SAND99-0539J

HEAD WAVE TRAVELTIMES IN A THREE-DIMENSIONAL MULTILAYERED EARTH

David F. Aldridge

Geophysical Technology Department
Sandia National Laboratories
Albuquerque, New Mexico, USA, 87185-0750

Phone: 505-284-2823
Fax: 505-844-7354
Email: dfaldri@sandia.gov

Abbreviated Title: **Head Wave Traveltimes**

Submitted to *Geophysical Journal International*

January 1999

DISCLAIMER

DISCLAIMER

SUMMARY

Traveltimes of head waves propagating within a three-dimensional (3D) multilayered earth are described by straightforward mathematical formulae. The earth model consists of a set of homogeneous and isotropic layers bounded by plane interfaces. Each interface (including the surface) may possess arbitrary strike and dip. In this model, the source-to-receiver raypath of a critically refracted wave consists of a set of straight line segments, not confined to a single plane. Algebraic derivations of the traveltime expressions are greatly simplified by using a novel 3D form of Snell's law of refraction. Various generalizations of the basic traveltime equation extend its applicability to arbitrary source-receiver recording geometries and/or mode-converted waves. Related expressions for the traveltimes of reflected waves and one-way transmitted waves propagating in the same layered earth model are obtained as byproducts of the analysis. The expressions contain a set of unit raypath orientation vectors that depend implicitly on source and receiver coordinates. Hence, the equations cannot be characterized as "closed-form" in the mathematical sense. However, for critically refracted waves, these vectors can be obtained by a minimal amount of numerical raytracing. The traveltime formulae are useful for a variety of forward modeling and inversion purposes.

Key Words: refraction seismology, traveltime, layered media, ray theory.

INTRODUCTION

The traveltimes of head waves propagating in layered earth models have been studied extensively since the inception of applied seismology in the 1920's. Head wave arrival times are particularly useful for inferring the seismic velocities of layered subsurface media. One- and two-dimensional (1D and 2D) multilayered earth models are commonly used for analysis and interpretation of critically refracted arrival times. However, there is a notable paucity of papers on the subject of head wave propagation in three-dimensional (3D) layered earth models. Chander (1977) examines a model consisting of uniform velocity layers separated by plane interfaces with arbitrary strike and dip, and describes a method for calculating head wave traveltimes between specified source and receiver positions on a horizontal surface. If an array of receivers is colinear with the source, the wave arrival time curve is a straight line. Hence, if two points on this line are established, then arrival times at all offsets can be determined simply by drawing the connecting straight line. Chander locates the two initial points via raytracing.

Chander's work is purely numerical and does not provide much insight into the dependence of head wave traveltime on the parameters that define the earth model. Moreover, it is restricted to conventional (i.e., line profile) data acquisition geometries. Buried sources and/or receivers as well as non-profile recording geometries require a more general treatment. Diebold's (1987) work constitutes the seminal contribution on this topic. He also considers a 3D multilayered earth, and derives traveltime formulae for both reflected and critically refracted waves. These formulae are logical extensions of more familiar traveltime expressions appropriate for 1D and 2D layered models. Thus, they offer the possibility of extending several known traveltime inversion techniques to accommodate 3D planar structure. Unfortunately, Diebold's derivations are very ambiguous. Furthermore, his generalization to arbitrary source-receiver geometries yields an incorrect traveltime formula. Finally, he does not present a numerical technique for computing the traveltimes. These deficiencies are addressed in the present work. Nevertheless, Diebold (1987) should be credited with an original contribution to traveltime analysis for this particular class of earth models.

Richards *et al.* (1991), using a rather convoluted geometric argument, "rederive Diebold's result from first principles". However, their traveltime equation retains the difficulty of accommodating arbitrary 3D recording situations. In particular, it yields an erroneous result when source and receiver are located on different, non-parallel interfaces of a multilayered earth model.

This work provides a rigorous derivation of the 3D head wave traveltime formula. An algebraic, rather than geometric, viewpoint is adopted for the analysis. Related expressions for traveltimes of reflected waves and one-way transmitted waves propagating in the same earth model are obtained as byproducts of the analysis. The mathematical proofs of the formulae are simplified by using a novel form of Snell's law of refraction and reflection. Various generalizations of the basic traveltime equation extend its applicability to arbitrary 3D recording geometries and/or mode-converted waves. Finally, a rapid numerical method for computing the arrival times of critical refractions is presented, and is illustrated with simulated examples from shallow refraction exploration and vertical seismic profiling (VSP).

EARTH MODEL

Consider an earth model consisting of a set of homogeneous and isotropic layers bounded by plane interfaces. In general, each interface may possess a 3D dipping attitude. The i^{th} interface of the model is illustrated in Figure 1. O is the origin of a right-handed, rectangular Cartesian coordinate system with orthonormal basis triad **ijk**. The xy plane is defined to be the horizontal plane and the depth coordinate z increases in the downward direction. The locus $\mathbf{r}$ of plane interface i satisfies the equation

$$\mathbf{r} \cdot \mathbf{n}_i = d_i, \tag{1}$$

where $\mathbf{n}_i$ is a unit vector normal to the interface, and d_i is the perpendicular distance from O to the interface. Figure 1 indicates that $\mathbf{n}_i$ is conveniently described by two interface orientation angles:

$$\mathbf{n}_i = (\sin\phi_i \cos\theta_i)\mathbf{i} + (\sin\phi_i \sin\theta_i)\mathbf{j} + (\cos\phi_i)\mathbf{k}. \tag{2}$$

ϕ_i ($0 \leq \phi_i < \pi/2$) is the dip angle and θ_i ($0 \leq \theta_i < 2\pi$) is the azimuth angle of the interface. If the $+x$ and $+y$ axes are taken to point toward geographic north and east, respectively, then $\theta_i + \pi/2$ (modulo 2π) is the interface strike angle. Although these angular coordinates are descriptive, a certain compactness in notation is achieved by specifying $\mathbf{n}_i$ in terms of its Cartesian components:

$$\mathbf{n}_i = n_{i,x}\mathbf{i} + n_{i,y}\mathbf{j} + n_{i,z}\mathbf{k}, \tag{3}$$

with $\|\mathbf{n}_i\| = 1$. This convention is followed in the sequel.

Solving equation (1) for z as a function of x and y yields the vertical depth of interface i:

$$z_i(x,y) = z_i(0,0) - x\left[\frac{n_{i,x}}{n_{i,z}}\right] - y\left[\frac{n_{i,y}}{n_{i,z}}\right], \tag{4}$$

where $z_i(0,0) \equiv d_i/n_{i,z}$ is the vertical depth of the i^{th} interface below the coordinate origin.

The surface (not necessarily horizontal) is interface 1, and subsequent interfaces are numbered sequentially in the downward direction. Interface i overlies layer i. The vertical thickness of layer i is defined to be $h_i(x,y) \equiv z_{i+1}(x,y) - z_i(x,y)$. Thus

$$h_i(x,y) = h_i(0,0) + x\left[\frac{n_{i,x}}{n_{i,z}} - \frac{n_{i+1,x}}{n_{i+1,z}}\right] + y\left[\frac{n_{i,y}}{n_{i,z}} - \frac{n_{i+1,y}}{n_{i+1,z}}\right], \tag{5}$$

where $h_i(0,0) = z_{i+1}(0,0) - z_i(0,0)$ is the vertical thickness of the i^{th} layer beneath the coordinate origin.

Finally, the seismic wave propagation speed assigned to layer i is given by v_i. This may be either the compressional wave speed α_i or the shear wave speed β_i. This flexibility allows the resulting traveltime equations to apply either to P, S, or mode-converted waves.

RAYPATH GEOMETRY

Initially, the analysis is restricted to the case where both source and receiver are located on the surface. Generalization to an arbitrary data acquisition geometry is straightforward and is given in a later section. The horizontal coordinates of the point source S and point receiver R are (x_S,y_S) and (x_R,y_R), respectively. Their vertical coordinates are easily obtained from equation (4): $z_S = z_1(x_S,y_S)$ and $z_R = z_1(x_R,y_R)$.

In order to facilitate derivation of the traveltime, the total head wave raypath is divided into three major portions: downgoing, critically refracted, and upgoing paths. In Figure 2, these correspond to raypath segments SP, PQ, and QR, respectively. The propagation time along each portion is calculated, and then all three are summed to obtain the surface-to-surface head wave traveltime.

