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THE SET CODES — IBM 704 CODES FOR THE CALCULATION OF THE STRESSES IN A PRESSURE VESSEL WITH AN ELLIPSOIDAL HEAD

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G.G. Bilodeau • J.B. Callaghan • H. Kraus

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A solution to the problem of stresses in a pressure vessel with an ellipsoidal head has long been sought by pressure vessel designers. To meet this need the codes described in this report were written. The codes are based on a finite-difference approximation to the Love-Meissner equations which are the basis of the bending theory of thin shells.

THE SET CODES—IBM 704 CODES FOR THE CALCULATION OF THE STRESSES IN A PRESSURE VESSEL WITH AN ELLIPSOIDAL HEAD

G. G. Bilodeau, J. B. Callaghan, and H. Kraus

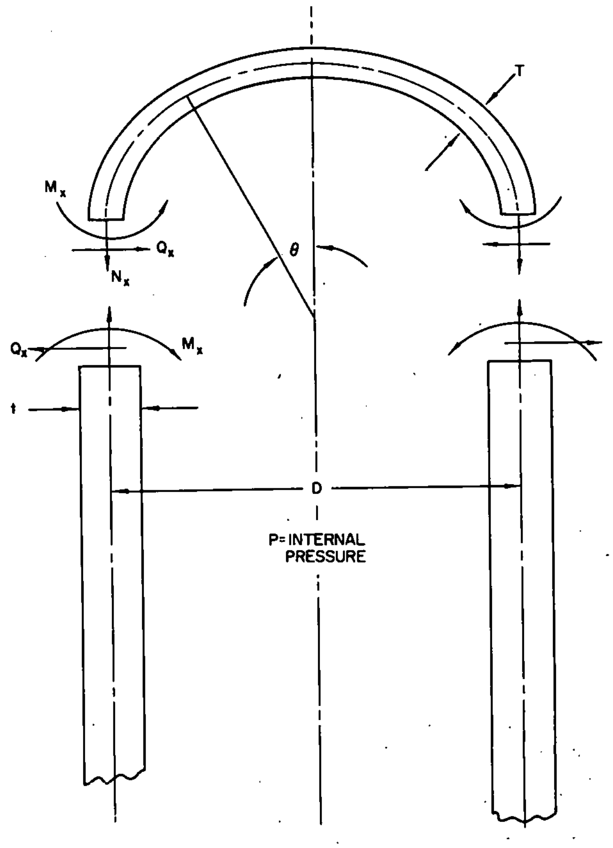
A means of determining the stresses in an ellipsoidal pressure vessel head has long been sought by pressure vessel designers. In the nuclear power plant industry, for example, the ellipsoidal shell appears frequently as the head of a steam generator or as the bottom of a reactor pressure vessel. In such applications there are discontinuity stresses set up because of the difference in the expansion experienced by the ellipsoidal shell and the cylindrical shell to which it is attached when the entire assembly is loaded by internal pressure. At present there is no method available for the determination of these stresses, and designers have been forced to make overly conservative assumptions to overcome this deficiency.

The basic differential equations of the problem have been available for many years, although their complexity has not permitted a convenient analytical solution. It was believed, however, that a high-speed electronic computing machine would be ideally suited for the solution of these equations. Hence, the IBM 704 codes described in this report were written.

THEORY

The configuration which will be considered in this discussion consists of an ellipsoid of revolution mounted on a long cylinder, as shown in Fig. 1.* The method of analysis generally applied to this type of shell is referred to as the bending theory of thin shells. A shell is considered thin when its diameter is more than ten times its thickness. This assumption allows analytical solutions to be found for many shells, although, for the ellipsoidal shell, analytical solutions have not been found in spite of this assumption. An exact theory for treating thick shells is not available at the present time.

*Note that the middle surfaces of the head and cylinder must coincide.



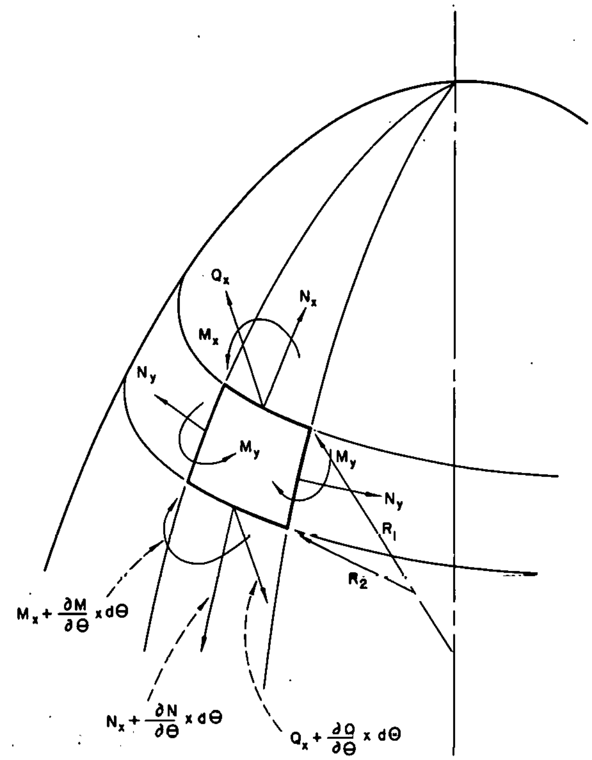


Fig. 2 A Differential Element in a Shell of Revolution Showing the Movements and Forces Acting on It (from Ref 2)

Fig. 1 Ellipsoidal Shell Mounted on a Cylinder Showing Loads at Junction

The bending theory of thin shells is very well described by Watts and Burrows (Ref 1) and by Timoshenko (Ref 2). The method presented in this report will draw largely upon the Watts and Burrows paper.

The analysis of any shell of revolution begins with the consideration of the forces and moments acting on a differential element of the shell as shown in Fig. 2. A summation of the forces and of the moments then yields three equations of equilibrium for a differential element in a shell under internal pressure:*

$$\frac{d}{d\theta}\left[N_x R_2 \sin\theta\right] - N_y R_1 \cos\theta + Q_x R_2 \sin\theta = 0 \;, \qquad (1)$$

$$\frac{d}{d\theta}\left[Q_x R_2 \sin\theta\right] - N_y R_1 \sin\theta - N_x R_2 \sin\theta = -PR_1 R_2 \sin\theta \;, \qquad (2)$$

and

$$\frac{d}{d\theta}\left[M_x R_2 \sin\theta\right] - M_y R_1 \cos\theta - Q_x R_2 R_1 \sin\theta = 0 \;, \qquad (3)$$

where

θ = angle between the axis of revolution and a perpendicular to the middle surface of the shell at any point

* Reference 1

R_1 = radius at any point of a middle surface section cut by a plane through the axis of revolution

R_2 = distance measured along a normal to any point on the middle surface, between the middle surface and the axis of revolution

P = internal pressure acting in the shell

N_x = axial membrane force

N_y = circumferential membrane force

M_x = axial bending moment

M_y = circumferential bending moment

Q_x = shear force acting along a normal to the middle surface.

