emission is from backwards slope.

$P^{\Psi}(x;0)$ = probability of leaving entrance without intervening collision on emission from forward slope at x = 0.

$P^{\phi}(x;0)$ = same probability as preceding for backward slope.

$K^{\Psi\Psi}(x,y)$ = probability per unit dx that emission from a forward slope at y will be followed by collision with a forward slope at x.

$K^{\Psi\phi}(x,y)$ = probability per unit dx that emission from a forward slope at y will be followed by collision with a backward slope at x.

$K^{\phi\Psi}(x,y)$ = probability per unit dx that emission from a backward slope at y will be followed by a collision with a forward slope at x.

$K^{\phi\phi}(x,y)$ = probability per unit dx that emission from a backward slope at y will be followed by a collision with a backward slope at x.

Explicit forms for all the functions defined above except, of course, $\Psi(x)$ and $\phi(x)$ have been evaluated by a straight forward application of the cosine law and are given below.

$$N_1(x) = \frac{1}{h}\left\{\sqrt{x^2 + h^2} - x\right\}$$

$$\Psi_1(x) = \frac{1}{2h\cos\alpha}\left[1 - \frac{x\cos\alpha + h\sin\alpha}{\sqrt{x^2 + h^2}}\right]$$

$$\phi_1(x) = \frac{1}{2h \cos \alpha} \left[1 + \frac{-x \cos \alpha + h \sin \alpha}{\sqrt{x^2 + h^2}} \right] - \frac{1 - \cos \alpha}{h \cos \alpha}$$

$$P^{\Psi}(x;L) = \frac{1}{2} \left[1 + \frac{h \sin \alpha - (L - x) \cos \alpha}{\sqrt{h^2 + (L - x)^2}} \right] - (1 - \cos \alpha)$$

$$P^{\phi}(x;L) = \frac{1}{2} \left[1 - \frac{(L - x) \cos \alpha + h \sin \alpha}{\sqrt{h^2 + (L - x)^2}} \right]$$

$$P^{\Psi}(x;0) = \frac{1}{2} \left[1 - \frac{x \cos \alpha + h \sin \alpha}{\sqrt{x^2 + h^2}} \right]$$

$$P^{\phi}(x;0) = \frac{1}{2} \left[1 + \frac{h \sin \alpha - x \cos \alpha}{\sqrt{h^2 + x^2}} \right] - (1 - \cos \alpha)$$

$$K^{\Psi\Psi}(x,y) = \frac{1}{4} \frac{h^2 \cos \alpha + (x - y) h \sin \alpha}{\left[h^2 + (x - y)^2 \right]^{3/2}} \left[1 - \frac{x - y}{h} \tan \alpha \right]$$

$$K^{\Psi\phi}(x,y) = \frac{1}{4} \frac{h^2 \cos \alpha + (x - y) h \sin \alpha}{\left[h^2 + (x - y)^2 \right]^{3/2}} \left[1 + \frac{x-y}{h} \tan \alpha \right] + (1-\cos \alpha)\delta(x-y)$$

$$K^{\phi\phi}(x,y) = \frac{1}{4} \frac{h^2 \cos \alpha + (y - x) h \sin \alpha}{\left[h^2 + (y - x)^2 \right]^{3/2}} \left[1 - \frac{y - x}{h} \tan \alpha \right]$$

$$K^{\phi\Psi}(x,y) = \frac{1}{4} \frac{h^2 \cos \alpha + (y - x) h \sin \alpha}{\left[(x - y)^2 + h^2 \right]^{3/2}} \left[1 + \frac{y-x}{h} \tan \alpha \right] + (1-\cos \alpha)\delta(x-y)$$

In the above, $\delta(x - y)$ is the usual Dirac delta function. It is also assumed that $(y - x)$, L, x are all less in absolute value than $h \operatorname{ctn} \alpha$ for all the functions above except $N_1(x)$.

The conditions for a steady state may now be written

$$\Psi(x) = \Psi_1(x) + (1 - \cos\alpha)\phi(x) + \int_0^L K^{\Psi\Psi}(x - y)\Psi(y)dy + \int_0^L \Gamma^{\phi\Psi}(x - y)\phi(y)dy$$

$$\phi(x) = \phi_1(x) + (1 - \cos\alpha)\Psi(x) + \int_0^L K^{\Psi\Psi}(x - y)\phi(y)dy + \int_0^L \Gamma^{\phi\Psi}(y - x)\Psi(y)dy$$

wherein

$$\Gamma^{\phi\Psi}(x - y) = \frac{1}{4}\,\frac{h^2\cos\alpha + (y - x)\sin\alpha}{(x^2 + h^2)^{3/2}}\left[1 + \frac{y - x}{h}\tan\alpha\right] .$$