Within each layer, the raypath is a straight line segment. On the downward portion of the raypath, the propagation direction within layer i of a wave critically refacted at subsurface interface k is described by the unit vector $\mathbf{p}_{ik}$:

$$\mathbf{p}_{ik} = p_{ik,x}\mathbf{i} + p_{ik,y}\mathbf{j} + p_{ik,z}\mathbf{k}. \tag{6a}$$

Similarly, the upward propagation direction within layer i of the wave critically refracted from interface k is specified by another unit vector $\mathbf{q}_{ik}$:

$$\mathbf{q}_{ik} = q_{ik,x}\mathbf{i} + q_{ik,y}\mathbf{j} + q_{ik,z}\mathbf{k}. \tag{6b}$$

The complete head wave raypath is described by the set of unit vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$ ($i = 1,2,\ldots,k$-1) together with a critically refracted propagation direction $\mathbf{p}_{kk} = \mathbf{q}_{kk}$.

At interface i in the overburden, the wave is refracted in accordance with Snell's law. The situation for the downgoing wave as it encounters the i^{th} interface from above is depicted in the upper portion of Figure 3. The plane of this diagram is the *plane of incidence* defined by the incident propagation direction $\mathbf{p}_{i-1,k}$ and the interface normal $\mathbf{n}_i$. Snells' law of refraction consists of the following two conditions:

(1) the unit propagation vector of the transmitted ray ($\mathbf{p}_{ik}$) is contained in the plane of incidence,

(2) $\sin\mu / v_{i-1} = \sin\nu / v_i$, where μ and ν are positive acute angles measured from the interface normal to the incident and transmitted propagation directions $\mathbf{p}_{i-1,k}$ and $\mathbf{p}_{ik}$, respectively.

Both conditions are contained in the single vector equation

$$\frac{\mathbf{n}_i \times \mathbf{p}_{i-1,k}}{v_{i-1}} = \frac{\mathbf{n}_i \times \mathbf{p}_{ik}}{v_i}. \tag{7}$$

The vector formed by the cross products points out of the plane of the diagram in Figure 3. In component form, equation (7) is

$$\frac{1}{v_{i-1}}(n_{i,y}p_{i-1,k,z} - n_{i,z}p_{i-1,k,y}) = \frac{1}{v_i}(n_{i,y}p_{ik,z} - n_{i,z}p_{ik,y}), \tag{8a}$$

$$\frac{1}{v_{i-1}}(n_{i,z}p_{i-1,k,x} - n_{i,x}p_{i-1,k,z}) = \frac{1}{v_i}(n_{i,z}p_{ik,x} - n_{i,x}p_{ik,z}), \tag{8b}$$

$$\frac{1}{v_{i-1}}(n_{i,x}p_{i-1,k,y} - n_{i,y}p_{i-1,k,x}) = \frac{1}{v_i}(n_{i,x}p_{ik,y} - n_{i,y}p_{ik,x}). \tag{8c}$$

Similarly, when the upgoing wave encounters interface i from below, Snell's law in form

$$\frac{\mathbf{n}_i \times \mathbf{q}_{i-1,k}}{v_{i-1}} = \frac{\mathbf{n}_i \times \mathbf{q}_{ik}}{v_i}, \tag{9}$$

holds (see bottom of Figure 3). In this case, angles μ and ν exceed $\pi/2$ radians. The component form of the vector expression (9) is analogous to equations (8a,b,c).

The 3D statement of Snell's law of refraction given by the above expressions is quite different from the form typically used in raytracing applications (e.g., Sorrells *et al.* 1971; Shah 1973; Chander 1977). However, it can be demonstrated that these expressions are equivalent to the raytracing formulae. The value of the current formulation is that it leads to a substantial simplification in the mathematical proof of the traveltime equations.

TRAVELTIME DERIVATION

Downgoing Traveltime

An expression for the traveltime increment of the downgoing wave as it traverses layer i is derived first. Let the position vectors $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ denote the intersection points of the downgoing ray with interfaces i and i+1, respectively. Then, $\mathbf{r}_{i+1} = \mathbf{r}_i + l_i\,\mathbf{p}_{ik}$ where l_i is the length of the (straight line) raypath segment within layer i. Solving for l_i gives $l_i = \mathbf{p}_{ik} \cdot (\mathbf{r}_{i+1} - \mathbf{r}_i)$. The traveltime increment is obtained by dividing this path length segment by the layer velocity v_i:

$$t_i = \frac{l_i}{v_i} = \frac{p_{ik,x}}{v_i}(x_{i+1} - x_i) + \frac{p_{ik,y}}{v_i}(y_{i+1} - y_i) + \frac{p_{ik,z}}{v_i}(z_{i+1} - z_i)\,.$$

The vertical (z) coordinates of the intersection points can be expressed in terms of the horizontal (x and y) coordinates by using the equation for a dipping plane interface. Equation (4) gives

$$z_{i+1} - z_i = h_i(0,0) + x_i\left[\frac{n_{i,x}}{n_{i,z}}\right] + y_i\left[\frac{n_{i,y}}{n_{i,z}}\right] - x_{i+1}\left[\frac{n_{i+1,x}}{n_{i+1,z}}\right] - y_{i+1}\left[\frac{n_{i+1,y}}{n_{i+1,z}}\right],$$

where $h_i(0,0) = z_{i+1}(0,0) - z_i(0,0)$ is used. Substituting this result into the equation for t_i and grouping terms yields the required expression for the traveltime increment:

$$t_i = \frac{h_i(0,0)p_{ik,z}}{v_i} + \frac{x_{i+1}}{n_{i+1,z}}\left[\frac{n_{i+1,z}p_{ik,x} - n_{i+1,x}p_{ik,z}}{v_i}\right] + \frac{y_{i+1}}{n_{i+1,z}}\left[\frac{n_{i+1,z}p_{ik,y} - n_{i+1,y}p_{ik,z}}{v_i}\right]$$

$$-\frac{x_i}{n_{i,z}}\left[\frac{n_{i,z}p_{ik,x} - n_{i,x}p_{ik,z}}{v_i}\right] - \frac{y_i}{n_{i,z}}\left[\frac{n_{i,z}p_{ik,y} - n_{i,y}p_{ik,z}}{v_i}\right].$$

The total downgoing traveltime is obtained by summing all of the layer traveltime increments t_i for $i = 1,2,\ldots,k\text{-}1$. Thus

$$T_{down} = \sum_{i=1}^{k-1}\frac{h_i(0,0)p_{ik,z}}{v_i} + \sum_{i=1}^{k-1}\left\{\frac{x_{i+1}}{n_{i+1,z}}\left[\frac{n_{i+1,z}p_{ik,x} - n_{i+1,x}p_{ik,z}}{v_i}\right] + \frac{y_{i+1}}{n_{i+1,z}}\left[\frac{n_{i+1,z}p_{ik,y} - n_{i+1,y}p_{ik,z}}{v_i}\right]\right\}$$

$$-\sum_{i=1}^{k-1}\left\{\frac{x_i}{n_{i,z}}\left[\frac{n_{i,z}p_{ik,x} - n_{i,x}p_{ik,z}}{v_i}\right] + \frac{y_i}{n_{i,z}}\left[\frac{n_{i,z}p_{ik,y} - n_{i,y}p_{ik,z}}{v_i}\right]\right\}.$$

The sum involving x_{i+1} and y_{i+1} is now re-indexed and combined with the other sum, yielding

$$T_{down} = \sum_{i=1}^{k-1}\frac{h_i(0,0)p_{ik,z}}{v_i} + \left[\frac{x_k(n_{k,z}p_{k-1,k,x} - n_{k,x}p_{k-1,k,z}) + y_k(n_{k,z}p_{k-1,k,y} - n_{k,y}p_{k-1,k,z})}{v_{k-1}n_{k,z}}\right]$$

$$-\left[\frac{x_1(n_{1,z}p_{1k,x} - n_{1,x}p_{1k,z}) + y_1(n_{1,z}p_{1k,y} - n_{1,y}p_{1k,z})}{v_1 n_{1,z}}\right]$$

$$+\sum_{i=2}^{k-1}\frac{x_i}{n_{i,z}}\left\{\left[\frac{n_{i,z}p_{i-1,k,x} - n_{i,x}p_{i-1,k,z}}{v_{i-1}}\right] - \left[\frac{n_{i,z}p_{ik,x} - n_{i,x}p_{ik,z}}{v_i}\right]\right\}$$

$$-\sum_{i=2}^{k-1}\frac{y_i}{n_{i,z}}\left\{\left[\frac{n_{i,y}p_{i-1,k,z} - n_{i,z}p_{i-1,k,y}}{v_{i-1}}\right] - \left[\frac{n_{i,y}p_{ik,z} - n_{i,z}p_{ik,y}}{v_i}\right]\right\}.$$