In Eqs (1), (2), and (3) there are five unknown quantities which are N_x, N_y, M_x, M_y, and Q_x. The first four quantities can be expressed in terms of the rotation and the deflection of the differential element (Ref 1) as follows:

$$N_x = \frac{ET}{R_1(1-\mu^2)}\left[\left(\frac{du}{d\theta} + w\right) + \mu\left(\frac{R_1}{R_2}\right)(w + u \cot\theta)\right] , \qquad (4)$$

$$N_y = \frac{ET}{R_1(1-\mu^2)}\left[\mu\left(\frac{du}{d\theta} + w\right) + \frac{R_1}{R_2}(w + u \cot\theta)\right] , \qquad (5)$$

$$M_x = \frac{ET^3}{m^4 R_1}\left[\frac{dW}{d\theta} + \mu\left(\frac{R_1}{R_2}\right) W \cot\theta\right] , \qquad (6)$$

and

$$M_y = \frac{ET^3}{m^4 R_1}\left[\mu\left(\frac{dW}{d\theta}\right) + \frac{R_1}{R_2} W \cot\theta\right] , \qquad (7)$$

where

T = thickness of the ellipsoidal shell

E = Young's modulus

μ = Poisson's ratio

$m^4 = 12\ (1 - \mu^2)$

W = rotation of a tangent to the middle surface, lying in a plane through the axis of revolution during deformation of the shell;

$W = \frac{1}{R_1}(u - dw/d\theta)$

u = deformation of any point on the middle surface measured along a tangent to the undeformed middle surface at the point and lying in a plane through the axis of revolution

w = deformation of any point on the middle surface measured along a normal to the undeformed middle surface at the point and lying in a plane through the axis of revolution

When the expressions (4), (5), (6), and (7) are substituted into Eqs (1), (2), and (3), the resulting expressions can be simplified to give the following differential equations (known as the Love-Meissner equations) relating the rotation and the shear:

$$L[V] + \frac{\mu}{R_1} V = -ETW + \Phi(\theta) \quad , \tag{8}$$

and

$$L[W] - \frac{\mu}{R_1} W = + \frac{V m^4}{ET^3} \quad . \tag{9}$$

In these equations, $V = R_2 Q_x$, $\Phi(\theta)$ is a function of load and shape, and $L[\ ]$ represents a differential operator which is defined as follows:

$$L[\dots] = \frac{R_2}{R_1^2} \frac{d^2(\dots)}{d\theta^2} + \frac{1}{R_1}\left[\frac{d}{d\theta}\left(\frac{R_2}{R_1}\right) + \frac{R_2}{R_1}\cot\theta\right]\frac{d(\dots)}{d\theta} - \frac{\cot^2\theta}{R_2}(\dots) \quad . \tag{10}$$

Up to this point the type of shell has not been specified. From the equations which have been presented it can be seen that, in order to make them apply to a particular shell, only a substitution of the appropriate expressions for the two radii of curvature R_1 and R_2 is needed.

Thus, for the ellipsoidal shell,

$$R_1 = 0.5\, D\beta v^3 \tag{11}$$

and

$$R_2 = 0.5\, D\beta v \quad , \tag{12}$$

where D = the major axis of the ellipse and is also the diameter of the cylinder, β = ratio of the major axis of the ellipse to the minor axis of the ellipse, and $(\frac{1}{v^2}) = [1 + (\beta^2 - 1)\sin^2\theta]$.

For the cylindrical shell,

$$R_1 = \infty \tag{13}$$

and

$$R_2 = 0.5\, D \quad . \tag{14}$$

After substitution of Eqs (11) and (12) into Eqs (4) through (10), the following equations (Ref 1) are obtained for the ellipsoidal shell:

$$M_x = \frac{2ET^3}{m^4 D\delta}\left[\frac{1}{v^2}\frac{dW}{d\theta} + \mu W \cot\theta\right] \quad , \tag{15}$$

$$M_y = \frac{2ET^3}{m^4 D\delta}\left[\frac{\mu}{v^2}\frac{dW}{d\theta} + W \cot\theta\right] \quad , \tag{16}$$

$$N_x = \frac{2V}{\delta D}\cot\theta + 0.25\, PD\delta \quad , \tag{17}$$

$$N_y = \frac{2}{D\delta'}\frac{dV}{d\theta} + 0.25\, PD\left[\frac{2\delta^2 - \beta^2}{\delta}\right] \quad , \tag{18}$$

$$L[V] + \mu V + 0.5\frac{DET}{\delta'} W = \frac{PD^2\beta^2}{8}(3v^2 + 1)(1 - v^2)\cot\theta \quad , \tag{19}$$

$$L[W] - \mu W - \frac{D}{2}\frac{\delta m^4}{ET^3} V = 0 \quad , \tag{20}$$

and the differential operator for the ellipse appears as

$$v^2 L[\ldots] = \frac{d^2(\ldots)}{d\theta^2} + \left[(3 - 2v^2) \cot \theta\right] \frac{d(\ldots)}{d\theta} - (\ldots)v^4 \cot^2\theta \quad , \tag{21}$$

where $\delta = \beta v$ and $\delta' = \beta v^3$.

Equations (19) and (20) are the equations which have caused so much difficulty for analysts seeking solutions to ellipsoidal shell problems. These equations must be solved for V and W which are used to determine M_x, M_y, N_x, and N_y. The latter quantities are combined by using the following relations to obtain the stresses:

$$\sigma_x = \frac{N_x}{T} \pm \frac{6M_x}{T^2} \tag{22}$$

and

$$\sigma_y = \frac{N_y}{T} \pm \frac{6M_y}{T^2} \quad , \tag{23}$$

where σ_x = axial stress, σ_y = circumferential stress, and the positive sign is used for stresses on the outer surface of the shell and the negative sign is used for the inner surface.

A similar procedure is followed for the cylinder to which the ellipsoidal shell is attached. However, the resulting expressions are far more simple than those presented previously for the ellipsoidal shell, and the basic differential equations can be solved easily. To obtain the cylindrical shell equations, expressions (13) and (14) are substituted into Eqs (4) through (10). The equations (Ref 1) for the cylinder then appear as

$$M_x = \frac{Et^3}{m^4} \frac{dW}{dx} \quad , \tag{24}$$

$$M_y = \mu M_x \quad , \tag{25}$$

$$N_x = 0.25 \text{ PD} \quad , \tag{26}$$

and

$$N_y = 0.5 D \frac{Et^3}{m^4} \frac{d^3W}{dx^3} + 0.5 \text{ PD} \quad , \tag{27}$$

where t = thickness of the cylinder, and x = axial coordinate of the cylinder.

For the cylinder, the Love-Meissner equations reduce to

$$\frac{d^4W}{dx^4} + 4 a^4 W = 0 \quad . \tag{28}$$

The solution of Eq (28) is given as

$$W = e^{-ax} (C_1 \sin ax + C_2 \cos ax) + e^{ax} (C_3 \sin ax + C_4 \cos ax) \quad , \tag{29}$$

where C_1, C_2, C_3, C_4 are arbitrary constants, and $a^4 = m^4/D^2t^2$.

Equations (22) and (23) are utilized to obtain the stresses in the cylinder after calculation of the moments and membrane forces from Eqs (24) through (27).

SOLUTION OF THE ELLIPSOIDAL SHELL EQUATIONS

Before proceeding to the solution, it is convenient to change Eqs (19) and (20) slightly by multiplying each equation by $-\sin \theta/P$ and by multiplying Eq (20) by E to give

$$M_\theta[\Gamma] - \mu(\sin\theta)\Gamma - g_1(\theta)(\sin\theta)\Omega = G(\theta) \;, \tag{30}$$

and

$$M_\theta[\Omega] + \mu(\sin\theta)\Omega + g_2(\theta)(\sin\theta)\Gamma = 0 \;, \tag{31}$$

where

$$M_\theta[Z] = -\frac{d}{d\theta}\left[(\sin\theta)v^{-2}\frac{dZ}{d\theta}\right] + (\cot^2\theta)(\sin\theta)v^2 Z \;,$$

$$G(\theta) = -1/8\, D^2\beta^2 (3v^2+1)(1-v^2)\cos\theta \;,$$

$$\Gamma = \Gamma(\theta) = V(\theta)/P \;,$$

$$\Omega = \Omega(\theta) = EW(\theta)/P \;,$$

$$g_1(\theta) = 1/2\, DT\beta\, v^3 = s_1(\theta)/E \;,$$

and

$$g_2(\theta) = 1/2\left[m^4 D\beta/T^3\right] v^3 = s_2(0)\, E \;.$$

The self-adjoint form of the operator M_θ has been used for general thin shells by Pöschl (Ref 3).

