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SOME EXACT RADIATING SOLUTIONS TO VLASOV'S EQUATIONS

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ABSTRACT

A class of exact solutions to the Vlasov equations which shows electromagnetic radiation is constructed, and a typical example discussed in some detail. Since velocities larger than c appear to be possibly of importance in these solutions, an exact radiating solution to the relativistic Vlasov equations is constructed, which, though much more specialized than the nonrelativistic solutions, shows that unphysically large velocities in the nonrelativistic solutions are not essential for the radiation there obtained.

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INTRODUCTION AND SUMMARY

There has been some discussion as to whether radiation can be emitted by an oscillating plasma which obeys the Vlasov equation. The available results on radiation in the Vlasov case do not provide a clear-cut answer to this problem. Previous proofs of the extence of this radiation such as that by Dawson and Oberman,[1] and by Harris,[2] have been restricted to the use of the linear approximation. Since the results obtained by Bernstein et al.[3] show the necessity for a very careful interpretation of any conclusions stemming from linearized equations, one might reasonably hold that the existence of radiating solutions to the Vlasov equations has as yet not been demonstrated.

The content of the work reported below is a construction of a class of exact solutions to the (nonlinear) Vlasov equations, and a demonstration that these solutions correspond to coherent radiation by the plasma as a whole. The plasma given by these solutions is infinite in extent, and the meaning to be attached to this radiation is appropriately specified.

While this counter example is logically sufficient to disprove the conjecture mentioned above, it is not completely satisfactory from a physical point of view. The reason is that the solutions involve radial velocity distributions centered about velocities greater than the velocity of light. This is a consequence of the use of nonrelativistic mechanics in the Vlasov equations, a limitation generally of no importance. However, the distributions do seem to involve velocities near c in possibly an essential way (as could conceivably be the situation if Cerenkov radiation were the radiation mechanism).

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1. J. Dawson and C. Oberman, The Physics of Fluids 2, 103 (1959).
2. E. G. Harris, private communication.
3. I. B. Bernstein, J. M. Green, and M. D. Kruskal, Phys. Rev. 108, 546 (1957).

To eliminate this objection a second exact solution to the Vlasov equations is constructed in which relativistic mechanics is used. This latter solution is more restricted than in the nonrelativistic case, for it is now required that the masses of the two plasma constituents be equal. Nevertheless, this solution shows that velocities larger than c are not essential to obtain radiation, and that this objection to the nonrelativistic solution is probably not of importance.

METHOD OF SOLUTION

We shall consider in the following a plasma composed of two species: one with mass m_i and charge (+e) ("ions"), and one with mass m_e and charge (-e) ("electrons"); collectively designated by i, as in m_i and e_i. This plasma is to be infinite in extent, but possessing cylindrical symmetry about the z-axis and displacement symmetry along the z-axis.

It is explicitly assumed that all quantities vary only with $r \left(= \sqrt{x^2 + y^2} \right)$ and t, and that the scalar potential is zero.

The Vlasov equations for the plasma are as usual:[4]

$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \vec{v} \cdot \frac{\partial f_i}{\partial \vec{r}} + \frac{d\vec{p}}{dt} \cdot \frac{\partial f_i}{\partial \vec{p}} = 0, \tag{1}$$

$$\Box^2 \vec{A} = - \mu_o \vec{j}, \tag{2a}$$

$$\phi \equiv 0, \tag{2b}$$

$$\frac{d\vec{p}}{dt} = e_i(\vec{E} + \vec{v} \times \vec{B}) = e_i \left[- \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\nabla \times A) \right]. \tag{3}$$

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4. E. G. Harris, Self-Consistent Field Theory for a Completely Ionized Gas, Naval Research Laboratory Report 4944 (May 1957).

