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Domain Decomposition Multigrid for Unstructured Grids

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SUMMARY

A two-level preconditioning method for the solution of elliptic boundary value problems using finite element schemes on possibly unstructured meshes is introduced. It is based on a domain decomposition and a Galerkin scheme for the coarse level vertex unknowns. For both the implementation and the analysis, it is not required that the curves of discontinuity in the coefficients of the PDE match the interfaces between subdomains. Generalizations to nonmatching or overlapping grids are made.

1.1 Introduction

The Black Box Multigrid method of [Den82] is considered robust for diffusion problems with possibly discontinuous coefficients on structured grids. More specifically, the application of this method requires that the coefficient matrix has a 3^d-coefficient stencil, where d is the dimension of the problem. Thus, Black Box Multigrid is not applicable for more complicated (e.g., unstructured) finite element schemes resulting from realistic engineering and applied science problems. Furthermore, it is pointed out in [Sha94a] [Sha95] that Black Box Multigrid stagnates for certain diffusion problems with high diffusion areas separated by a thin strip. Surprisingly, this stagnation occurs when the discontinuity curves are aligned with all the coarse grids, case which can be handled easily by either standard multigrid or the method of [BPS86]. The AutoMUG method introduced there avoids this stagnation but diverges for other examples. In [Sha94b] this stagnation is explained and a modified version of Black Box Multigrid which avoids it is introduced. This version is related to the method of [Den80] [KM81] and based on 'throwing' certain matrix elements to the main diagonal when constructing the prolongation operator from coarse to fine grids. It is shown in

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[Sha94b] that this version is robust both theoretically (for a certain class of problems) and numerically (for the above example and others). In [Sha96] the robustness of the method is shown for locally refined finite element schemes.

In this work, a method in the spirit of the above version is applied to general finite element schemes which not necessarily arise from local mesh refinement. It is assumed that a suitable domain decomposition is available (see Figure 1.1). The idea is to choose suitable vertex variables on the interface between subdomains to serve as the coarse grid. The prolongation operator extending a coarse grid function to the whole grid consists of two steps: first solve low order systems (resulting from the coefficient matrix by 'throwing' certain elements to the main diagonal) to extend the function to the interface unknowns; then extend the function to the interior of the subdomains by solving the original scheme on each subdomain (for simplicity we consider the 2-d case; in 3-d, three steps are needed). The coarse grid equation is obtained from a Galerkin scheme; it is solved either directly or iteratively using some preconditioning method or multigrid. Both the formulation of the coarse grid equation and the actual restriction and prolongation involve mainly local operations which are well parallelizable. The method can be supplemented with presmoothing and postsmoothing as in multigrid or with an outer acceleration scheme. We call the method Domain Decomposition Multigrid (DDMG).

Note that the method analyzed in [Sha94b] for the structured grid case may be obtained from DDMG by considering four-cell unions in a uniform grid as subdomains. The present analysis is a modification of that of [Sha94b] in which matrix elements are replaced by suitable submatrices. Unlike in [BPS86], it is not necessary to assume neither for the implementation nor for the analysis that the curves of discontinuity in the coefficients of the PDE match the interfaces between subdomains. Also, it is not assumed here that the domain is polygonal; for simplicity, however, we use polygonal domains in the present example.

1.2 The Domain Decomposition Multigrid Method

Consider a finite element scheme for an elliptic boundary value problem on a mesh of the type used in [BPS86] (illustrated in Figure 1.1). Assume that the underlying linear system is given by

$$Ax = \mathcal{B},$$

where x is the vector of unknowns corresponding to the nodes in the mesh and $\mathcal{B}$ is the right hand side vector and A is the coefficient matrix.

Consider a domain decomposition as in Figure 1.1, where nodes on the thick lines correspond to interface or boundary unknowns. Let some of these unknowns (typically, vertex variables such as those denoted by '$\bullet$' in Figure 1.2) serve as coarse grid variables. In the following we denote by c the set of coarse grid variables, by b the set of the other boundary and interface variables and by s the set of all other variables (corresponding to nodes in subdomain interiors). This induces a partitioning of the coefficient matrix A as

$$A = \begin{pmatrix} A_{ss} & A_{sb} & A_{sc} \\ A_{bs} & A_{bb} & A_{bc} \\ A_{cs} & A_{cb} & A_{cc} \end{pmatrix}. \qquad (1.1)$$

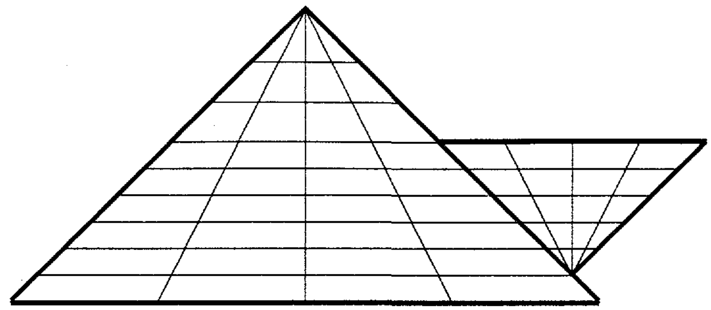

Figure 1.1 The unstructured grid and the domain decomposition.