From Eqs (29), (30), and (31), it is evident that there will be eight arbitrary constant to be determined in the solution of the problem of an ellipsoidal shell mounted on a long cylinder, since the solution of each of the two parts involves four arbitrary constants. Thus, eight boundary conditions are needed, which are as follows: At the top of the ellipsoidal shell (at $\theta = 0°$) the shear and rotation are zero. Furthermore, the cylinder will be assumed to be semi-infinite in extent. Inspection of Eq (29) will show that, to insure finite results at the remote end of such a cylinder ($x\to\infty$), the terms involving the positive exponential e^{ax} must be dropped. This is done by setting C_3 and C_4 each equal to zero. The remaining four boundary conditions are obtained from the junction between the shell and the cylinder where continuity of axial moment, axial shear, deflection, and rotation must be maintained across the boundary.

Boundary Conditions

The boundary conditions can be summarized as follows:

For the Ellipse

$$\Gamma(\theta),\ \Omega(\theta) = 0 \text{ at } \theta = 0 \tag{32}$$

For the Cylinder

All quantities must be finite as $x\to\infty$ (33)

For the Junction

Equal Radial Moments:

$$\frac{2T^3\beta^2}{D}\frac{d\Omega}{d\theta}\bigg|_{\theta=\pi/2} = \alpha = t^3 a(C_1 - C_2) \tag{34}$$

Equal Radial Deformation:

$$T^{-1}\left[2\beta^2\frac{d\Gamma}{d\theta}\bigg|_{\theta=\pi/2} + \frac{D^2}{4}(2-\beta^2-\mu)\right] = \gamma = t^{-1}\left[\frac{D^2t^3a^3}{m^4}(C_1+C_2) + \frac{D^2}{4}(2-\mu)\right] \tag{35}$$

Equal Axial Shear Forces:

$$\Gamma\bigg|_{\theta=\pi/2} = -\frac{Da}{2m^4}\left\{\alpha + \frac{m^4}{D^2a^2}\left[\gamma t - \frac{D^2(2-\mu)}{4}\right]\right\} \tag{36}$$

Equal Rotations:

$$\Omega\Big|_{\theta=\pi/2} = -\frac{1}{2t^3 a}\left\{\alpha - \frac{m^4}{D^2 a^2}\left[\gamma t - \frac{D^2(2-\mu)}{4}\right]\right\} \tag{37}$$

For the junction equations, α and γ are defined by Eqs (34) and (35) and are considered unknowns until their values are determined toward the end of the problem. In Eqs (34) through (37) the left side of each equation represents the quantity for the ellipse at $\theta = \pi/2$, and the right side of each equation represents the corresponding quantity for the cylinder at $x = 0$.

As a result of the introduction of the quantities α and γ, the following expressions hold for the arbitrary constants in the cylinder equations:

$$C_1 = \frac{1}{2t^3 a}\left\{\alpha + \frac{m^4}{D^2 a^2}\left[\gamma t - \frac{D^2(2-\mu)}{4}\right]\right\}$$

$$C_2 = \frac{-1}{2t^3 a}\left\{\alpha + \frac{m^4}{D^2 a^2}\left[\gamma t - \frac{D^2(2-\mu)}{4}\right]\right\}$$

$C_3\ C_4 = 0$ by condition (33).

DIFFERENCE EQUATIONS

An approximate solution for the ellipsoidal shell is obtained by using finite difference methods. The method of finite differences has been used in previous work in shell theory—notably by Au, Goodman, and Newmark (Ref. 4). However, their work was primarily by hand computation; consequently, the methods of solution mentioned in this report (as well as the difference equations) are different.

Difference equations corresponding to the differential equations of the ellipsoidal shell [Eqs (30) and (31)] are derived according to a method introduced by Varga (Ref 5). The boundary conditions [Eq (32)] are needed for this derivation as well as two of the set of Eqs (34) through (37). The two chosen are Eqs (34) and (35). This introduces two unknowns, α and γ, which will be evaluated at a later time in the code with the use of the two remaining Eqs (36) and (37).

The interval $0 \le \theta \le \pi/2$ is divided into N subintervals (not necessarily equal) where

$$h_i = \theta_i - \theta_{i-1}$$

and

$$0 - \theta_o < \theta_1 < \cdots < \theta_N - \pi/2 \ .$$

By way of notation, let

$$\Gamma(\theta_n) = \Gamma_n ,\ \Omega(\theta_n) = \Omega_n ,\ \text{etc.}$$

Case I: θ_n an Interior Point

This situation (θ_n an interior point) occurs when $n \neq N$. (There is no equation corresponding to $n = 0$.) Then, at this point, the difference equation corresponding to the first differential Eq (30) is

$$-\left(\frac{a_n + a_{n-1}}{h_n}\right)\Gamma_{n-1} + \left[\frac{a_n + a_{n-1}}{h_n} + \frac{a_{n+1} + a_n}{h_{n+1}} + (h_{n+1} + h_n)\, b_n\right]\Gamma_n$$

$$-\left(\frac{a_{n+1} + a_n}{h_{n+1}}\right)\Gamma_{n+1} - \mu(\sin\theta_n)(h_{n+1} + h_n)\ \Gamma_n$$

$$- g_1(\theta_n)(\sin\theta_n)(h_{n+1} + h_n)\,\Omega_n = (h_{n+1} + h_n)\,\tilde{G}_n \,, \tag{38}$$

where

$$a(\theta) = (\sin\theta)\,v^{-2}$$

and

$$b(\theta) = (\cot^2\theta)(\sin\theta)\,v^2 \;;$$

thus

$$a(\theta_n) = a_n \,,$$

and

$$b(\theta_n) = b_n \,.$$

The corresponding equation for (31) is

$$-\left(\frac{a_n + a_{n-1}}{h_n}\right)\Omega_{n-1} + \left[\frac{a_n + a_{n-1}}{h_n} + \frac{a_{n+1} + a_n}{h_{n+1}} + (h_{n+1} + h_n)\,b_n\right]\Omega_n$$

$$-\left(\frac{a_{n+1} + a_n}{h_{n+1}}\right)\Omega_{n+1} + \mu\,(\sin\theta_n)\,\Omega_n + g_2\,(\theta_n)(\sin\theta_n)(h_{n+1} + h_n)\;\Gamma_n = 0 \,. \tag{39}$$

As a special case of the equations, if $n = 1$, then the boundary conditions [Eq (32)], as well as the definition of $a(\theta)$, lead to

$$\left[\frac{a_1}{h_1} + \frac{a_2 + a_1}{h_2} + (h_2 + h_1)\,b_1\right]\Gamma_1 - \left(\frac{a_2 + a_1}{h_2}\right)\Gamma_2 - \mu\,(\sin\theta_1)(h_2 + h_1)\;\Gamma_1$$

$$- g_1(\theta_1)(\sin\theta_1)(h_2 + h_1)\;\Omega_1 = (h_2 + h_1)\;G_1 \,, \tag{40}$$

and a similar expression for $\Omega(\theta)$, $\Gamma(\theta)$ from Eq (39).