For convenience in treating the relativistic case to follow, we use momentum space in place of velocity space. Thus the distribution functions for the plasma constituents are of the form:

$$f_i = f_i(r,t,\vec{p}), \tag{4}$$

and the charge and current densities which are the sources of the fields are:

$$\rho \equiv 0 = \sum_i \int e_i f_i \vec{dp} \tag{5}$$

$$\vec{j} = \sum_i \int e_i \vec{v} f_i \vec{dp} \tag{6}$$

It follows from the symmetry requirements on the system that the vector potential has only a component along z, and that the radial and tangential currents are identically zero. That is:

$$\vec{A} = \hat{k}\, A_z(r,t), \tag{7}$$

$$\vec{j} = \hat{k}\, j(r,t). \tag{8}$$

The method of constructing solutions to Eqs. (1) and (2) follows Harris. The characteristics of the partial differential equations of (1) are given by solutions to the mechanical equations (3). It is necessary to have sufficient integrals to these equations to fix the length of the momentum vector, and we shall obtain three integrals below.

Calling these integrals α_1, α_2, and α_3, it follows that any function $f_i(\alpha_r)$ satisfies Eq. (1). In terms of these distribution functions, f_i, it is necessary then to satisfy Eq. (2) as restricted by Eqs. (5) and (8). The final restriction we shall inpose on the vector potential is that A be a function of the single variable $u = r - ct$.

CONSTRUCTION OF INTEGRALS

Two integrals of Eq. (3) can be found at once from the symmetry of the problem. Using Eq. (3) one has:

$$\frac{dp_z}{dt} = e_i \left[- \frac{\partial A}{\partial t} - \left(\frac{xv_x + yv_y}{r} \right) \frac{\partial A}{\partial r} \right] = - e_i \left(\frac{dA}{dt} \right) \qquad (9)$$

so that one integral is:

$$p_z + e_i A(r,t) \equiv \alpha_1 = \text{constant}. \qquad (10)$$

The remaining two equations,

$$\frac{dp_y}{dt} = e_i \; yv_z \frac{1}{r} \frac{\partial A}{\partial r} , \qquad (11)$$

$$\frac{dp_x}{dt} = e_i \; xv_z \frac{1}{r} \frac{\partial A}{\partial r} , \qquad (12)$$

have the z-component of the angular momentum as an integral. That is:

$$xp_y - yp_x \equiv \alpha_2 = \text{constant}. \qquad (13)$$

Since the radial acceleration (nonrelativistically) has the form:

$$\frac{d}{dt} \left(\frac{dr}{dt} \right) = \frac{(yv_x - xv_y)^2}{r^3} + \frac{x}{r} \frac{dv_x}{dt} + \frac{y}{r} \frac{dv_y}{dt} , \qquad (14)$$

it follows from Eqs. (11) and (12) that

$$\frac{d}{dt}(p_r) = \frac{\alpha_2^2}{m_i r^3} + e_i v_z \frac{\partial A}{\partial r} . \tag{15}$$

Explicitly introducing the assumption that $A = A(r - ct)$, it follows from Eq. (15) that:

$$p_r \frac{d}{dt}(p_r) - m_i c \frac{d}{dt}(p_r) = \frac{\alpha_2^2}{m_i r^3}(p_r - m_i c) - \frac{1}{2}\left[(p_r - m_i c)\frac{\partial}{\partial r}(\alpha_1 - e_i A)^2\right],$$

$$= \frac{\alpha_2^2}{m_i r^3}(p_r - m_i c) - \frac{m_i}{2}\frac{d}{dt}(\alpha_1 - e_i A)^2, \tag{17}$$

so that, for $\alpha_2 = 0$, a third constant is:

$$\alpha_3 = (p_r - m_i c)^2 + p_z^2 . \tag{18}$$

The three integrals given by Eqs. (10), (13), and (18) are sufficient for constructing suitable distribution functions that satisfy Eq. (1). By reason of the restriction of Eq. (18) to zero angular momentum, the distribution functions all involve $\delta(\alpha_2)$.

CONSTRUCTION OF A TYPICAL SOLUTION

It will be clear from the following that a great many solutions to Eqs. (1) and (3) can be constructed from the integrals obtained above. It will suffice, however, for the purpose at hand to give a single example.