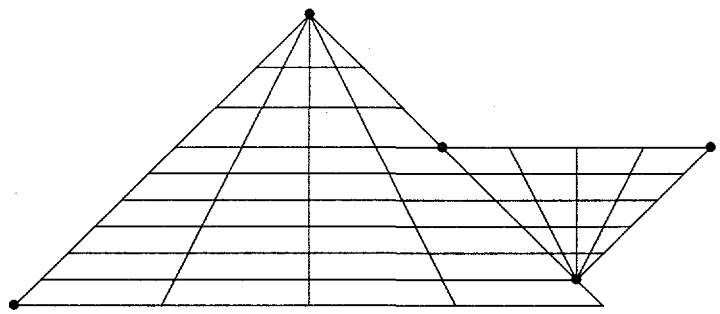

Figure 1.2 The coarse grid variables are denoted by '•'.

In the following we use this partitioning also for other matrices of the same order and (unless specified otherwise) refer by 'blocks' to the blocks in such partitionings. We also denote $f = s \cup b$ (the set of fine grid points, namely, all variables but the coarse grid ones). This induces another block partitioning for A:

$$A = \begin{pmatrix} A_{ff} & A_{fc} \\ A_{cf} & A_{cc} \end{pmatrix}$$

where, for example,

$$A_{ff} = \begin{pmatrix} A_{ss} & A_{sb} \\ A_{bs} & A_{bb} \end{pmatrix}. \qquad (1.2)$$

This partitioning will also be used for other matrices of the same order.

For any set g, let $|g|$ denote its intensity. For any positive integer k, let I_k denote the identity matrix of order k. For any set $g \subset c \cup f$, let $J_g : l_2(c \cup f) \to l_2(g)$ denote the injection

$$(J_g w)_j = w_j, \; w \in l_2(c \cup f), \; j \in g.$$

For any matrix M, $M = (m_{i,j})_{1\le i\le K,\ 1\le j\le L}$, define $|M| = (|m_{i,j}|)_{1\le i\le K,\ 1\le j\le L}$; define also the diagonal matrix of row-sums of M by

$$rs(M) = diag(\sum_{j=1}^{L} m_{i,j})_{1\le i\le K}.$$

Let us now define a matrix $T(A)$ which is obtained from A by 'throwing' certain matrix elements to the main diagonal. More specifically, $T(A)$ is of the same order as A and

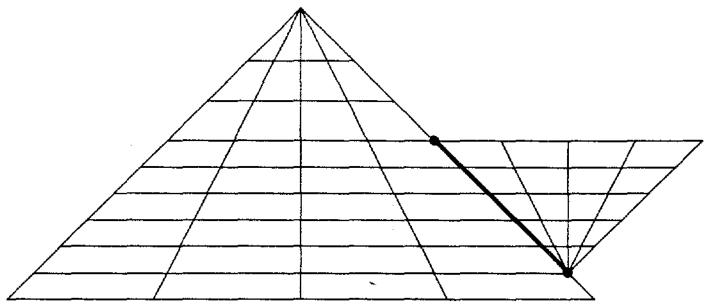

Figure 1.3 First prolongation step; from C, the set of variables denoted by '•', into B, the set of all the other variables on the thick line.

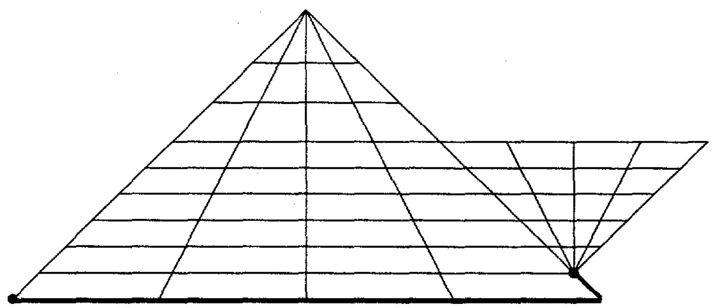

Figure 1.4 First prolongation step; from C, the set of variables denoted by '•', into B, the set of all the other variables on the thick lines.