<u>Case II: $n = N$</u>

In this case, $\theta_n = \theta_N = \pi/2$, and use is made of Eqs (34) and (35) to obtain

$$-\left(\frac{a_N + a_{N-1}}{h_N}\right)\Gamma_{N-1} + \left(\frac{a_N + a_{N-1}}{h_N} + b_N h_N\right)\Gamma_N - \mu\, h_N\,\Gamma_N$$

$$- g_1(\theta_N)\,h_N\Omega_N = h_N G_N + \frac{T a_N}{\beta^2}\gamma - \frac{D^2 a_N}{4\beta^2}(2 - \beta^2 - \mu) \tag{41}$$

and

$$-\left(\frac{a_N + a_{N-1}}{h_n}\right)\Omega_{N-1} + \left(\frac{a_N + a_{N-1}}{h_N} + b_N h_N\right)\Omega_N + \mu\, h_N\Omega_N$$

$$+ g_2(\theta_N)\,h_N\,\Gamma_N = a_N\,(T^3\beta^2)^{-1}\,D\alpha \,. \tag{42}$$

Thus, the unknowns which occur in these equations are

$$\left\{\Gamma_i\right\}_{i=1}^{N}\,,\quad \left\{\Omega_i\right\}_{i=1}^{N}\,,\quad \alpha,\ \text{and}\ \gamma\,.$$

The difference equations just derived can be written in matrix notation as follows:

$$\left.\begin{aligned} A\vec{\Gamma} - D_1\vec{\Gamma} - D_2\vec{\Omega} &= \gamma\,\vec{P} + \vec{Q} \\ A\vec{\Omega} + D_1\vec{\Omega} + e\,D_2\,\vec{\Gamma} &= \alpha\,\vec{R} \end{aligned}\right\} \qquad (43)$$

where

A = NxN matrix corresponding to the operator M_θ occurring in Eqs (30) and (31)

D_1 = NxN diagonal matrix with positive diagonal elements whose n^{th} element (along the major diagonal $n \neq N$) is $\mu(\sin\theta_n)(h_{n+1} + h_n)$. The N^{th} element is μh_N.

D_2 = NxN diagonal matrix with positive diagonal elements whose n^{th} element (along the major diagonal $n \neq N$) is $(\sin\theta_n)g_1(\theta_n)(h_{n+1} + h_n)$. The N^{th} element is $g_1(\theta_N)h_N$.

e = $g_2(\theta)/g_1(\theta)$ is a positive constant.

$\vec{Q}$ = Nx1 matrix whose n^{th} component ($n \neq N$) is $G_n(h_{n+1} + h_n)$. The N^{th} component is

$$h_N G_N - \frac{D^2 a_N}{4\beta^2}(2 - \beta^2 - \mu) .$$

$\vec{P}$ = Nx1 matrix, all of whose components are zero except the N^{th} one. This one is

$$\frac{T\,a_N}{\beta^2}.$$

$\vec{R}$ = Nx1 matrix, all of whose components are zero except the N^{th} one. This one is

$$a_N(T^3\beta^2)^{-1}\,D .$$

α, γ = previously defined scalars (unknown quantities at this point).

$\vec{\Gamma}, \vec{\Omega}$ = unknown Nx1 vectors whose components are, respectively,

$$\left\{\Gamma_i\right\}_{i=1}^{N} \quad \text{and} \quad \left\{\Omega_i\right\}_{i=1}^{N} .$$

GENERAL METHODS OF SOLUTION

From the second equation of (43),

$$\vec{\Omega} = -e(A + D_1)^{-1} D_2\,\vec{\Gamma} + \alpha(A + D_1)^{-1}\,\vec{R} \;; \qquad (44)$$

thus,

$$\left[A - D_1 + e\,D_2\,(A + D_1)^{-1}\,D_2\right]\vec{\Gamma} = \gamma\,\vec{P} + \alpha\,D_2\,(A + D_1)^{-1}\vec{R} + \vec{Q} \;,$$

which will be written as

$$T\,\vec{\Gamma} = \gamma\,\vec{P} + \alpha\,\vec{S} + \vec{Q} \qquad (45)$$

with

$$T = \left[A - D_1 + e\,D_2\,(A + D_1)^{-1}\,D_2\right] \;, \text{ and } \vec{S} = D_2\,(A + D_1)^{-1}\,\vec{R} \;. \qquad (46)$$

The matrix Eq (45) may also be changed into another useful form by premultiplying by $(A + D_1)D_2^{-1}$.

Then, Eq (45) becomes

$$T'\,\vec{\Gamma} = \gamma\,\vec{P}' + \alpha\,\vec{S}' + \vec{Q}' \;, \qquad (47)$$

where

$$T' = \left[(A + D_1)D_2^{-1} (A - D_1) + e D_2 \right] \tag{48}$$

and

$$\left.\begin{aligned} \vec{P'} &= (A + D_1)D_2^{-1} \vec{P} \\ \vec{S'} &= (A + D_1)D_2^{-1} \vec{S} = \vec{R} \\ \vec{Q'} &= (A + D_1)D_2^{-1} \vec{Q} . \end{aligned}\right\} \tag{49}$$

There are two distinct SET codes, SET02 and SET03. Up to this point, no distinction has been made between these codes.

The difference in the two occurs in the methods used to solve the system (45) or (47) and is a result of the nature of the matrices T and T'.

Method (A): The SET02 Code

This method is based on Eq (47) with T' as defined in Eq (48). Three vectors must be found—

$$(T')^{-1} \vec{P'} , \quad (T')^{-1} \vec{S'} , \text{ and } (T')^{-1} \vec{Q'}$$

so that the system $T' \vec{x} = \vec{k}$ must be solved for $\vec{x}$ for three different values of $\vec{k}$. Following a suggestion of G. Birkhoff, a direct method is used to solve the system $T' \vec{x} = \vec{k}$ for $\vec{x}$, in contrast to an iterative method (about which more will be said later). A direct method seems particularly well suited to the matrix T' since "most" of the elements of T' are zero. A is tri-diagonal; thus, the same is true of $(A + D_1)D_2^{-1}$ and of $A - D_1$, so that the product $(A + D_1)D_2^{-1} (A - D_1)$ is a matrix whose only nonzero elements are on its main diagonal or on one of its nearest four parallel (to the main diagonal) diagonals. The same is consequently true also of

$$T' = \left[(A + D_1)D_2^{-1} (A - D_1) + e D_2 \right] .$$

Thus, $T' = (t_{i,j})$, with $t_{i,j} = 0$ if $j \neq i-2,\ i-1,\ i,\ i+1,\ 1+2$ and $i=3, \ldots, N-2$. A similar fact holds for $i=1,\ 2,\ N-1$, and N.

The system $T' \vec{x} = \vec{k}$ is now solved (for $\vec{x}$) by a method of elimination defined as follows:

1) The matrix T' is transformed into the matrix U defined by $U = (u_{i,j})$, where

$$u_{i,j} = t_{i,j} \qquad i = 1, 2$$

$$u_{i,j} = t_{i,j} - \frac{u_{i-1,j}\, t_{i,i-2}}{u_{i-1,i-2}}$$

for $i \geq 3$ and $j = i-1,\ i,\ i+1,\ i+2$.

2) The U is transformed into $U' = (u'_{i,j})$ with

$$u'_{i,j} = u_{i,j} \qquad \text{for } i=N-1, N$$

$$u'_{i,j} = u_{i,j} - \frac{u'_{i+1,j}\, u_{i,i+2}}{u'_{i+1,i+2}}$$

for $i \leq N-2$ and $j = i-1,\ i,\ i+1$.

3) The vector $\vec{k}$, with components $\left\{k_i\right\}_{i=1}^{N}$, is first transformed into the vector $\vec{\ell}$, with components $\left\{\ell_i\right\}_{i=1}^{N}$, where

$$\ell_i = \begin{cases} k_i & i = 1, 2 \\ k_i - \dfrac{t_{i,i-2}}{u_{i-1,i-2}} \ell_{i-1} , & i \geq 3 , \end{cases}$$

and then $\vec{\ell}$ is transformed into the vector $\vec{m}$, with components $\left\{m_i\right\}_{i=1}^{N}$, with

$$m_i = \begin{cases} \ell_i & i = N-1, N \\ \ell_i - \dfrac{u_{i,i+2}}{u'_{i+1,i+2}} m_{i+1} , & i \leq N-2 . \end{cases}$$

The result of these steps is the reduction of the original system $T' \vec{x} = \vec{k}$ into the system

$$U' \vec{x} = \vec{m} \tag{50}$$

where U' is now a tri-diagonal matrix. Thus, the system can now be solved for $\vec{x}$ by the method which will be mentioned later in connection with the SET03 code.