Let us take the distribution functions:

$$f_i(r,t,p_x,p_y,p_z) = \beta_i N_i \pi^{-1/2} \delta(xp_y - yp_x)\, \delta\left[p_z + e_i A(r - ct)\right]$$

$$\cdot \exp\left\{- \beta_i^2 \left[r^{-1}(xp_x + yp_y) - m_i c\right]^2 + p_z^2\right\}, \qquad (19)$$

where N_i and β_i are positive constants.

It is necessary that the distribution functions of Eq. (19) satisfy the conditions that ρ, j_r, and j_θ vanish identically. Consider first the charge density defined by:

$$\rho(r,t) = \sum_i \int e_i f_i \, dp_x dp_y dp_z \,. \qquad (5)$$

To carry out the integral over $dp_x dp_y$ we make an orthogonal transformation to:

$$\eta = yp_x - xp_y$$

$$\xi = r^{-1}(xp_x + yp_y), \qquad (20)$$

and

$$dp_x dp_y = J \, d\eta \, d\xi, \qquad (21)$$

with the Jacobian $J = 1/r$. (The integration limits is still $-\infty$ to $+\infty$ for both ξ and η.) The charge density is easily found to be:

$$\rho(r,t) = \sum_i e_i N_i r^{-1} \exp\left[- \beta_i^2 e^2 A^2 (r - ct)\right] . \qquad (22)$$

To satisfy the condition that $\rho \equiv 0$ it is necessary that $\beta_i = \beta$, and $N_i = N$.

The condition that the aximuthal current (j_θ) be zero is seen to be satisfied in view of $\delta(xp_y - yp_x)$. The radial current is given by:

$$j_r(r,t) \equiv \hat{r}\cdot\vec{j} = \sum_i \int e_i \hat{r}\cdot\vec{v}f_i\,d\vec{p} = \sum_i \left(\frac{e_i N\beta\pi^{-1/2}}{rm_i}\right) \iiint_{-\infty}^{\infty} d\eta\, d\xi\, dp_z\, \xi \cdot$$

$$\cdot\, \delta(\eta)\delta(p_z + e_i A) \cdot \exp\left\{-\beta^2 \left[(\xi - m_i c)^2 + p_z^2\right]\right\} = c\rho(r,t) \equiv 0. \qquad (23)$$

It remains to calculate the current in the z-direction. This is found to be:

$$j_z(r,t) = \sum_i \left(\frac{e_i N\beta\pi^{-1/2}}{rm_i}\right) \iiint_{-\infty}^{\infty} d\eta\, d\xi\, dp_z (p_z)\delta(\eta)\cdot\delta(p_z + e_i A)$$

$$\exp\left\{-\beta^2 \left[(\xi - m_i c)^2 + p_z^2\right]\right\}$$

$$= -\left(\frac{N_e^2}{r}\right)\left(\frac{1}{m_i} + \frac{1}{m_e}\right) A(r - ct)\, e^{-\beta^2 e^2 A^2 (r-ct)} \qquad (24)$$

To complete the calculation it is required to find the field generated by the current in Eq. (24), and show that it is a function only of $u = r - ct$.

The equation determining A is given by:

$$\Box^2 A_z = -\mu_o j_z = \frac{(N\mu_o e^2)\left(\frac{1}{m_e} + \frac{1}{m_i}\right) A e^{-\beta^2 e^2 A^2}}{r}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A = \frac{1}{r}\frac{\partial A}{\partial r} = \frac{1}{r}\frac{dA}{du} \;. \qquad (25)$$

Putting this in a more transparent form we have the equation:

$$\frac{da}{dw} = a e^{-a^2}, \qquad (26)$$

where

$$w = \frac{1}{\beta}(N\mu_o e)\left(\frac{1}{m_e} + \frac{1}{m_i}\right)(r - ct),$$

$$a = \beta e A(r - ct). \qquad (27)$$

The solution to Eq. (26) is:

$$w = \frac{1}{2}\int_{a_o^2}^{a^2} x^{-1} e^{x} dx \qquad (28)$$

or,

$$2w = \overline{E_i}(a^2) \qquad (28')$$

where $\overline{E_i}(x)$ is the exponential integral of Jahnke-Emde, and a_o^2 is defined such that $\overline{E_i}(a_o^2) = 0$.