At interfaces 2,3,...,k-1 in the overburden, Snell's law of refraction must apply. Equations (8a,b) then imply that all terms in the summations involving x_i and y_i vanish! The downgoing traveltime from $(x_1,y_1) = (x_S,y_S)$ to $(x_k,y_k) = (x_P,y_P)$ reduces to

$$T_{down} = \sum_{i=1}^{k-1} \frac{h_i(0,0)p_{ik,z}}{v_i} + \left[\frac{x_P(n_{k,z}p_{k-1,k,x} - n_{k,x}p_{k-1,k,z}) + y_P(n_{k,z}p_{k-1,k,y} - n_{k,y}p_{k-1,k,z})}{v_{k-1}n_{k,z}}\right]$$

$$-\left[\frac{x_S(n_{1,z}p_{1k,x} - n_{1,x}p_{1k,z}) + y_S(n_{1,z}p_{1k,y} - n_{1,y}p_{1k,z})}{v_1 n_{1,z}}\right]. \qquad (10)$$

Upgoing Traveltime

The traveltime along the upward propagating portion of the total raypath is derived by similar techniques. Snell's law in form (9) is used at each interface in the overburden. The result is

$$T_{up} = -\sum_{i=1}^{k-1} \frac{h_i(0,0)q_{ik,z}}{v_i} - \left[\frac{x_Q(n_{k,z}q_{k-1,k,x} - n_{k,x}q_{k-1,k,z}) + y_Q(n_{k,z}q_{k-1,k,y} - n_{k,y}q_{k-1,k,z})}{v_{k-1}n_{k,z}}\right]$$

$$+\left[\frac{x_R(n_{1,z}q_{1k,x} - n_{1,x}q_{1k,z}) + y_R(n_{1,z}q_{1k,y} - n_{1,y}q_{1k,z})}{v_1 n_{1,z}}\right]. \qquad (11)$$

Equations (10) and (11) give one-way transmission times of a wave propagating through a stack of homogeneous and isotropic layers with plane interfaces. Mode conversions are allowed at the layer boundaries. Hence, these expressions (and their generalizations to equations (19) and (20) below) could be used for studying propagation times of seismic waves through layered crustal structure to or from remote (teleseismic) sites.

Critically Refracted Traveltime

The final traveltime increment needed for the derivation corresponds to the critically refracted segment of the total raypath. Let position vectors $\mathbf{r}_P$ and $\mathbf{r}_Q$ refer to the intersection points of the downgoing and upgoing portions of the raypath with interface k, respectively. The critically refracted segment is a straight line connecting these two points (and thus lying entirely within the plane of interface k). Then $\mathbf{r}_Q = \mathbf{r}_P + l_k\, \mathbf{p}_{kk}$ and hence $l_k = \mathbf{p}_{kk} \cdot (\mathbf{r}_Q - \mathbf{r}_P)$. The propagation time along this path length is

$$t_k = \frac{l_k}{v_k} = \frac{p_{kk,x}}{v_k}(x_Q - x_P) + \frac{p_{kk,y}}{v_k}(y_Q - y_P) + \frac{p_{kk,z}}{v_k}(z_Q - z_P) \; .$$

Since points P and Q reside on the same plane interface, equation (4) gives

$$z_Q - z_P = z_k(x_Q, y_Q) - z_k(x_P, y_P) = -(x_Q - x_P)\left[\frac{n_{k,x}}{n_{k,z}}\right] - (y_Q - y_P)\left[\frac{n_{k,y}}{n_{k,z}}\right].$$

The expression for the critically refracted traveltime increment then reduces to

$$T_{crit} \equiv t_k = \left[\frac{(x_Q - x_P)(n_{k,z}p_{kk,x} - n_{k,x}p_{kk,z}) + (y_Q - y_P)(n_{k,z}p_{kk,y} - n_{k,y}p_{kk,z})}{v_k n_{k,z}}\right]. \qquad (12)$$

Total Traveltime

The total surface-to-surface traveltime of the wave critically refracted on interface k is obtained by adding the traveltime contributions of the downgoing, critically refracted, and upgoing raypath portions: $T_{total} = T_{down} + T_{crit} + T_{up}$. Summing equations (10), (11), and (12) yields

$$T_k(x_S, y_S, x_R, y_R) = \sum_{i=1}^{k-1} \frac{h_i(0,0)(p_{ik,z} - q_{ik,z})}{v_i} + \left[\frac{x_R(n_{1,z}q_{1k,x} - n_{1,x}q_{1k,z}) + y_R(n_{1,z}q_{1k,y} - n_{1,y}q_{1k,z})}{v_1 n_{1,z}}\right]$$

$$- \left[\frac{x_S(n_{1,z}p_{1k,x} - n_{1,x}p_{1k,z}) + y_S(n_{1,z}p_{1k,y} - n_{1,y}p_{1k,z})}{v_1 n_{1,z}}\right]$$

$$+ F(x_P, y_P, x_Q, y_Q).$$

The quantity F depends on the horizontal coordinates of the two points of critical refraction, and is given by

$$F(x_P, y_P, x_Q, y_q) = \frac{x_P}{n_{k,z}}\left[\frac{1}{v_{k-1}}(n_{k,z}p_{k-1,k,x} - n_{k,x}p_{k-1,k,z}) - \frac{1}{v_k}(n_{k,z}p_{kk,x} - n_{k,x}p_{kk,z})\right]$$

$$-\frac{y_P}{n_{k,z}}\left[\frac{1}{v_{k-1}}(n_{k,y}p_{k-1,k,z} - n_{k,z}p_{k-1,k,y}) - \frac{1}{v_k}(n_{k,y}p_{kk,z} - n_{k,z}p_{kk,y})\right]$$

$$-\frac{x_Q}{n_{k,z}}\left[\frac{1}{v_{k-1}}(n_{k,z}q_{k-1,k,x} - n_{k,x}q_{k-1,k,z}) - \frac{1}{v_k}(n_{k,z}p_{kk,x} - n_{k,x}p_{kk,z})\right]$$

$$+\frac{y_Q}{n_{k,z}}\left[\frac{1}{v_{k-1}}(n_{k,y}q_{k-1,k,z} - n_{k,z}q_{k-1,k,y}) - \frac{1}{v_k}(n_{k,y}p_{kk,z} - n_{k,z}p_{kk,y})\right].$$

Since the wave is critically refracted at interface k, the propagation directions $\mathbf{p}_{kk}$ and $\mathbf{q}_{kk}$ are identical. Then, requiring Snells's law [equations (8a,b)] to be satisfied at points P and Q results in $F = 0$. The final formula for surface-to-surface head wave traveltime thus becomes

$$T_k(x_S, y_S, x_R, y_R) = \sum_{i=1}^{k-1}\frac{h_i(0,0)(p_{ik,z} - q_{ik,z})}{v_i} + \left[\frac{x_R(n_{1,z}q_{1k,x} - n_{1,x}q_{1k,z}) + y_R(n_{1,z}q_{1k,y} - n_{1,y}q_{1k,z})}{v_1 n_{1,z}}\right]$$

$$-\left[\frac{x_S(n_{1,z}p_{1k,x} - n_{1,x}p_{1k,z}) + y_S(n_{1,z}p_{1k,y} - n_{1,y}p_{1k,z})}{v_1 n_{1,z}}\right]. \qquad (13)$$

This completes the derivation. Although equation (13) conveys an impression that head wave traveltime depends explicitly on the source and receiver position coordinates, it should be emphasized that the raypath vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$ *also* depend on the recording geometry. Hence, there is implicit dependence on (x_S, y_S) and (x_R, y_R) as well. This issue is discussed more fully in the section regarding numerical computation of traveltimes.

Variants of the Basic Formula

Obviously, if the source or receiver is located at the coordinate origin, then the traveltime formula (13) simplifies considerably. Another simplification arises with a horizontal surface ($n_{1,x} = n_{1,y} = 0$, $n_{1,z} = 1$); expression (13) reduces to

$$T_k(x_S, y_S, x_R, y_R) = \sum_{i=1}^{k-1} \frac{h_i(0,0)(p_{ik,z} - q_{ik,z})}{v_i} + \left[\frac{(x_R q_{1k,x} + y_R q_{1k,y}) - (x_S p_{1k,x} + y_S p_{1k,y})}{v_1} \right]. \quad (14)$$

This is equivalent to the surface-to-surface traveltime formulae given by Diebold (1987) and Richards *et al.* (1991).