Method (B): The SET03 Code

This method is based on Eq (45). It is clear that the vectors

$$T^{-1} \vec{P} \ , \ T^{-1} \vec{S} \ , \text{ and } T^{-1} \vec{Q}$$

must be known to solve for $\vec{\Gamma}$. After these vectors are obtained, the relation

$$\vec{\Gamma} = \gamma (T^{-1} \vec{P}) + \alpha (T^{-1} \vec{S}) + (T^{-1} \vec{Q})$$

yields each component of $\vec{\Gamma}$ as a simple function of α, and γ. Equation (44) can be utilized to obtain $\vec{\Omega}$.

The process just described is based on the assumption that the two matrices, $A+D_1$ and T, can be inverted. By construction, A is symmetric and positive definite, so that the addition of a diagonal matrix D_1 with positive diagonal elements results in a matrix $A+D_1$ which is also positive definite. In particular, $(A+D_1)^{-1}$ exists. Moreover, by construction, A has nonzero elements only along its main diagonal and the two diagonals parallel to and nearest the main diagonal. A is said to be tri-diagonal. The same is true of $(A+D_1)$. Such a matrix leads to a system of equations $(A+D_1) \vec{x} = \vec{k}$ which can be solved for $\vec{x}$ by a simple recursion relation (see, for example, Ref 6, p 34). Thus, effectively, $(A+D_1)^{-1}$ of Eq (46) can be obtained in a fairly simple way.

On the other hand, T is symmetric because, with T* as the transpose of T,

$$T^* = \left[A - D_1 + e\, D_2 (A + D_1)^{-1} D_2 \right]^* = A - D_1 + e\, D_2 (A + D_1)^{-1} D_2 = T$$

since A, D_1, and D_2 are symmetric.

Moreover,

$$T = A + D_1 + e D_2(A + D_1)^{-1} D_2 - 2D_1$$
$$= e^{1/2} D_2^{1/2} \left[e^{-1/2} D_2^{-1/2} (A + D_1) D_2^{-1/2} + e^{1/2} D_2^{1/2} (A + D_1)^{-1} D_2^{1/2} - 2e^{-1/2} D_1 D_2^{-1} \right] D_2^{1/2}$$
$$= e^{1/2} D_2^{1/2} \left[M + M^{-1} - 2e^{-1/2} D_1 D_2^{-1} \right] D_2^{1/2} ,$$

where

$$M = e^{-1/2} D_2^{-1/2} (A + D_1) D_2^{-1/2}$$

is symmetric and positive definite. Now if $K = M + M^{-1}$, then K is symmetric and positive definite and its eigenvalues are of the form $\lambda + \lambda^{-1}$, where λ is an eigenvalue of M. Thus, since $\lambda > 0$, the minimum of $\lambda + \lambda^{-1}$ is 2 and, therefore, a lower bound for the eigenvalues of K is 2. Now,

$$(T\vec{x}, \vec{x}) = e^{1/2} \left\{ (K\vec{y}, \vec{y}) - 2e^{-1/2} (D_1 D_2^{-1} \vec{y}, \vec{y}) \right\} ,$$

where

$$\vec{y} = D_2^{1/2} \vec{x} .$$

Again,

$$(K\vec{y}, \vec{y}) \geq 2(\vec{y}, \vec{y})$$

and

$$e^{-1/2} (D_1 D_2^{-1} \vec{y}, \vec{y}) \leq \zeta(\vec{y}, \vec{y}) ,$$

where ζ is the largest eigenvalue of $e^{-1/2} D_1 D_2^{-1}$. However, $e^{-1/2} D_1 D_2^{-1}$ (a diagonal matrix) has elements $2\mu/m^2 (\frac{D}{T}) v_n^3 \beta$. The minium of v^3 is $1/\beta^3$, so that (assuming $\beta \geq 1$):

$$\zeta \leq \frac{2\mu\beta^2}{m^2 (\frac{D}{T})} .$$

Thus,

$$(T\vec{x}, \vec{x}) \geq e^{1/2} \left[2 - \frac{4\mu\beta^2}{m^2 (\frac{D}{T})} \right] (\vec{y}, \vec{y}) .$$

Now,

$$e^{1/2} = \frac{m^2}{T^2}$$

and

$$(\vec{y}, \vec{y}) = (D_2 \vec{x}, \vec{x}) \geq \eta(\vec{x}, \vec{x}) ,$$

where η is a lower bound for the eigenvalues of D_2. Now D_2 is a diagonal matrix with elements

$$\frac{1}{2} (\sin \theta_n) DT\beta v_n^3 (h_{n+1} - h_n)$$

except for the N^{th} element which is $\frac{1}{2} DT\beta v_N^3 h_N$. Since

$$\sin \theta_1 \simeq \theta_1 = \theta_1 - \theta_o = h_1 ,$$

it can be assumed, in general*, that η can be chosen as $DT\beta h_1^2$.

*This is not precisely so in all cases. We will, however, make this assumption.

Thus,

$$(T\vec{x}, \vec{x}) \geq 2\beta h_1^2 \left[m^2(\frac{D}{T}) - 2\mu\beta^2 \right] (x, x) \ .$$

Clearly then, T is positive definite when

$$\Lambda = m^2(\frac{D}{T}) - 2\mu\beta^2 > 0 \qquad (51)$$

or

$$1 \leq \beta^2 < \sqrt{3} \left(\frac{1 - \mu^2}{\mu^2} \right)^{1/2} (\frac{D}{T}) \ .$$

This is not a severe restriction. For example, for $\mu = 0.3$, this states that

$$1 \leq \beta^2 < \sqrt{30}\ (\frac{D}{T}) \ ,$$

and $\frac{D}{T}$ is usually ≥ 10. Moreover, if a is the smallest eigenvalue of T, then it can be said that, from Eq (51),

$$a \geq 2\beta h_1^2 \Lambda \ . \qquad (52)$$

It is not clear what happens for values of β other than those satisfying Eq (51). Although the code SET03 is not internally restricted to these values of β, it will nevertheless be assumed for the following theory that β does satisfy Eq (51).

The method used in SET03 to solve the system $T\vec{x} = \vec{k}$ is an iterative method in contrast to the direct method used in SET02. The primary advantage of this iterative method is its inherent stability with respect to round-off errors. Its primary disadvantage (a major one) is the time necessary to complete a problem in contrast to a direct method. This will be discussed in detail later.

The method to be used depends strongly on the fact that T is positive definite. Let

b = largest eigenvalue of T

and

a = smallest eigenvalue of T (as before).

Then, the new matrix

$$\left[I - \frac{2}{a + b} T \right]$$

is formed which now has eigenvalues in the region $-1 < \frac{a - b}{a + b} \leq \lambda \leq \frac{b - a}{a + b} < 1$. With this new matrix, polynomial operators are used with Chebyshev polynomials in the manner introduced by Shortley (Ref 7), which leads to the following iterative scheme. Let

$$\vec{u}_o = \vec{x}_o = \text{initial guess},$$

$$\vec{u}_1 = \left[I - \frac{2}{a + b} T \right] \vec{u}_o + \frac{2}{a + b} \vec{k} \ ,$$

and

$$T_n(d)\,\vec{u}_n = 2d\,T_{n-1}(d) \left[I - \frac{2}{a + b} T \right] \vec{u}_{n-1} - T_{n-2}(d)\,\vec{u}_{n-2} + \frac{4d\,T_{n-1}(d)}{a + b} \vec{k}$$

for n = 2, 3, ... Now,

$$d = \frac{a + b}{b - a} > 1$$

and

$$T_o(d) = 1 \ , \quad T_1(d) = d$$

with

$$T_n(d) = 2d\, T_{n-1}(d) - T_{n-2}(d)$$

for n = 2, 3, ... Let $\|\vec{x}\| = \sqrt{\sum_i x_i^2}$ where $\{x_i\}$ is the set of components of the vector $\vec{x}$. Also let

$$\vec{\epsilon}_n = \vec{x} - \vec{u}_n \ .$$

Then it can be shown that*

$$\|\vec{\epsilon}_n\| \le \left[T_n(d)\right]^{-1} \|\vec{\epsilon}_o\| \ ,$$

so that if $T_N(d) \ge 50$, then $\vec{u}_N$ will be less than 1% from the answer, provided the initial vector $\vec{u}_o = \vec{x}_o$ is within 50% of the answer.