is upper block-triangular (with respect to the partitioning (1.1)) with $T(A)_{cc} = I_{|c|}$. The structure of $T(A)$ is thus

$$T(A) = \begin{pmatrix} T(A)_{ss} & T(A)_{sb} & T(A)_{sc} \\ 0 & T(A)_{bb} & T(A)_{bc} \\ 0 & 0 & I_{|c|} \end{pmatrix}.$$

Furthermore, $T(A)_{bb}$ is block diagonal, with blocks corresponding to portions of interfaces or boundaries. Consider, for example, the unknowns on the thick line in Figure 1.3. Denote by C the set of the two variables denoted by '•', by B the set of the other variables on the thick line and by F the set of all other variables. The rows in A corresponding to B can be partitioned in the form

$$\begin{pmatrix} A_{BF} & A_{BB} & A_{BC} \end{pmatrix}.$$

The corresponding rows in $T(A)$ are defined by

$$\begin{pmatrix} 0 & A_{BB} - rs(|A_{BF}|) & A_{BC} \end{pmatrix}.$$

Consequently, the unknowns in B are coupled in $T(A)$ only with themselves and possibly with those in C. The rows of $T(A)$ corresponding to other boundary portions of the form $B \subset b$ (such as that of Figure 1.4) are defined in a similar way.

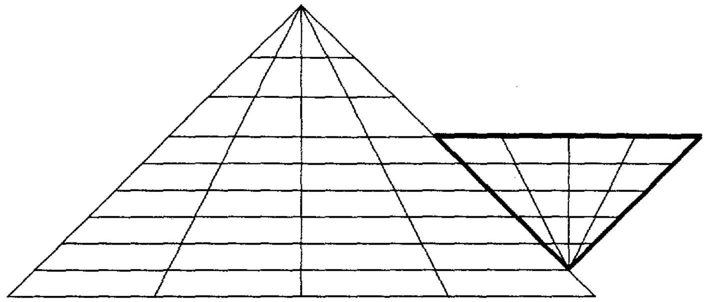

Figure 1.5 Second prolongation step; from B, the set of variables on the thick lines, into S, the set of variables in the interior of the subdomain bounded by the thick lines.

Similarly, $T(A)_{ss}$ is block diagonal, with blocks corresponding to sets $S \subset s$ corresponding to interiors of subdomains. For example, denote by B the unknowns corresponding to nodes on the thick lines in Figure 1.5, by S the unknowns corresponding to the nodes in the interior of the subdomain bounded by these lines and by F the rest of the unknowns. The rows in A corresponding to S can be partitioned in the form

$$\begin{pmatrix} A_{SF} & A_{SS} & A_{SB} \end{pmatrix}.$$

The corresponding rows in $T(A)$ are defined by

$$\begin{pmatrix} 0 & A_{SS} - rs(|A_{SF}|) & A_{SB} \end{pmatrix}.$$

Consequently, the unknowns in S are coupled in $T(A)$ only with themselves and possibly with those in B. Rows of $T(A)$ corresponding to other subdomains of the form $S \subset s$ are defined in a similar way. For usual finite element schemes (such as that of Figure 1.1) $A_{SF} \equiv 0$ and, hence,

$$T(A)_{ss} = A_{ss}, \quad T(A)_{sb} = A_{sb} \quad \text{and} \quad T(A)_{sc} = A_{sc}. \tag{1.3}$$

However, by using the above definition we handle also the more general case, where interiors of different subdomains might be coupled in A (e.g., when subdomains are not aligned with finite elements). The above definition of $T(A)$ ensures that such a coupling cannot exist in $T(A)$; this allows efficient and parallelizable restriction and prolongation operations.

Define the prolongation operator P and the restriction operator R by

$$P = T(A)^{-1} \quad \text{and} \quad R = (T(A^t) - blockdiag(T(A^t)) + blockdiag(T(A)))^{-t}.$$

Here '*blockdiag*' corresponds to the partitioning (1.1). Since $T(A)$ is block triangular, the application of P and R is performed easily by block back substitution and block forward elimination, respectively. Furthermore, since the subdomain interiors (such as S in Figure 1.5) are decoupled from each other in $T(A)$ and similarly for the interface portions (such as B in Figure 1.3), the applications of R and P are highly parallelizable.