Conventionally, seismic refraction traveltime is expressed as a function of the source-receiver offset distance. The current equation is easily converted to this form by specifying the receiver position in terms of an offset distance X ($X \geq 0$) and an azimuth angle Ψ ($0 \leq \Psi < 2\pi$) relative to the source. The receiver coordinates are given by

$$x_R = x_S + X \cos\Psi, \qquad\qquad y_R = y_S + X \sin\Psi.$$

Substituting these expressions into equation (13) yields

$$T_k(x_S, y_S, X, \Psi) = \sum_{i=1}^{k-1} \frac{h_i(0,0)(p_{ik,z} - q_{ik,z})}{v_i}$$

$$- x_S \left[\frac{n_{1,z}(p_{1k,x} - q_{1k,x}) - n_{1,x}(p_{1k,z} - q_{1k,z})}{v_1 n_{1,z}} \right] - y_S \left[\frac{n_{1,z}(p_{1k,y} - q_{1k,y}) - n_{1,y}(p_{1k,z} - q_{1k,z})}{v_1 n_{1,z}} \right]$$

$$+ X \left[\frac{\cos\Psi(n_{1,z} q_{1k,x} - n_{1,x} q_{1k,z}) + \sin\Psi(n_{1,z} q_{1k,y} - n_{1,y} q_{1k,z})}{v_1 n_{1,z}} \right]. \quad (15)$$

Note that X is the horizontal distance between source and receiver; the actual distance may be larger since it is measured within the plane of interface 1. It is straightforward to demonstrate that the true source-receiver distance is $L = X\sqrt{1+\tan^2\phi_1 \cos^2(\Psi - \theta_1)}$, where ϕ_1 and θ_1 are the dip and azimuth angles of the surface.

Equation (15) is an extension of the common "slope and intercept" head wave traveltime formula to 3D multilayered earth models. For the particular case of a model consisting of only two layers and a horizontal surface, it can be shown that (15) reduces to a simple closed-form expression derived by Aldridge (1989). This serves as an important check on the validity of the general formula. The proof entails some cumbersome algebra, and thus is not reproduced here; mathematical details are contained in Aldridge (1992).

GENERALIZATIONS OF THE TRAVELTIME FORMULA

Heretofore, both the source and the receiver have been restricted to the surface. More versatile formulae are needed to model data acquisition geometries with buried sources and/or receivers. These situations arise in surface-to-borehole, borehole-to-surface, and borehole-to-borehole seismic experiments, as well as with placement of sources and/or receivers in underground mines, tunnels, on the seabed, etc.

Source and Receiver on Separate Interfaces

Let the source S be located on the j^{th} interface with $1 \le j < k$. The downgoing traveltime is obtained by summing the layer traveltime increments t_i, $i = j, j+1, \ldots, k-1$. Equation (10) retains its form except that the index 1 is replaced by j throughout. Similarly, if the receiver R is located on the l^{th} interface ($1 \le l < k$), then the upgoing traveltime is given by equation (11) with the index 1 replaced by l. The critically refracted traveltime increment is still given by (12). Summing these three components of the total traveltime gives

$$T_k(x_S, y_S, z_S, x_R, y_R, z_R) = \sum_{i=j}^{k-1} \frac{h_i(0,0)p_{ik,z}}{v_i} - \sum_{i=l}^{k-1} \frac{h_i(0,0)q_{ik,z}}{v_i}$$

$$+\left[\frac{x_R(n_{l,z}q_{lk,x} - n_{l,x}q_{lk,z}) + y_R(n_{l,z}q_{lk,y} - n_{l,y}q_{lk,z})}{v_l n_{l,z}}\right]$$

$$-\left[\frac{x_S(n_{j,z}p_{jk,x} - n_{j,x}p_{jk,z}) + y_S(n_{j,z}p_{jk,y} - n_{j,y}p_{jk,z})}{v_j n_{j,z}}\right], \qquad (16)$$

where the vertical coordinates of source and receiver are $z_S = z_j(x_S,y_S)$ and $z_R = z_l(x_R,y_R)$. This is the proper expression for head wave traveltime when source and receiver are located on different interfaces of the model. It differs significantly from the analogous formula published by Diebold (1987) and Richards *et al.* (1991). Their expression is actually a special case of the general equation (16); in particular, it is valid only if *both* the source and receiver interfaces are horizontal ($n_{j,x} = n_{j,y} = n_{l,x} = n_{l,y} = 0$, $n_{j,z} = n_{l,z} = 1$). The difference between these two formulae is clearly revealed by examining an earth model for which head wave traveltime can be derived by independent techniques. The analysis demonstrates that (16) reduces to the known traveltime solution in this situation, whereas Diebold's equation (21) yields an erroneous result (Aldridge 1992, Appendix C).

Arbitrary Source and Receiver Locations

A further generalization is obtained by allowing source and receiver to be located within designated layers. Assume that the source is located in layer j at a vertical depth d_S below the immediately overlying interface (the j^{th}). Similarly, let the receiver be located within layer l at a depth d_R beneath interface l. These incremental source and receiver depths must satisfy $0 \le d_S \le h_j(x_S,y_S)$ and $0 \le d_R \le h_l(x_R,y_R)$, respectively. The previously developed techniques can be used to derive the head wave traveltime for this situation. Traveltime increments induced by the source layer j on the downward path and the receiver layer l on the upward path must be treated separately, because the wave does not propagate across the full thickness of each layer. The result of the analysis is

$$T_k(x_S, y_S, z_S, x_R, y_R, z_R) = \sum_{i=j}^{k-1} \frac{h_i(0,0)p_{ik,z}}{v_i} - \frac{d_S p_{jk,z}}{v_j} - \sum_{i=l}^{k-1} \frac{h_i(0,0)q_{ik,z}}{v_i} + \frac{d_R q_{lk,z}}{v_l}$$

$$+ \left[\frac{x_R(n_{l,z}q_{lk,x} - n_{l,x}q_{lk,z}) + y_R(n_{l,z}q_{lk,y} - n_{l,y}q_{lk,z})}{v_l n_{l,z}} \right]$$

$$- \left[\frac{x_S(n_{j,z}p_{jk,x} - n_{j,x}p_{jk,z}) + y_S(n_{j,z}p_{jk,y} - n_{j,y}p_{jk,z})}{v_j n_{j,z}} \right], \qquad (17)$$

where source and receiver depths are now $z_S = z_j(x_S,y_S) + d_S$ and $z_R = z_l(x_R,y_R) + d_R$. Note that the prior expression (16) is recovered in the limit as $d_S \to 0$ and $d_R \to 0$, as expected. Moreover, it is possible to demonstrate that (17) reduces to the proper form when source and/or receiver approach the basal interfaces of their respective layers, that is $d_S \rightarrow h_j(x_S, y_S)$ and/or $d_R \rightarrow h_l(x_R, y_R)$ (Aldridge 1992).

An additional benefit accrues from separating the downward and upward sums in the traveltime formule: asymmetric wave propagation paths can be treated. An asymmetric raypath is defined as one where the mode of *downgoing* wave propagation in the i^{th} layer differs from the mode of *upgoing* wave propagation across the same layer. Strictly, different symbols should be used to designate the wave speeds within layer i in the downward and upward sums (e.g., v_i^d and v_i^u for the velocities of the downgoing and upgoing waves, respectively). However, this complication is avoided for the time being in order to maintain notational simplicity. The velocity v_i appearing in each sum is simply interpreted as the propagation speed of the desired mode (P or S) across layer i. Equation (17) then constitutes a general formula for point-to-point traveltimes of head waves propagating in a 3D layered earth model.

Arbitrary Reference Points for Layer Thickness

Individual layer thicknesses enter the traveltime expressions evaluated at the coordinate origin O. An alternate form of the traveltime equation is characterized by layer thicknesses specified below the source and receiver. This variant is particularly suitable for the time term, delay time, and reciprocal time inversion methods. Hence, the previous derivation is now modified to incorporate layer thicknesses prescribed at *arbitrary* reference locations; these points can then be specialized to the source and receiver positions. The resulting traveltime expression forms the point of departure for a 3D extension of the aforementioned inversion techniques.