This code makes use of the result of SET02 as the initial vector $\vec{u}_o$. An estimate of the largest eigenvalue of T is obtained from the recursion relation

$$\left.\begin{aligned} T\vec{x}_{n-1} &= \vec{S}_n \\ \lambda_n &= (\vec{S}_n, \vec{x}_{n-1}) \big/ (\vec{x}_{n-1}, \vec{x}_{n-1}) \\ \vec{x}_n &= \vec{S}_n / \lambda_n \end{aligned}\right\} \qquad (53)$$

for n = 1, 2, 3, ... M. (M is set at 50.) The symbol $(\vec{x}, \vec{y})$ indicates the inner product of $\vec{x}$ and $\vec{y}$ and is defined as

$$(\vec{x}, \vec{y}) = \sum_i x_i y_i \ ,$$

where $\{x_i\}$, $\{y_i\}$ are, respectively, the sets of components of $\vec{x}$ and $\vec{y}$. The number used for b, b_1, is then set at $(1 \cdot 1)\lambda_M$. It is important that this number be greater than or equal to b. An estimate, a_1, of the number a (smallest eigenvalue of T) is obtained from the Eqs (53) by using the matrix $T - b_1 I$ instead of T. The number obtained at the end of M iterations, call it λ'_M, gives rise to the estimate

$$a_1 = \lambda'_M + b_1 \ .$$

This value is used unless it is smaller than the estimate of Eq (52), in which case this latter value is used. Unlike the estimate for the largest eigenvalue, which had to be larger than the actual value, no such restriction exists for the estimate a_1.

CALCULATION OF THE STRESSES

From Eqs (15) through (18), the equations for the resultant stresses and stress couples for the ellipsoidal shell can be found in terms of $\Gamma(\theta)$ and $\Omega(\theta)$ in the following equations. It should be mentioned that only normalized resultant stresses and stress couples are obtained, namely N_x/P, N_y/P, M_x/P, and M_y/P. This will lead to the normalized stresses σ_x/P and σ_y/P.

$$\left(\frac{N_x}{P}\right)_n = \frac{2}{D\delta_n} (\cot \theta_n)\, \Gamma_n + \frac{1}{4} D\delta_n \qquad (54)$$

$$\left(\frac{N_y}{P}\right)_n = \frac{2}{D\delta'_n} \left(\frac{d\Gamma}{d\theta}\right)_n + \frac{1}{4} D \left[\frac{2\delta_n^2 - \beta^2}{\delta_n}\right] \qquad (55)$$

---

* See, for example, a similar analysis by D. Young in Ref 8.

$$\left(\frac{M_x}{P}\right)_n = \frac{2T^3}{m^4 D\delta_n}\left[\frac{1}{v_n^2}\left(\frac{d\Omega}{d\theta}\right)_n + \mu (\cot \theta_n)\,\Omega_n\right] \tag{56}$$

$$\left(\frac{M_y}{P}\right)_n = \frac{2T^3}{m^4 D\delta_n}\left[\frac{\mu}{v_n^2}\left(\frac{d\Omega}{d\theta}\right)_n + \Omega_n (\cot \theta_n)\right] \tag{57}$$

where the subscript n indicates, as before, the evaluation of the function at $\theta = \theta_n$.

The functions $\left(\frac{d\Omega}{d\theta}\right)_n$ and $\left(\frac{d\Gamma}{d\theta}\right)_n$ are evaluated from the following formulas:

For $n \neq N$,

$$\left(\frac{d\Omega}{d\theta}\right)_n = \frac{h_n}{h_{n+1}^2 + h_n h_{n+1}}\,\Omega_{n+1} + \frac{h_{n+1} - h_n}{h_{n+1}\, h_n}\,\Omega_n - \frac{h_{n+1}}{h_n^2 + h_n h_{n+1}}\,\Omega_{n-1} .$$

For $n = N$,

$$\left(\frac{d\Omega}{d\theta}\right)_N = \frac{2h_N + h_{N-1}}{h_N^2 + h_N h_{N-1}}\,\Omega_N - \frac{h_N + h_{N-1}}{h_N h_{N-1}}\,\Omega_{N-1} + \frac{h_N}{h_{N-1}^2 + h_N h_{N-1}}\,\Omega_{N-2} .$$

Similar expressions are used for $\left(\frac{d\Gamma}{d\theta}\right)_n$.

The stresses are now obtained by using the formulas of Eqs (22) and (23).

For the cylinder, the following relations hold:

$$\frac{M_x}{P} = \frac{at^3}{m^4}\, e^{-ax}\left[C_1 (\cos ax - \sin ax) - C_2 (\cos ax + \sin ax)\right] ,$$

$$\frac{M_y}{P} = \mu \frac{M_x}{P} ,$$

$$\frac{N_y}{P} = \frac{Dt^3 a^3}{m^4}\, e^{-ax}\left[C_1 (\cos ax + \sin ax) + C_2 (\cos ax - \sin ax)\right] + \frac{D}{2} ,$$

and

$$\frac{N_x}{P} = \frac{D}{4} ,$$

where

$$C_1 = \frac{1}{2t^3 a}\left\{\alpha + \frac{m^4}{D^2 a^2}\left[\gamma t - \frac{D^2(2 - \mu)}{4}\right]\right\}$$

and

$$C_2 = \frac{1}{2t^3 a}\left\{\alpha - \frac{m^4}{D^2 a^2}\left[\gamma t - \frac{D^2(2 - \mu)}{4}\right]\right\}.$$

The stresses in the cylinder can then be obtained from the formulas of Eqs (22) and (23), with T now replaced by t.

The cylinder has been assumed infinitely long in the previous equations. Since the discontinuity stresses caused by the juncture reach steady values in one period of the trigonometric functions shown previously, the stresses will only be calculated for a distance from the juncture corresponding to one period. This distance is calculated from $aL = 2\pi$. Since a has been shown to depend on the dimensions and material of the cylinder, the length L is different for each problem. L will be determined for each problem, and the axial and radial stresses on the inner and outer surfaces of the cylinder will be calculated for any specified number of points, not necessarily equally spaced along its length.

INPUT PREPARATION

The input for the two codes is the same except for the code designation. A sample input is shown in the Appendix. The input is as follows:

TITLE Card: Columns 1-67 are available for problem identification. Columns 68-72 must contain either SET02 or SET03, where the next to last character is a zero.

Card 1001: This card contains Poissons' ratio.

Card 2001: This card is used to specify the following:

1) The number of intervals, an integer, into which the ellipse is divided by the mesh: $5 \le n \le 500$.

2) β, the ratio of the major diameter to the minor diameter for the ellipse.

3) D, the cylinder diameter.

4) T, the ellipse thickness.

Card 3001: This card is used to define the mesh in the ellipse by pairs of numbers. The first number of each pair, an integer, indicates the number of intervals in a given region. The second number of each pair indicates the angle, in degrees, at which the region terminates. Each region is assumed to start at the angle where the last region terminated; the first region is assumed to start at zero. The angle used for input is that formed by the major axis of the ellipse and a line through the center of the ellipse intersecting the shell. Thus, this angle is not the angle θ used in the difference equations. In fact, if φ is the angle used in the input, then φ and θ are connected by the relation

$$\varphi = \arctan\left[\frac{1}{\beta^2}\cot\theta\right].$$

Thus, for example, $\varphi = 0$ and $\theta = \pi/2$ are corresponding angles. There can be a maximum of ten regions in the ellipse, and the last one must terminate with an angle of 90 degrees. The sum of the intervals on this card must equal the number of intervals specified on card 2001.