The function $\overline{E_i}(x)$ has the behavior:

$$(1) \quad \overline{E_i}(x) \to \ln \gamma x + \sum_{i}^{\infty} \frac{x^n}{n \cdot n!}$$

$$x \longrightarrow 0$$

$$(2) \quad \overline{E_i}(x) \sim \frac{e^x}{x} \left(1 + \frac{1!}{x} + \ldots\right)$$

The vector potential is a monotonically increasing function of $u = r - ct$. For $u \to -\infty$, A becomes small exponentially, i.e.,

$$a \cong e^{-|w|} \qquad \text{as } w \longrightarrow -\infty .$$

For $u \longrightarrow +\infty$, A becomes infinite, but relatively slowly,

$$a \sim (\log 2w)^{1/2}. \qquad \text{as } w \longrightarrow +\infty$$

PROPERTIES OF THE NONRELATIVISTIC SOLUTION

The solution found in the previous section is given by:

$$(1) \qquad f_i(r,t,\vec{p}) = (\beta N \pi^{-1/2}) \delta(xp_y - yp_x) \delta(p_z + e_i A)$$

$$\cdot \exp\left\{ -\beta^2 \left[\left(\frac{xp_x + yp_y}{r} - m_i c \right)^2 + p_z^2 \right] \right\}, \qquad (29)$$

(2) $\vec{A} = \hat{k}\, A(r - ct),$ (30)

where:

$$2\mu_o Ne^2 \left(\frac{1}{m_e} + \frac{1}{m_i}\right) (r - ct) = \overline{E_i}(\beta^2 e^2 A^2). \qquad (31)$$

In order to determine the properties of this solution let us first consider the particle density and particle current. Let us define:

(a) $n(r,t) \equiv$ "particle density" $= \int d\vec{p}\, f_i = \frac{N}{r} \exp(-\beta^2 e^2 A^2).$ (32)

The particle density is the same for each species.

(b) $\vec{j}(r,t) \equiv$ "particle current" $= \int dp_x dp_y dp_x \vec{v} f = \left[c\hat{r} - \left(\frac{e_i}{m_i}\right) \hat{k} A\right] n(r,t).$ (33)

Only the radial particle current is independent of the type of particle.

It follows from Eqs. (32) and (33) that:

$$\text{div}\, \vec{j} + \frac{\partial n}{\partial t} = 0, \quad \text{for } r \neq 0. \qquad (34)$$

The radial flow across a cylinder coaxial with the symmetry axis is given by:

$$J = 2\pi \cdot Nc \exp(-\beta^2 e^2 A^2), \qquad (35)$$

so that <u>there exists a net inflow of particles from the axis</u>. That is:

$$\text{div}\, \vec{j} + \frac{\partial n}{\partial t} = \delta(r) \cdot \frac{Nc}{r} \exp(-\beta^2 e^2 A^2). \qquad (36)$$

The distribution function given in Eq. (29) is not well defined in the limit $r \rightarrow 0$, for the three constants α_1, α_2, α_3 do not map all of momentum space in this limit. For any $r \geq \mathcal{E} > 0$, where $\mathcal{E}$ is a small constant, the distribution function is well defined, and the solution developed previously defines the meaning to be associated with the limit $r \rightarrow 0$. Thus the solution given by Eqs. (29), (30), (31) is a solution to the Vlasov equations for $r \neq 0$, but for $r = 0$ there exists a line source of particles given by the right hand side of Eq. (36).