The depth of the i^{th} interface, referred to an arbitrary location A with horizontal coordinates (x_A, y_A) is

$$z_i(x,y) = z_i(x_A, y_A) - (x - x_A)\left[\frac{n_{i,x}}{n_{i,z}}\right] - (y - y_A)\left[\frac{n_{i,y}}{n_{i,z}}\right]. \qquad (18)$$

Consider the downgoing raypath first. The previous expression for the traveltime increment t_i induced by wave propagation across the i^{th} layer is modified to

$$t_i = \frac{p_{ik,x}}{v_i}\left[(x_{i+1} - x_A) - (x_i - x_A)\right] + \frac{p_{ik,y}}{v_i}\left[(y_{i+1} - y_A) - (y_i - y_A)\right] + \frac{p_{ik,z}}{v_i}(z_{i+1} - z_i).$$

The vertical (z) coordinates of the ray intersection points are expressed in terms of the corresponding horizontal coordinates via equation (18):

$$z_{i+1} - z_i = h_i(x_A, y_A) + (x_i - x_A)\left[\frac{n_{i,x}}{n_{i,z}}\right] + (y_i - y_A)\left[\frac{n_{i,y}}{n_{i,z}}\right] - (x_{i+1} - x_A)\left[\frac{n_{i+1,x}}{n_{i+1,z}}\right] - (y_{i+1} - y_A)\left[\frac{n_{i+1,y}}{n_{i+1,z}}\right],$$

where $h_i(x_A,y_A) = z_{i+1}(x_A,y_A) - z_i(x_A,y_A)$ is the vertical thickness of layer i below reference point A. Substituting this expression into the equation for the traveltime increment t_i gives

$$t_i = \frac{h_i(x_A, y_A)p_{ik,z}}{v_i}$$

$$+ \frac{(x_{i+1} - x_A)}{n_{i+1,z}}\left[\frac{n_{i+1,z}p_{ik,x} - n_{i+1,x}p_{ik,z}}{v_i}\right] + \frac{(y_{i+1} - y_A)}{n_{i+1,z}}\left[\frac{n_{i+1,z}p_{ik,y} - n_{i+1,y}p_{ik,z}}{v_i}\right]$$

$$- \frac{(x_i - x_A)}{n_{i,z}}\left[\frac{n_{i,z}p_{ik,x} - n_{i,x}p_{ik,z}}{v_i}\right] - \frac{(y_i - y_A)}{n_{i,z}}\left[\frac{n_{i,z}p_{ik,y} - n_{i,y}p_{ik,z}}{v_i}\right].$$

The traveltime increments t_i are now summed over all layers in the overburden (assumed here to include all layers from the surface ($i = 1$) down to the critical refractor). Applying Snell's law at each plane interface results in

$$T_{down} = \sum_{i=1}^{k-1} \frac{h_i(x_A, y_A)p_{ik,z}}{v_i}$$

$$+ \left[\frac{(x_P - x_A)(n_{k,z}p_{k-1,k,x} - n_{k,x}p_{k-1,k,z}) + (y_P - y_A)(n_{k,z}p_{k-1,k,y} - n_{k,y}p_{k-1,k,z})}{v_{k-1}n_{k,z}}\right]$$

$$- \left[\frac{(x_S - x_A)(n_{1,z}p_{1k,x} - n_{1,x}p_{1k,z}) + (y_S - y_A)(n_{1,z}p_{1k,y} - n_{1,y}p_{1k,z})}{v_1 n_{1,z}}\right]. \qquad (19)$$

A similar analysis yields the upgoing traveltime. Layer thicknesses are now referred to a *different* arbitrary position B with horizontal coordinates (x_B, y_B):

$$T_{up} = -\sum_{i=1}^{k-1} \frac{h_i(x_B, y_B) q_{ik,z}}{v_i}$$

$$-\left[\frac{(x_Q - x_B)(n_{k,z} q_{k-1,k,x} - n_{k,x} q_{k-1,k,z}) + (y_Q - y_B)(n_{k,z} q_{k-1,k,y} - n_{k,y} q_{k-1,k,z})}{v_{k-1} n_{k,z}}\right]$$

$$+\left[\frac{(x_R - x_B)(n_{1,z} q_{1k,x} - n_{1,x} q_{1k,z}) + (y_R - y_B)(n_{1,z} q_{1k,y} - n_{1,y} q_{1k,z})}{v_1 n_{1,z}}\right]. \qquad (20)$$

As indicated previously, equations (19) and (20) can be exploited to study one-way transmission times of waves propagating through layered media. Finally, the critically refracted traveltime increment is given by a simple alteration to the previous equation (12):

$$T_{crit} = \frac{\left[(x_Q - x_B) - (x_P - x_A)\right](n_{k,z} p_{kk,x} - n_{k,x} p_{kk,z})}{v_k n_{k,z}}$$

$$+ \frac{\left[(y_Q - y_B) - (y_P - y_A)\right](n_{k,z} p_{kk,y} - n_{k,y} p_{kk,z})}{v_k n_{k,z}}$$

$$-\left[\frac{(x_A - x_B)(n_{k,z} p_{kk,x} - n_{k,x} p_{kk,z}) + (y_A - y_B)(n_{k,z} p_{kk,y} - n_{k,y} p_{kk,z})}{v_k n_{k,z}}\right]. \qquad (21)$$

The total head wave traveltime is obtained in the usual manner by summing the contributions along all three raypath portions. Adding expressions (19), (20), and (21) yields

$$T_k(x_S,y_S,x_R,y_R)=\sum_{i=1}^{k-1}\frac{h_i(x_A,y_A)p_{ik,z}-h_i(x_B,y_B)q_{ik,z}}{v_i}$$

$$+\left[\frac{(x_R-x_B)(n_{1,z}q_{1k,x}-n_{1,x}q_{1k,z})+(y_R-y_B)(n_{1,z}q_{1k,y}-n_{1,y}q_{1k,z})}{v_1n_{1,z}}\right]$$

$$-\left[\frac{(x_S-x_A)(n_{1,z}p_{1k,x}-n_{1,x}p_{1k,z})+(y_S-y_A)(n_{1,z}p_{1k,y}-n_{1,y}p_{1k,z})}{v_1n_{1,z}}\right]$$

$$+\left[\frac{(x_B-x_A)(n_{k,z}p_{kk,x}-n_{k,x}p_{kk,z})+(y_B-y_A)(n_{k,z}p_{kk,y}-n_{k,y}p_{kk,z})}{v_kn_{k,z}}\right]$$

$$+F(x_P-x_A,y_P-y_A,x_Q-x_B,y_Q-y_B), \qquad (22)$$

where the function F has been previously defined. Once again, applying Snell's law at the critically refracting horizon demonstrates that F vanishes. Note that the above expression reduces to the previous equation (13) if the reference points A and B are both situated at the coordinate origin O. Furthermore, it is common (but not mandatory) practice to select points A and B to coincide with the source S and receiver R, respectively. With these substitutions, the above expression simplifies dramatically to

$$T_k(x_S,y_S,x_R,y_R)=\sum_{i=1}^{k-1}\frac{h_i(x_S,y_S)p_{ik,z}-h_i(x_R,y_R)q_{ik,z}}{v_i}$$

$$+\left[\frac{(x_R-x_S)(n_{k,z}p_{kk,x}-n_{k,x}p_{kk,z})+(y_R-y_S)(n_{k,z}p_{kk,y}-n_{k,y}p_{kk,z})}{v_kn_{k,z}}\right]. \qquad (23)$$

This is the desired result. Since vertical layer thicknesses beneath source and receiver enter the expression, equation (23) forms the logical point of departure for extending the time-term, delay time, and reciprocal time inversion approaches to 3D layered models.

Reflection Traveltime

The previous analysis has concentrated on waves that are critically refracted at the k^{th} subsurface horizon. However, it is straightforward to demonstrate that the traveltime formulae also apply to waves that are *reflected* from interface k of the model. Snell's law of reflection for this situation is illustrated in Figure 4 and is compactly expressed as

$$\frac{\mathbf{n}_k \times \mathbf{p}_{k-1,k}}{v^d_{k-1}} = \frac{\mathbf{n}_k \times \mathbf{q}_{k-1,k}}{v^u_{k-1}}. \tag{24}$$

Note that equation (24) allows for a possible mode conversion upon reflection, i.e., v^d_{k-1} is not necessarily equal to v^u_{k-1}.

For a wave reflected at interface k, points P and Q are coincident; there is no intervening critically refracted raypath segment. The total traveltime is obtained simply by adding the downgoing and upgoing components. Hence, setting $(x_Q,y_Q) = (x_P,y_P)$ and summing equations (10) and (11) yields

$$T_{down} + T_{up} = \sum_{i=j}^{k-1} \frac{h_i(0,0)p_{ik,z}}{v^d_i} - \sum_{i=l}^{k-1} \frac{h_i(0,0)q_{ik,z}}{v^u_i}$$

$$+ \left[\frac{x_R(n_{l,z}q_{lk,x} - n_{l,x}q_{lk,z}) + y_R(n_{l,z}q_{lk,y} - n_{l,y}q_{lk,z})}{v^u_l n_{l,z}} \right]$$

$$- \left[\frac{x_S(n_{j,z}p_{jk,x} - n_{j,x}p_{jk,z}) + y_S(n_{j,z}p_{jk,y} - n_{j,y}p_{jk,z})}{v^d_j n_{j,z}} \right] + G(x_P,y_P), \tag{25}$$

where the quantity $G(x_P,y_P)$ depends on the horizontal coordinates of the reflection point and is given by

$$G(x_P,y_P) = \frac{x_P}{n_{k,z}} \left[\frac{1}{v^d_{k-1}}(n_{k,z}p_{k-1,k,x} - n_{k,x}p_{k-1,k,z}) - \frac{1}{v^u_{k-1}}(n_{k,z}q_{k-1,k,x} - n_{k,x}q_{k-1,k,z}) \right]$$

$$+ \frac{y_P}{n_{k,z}} \left[\frac{1}{v^d_{k-1}}(n_{k,z}p_{k-1,k,y} - n_{k,y}p_{k-1,k,z}) - \frac{1}{v^u_{k-1}}(n_{k,z}q_{k-1,k,y} - n_{k,y}q_{k-1,k,z}) \right].$$

Distinct downgoing and upgoing layer velocities are incorporated into the above expressions in order to emphasize the possibility of asymmetric mode-converted raypaths. Also, source and receiver are assumed to be located on different interfaces. The component form of Snell's law of reflection implies that G vanishes. Equation (25) is then identical in form to the previous traveltime expression (16) that was derived for head waves!