Card 4001: This card contains two quantities: the number of intervals (an integer) to be used in the cylinder in the first period of behavior, and the cylinder thickness.

Card 5001: This card is used to define the mesh in the cylinder by pairs of numbers in a manner similar to that used in the ellipse. The first number of each pair, an integer, gives the number of intervals in a region. The second number of each pair indicates the fraction of the first period of behavior to which the region extends. As in the ellipse, each region is assumed to start where the previous region terminated, the first region starting at zero. There can be a maximum of ten regions, the last of which must have a termination fraction of 1.0. The sum of the intervals must equal the number of intervals specified on card 4001.

All of the card numbers, as well as those quantities designated as integers, must not contain a decimal point. All other input quantities must have a decimal point somewhere in the number. For example, the value .00125 can be written as .00125, .125E-2, or 1.25E-3.

Values on each card, other than the title card, must start in column 1 and must be separated by commas. The last number on a card must not be followed by a comma. The first blank column on any card other than the title card indicates the end of data on that card.

OUTPUT DESCRIPTION

A sample output is shown in Appendix I. The output tape must be printed on the 717 tape-to-printer under program control. Each page of output contains the problem identification, the code designation, and the page number.

On page one, all of the input data except mesh description is printed along with α and γ. Page one will also contain the estimate for the upper bound of the eigenvalues and the number of terms of the recursion formula needed for convergence for each of the three vectors.

On page two, the normalized stress values for the ellipse begin. The axial and circumferential components on the outer and inner surfaces are tabulated as functions of angle, as specified by the input, from the top of the ellipse down to the juncture.

The cylinder stresses are tabulated in much the same manner except that they are a function of distance from the juncture rather than angle.

CONCLUSIONS

The two codes have certain characteristics which are very useful in obtaining results which can be regarded as reasonably accurate. In the first place, a typical problem is run on the SET02 code much faster than on the SET03 code. On the other hand, the SET02 code is subject to round-off errors when the mesh is sufficiently refined, while the method used in the SET03 code is inherently "stable" in the sense that an error introduced at the m^{th} step will decrease to zero as the number of iterations is increased. This fact is particularly useful when a mesh is being determined for a certain problem or a class of problems to be run on SET02. On the one hand, the mesh should be fine enough to insure a reasonable approximation to the solution of the differential equation; on the other hand, it should not introduce so many points that round-off will play a significant role in the results.

Care should be exercised in choosing a mesh because of the transformation of the independent variable in the code—namely, the change from φ to θ. If h_i' is the mesh in the angle φ and h_i is the mesh in the angle θ, then,

$$h_i' \left[\frac{\beta^2}{1 + (\beta^4 - 1)\sin^2\varphi} \right] = h_i \ ,$$

where

$$\varphi_i \le \varphi \le \varphi_{i-1}$$

with

$$\theta_i - \theta_{i-1} = h_i \ , \ \varphi_{i-1} - \varphi_i = h_i' \ , \ \text{and } \varphi_i = \text{arc tan}\left[\frac{1}{\beta^2} \cot \theta_i \right] .$$

Thus, at the top, $\varphi = 90°$ and $0 = 0°$ so that

$$h_i \simeq \frac{1}{\beta^2} h_i' \ ,$$

and at the juncture, $\theta = 90°$ and $\varphi = 0°$,

$$h_i \simeq \beta^2 h_i' \ ;$$

all of which means that the mesh the code is using may be very much finer or coarser than the user might have intended. It has been the experience of the authors that the mesh in the neighborhood of the juncture should be finer than that at the top of the ellipsoid. The two codes have also been used to check each other's results for given problems. This was necessary inasmuch as analytic results were not available for any but hemispherical heads. Hemispherical heads correspond to $\beta = 1$, and some results for this type of shell are available from Watts and Lang (Ref 9). To illustrate the type of accuracy obtainable in the code SET02, and also to check the code itself, some problems from Ref 9 were run, and the comparison is presented in Table I.

The mesh used consists of 54 points distributed as follows (measured in the angle φ):

1/2 degree intervals to 10 degrees
1 degree intervals to 16 degrees
2 degree intervals to 24 degrees
4 degree intervals to 72 degrees
2 degree intervals to 84 degrees
1 degree intervals to 90 degrees

The stresses given are for both the hemisphere and the cylinder at the juncture ($\varphi = 0°$) and are normalized, as is done by Watts and Lang, by dividing through by D/2t. By way of notation, a subscript o or i is used in the stresses to indicate the outside or the inside surfaces, respectively.

TABLE I - COMPARISON OF PROBLEM RESULTS

Hemisphere			Watts and Lang				SET02 Results			
D	T	t	$\frac{2t\sigma_{xo}}{DP}$	$\frac{2t\sigma_{xi}}{DP}$	$\frac{2t\sigma_{yo}}{DP}$	$\frac{2t\sigma_{yi}}{DP}$	$\frac{2t\sigma_{xo}}{DP}$	$\frac{2t\sigma_{xi}}{DP}$	$\frac{2t\sigma_{yo}}{DP}$	$\frac{2t\sigma_{yi}}{DP}$
32	.8	1.0	.6767	.5733	.8583	.8273	.6737	.5763	.8585	.8293
16	.8	1.0	.6772	.5728	.8582	.8268	.6822	.5678	.8569	.8226
20	1.0	1.0	.5007	.4993	.7496	.7492	.5048	.4952	.7477	.7449
16	1.6	1.0	.2359	.3891	.5619	.6079	.2388	.3862	.5587	.6029
Cylinder										
32	.8	1.0	.5331	.4669	.8152	.7954	.5318	.4682	.8159	.7968
16	.8	1.0	.5334	.4666	.8150	.7950	.5369	.4631	.8133	.7911
20	1.0	1.0	.5007	.4993	.7496	.7492	.5053	.4947	.7479	.7447
16	1.6	1.0	.3040	.6960	.5824	.7000	.3121	.6879	.5807	.6934

OPERATING INSTRUCTIONS

Either program may be operated from cards, a program tape, or a service tape.

1) Ready program in one of the given forms.
2) Ready a tape on logical 5 for output.
3) Ready all problems in reader (a blank card must follow each problem).
4) All sense switches on console must be up.
5) Press CLEAR on the console and load the program.

There is only one stop in the code at 110 octal; the on-line comment will instruct the operator.

The output tape which must be printed on program control is not rewound nor is an end-of-file written on it at any time by the program.