The total number of particles in a coaxial cylinder of radius R and length L is given by:

$$N(R,t) = \frac{2\pi L\beta}{\mu_o |e| \left(\frac{1}{m_e} + \frac{1}{m_i}\right)} \ \ell n \left[A(R - ct)/A(- ct)\right] . \tag{37}$$

The number of particles that flow in from the source from time - T to t is given by:

$$N_{source}(-T,t) = \frac{2\pi L\beta}{\mu_o \, e \left(\frac{1}{m_e} + \frac{1}{m_i}\right)} \ \ell n \left[A(- cT)/A(- ct)\right] \tag{38}$$

Equations (37) and (38) verify the rather obvious point that the number of particles that flow in from time $-\infty$ to t are found within a cylinder of infinite radius. The number of particles (per unit length) in the system at any finite time is, however, infinite.

From Eq. (32) one sees that the particle density for fixed r ($\neq 0$) is zero for $t \rightarrow -\infty$, rises as t increases, and asymptotically approaches a constant final value of $n(r,t \rightarrow \infty) = N/r$.

From Eq. (33) one sees that the particle currents in the z direction are oppositely directed for the two species, and are zero at $t \rightarrow -\infty$, rise to a maximum and again approach zero for $t \rightarrow +\infty$. The average particle speed (in the z direction) on the other hand monotonically decreases, corresponding to a transfer of energy to the electromagnetic field.

Let us consider now the electromagnetic fields. From the vector potential given by (30), it follows that:

$$\vec{E} = \hat{k} \left(\frac{N\mu_o ce}{\beta} \right) \left(\frac{1}{m_c} + \frac{1}{m_i} \right) A \, e^{-\beta^2 e^2 A^2}, \tag{39}$$

$$\vec{B} = \hat{\theta} \left(\frac{N\mu_o ce}{\beta} \right) \left(\frac{1}{m_e} + \frac{1}{m_i} \right) A \, e^{-\beta^2 e^2 A^2}. \tag{40}$$

The electromagnetic energy flow is therefore radially outward at all times, and the energy flowing through a coaxial cylinder of radius r and length L per unit time is:

$$\frac{dW}{dt} = 2\pi\mu_o \left[\frac{Nce}{\beta} \left(\frac{1}{m_e} + \frac{1}{m_i} \right) \right]^2 (rL) \, A^2 \, e^{-2\beta^2 e^2 A^2}. \tag{41}$$

For fixed r, $\frac{dW}{dt}$ is zero for $t \rightarrow -\infty$, rises a maximum and goes to zero again for $t \rightarrow +\infty$. This energy flow does not all represent radiation, however, for some of the flow only represents a change in the local energy density. According to Poynting's theorem, however, the radiation results from the work done on the current sources, that is:

$$\mathrm{div}\,(\vec{E} \times \vec{H}) + \frac{\partial}{\partial t} \left[\frac{1}{2} (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \right] = - \vec{j} \cdot \vec{E}$$

$$= \frac{\mu_o ce^3}{\beta} \left(\frac{1}{m_e} + \frac{1}{m_i} \right)^2 N^2 \frac{A^2}{r} e^{-2\beta^2 e^2 A^2}. \tag{42}$$

One sees that energy is monotonically put into the electromagnetic field at every point in space and since the electromagnetic energy density is zero at both $t = -\infty$ and $t = +\infty$, it is correct to say that the transfer of mechanical energy (from motion in the z-direction) to the electromagnetic field represents radiation by the plasma.