Finally, define the coarse grid operator Q by

$$Q = \begin{pmatrix} W & 0 \\ 0 & J_c RAPJ_c^t \end{pmatrix},$$

where W is a nonsingular matrix of order $|f|$. For the analysis in Section 1.3 to be valid W should be symmetric positive definite (SPD) whenever A is. The reasonable choices are

$$W = I_{|f|}$$

or, in the spirit of [Den82],

$$W = R_{ff} diag(A_{ff}) blockdiag(P_{ff}) \tag{1.4}$$

(where '*blockdiag*' corresponds to the partitioning (1.2)). The choice (1.4) yields better numerical results in [Sha96]. Although W in (1.4) is not SPD, it is spectrally equivalent to the SPD matrix $blockdiag(W)$. Hence, the application of the proof of Theorem 1 below is essentially unchanged (see [Sha94b]).

It is assumed here that $T(A)$ is nonsingular. It is also assumed that $J_c RAPJ_c^t$ is nonsingular; this is guaranteed when A is SPD and holds in most cases.

The two-level method is defined by

$$x_{out} = x_{in} + PQ^{-1}R(b - Ax_{in}). \tag{1.5}$$

(1.5) may be supplemented with relaxations before and after it in the spirit of multigrid methods. This approach is used in [Sha94b] for uniform grids. Alternatively, a Lanczos type acceleration may be applied to it. This approach is used in [Sha96]. For both approaches, the condition number of the preconditioned matrix $PQ^{-1}RA$ is an important measure for the rate of convergence. In the following, this condition number is estimated for SPD problems.

1.3 Analysis in the SPD Case

Here $(\cdot,\cdot)$ denotes the usual inner product in $l_2(c \cup f)$ and $\|\cdot\|$ denotes the corresponding vector and matrix norms. The following lemma is used in the proof of Theorem 1.

Lemma 1 *Let M be a symmetric and positive semi-definite matrix of the same order as A. Then, for any vector $x \in l_2(c \cup f)$,*

$$(x, Mx) \le 2(x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x).$$

Proof: Let $\tilde{x} = J_f^t J_f x - J_c^t J_c x$. Then we have

$$0 \le (\tilde{x}, M\tilde{x}) = (x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x) - (x, (J_f^t J_f M J_c^t J_c + J_c^t J_c M J_f^t J_f)x).$$

The lemma follows from

$$\begin{aligned}(x, Mx) &= (x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x) + (x, (J_f^t J_f M J_c^t J_c + J_c^t J_c M J_f^t J_f)x) \\ &\le 2(x, (J_f^t J_f M J_f^t J_f + J_c^t J_c M J_c^t J_c)x).\end{aligned}$$

Theorem 1 *Assume that A is symmetric and (possibly weakly) diagonally dominant, $T(A)$ is nonsingular and W is SPD. Then the condition number of the preconditioned coefficient matrix $PQ^{-1}RA$ is bounded by*

$$2\max(\|W^{-1}J_f RAPJ_f^t\|, 1)(1 + 2\|P\|\sqrt{\eta\|A\|} + \eta\|RAP\| + \eta\|W\|),$$

with $\eta = (\sqrt{2}+1)^2\|A\|$.

Proof: Since A is symmetric, $R = P^t$. Since A is symmetric and diagonally dominant, it follows from Gershgorin's theorem that it is positive semi definite. Let $x \in l_2(c \cup f)$ satisfy $\|x\| = 1$ and denote $\varepsilon = (x, Ax)$. Since A is symmetric and positive semi-definite, x may be written as a linear combination of the orthogonal eigenvectors of A. Consequently, $\|Ax\|^2 \leq \|A\|\varepsilon$.

Define

$$A_1 = \begin{pmatrix} (A - T(A))_{ss} & (A - T(A))_{sb} & (A - T(A))_{sc} \\ (A - T(A)^t)_{bs} & rs(|(A - T(A)^t)_{bs}|) & 0 \\ (A - T(A)^t)_{cs} & 0 & rs(|(A - T(A)^t)_{cs}|) \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} rs(|(A - T(A)^t)_{sb}|) & (A - T(A)^t)_{sb} & 0 \\ (A - T(A))_{bs} & (A - T(A))_{bb} & (A - T(A))_{bc} \\ 0 & (A - T(A)^t)_{cb} & rs(|(A - T(A)^t)_{cb}|) \end{pmatrix}$$