SPECIALIZATION TO TWO DIMENSIONS

Specializing the above traveltime formulae to two spatial dimensions provides useful checks on the correctness of the derivations. These 2D equations underpin several head wave inversion techniques in current use. In the 2D layered earth model, all of the y-components of the interface normal vectors vanish (equivalently, all interface azimuth angles are restricted to $\theta_i = 0$ or $\theta_i = \pi$). Additionally, the recording profile must be oriented perpendicular to the strike directions of the subsurface horizons. Hence, the azimuth $\Psi = 0$ or $\Psi = \pi$ as well. If this second condition is not satisfied, the unit wave propagation vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$ are not confined to the xz plane, and a full 3D treatment is necessary.

Since many previous investigators have assumed a horizontal surface, equation (14) is used as the point of departure for the analysis. Setting $y_S = y_R = 0$ gives

$$T_k(x_S, x_R) = \sum_{i=1}^{k-1} \frac{h_i(0,0)(p_{ik,z} - q_{ik,z})}{v_i} + \left[\frac{x_R q_{1k,x} - x_S p_{1k,x}}{v_1} \right], \tag{26}$$

This expression is compatible with analogous equations developed by Diebold & Stoffa (1981) and Diebold (1987). If the source is located at the coordinate origin, then (26) is also consistent with earlier 2D head wave traveltime formulae published by Dooley (1952), Adachi (1954), Ocola (1972), and Johnson (1976). All of these investigators prescribe vertical layer thicknesses, either beneath the origin or the shotpoint. In contrast, Ewing *et al.* (1939) and Mota (1954) measure thickness normal to the basal interface bounding a layer. Hence, their traveltime equations, although designed to treat an equivalent situation, differ in mathematical detail.

Specializing the traveltime variant (23) above to 2D yields a new and advantageous expression on which the generalized reciprocal method (GRM) of refraction analysis can be based. The GRM is a technique for delineating a subsurface horizon from head wave arrivals recorded by inline forward and reverse profiles (Palmer 1980, 1981).

In the 3D earth model, interface dip is characterized by a positive angle ϕ_i ($0 \le \phi_i < \pi/2$). For subsequent analysis of the 2D refraction problem, it is convenient to reparameterize the model in terms of interface dip angles that may be positive *or* negative. Hence, let the symbol φ_i refer to an interface dip in a 2D earth model. φ_i is an acute angle ($0 \leq |\varphi_i| < \pi/2$) measured with respect to the $+x$ axis; it is considered positive (negative) if the angle opens in the clockwise (counterclockwise) sense. This angle is related to the 3D interface orientation angles via $\varphi_i = (-\cos\theta_i)\phi_i$, where the azimuth θ_i is restricted to the two values 0 or π. Then, familiar 2D expressions for the interface unit normal vector, interface depth, and layer thickness in terms of the 2D dip angle are

$$\mathbf{n}_i = (-\sin\varphi_i)\mathbf{i} + (\cos\varphi_i)\mathbf{k},$$

$$z_i(x) = z_i(0) + x\tan\varphi_i,$$

$$h_i(x) = h_i(0) + x\left[\frac{\sin(\varphi_{i+1} - \varphi_i)}{\cos\varphi_{i+1}\cos\varphi_i}\right],$$

respectively. Note that $\mathbf{n}_i$ has no y-component, and $z_i(x)$ and $h_i(x)$ are independent of the y-coordinate.

The head wave traveltime formula relevant to GRM analysis is equation (23). This is specialized to a 2D model by setting $y_S = y_R = 0$, $n_{k,x} = -\sin\varphi_k$, and $n_{k,z} = \cos\varphi_k$. Equation (23) then becomes

$$T_k(x_S, x_R) = \sum_{i=1}^{k-1} \frac{h_i(x_S)p_{ik,z} - h_i(x_R)q_{ik,z}}{v_i} + \left[\frac{(x_R - x_S)(p_{kk,x}\cos\varphi_k + p_{kk,z}\sin\varphi_k)}{v_k\cos\varphi_k}\right]. \qquad (27)$$

Next, the 2D unit propagation vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$ are expressed in terms of raypath orientation angles as follows:

$$\mathbf{p}_{ik} = (\pm\sin\alpha_{ik})\mathbf{i} + (\cos\alpha_{ik})\mathbf{k}, \qquad \mathbf{q}_{ik} = (\pm\sin\beta_{ik})\mathbf{i} + (\cos\beta_{ik})\mathbf{k}. \qquad (28a,b)$$

Angles α_{ik} and β_{ik} are polar angles measured from the $+z$ axis ($0 \leq \alpha_{ik}, \beta_{ik} \leq \pi$). The propagation direction of the critically refracted raypath segment is obtained by straightforward geometric analysis:

$$\mathbf{p}_{kk} = \pm[(\cos\varphi_k)\mathbf{i} + (\sin\varphi_k)\mathbf{k}]. \qquad (28c)$$

In the above expressions, plus signs are used for $x_R > x_S$ and negative signs are used for $x_R < x_S$. Substituting (28a,b,c) into (27) yields the remarkably simple result

$$T_k(x_S, x_R) = \sum_{i=1}^{k-1} \frac{h_i(x_S)\cos\alpha_{ik} - h_i(x_R)\cos\beta_{ik}}{v_i} + \frac{|x_S - x_R|}{v_k \cos\varphi_k}. \quad (29)$$

This is a novel expression for 2D surface-to-surface head wave traveltime that forms the basis for an alternative development of the GRM. It differs from the analogous equation given by Palmer (1980, 1981) in several important ways: (i) layers are characterized by vertical thicknesses below source and receiver, (ii) raypath orientation angles are measured with respect to the vertical, (iii) the coefficient multiplying the source-receiver offset distance depends only on critical refractor quantities (i.e., velocity v_k and dip φ_k), and (iv) the earth's surface may be nonhorizontal. These attributes facilitate a straightforward derivation of the GRM analysis tools (Aldridge 1992). Perhaps more importantly, GRM inversion of refraction arrival times utilizing equation (29) yields *point depth estimates* of a critically refracting horizon. Construction of the refractor depth profile then reduces to an interpolation problem. In contrast, Palmer's time-depth function yields a circular locus of possible refractor positions. Depth profile calculation then involves the more complicated task of constructing an envelope to a set of circular arcs with varying radii.

The raypath angles in equation (29) depend on the recording profile azimuth Ψ, which in turn is restricted to the two angles 0 and π. However, raypath reciprocity requires that $\alpha_{ik}(\pi) = \pi - \beta_{ik}(0)$ and $\beta_{ik}(\pi) = \pi - \alpha_{ik}(0)$, implying $\cos\alpha_{ik}(\pi) = -\cos\beta_{ik}(0)$ and $\cos\beta_{ik}(\pi) = -\cos\alpha_{ik}(0)$. Using these relations, it is easy to demonstrate the the traveltime formula (29) satisfies source-receiver reciprocity: $T_k(x_S, x_R) = T_k(x_R, x_S)$, as expected.

RAPID TRAVELTIME COMPUTATION

In order to compute traveltimes via the above formulae, the unit propagation vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$ overlying the refracting/reflecting interface must be determined. For the reflection problem, this set of vectors depends on both the offset distance X and the azimuth angle Ψ of the receiver relative to the source. However, the critical refraction problem is qualitatively different; the propagation vectors depend *only* on the azimuth Ψ. This particular feature can be exploited to yield a rapid computational procedure for head wave traveltimes.

Since the propagation vectors depend on the recording azimuth, they should be written as $\mathbf{p}_{ik}(\Psi)$ and $\mathbf{q}_{ik}(\Psi)$, although explicit dependence on Ψ is often suppressed for notational convenience. The functional form of this dependence is not known. However, with a minimal amount of raytracing, it is possible to numerically generate the function $\Psi(\mathbf{p}_{ik},\mathbf{q}_{ik})$ over the full range of possible recording azimuths (2π radians). Inversion of this function then yields the propagation vectors for a prescribed value of the source-receiver azimuth angle. This technique is discussed in general terms in the following subsection.