The final point to investigate is the energy balance. Rather than consider the various terms separately we shall simply establish local energy conservation for the Vlasov equations. The mechanical energy density is given by the expression: (Nonrelativistically)

$$U_M = \frac{1}{2} \sum_i \int d^3p \, \frac{p^2}{m_i} f_i, \text{ and therefore,} \tag{43}$$

$$\frac{\partial U_M}{\partial t} = -\frac{1}{2} \sum_i \int d^3p \left(\frac{p^2}{m_i}\right)\left[\frac{\vec{p}}{m_i} \cdot \nabla f_i + e_i\left(\vec{E} + \frac{\vec{p}}{m_i} \times \vec{B}\right) \cdot \nabla_p f_i\right], \tag{44}$$

using Eq. (1) to eliminate $\frac{\partial f_i}{\partial t}$. It follows that:

$$\frac{\partial U_M}{\partial t} = -\nabla \cdot \left(\sum_i \int d^3p \, \frac{p^2}{2m_i} \frac{\vec{p}}{m_i} f_i\right) + \sum_i \frac{e_i}{m_i} \vec{E} \cdot \int d^3p \, \vec{p} \, f_i. \tag{45}$$

Letting the mechanical energy flux be defined by $J_M = \sum_i \int d^3p \left(\frac{p^2}{2m_i}\right) \frac{\vec{p}}{2m_i} f_i$, we obtain the obvious result:

$$\frac{\partial U_M}{\partial t} + \text{div } \vec{J}_M = \vec{j} \cdot \vec{E}. \tag{46}$$

Upon combining this with the Poynting integral (Eq. 42) the final result is obtained:[5]

$$\frac{\partial}{\partial t}\left(U_M + \frac{1}{2}\vec{E}\cdot\vec{D} + \frac{1}{2}\vec{B}\cdot\vec{H}\right) + \mathrm{div}(\vec{J}_M + \vec{E}\times\vec{H}) \equiv 0. \qquad (47)$$

Equation (47) shows that the Vlasov equations explicitly conserve the sum of the mechanical and electromagnetic energies (including radiation) locally.

Since the solution given in Eqs. (29) through (31) satisfies the Vlasov equations everywhere except $r = 0$, Eq. (47) establishes energy conservation everywhere, except for $r = 0$. For $r = 0$, however, it is clear from Eqs. (39) and (40) the net flux of electromagnetic energy vanishes. The source terms at $r = 0$ are therefore sources only of particles and mechanical energy. This justifies once again the designation of the solution given by Eqs. (29) through (31) as an exact radiating solution to Vlasov's equations.

EXTENSION TO A RELATIVISTIC SOLUTION

The radiating solution discussed in the previous section has the disturbing feature that the radial motion involves velocities whose mean is the velocity of light. While it is the z-motion and not the radial motion that is coupled directly to the electromagnetic field, this feature of the solution is nonetheless sufficiently unphysical as to cast doubt upon its usefulness as a counter-example. To remedy this defect, we shall in this section utilize relativistic mechanics in Vlasov's equation. To be precise, we shall construct the characteristics to Eq. (1) using $\vec{p} = \dfrac{m_o\vec{v}}{\sqrt{1-(v/c)^2}}$ in determining $\dfrac{d\vec{p}}{dt}$.

The two constants α_1 and α_2 [Eqs. (10) and (13)], are once again constants of the motion, as is to be expected since these are obtained from considerations of symmetry that are valid relativistically.

5. A similar, but less general, result has been given by I. B. Bernstein, Phys. Rev. 109, 10 (1958).

The constant α_3 is replaced by the new functional:

$$\alpha^3 = \left[(m_i c)^2 + p_r^2 + p_z^2\right]^{1/2} - p_r, \tag{48}$$

which, as can be shown by direct calculation (see Appendix), is a constant of the relativistic equations of motion if $\alpha_1 = \alpha_2 = 0$. [The symbol p_r is a shorthand for $\hat{r} \cdot \vec{p} = r^{-1}(xp_x + yp_y)$] .

The nonrelativistic constant, Eq. (18) (for the special case where α_1 is also zero), is a limiting case of Eq. (48).

It is clear from the form of Eq. (48) that $\alpha_3 \geq 0$ if the square root is chosen with a positive sign. The sign of the square root, however, is fixed from the derivation (see Appendix) to be positive. This is an essential limitation which is imposed on the admissible distribution functions.