If we consider instead the more general problem

$$\Psi^*(x) = \Psi_1(x) + \phi_1(L - x) + (1 - \cos\alpha)\phi^*(x) + \int_0^L K^{\phi\Psi}(x - y)\Psi^*(y)dy + \int_0^L \Gamma^{\phi\Psi}(x - y)\phi^*(y)dy$$

$$\phi^*(x) = \phi_1(x) + \Psi_1(L - x) + (1 - \cos\alpha)\Psi^*(x) + \int_0^L K^{\Psi\Psi}(x - y)\phi^*(y)dy + \int_0^L K^{\phi\Psi}(y - x)\Psi^*(y)dy ,$$

the solution can be written down from the fact that equilibrium must ensue, and, hence,

$$\phi^* = \Psi^* = \frac{1}{2h\cos\alpha} .$$

But Ψ^* is the sum of $\Psi_a(x) + \Psi_b(x)$, and ϕ^* is the sum of $\phi_a(x) + \phi_b(x)$

where

$$\Psi_a(x) = \Psi_1(x) + (1 - \cos\alpha)\phi_a(x) + \int_0^L K^{\Psi\Psi}(x - y)\Psi_a(y)dy$$

$$+ \int_0^L K^{\phi\Psi}(x - y)\phi_a(y)dy$$

$$\phi_a(x) = \phi_1(x) + (1 - \cos\alpha)\Psi_a(x) + \int_0^L K^{\Psi\Psi}(x - y)\phi_a(y)dy$$

$$+ \int_0^L K^{\phi\Psi}(x - y)\Psi_a(y)dy$$

$$\Psi_b(x) = \phi_1(L - x) + (1 - \cos\alpha)\phi_b(x) + \int_0^L K^{\Psi\Psi}(x - y)\Psi_b(y)dy$$

$$+ \int_0^L K^{\phi\Psi}(x - y)\phi_b(y)dy$$

$$\phi_b(x) = \Psi_1(L - x) + (1 - \cos\alpha)\Psi_b(x) + \int_0^L K^{\Psi\Psi}(x - y)\phi_b(y)dy$$

$$+ \int_0^L K^{\phi\Psi}(y - x)\Psi_b(y)dy$$

In the last two equations, put $\xi = L - x$, $\eta = L - y$. Then they become

$$\Psi_b(L - \xi) = \phi_1(\xi) + (1 - \cos\alpha)\phi_b(L - \xi) + \int_0^L K^{\Psi\Psi}(\xi - \eta)\Psi_b(L - \eta)d\eta$$

$$+ \int_0^L K^{\phi\Psi}(\eta - \xi)\phi_b(L - \eta)d\eta$$

$$\phi_b(L - \xi) = \Psi_1(\xi) + (1 - \cos\alpha)\Psi_b(L - \xi) + \int_0^L K^{\Psi\Psi}(\xi - \eta)\phi_b(L - \eta)d\eta$$

$$+ \int_0^L K^{\phi\Psi}(\xi - \eta)\Psi_b(L - \eta)d\eta$$

whence

$\Psi_b(L - \xi) = \phi_a(\xi)$, $\phi_b(L - \xi) = \Psi_a(\xi)$, and hence,

$$\Psi(x) + \phi(L - x) = \frac{1}{2h \cos \alpha}$$

and, also,

$$\phi(x) + \Psi(L - x) = \frac{1}{2h \cos \alpha} .$$

By subtraction,

$$\Psi(x) - \phi(x) = \Psi(L - x) - \phi(L - x).$$

But these results hold independently of x and L. Hence,

$$\Psi(x) - \phi(x) = \text{const.}$$

RESULTS

Since the problem consists of a set of coupled linear integral equations, explicit solutions appear out of the question. Accordingly, the solution was obtained approximately for two cases by satisfying the integral equations at a finite number of equally spaced points assuming that the value of $\Psi(x)$ or $\phi(x)$ at the mid-point of a given interval held throughout the interval. The transmission probability can be written in the form

$$Q = N_1(L) + \Delta Q$$

with

$$\Delta Q = \int_0^L P^{\Psi}(x;L)\Psi(x)dx + \int_0^L P^{\phi}(x;L)\phi(x)dx$$

and ΔQ is evaluated by the same numerical technique as used to find $\Psi(x)$ and $\phi(x)$. The results obtained to date are given in tables I and II.

TABLE I

$L/h = 0.1$
$N_1(L) = 0.90499$

α	ΔQ	Q
0°	0.04751	0.95250
22.5°	0.04364	0.94863
45°	0.04020	0.94519

TABLE II

$L/h = 0.3$
$N_1(L) = 0.74403$

α	ΔQ	Q
0°	0.1271	0.8711
22.5°	0.1133	0.8573
45°	0.1002	0.8442

Some indication of the accuracy of the results can be obtained by comparing with the results given by the writer in a previous report[2] which correspond to $\alpha = 0°$. For $L/h = 0.1$ and 0.3, these are, respectively, 0.95245 and 0.87097 or, in terms of ΔQ where all the error must lie, 0.04746 and 0.1269 so that the error in ΔQ seems to be only of the order 0.1%.

[2]DeMarcus, W. C., The Problem of Knudsen Flow, Part III, Union Carbide Nuclear Company, Oak Ridge Gaseous Diffusion Plant, March 19, 1959, (K-1302).