Algorithm Description

The computational algorithm is based on the close relationship between a critically reflected and critically refracted raypath. For a given source-receiver azimuth angle, both raypath types possess the same unit propagation vectors $\mathbf{p}_{ik}(\Psi)$ and $\mathbf{q}_{ik}(\Psi)$ for $i < k$. Consider the following six-step calculation procedure, with reference to the critically refracted/reflected raypath segments depicted in Figure 5:

1) Select a point P on the critically refracting interface. Since the location of P is arbitrary, it is convenient to position it beneath the coordinate origin O.

2) Choose a critically refracted propagation direction $\mathbf{p}_{kk}$ through point P. The orientation of this vector within the plane of interface k is defined by an angle χ measured from an arbitrary reference line.

3) Forward raytrace from P along the set of upward unit propagation vectors $\mathbf{q}_{ik}$ ($i = k$-1,k-2,...,2,1) to establish the position of a receiving point R on the surface. The departing vector $\mathbf{q}_{k-1,k}$ is oriented at the critical angle $i_k = \sin^{-1}(v_{k-1}/v_k)$ relative to the interface normal, and is contained in the plane defined by $\mathbf{p}_{kk}$ and $\mathbf{n}_k$. The appropriate 3D form of Snell's law is applied at all interfaces intervening between the refractor and the surface.

4) Reverse raytrace from P along the set of downward unit propagation vectors $\mathbf{p}_{ik}$ ($i = k$-1,k-2,...,2,1) to establish the position of a source point S on the surface. The incidence angle of the arriving vector $\mathbf{p}_{k-1,k}$ (in the plane defined by $\mathbf{p}_{kk}$ and $\mathbf{n}_k$) also equals the critical angle. A 3D "backward propagating" form of Snell's law is applied at all refracting interfaces.

5) Calculate the azimuth angle Ψ of R with respect to S.

6) Increment angle χ by a small amount and repeat steps 1) through 5). Stop after χ has been incremented by a total of π radians.

This procedure numerically defines a function $\Psi = f(\chi)$ over an interval $[\chi_0, \chi_0+\pi]$. It is not necessary to perform raytracing to determine the azimuth angle for $\chi \in [\chi_0+\pi, \chi_0+2\pi]$. Rather, values are easily generated from the symmetry relation $\Psi = f(\chi-\pi) + \pi$. This symmetry condition arises from raypath reciprocity: reversing the direction of the critically refracted propagation vector $\mathbf{p}_{kk}$ merely interchanges the positions of the source and receiver on the surface.

If the propagation vector $\mathbf{p}_{kk}$ makes one complete rotation on the critically refracting interface (i.e., χ increments by 2π radians), points R and S make one complete closed circuit on the surface (i.e., Ψ also increments by 2π). This functional dependence is designated $\Psi = g(\chi)$. This inverse function $\chi = g^{-1}(\Psi)$

can then be used to determine the appropriate value of χ for a specified source-receiver azimuth angle. Finally, unit propagation vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$ corresponding to this value of χ are regenerated via points 3) and 4) above.

Note that the function $\Psi = g(\chi)$ needs to be calculated only once for a given critical refractor in an earth model. All recording azimuths contained in the data acquisition geometry are treated by this same function.

Rapid raytracing through a set of homogeneous and isotropic layers is readily achieved using the formulae in Shah (1973); mathematical details are described more fully in Aldridge (1992).

MODELING EXAMPLES

The two examples presented in this section illustrate the utility, as well as some of the limitations, of the head wave traveltime formulae for forward modeling applications. The recording geometry for the first example is the common reversed profile; all sources and receivers are located on a horizontal surface. The second example considers a typical offset VSP geometry (surface source and downhole receivers).

Profile Geometry

Equation (15) expresses head wave traveltime in terms of the horizontal offset distance between a surface source and a surface receiver. It is written in condensed form as

$$T_k(x_S, y_S, X, \Psi) = m_k(\Psi)X + b_k(x_S, y_S, \Psi), \tag{30}$$

where the definitions of the slope $m_k(\Psi)$ and intercept $b_k(x_S, y_S, \Psi)$ are obvious. This expression is a 3D extension of the slope/intercept formulae that are commonly used to describe head wave traveltimes. Equation (30) is evaluated for a shallow three-layer model defined by the parameters

$$\phi_1 = 0^\circ, \quad \theta_1 = 0^\circ, \quad z_1(0,0) = 0\,\text{m}, \quad v_1 = 1000\,\text{m/s},$$
$$\phi_2 = 8^\circ, \quad \theta_2 = 0^\circ, \quad z_2(0,0) = 5\,\text{m}, \quad v_2 = 1800\,\text{m/s},$$
$$\phi_3 = 5^\circ, \quad \theta_3 = 60^\circ, \quad z_3(0,0) = 12\,\text{m}, \quad v_3 = 3200\,\text{m/s}.$$

Figure 6 displays simulated head wave arrival times observed on a set of four reversed refraction profiles. Direct wave traveltimes are also included in each panel. Shots are located at each end of the recording spreads and the maximum source-receiver offset distance is 50 m. The profile pairs are all centered at the coordinate origin and are oriented in the N-S, NE-SW, E-W, and SE-NW geographic directions.

The straight line arrival time curves plotted in each panel convey an impression of a 2D earth model. The full three-dimensionality of the subsurface is only appreciated by comparing traveltime curves recorded along several separate azimuths. Slopes and intercepts of the refraction arrival lines change as the profile direction is reoriented. Thus, the crossover distances shift as well. Note that reciprocal times (the shot-to-shot traveltimes) for forward and reverse arrivals in all panels agree, as expected. Refraction traveltime curves are extended to zero offset distance, even though head waves do not exist in the precritical offset zones. Since the traveltime computation method does not locate the critical offset distance, equation (30) is evaluated over the full offset range covered by the receiver array. If traveltime analysis is concerned solely with first arrivals, then these calculated nonphysical traveltimes do not pose any problems, because they are always associated with later arrivals. However, precritical offset arrivals do have interpretive significance (Ackermann *et al.* 1986) and thus their inclusion in the current algorithm may be useful for some studies.

Note that the intercept time in equation (30) depends on the recording azimuth angle Ψ (through the propagation vectors $\mathbf{p}_{ik}$ and $\mathbf{q}_{ik}$) in addition to the source coordinates (x_S, y_S). This unusual feature appears to be peculiar to 3D multilayered earth models. Intercept time is obviously independent of profile azimuth for all 1D earth models. For 2D earth models (recorded normal to strike), the identity of intercept times observed on split spread profiles is an interpretive rule (Johnson 1976; Merrick *et al.* 1978; Ackermann *et al.* 1986; Brückl 1987). Finally, in the case of the simplest 3D model consisting of a single layer overlying a halfspace, the intercept time is also independent of profile azimuth (Aldridge 1989). Dependence on the azimuth angle Ψ only arises when multiple layers in three dimensions are analyzed. However, raypath reciprocity requires that the propagation direction vectors satisfy

$$\mathbf{p}_{ik}(\Psi + \pi) = -\mathbf{q}_{ik}(\Psi), \qquad \mathbf{q}_{ik}(\Psi + \pi) = -\mathbf{p}_{ik}(\Psi).$$

Inserting these results into the expression for the intercept time yields

$$b_k(x_S, y_S, \Psi + \pi) = b_k(x_S, y_S, \Psi).$$

Hence, the 2D interpretive rule also holds for the class of 3D models examined here (uniform velocity layers bounded by plane interfaces). However, it is not true that intercept times recorded on *all* line profiles emanating from the same shotpoint are identical. For earth models with small dips ($\leq 10°$) this variation in intercept time with profile azimuth appears to be minute. In the current example, $b_3(0,0,\Psi)$ varies by only 0.03% as the azimuth Ψ increases from 0° to 360°.

VSP Geometry

The final example examines head wave traveltimes recorded in a simulated VSP experiment. The computation procedure described in the previous section is readily generalized to a situation where source and receiver are located on different interfaces of the model. An azimuth angle function $\Psi = g(\chi)$ can be calculated for a hypothetical source located on interface j and and hypothetical receiver located on interface l. In an offset VSP survey, the source is located on the surface ($j = 1$) and a borehole geophone is lowered continuously down the well. Equation (17) is used to calculate the traveltimes.