The next point to establish is that there exists no distribution function constructed from the constants $\alpha_1 = \alpha_2 = 0$ and α_3 which satisfies simultaneously the two conditions that $\rho \equiv 0$ and $j_r \equiv 0$, unless $m_i = m_e$. Let the distribution functions be defined by:

$$f_i(r,t,\vec{p}) = N_i \delta(\alpha_1)\delta(\alpha_2)F_i(\alpha_3). \tag{49}$$

The condition that the charge density vanish identically requires that:

$$N_i \int_{-\infty}^{\infty} d\zeta\, F_i\left(\left|\left[(m_i c)^2 + (eA)^2 + \zeta^2\right]^{1/2}\right| - \zeta\right)$$

$$= N_e \int_{-\infty}^{\infty} d\zeta\, F_e\left(\left|\left[(m_e c)^2 + e^2A^2 + \zeta^2\right]^{1/2}\right| - \zeta\right). \tag{50}$$

The condition that the radial current density vanish identically takes a little manipulation. The radial current involves the radial velocities, and for each species, is given by:

$$j_{r,i} = \frac{e_i N_i c}{r} \int_{-\infty}^{\infty} d\zeta \, \frac{\zeta}{\left| \left[(m_i c)^2 + e^2 A^2 + \zeta^2 \right]^{1/2} \right|}$$

$$F_i \left(\left| \left[(m_i c)^2 + e^2 A^2 + \zeta^2 \right]^{1/2} \right| - \zeta \right) \qquad (51)$$

Introducing α_3 as the variable in Eq. (51), one finds that:

$$j_{r,i} = c \, \rho_i - \frac{e_i N_i c}{r} \int_0^{\infty} d\alpha_3 \, F_i(\alpha_3) \qquad (52)$$

If Eq. (50) is similarly written in terms of α_3, one finds that:

$$N_i \left\{ \left[(m_i c)^2 + e^2 A^2 \right] \int_0^{\infty} \alpha_3^{-2} \, d\alpha_3 \, F_i(\alpha_3) + \int_0^{\infty} d\alpha_3 \, F_i(\alpha_3) \right\}$$

$$= N_e \left\{ \left[m_e^2 c^2 + e^2 A^2 \right] \int_0^{\infty} \alpha_3^{-2} \, d\alpha_3 \, F_e(\alpha_3) + \int_0^{\infty} d\alpha_3 \, F_e(\alpha_2) \right\} . \qquad (50')$$

Using Eqs.(50') and (52), and noting that the radial current and charge densities must vanish identically, the required relations are:

$$N_i m_i^2 \int_0^{\infty} \alpha^{-2} \, d\alpha \, F_i(\alpha) = N_e m_e^2 \int_0^{\infty} \alpha^{-2} \, d\alpha \, F_e(\alpha) \tag{53a}$$

$$N_i \int_0^{\infty} \alpha^{-2} \, d\alpha \, F_i(\alpha) = N_e \int_0^{\infty} \alpha^{-2} \, d\alpha \, F_e(\alpha), \tag{53b}$$

$$N_i \int_0^{\infty} d\alpha \, F_i(\alpha) = N_e \int_0^{\infty} d\alpha \, F_e(\alpha) \tag{53c}$$

Equations (53a) and (53b) are inconsistent unless $m_i = m_e$.

The distinction between this situation and the nonrelativistic cases treated earlier, lies in the fact that in the nonrelativistic case the radial current was proportional to the charge density, unlike Eq. (52).

Since our aim is only to demonstrate that the use of relativistic mechanics does not in itself prevent a radiating solution similar to that discussed for the nonrelativistic case, we shall simply take the otherwise uninteresting case $m_i = m_e = m$, for which solutions can indeed be found.

A particularly simple solution results from the distribution functions:

$$f_i(r,t,\vec{p}) = N \, \delta(\alpha_1)\delta(\alpha_2)\delta(\alpha_3 - a), \tag{54}$$

where a, N are positive constants. It is easily seen that ρ, j_r, and j_θ are now all identically zero.