(note that $A_1 \equiv 0$ when (1.3) is satisfied). Since A_1, A_2, $A - A_1$ and $A - A_2$ are symmetric and diagonally dominant, it follows from Gershgorin's theorem that they are positive semi definite. Using the same argument as in the beginning of the proof, we obtain

$$\|A_n x\|^2 \leq \|A_n\|(x, A_n x) \leq \|A_n\|\varepsilon, \quad n = 1, 2.$$

For convenience we use here the notation $f_1 = s$ and $f_2 = b$. Note that

$$J_{f_n} A_n = J_{f_n}(A - P^{-1}), \quad n = 1, 2.$$

Consequently,

$$\begin{aligned} |\, \|J_f A x\| - \|J_f P^{-1} x\| \,|^2 &\leq \|J_f(A - P^{-1})x\|^2 \\ &= \sum_{n=1}^{2} \sum_{i \in f_n} |(A_n x)_i|^2 \\ &\leq \sum_{n=1}^{2} \|A_n x\|^2 \leq (\|A_1\| + \|A_2\|)\varepsilon, \end{aligned}$$

which implies that

$$\|J_f P^{-1} x\| \leq \sqrt{\eta \varepsilon}, \quad \text{where } \eta = (\sqrt{\|A_1\| + \|A_2\|} + \sqrt{\|A\|})^2 \leq (\sqrt{2} + 1)^2 \|A\|.$$

As a result, we have

$$\begin{aligned} (x, R^{-1} Q P^{-1} x) &= (P^{-1} x, Q P^{-1} x) \\ &= (J_c^t J_c x + J_f^t J_f P^{-1} x, Q(J_c^t J_c x + J_f^t J_f P^{-1} x)) \\ &\leq (J_c^t J_c x, RAP(J_c^t J_c x)) + \eta \|W\| \varepsilon \\ &= (P^{-1} x - J_f^t J_f P^{-1} x, RAP(P^{-1} x - J_f^t J_f P^{-1} x)) + \eta \|W\| \varepsilon \\ &\leq (1 + 2\|P\| \sqrt{\eta \|A\|} + \eta \|RAP\| + \eta \|W\|)\varepsilon, \end{aligned}$$

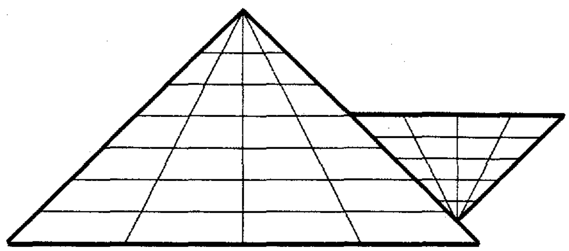

Figure 1.6 A domain decomposition with nonmatching grids.

which implies that the function $(x, R^{-1}QP^{-1}x)/(x, Ax)$ is bounded. On the other hand, we have from Lemma 1 that, for any $y \in l_2(c \cup f)$,

$$\begin{aligned}(y, RAPy) &\leq 2(y, (J_f^t J_f RAPJ_f^t J_f + J_c^t J_c RAPJ_c^t J_c)y) \\ &\leq 2\max(\|W^{-1}J_f RAPJ_f^t\|, 1)(y, Qy),\end{aligned}$$

which implies that the function $(x, Ax)/(x, R^{-1}QP^{-1}x)$ is bounded. This completes the proof of the theorem.

1.4 Discussion

The bound derived in Theorem 1 for the condition number of the preconditioned coefficient matrix depends mainly on $\|A\|$ and $\|P\|$. In order to minimize the bound, $\|A\|$ and $\|P\|$ should be minimized. Hence, A should be taken in the undivided form, as is usually the case for finite element schemes. On the other hand, $\|P\|$ may be large if one of the submatrices $T(A)_{BB}$ or $T(A)_{SS}$ (corresponding to variable sets such as B in Figure 1.3 or S in Figure 1.5) are nearly singular. This might happen if the variables in such a set B are nearly decoupled in the coefficient matrix A (which is unlikely to happen for usual elliptic problems, see [Sha96]) or if one of the submatrices A_{SS} is nearly singular. The latter case might happen when the number of nodes in a subdomain is large or when there is a large jump in the coefficients in the PDE. However, for a bounded number of nodes per subdomain and a bounded range of jumps in the coefficients the bound on the condition number is robust. Furthermore, the assumptions made in [BPS86] (polygonal domains and discontinuity curves aligned with the coarse grid) are not needed here. In practice it is expected that the convergence rates for DDMG are independent of the jump in the coefficients as well, as is the case for Black Box Multigrid for uniform grids.

Since the definition of DDMG relies on the domain decomposition and the coefficient matrix only, it is applicable also for nonmatching grids (such as those of Figure 1.6), provided that the underlying linear system is given. Furthermore, it can be extended in a natural way to the case of overlapping subdomains. The key is the definition of the prolongation operator from the coarse grid to the rest of the nodes. The prolongation to the interface and boundary unknown is done as before. The prolongation to the interior may be also done as before in each subdomain separately, and then taking the average of the multiple definitions in the overlapping area.

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