Code Restrictions

1) Number of intervals in ellipse: $5 \leq n \leq 500$
2) Number of regions in ellipse ≤ 10
3) Number of regions in cylinder ≤ 10

Machine Requirements

Core Size	32768 words
Tapes	2
Drums	none
Punch	none
On-Line Printer	with SHARE 2 or GL OUT2 Board
Off-Line Printer	yes

```
                                                              SAMPLE PROBLEM                    SET02  PAGE    1
    INPUT PARAMETERS
               POISSONS RATIO (MU)   =     .30000000
               INTERVALS IN ELLIPSE  =    20
               BETA                  =    2.00000000
               HEAD THICKNESS        =    1.00000000

               POINTS IN CYLINDER    =    20
               CYLINDER DIAMETER     = 110.00000000
               CYLINDER THICKNESS    =    2.00000000




    ALPHA =  54.7682767  ...... GAMMA =-109.9534988
```

ELLIPSE
DIMENSIONLESS STRESS TO PRESSURE RATIO

AXIAL			CIRCUMFERENTIAL	
OUTER SURFACE	INNER SURFACE	ANGLE IN DEGREES	OUTER SURFACE	INNER SURFACE
54.1114159	45.8949080	87.0000	54.0864067	45.8694296
54.0725322	45.8572040	84.0000	53.9709287	45.7547526
53.8827205	45.6597476	77.0000	53.3903599	45.1757050
53.5512009	45.2859087	70.0000	52.3325019	44.1055303
53.0528688	44.6960726	63.0000	50.6935630	42.4346385
52.3551993	43.8182740	56.0000	48.3020258	39.9747410
51.4050722	42.5434599	49.0000	44.8753471	36.4183884
50.0963483	40.7320704	42.0000	39.9397411	31.2731352
48.1519513	38.3008847	35.0000	32.6745563	23.7858272
44.7357025	35.6063566	28.0000	21.6529596	12.9932983
37.4458199	34.6252193	21.0000	4.7261970	-1.4969602
19.9464619	42.1763496	14.0000	-18.8460364	-15.3895397
8.5491543	49.6124163	11.0000	-28.3077750	-17.5239973
-.2789533	55.1768484	8.0000	-32.7950678	-16.0352845
.2174754	52.3725376	5.0000	-27.6660759	-10.9610527
22.9243615	28.0881922	2.0000	-9.7108350	-7.4557920
29.5588732	21.2105889	1.5000	-5.5297040	-7.5044747
37.0210795	13.5011666	1.0000	-1.0615098	-7.7722428
45.5236349	4.7432911	.5000	3.6217575	-8.4483032
54.5388622	-4.5388911	.0000	8.3941602	-9.3291816

CYLINDER
DIMENSIONLESS STRESS TO PRESSURE RATIO

AXIAL			CIRCUMFERENTIAL	
OUTER SURFACE	INNER SURFACE	DISTANCE FROM JUNCTION	OUTER SURFACE	INNER SURFACE
20.0231147	4.9753852	.0000	-2.0925989	-6.6064677
29.7537673	-4.7537673	2.4441	10.7235695	.3713092
32.4572954	-7.4572954	4.8881	19.6192107	7.6448337
31.0221460	-6.0221460	7.3322	25.1407256	14.0274382
27.5859871	-2.5859871	9.7762	28.0660937	19.0145018
23.5777397	1.4222602	12.2203	29.1846704	22.5380268
19.3421996	5.1578003	14.6643	29.1730705	24.7727509
16.7907372	8.2092628	17.1084	28.5721545	25.9977126
14.5441810	10.4558190	19.5524	27.7338719	26.5073633
13.0509418	11.9490582	21.9965	26.8924890	26.5619240
12.4748828	12.5251172	24.4405	26.1707625	26.3658328
11.7543919	13.2456081	26.8846	25.5169295	26.0642943
11.6375672	13.3624328	29.3286	25.2325130	25.7499774
11.6995883	13.3004117	31.7727	24.9939146	25.4741616
11.8480793	13.1519207	34.2167	24.8675001	25.2586524
12.4212917	12.5787083	36.6608	24.8191638	25.1063886
12.6827183	12.3172817	39.1049	24.8194499	25.0093186
12.4145832	12.5854163	41.5489	24.8456342	24.9568841
12.4116648	12.5883352	43.9929	24.8818598	24.9348607
12.4761928	12.5238072	46.4370	24.9182191	24.9325032
12.5140501	12.4859499	48.8810	24.9494076	24.9409776

END OF PROBLEM

```
INPUT PARAMETERS
            POISSONS RATIO (MU)  =    .30000000
            INTERVALS IN ELLIPSE =    20
            BETA                 =   2.00000000
            HEAD THICKNESS       =   1.00000000

            POINTS IN CYLINDER   =    20
            CYLINDER DIAMETER    = 100.00000000
            CYLINDER THICKNESS   =   2.00000000




ESTIMATE FOR LARGEST EIGENVALUE =     463.7543869

 FORMULA ESTIMATE FOR SMALLEST EIGENVALUE             =    .2253570

  ITERATION FORMULA ESTIMATE FOR SMALLEST EIGENVALUE  =   2.8457146

 FIRST OF 3 VECTORS COMPUTED. N =    30

 SECOND OF 3 VECTORS COMPUTED.

 LAST OF 3 VECTORS COMPUTED.

ALPHA =  53.0912409  ...... GAMMA =-784.9936295
```

ELLIPSE
DIMENSIONLESS STRESS TO PRESSURE RATIO

AXIAL			CIRCUMFERENTIAL	
OUTER SURFACE	INNER SURFACE	ANGLE IN DEGREES	OUTER SURFACE	INNER SURFACE
54.2201042	45.7857595	87.0000	54.1951127	45.7602572
54.1811323	45.7481375	84.0000	54.0795808	45.6456285
53.9910164	45.5509834	77.0000	53.4988065	45.0668159
53.6591511	45.1775112	70.0000	52.4406872	43.9970331
53.1603737	44.5881891	63.0000	50.8014379	42.3267837
52.4622550	43.7109914	56.0000	48.4096127	39.8678570
51.5119863	42.4366088	49.0000	44.9828472	36.3128738
50.2042837	40.6247149	42.0000	40.0477800	31.1693649
48.2642779	38.1900258	35.0000	32.7842464	23.6833189
44.8599262	35.4846420	28.0000	21.7644043	12.8877215
37.5903072	34.4827690	21.0000	4.8307996	-1.6191917
20.0728560	42.0457768	14.0000	-18.8026834	-15.5531468
8.6059723	49.5459423	11.0000	-28.3310194	-17.6981592
-.3715415	55.2553611	8.0000	-32.9003062	-16.1763732
-.1345537	52.7107263	5.0000	-27.8446231	-10.9829072
22.2283425	28.7775660	2.0000	-9.8747373	-7.2179494
28.9032207	21.9613428	1.5000	-5.6597528	-7.1918335
36.2067304	14.3125991	1.0000	-1.1678276	-7.3961839
44.6533804	5.6120718	.5000	3.5490099	-8.0015255
53.6153455	-3.6153736	.0000	8.3948201	-8.7744111

CYLINDER

DIMENSIONLESS STRESS TO PRESSURE RATIO

AXIAL			CIRCUMFERENTIAL	
OUTER SURFACE	INNER SURFACE	DISTANCE FROM JUNCTION	OUTER SURFACE	INNER SURFACE
19.7927527	5.2072473	.0000	-1.9121090	-6.2877606
29.4914567	-4.4914567	2.4441	10.7896245	.5947506
32.2157612	-7.2157612	4.8831	19.6146927	7.7852361
30.8265195	-5.8265195	7.3222	25.0992086	14.1032974
27.4430573	-2.4430574	9.7762	28.0108244	19.0449903
23.4835310	1.5164689	12.2203	29.1300333	22.5399148
19.7875657	5.2124342	14.6643	29.1316481	24.7591090
16.7650931	8.2349069	17.1084	28.5368080	25.9777522
14.5377001	10.4622999	19.5524	27.7095304	26.4869103
13.0556185	11.9443815	21.9965	26.8775277	26.5441570
12.1348377	12.8151623	24.4405	26.1629629	26.3520603
11.7657274	13.2342726	26.8846	25.6140754	26.0546386
11.6480048	13.3519952	29.3286	25.2327132	25.7439103
11.7080420	13.2919580	31.7727	24.9957089	25.4708834
11.8542558	13.1457442	34.2167	24.9698885	25.2573349
12.0253627	12.9746373	36.6608	24.9215251	25.1063070
12.1850792	12.8149208	39.1048	24.8214560	25.0104084
12.3156912	12.6843088	41.5489	24.8471615	24.9577467
12.4119449	12.5880551	43.9929	24.8829117	24.9357448
12.4759908	12.5240092	46.4370	24.9188657	24.9332709
12.5136200	12.4863800	48.8810	24.9497445	24.9415729

END OF PROBLEM

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