The current in the z-direction is now given by:

$$j_z(r,t) = - \left(\frac{2e^2 N}{am}\right) \cdot \frac{A(r - ct)}{r} \, . \tag{55}$$

From the equation for the vector potential, $\Box^2 A_z = -\mu_o j_z$, one finds that:

$$A = A_o \exp\left[\left(\frac{2\mu_o e^2 N}{am}\right)(r - ct)\right] \tag{56}$$

The solution to the "relativistic" Vlasov equations given in Eqs. (54) and (56) is qualitatively similar to that discussed earlier for the nonrelativistic case, and the discussion need not be repeated.

We can conclude from this example that the radiation found and discussed in a nonrelativistic exact solution to Vlasov's equations is not dependent in any essential way on the occurrence of unphysically large radial velocities.

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Appendix

DERIVATION OF THE RELATIVISTIC CONSTANTS

From the equation $\frac{d\vec{p}}{dt} = e_i\ (\vec{E} + \vec{v} \times \vec{B})$, the equations $\vec{E} = -\frac{\partial A}{\partial t}$, $\vec{B} = \nabla \times \vec{B}$, $\vec{A} = \hat{k}\ A(r - ct)$, it follows that:

$$\frac{dp_z}{dt} = -\ e_i \left(\frac{\partial A}{\partial t} + \frac{x}{r} v_x \frac{\partial A}{\partial r} + \frac{y}{r} v_y \frac{\partial A}{\partial r} \right) = -\ e_i \frac{dA}{dt},$$

or: $\alpha_1 = p_z + e_i A$ = constant, and: (A-1)

$$\frac{dp_y}{dt} = e_i v_z \frac{y}{r} \frac{\partial A}{\partial r} \qquad \text{(A-2)}$$

$$\frac{dp_x}{dt} = e_i v_z \frac{x}{r} \frac{\partial A}{\partial r}. \qquad \text{(A-3)}$$

From Eqs. (2) and (3) it follows that $\alpha_2 = xp_y - yp_x$ = constant.

Using the definition $p_r = \hat{r} \cdot \vec{p} = (xp_x + yp_y)/r$, one finds the identity:

$$\frac{dp_r}{dt} = \frac{x}{r} \frac{dp_x}{dt} + \frac{y}{r} \frac{dp_y}{dt} + r^{-3}(yp_x - xp_y)(yv_x - xv_y). \qquad \text{(A-4)}$$

From Eqs. (2), (3), and (4) it follows that:

$$p_r \frac{dp_r}{dt} = e_i(\alpha_1 - e_i A) \left(\frac{dr}{dt}\right) \frac{\partial A}{\partial r} + \alpha_2^2\, r^{-3} \left(\frac{dr}{dt}\right), \qquad \text{(A-5)}$$

and, using the fact that $A = A(r - ct)$, one finds:

$$\frac{m_o c}{\sqrt{1 - (v^2/c^2)}} \frac{dp_r}{dt} = - e_i(\alpha_1 - e_i A) \frac{\partial A}{\partial t} + c\alpha_2^2/r^3. \qquad (A-6)$$

Taking $\alpha_1 = \alpha_2 = 0$, (5) and (6) show that:

$$p_r \frac{dp_r}{dt} - \frac{m_o c}{\sqrt{1 - (v^2/c^2)}} \frac{dp_r}{dt} + \frac{e^2}{2} \frac{d}{dt} (A^2) = 0. \qquad (A-7)$$

For $\alpha_1 = \alpha_2 = 0$, the square root in (7) may be written in terms of p_r and A. That is:

$$m_o c(1 - v^2/c^2)^{-1/2} = \left[(m_o c)^2 + p_r^2 + e^2 A^2\right]^{1/2} \qquad (A-8)$$

Introducing (8) into (7), it is an immediate result that a third integral is given by:

$$\alpha_3 = \left[(m_o c)^2 + p_r^2 + p_z^2\right]^{-1/2} - p_r \qquad (A-9)$$

for the case where $\alpha_1 = \alpha_2 = 0$.

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