This example considers a four-layer model defined by the parameters

$$\begin{array}{llll}
\phi_1 = 0^\circ, & \theta_1 = 0^\circ, & z_1(0,0) = 0\,\text{m}, & v_1 = 1800\,\text{m/s}, \\
\phi_2 = 3^\circ, & \theta_2 = 90^\circ, & z_2(0,0) = 50\,\text{m}, & v_2 = 2500\,\text{m/s}, \\
\phi_3 = 4^\circ, & \theta_3 = 270^\circ, & z_3(0,0) = 120\,\text{m}, & v_3 = 3200\,\text{m/s}, \\
\phi_4 = 0^\circ, & \theta_4 = 0^\circ, & z_4(0,0) = 155\,\text{m}, & v_4 = 3900\,\text{m/s}.
\end{array}$$

The model is strictly 2D; the strike directions of the two non-horizontal interfaces are north-south. However, since sources are deployed at various azimuths around the well, the 3D formulae are needed to compute accurate arrival times. Figure 7 displays head wave traveltime curves as a function of geophone depth within a borehole positioned at the coordinate origin. The maximum geophone depth is 160 m, which places the deepest receiver in layer 4. Surface sources are offset from the well by 500 m to the south, west, and east in separate panels. [For this 2D model, a source offset 500 m to the north generates traveltimes identical to those for the south source]. The arrival time of a given head wave *decreases* as the geophone is lowered in the well, because the receiver approaches the critically refracting horizon more closely. As the geophone passes through an interface, the slope of the traveltime curve changes. An arrival time curve thus consists of straight line segments joined end-to-end, as predicted by equation (17). Each curve terminates at the depth of the critical refractor in the well; the associated head wave is not observable below this level. Note that the critically refracted waves depicted in Figure 7 are not necessarily initial arrivals. Other waves neglected in the modeling procedure (e.g., direct and/or reflected waves) may actually arrive first over certain ranges of receiver depth.

CONCLUSION

The traveltime formulae derived in this study are useful for a variety of forward modeling applications involving head waves propagating in 3D multilayered earth models. Obviously, construction of traveltime curves for a trial earth model can assist in the interpretation of field recorded data. The equations also provide means for evaluating the importance of 3D effects on head wave traveltimes in various seismological contexts. For example, Merrick *et al.* (1978) investigate the hidden layer phenomenon using a 2D layered earth model. Hunter & Pullan (1990), using a 1D model, compare the sensitivities of vertical and horizontal receiver arrays for discriminating layer velocities. Many investigators are concerned about the possibility that head waves might constitute first arrivals in crosswell traveltime tomography experiments. All of these studies can benefit from a full 3D analysis.

Several straightforward extensions of the results described herein can enhance the utility of the formulae for forward modeling purposes. Inclusion of multiple reflections within the overburden of the critically refracting horizon does not pose any special problems (note that multiple raypath segments along the critical horizon are *not* allowed; see Cervený & Ravindra 1971, page 210). Also, well known tools of asymptotic ray theory can be applied to calculate the amplitude and waveform of head wave particle displacement in this 3D situation. Care must be exercised in treating shear wave propagation, because SV and SH modes are not globally decoupled in 3D. Richards *et al.* (1991) examine some of these phenomena and propose a particular computing procedure.

A more substantial improvement on the current work entails assigning anisotropic material properties to each layer of the model. The appropriate 3D anisotropic form of Snell's law of refraction must be applied at each plane interface. Frederiksen & Bostock (1998) have recently undertaken this investigation, in the context of upward propagation of incident plane waves through layered continental crust.

Finally, since head wave traveltime can be expressed by a simple mathematical formula, inverse methods designed to recover the earth model parameters from measured data are facilitated. Aldridge (1992) and Aldridge & Oldenburg (1999) describe an inversion procedure that exploits the rapid forward modeling capability developed here. Other head wave traveltime inversion schemes are probably possible. In particular, the "slope and intercept" equation (30) may allow the extension of the 2D methods of Dooley (1952), Adachi (1954), and Johnson (1976) to 3D models.

ACKNOWLEDGMENTS

This research was conducted while the author was a graduate student in the Department of Geophysics and Astronomy at the University of British Columbia. Support was provided by operating grant 5-84270 from the Natural Sciences and Engineering Research Council of Canada (principal investigator: Professor Douglas W. Oldenburg) and a Killam Predoctoral Fellowship from the University of British Columbia. Sandia National Laboratories, a multiprogram science and engineering laboratory operated by the US Department of Energy under contract DE-AC04-94AL85000, is acknowledged for providing resources necessary for publishing this material.

Sandia is a multiprogram laboratory operated by Sandia Corporation, a Lockheed Martin Company, for the United States Department of Energy under contract DE-AC04-94AL85000.

REFERENCES

Ackermann, H.D., Pankratz, L.W. & Dansereau, D., 1986. Resolution of ambiguities of seismic refraction traveltime curves, *Geophysics*, **51**, 223-235.

Adachi, R., 1954. On a proof of fundamental formula concerning refraction method of geophysical prospecting and some remarks, *Kumamoto J. Sci., Ser. A*, **2**, 18-23.

Aldridge, D.F., 1989. Attitude, velocity, and depth of a plane refractor from two line profiles, *Geophysics*, **54**, 1199-1203.

Aldridge, D.F., 1992. Analysis and inversion of seismic refraction traveltimes, *PhD thesis*, University of British Columbia.

Aldridge, D.F. & Oldenburg, D.W., 1999. Inversion of head wave traveltimes for three-dimensional planar structure, *Geophys. J. Int.*, submitted.

Brückl, E., 1987. The interpretation of traveltime fields in refraction seismology, *Geophys. Prospect.*, **35**, 973-992.

Cervený, V. & Ravindra, R., 1971. *Theory of seismic head waves*, University of Toronto Press.

Chander, R., 1977. Numerical tracing of critically refracted rays, *Geophysics*, **42**, 1048-1052.

Diebold, J.B., 1987. Three-dimensional traveltime equation for dipping layers, *Geophysics*, **52**, 1492-1500.

Diebold, J.B. & Stoffa, P.L., 1981. The traveltime equation, tau-p mapping, and inversion of common midpoint data, *Geophysics*, **46**, 238-254.

Dooley, J.C., 1952. Calculation of depth and dip of several layers by refraction seismic method, *Appendix* 1 in Thyer, R.F. & Vale, K.R., Geophysical surveys, Oaklands-Coorabin coalfield, New South Wales, *Bull. Austral. Bur. Min. Res. Geol. Geophys.*, **19**, 1-12.

Ewing, M., Woollard, G.P. & Vine, A.C., 1939. Geophysical investigations in the emerged and submerged Atlantic coastal plain, part III: Barnegat Bay, New Jersey, section, *Bull. Geol. Soc. Am.*, **50**, 257-296.

Frederiksen, A.W. & Bostock, M.G., 1998. Modeling of teleseismic waves in dipping anisotropic structures, *EOS, Trans., Am. Geophys. Union.*, **79**, *1998 Fall Meeting Supplement*, F645.

Hunter, J.A. & Pullan, S.E., 1990. A vertical array method for shallow seismic refraction surveying of the sea floor, *Geophysics*, **41**, 233-242.

Johnson, S.H., 1976. Interpretation of split-spread refraction data in terms of plane dipping layers, *Geophysics*, **41**, 418-424.

Merrick, N.P., Odins, J.A. & Greenhalgh, S.A., 1978. A blind zone solution to the problem of hidden layers within a sequence of horizontal or dipping refractors, *Geophys. Prospect.*, **26**, 703-721.

Mota, L., 1954. Determination of dips and depths of geological layers by the seismic refraction method, *Geophysics*, **19**, 242-254.

Ocola, L.C., 1972. A nonlinear least-squares method for seismic refraction mapping – part II: model studies and performance of reframap method, *Geophysics*, **37**, 273-287.

Palmer, D., 1980. *The generalized reciprocal method of seismic refraction interpretation*, Society of Exploration Geophysicists.

Palmer, D., 1981. An introduction to the generalized reciprocal method of seismic refraction interpretation, *Geophysics*, **46**, 1508-1518.

Richards, P.G., Witte, D.W. & Ekström, G., 1991. Generalized ray theory for seismic waves in structures with planar nonparallel interfaces, *Bull. Seism. Soc. Am.*, **81**, 1309-1331.

Shah, P.M., 1973. Ray tracing in three dimensions, *Geophysics*, **38**, 600-604.

Sorrells, G.G., Crowley, J.B. & Veith, K.F., 1971. Methods for computing ray paths in complex geological structures, *Bull. Seism. Soc. Am.*, **61**, 27-53.

LIST OF FIGURE CAPTIONS

Figure 1. Orientation of the i^{th} interface of a multilayered earth model. Angles ϕ_i and θ_i describe the dip and azimuth of an interface located a distance d_i from the coordinate origin. $\mathbf{n}_i$ is a unit vector normal to the plane interface. The interface separates media with seismic wavespeeds v_{i-1} (above) and v_i (below).

Figure 2. Schematic representation of the raypath of a wave critically refracted on interface k of a multilayered earth model. S and R denote a surface source and receiver, respectively. P and Q denote the two points of critical refraction. Layers are characterized by seismic wavespeeds v_i, i=1,2,...,k. In the 3D situation, the raypath is not confined to a single plane.

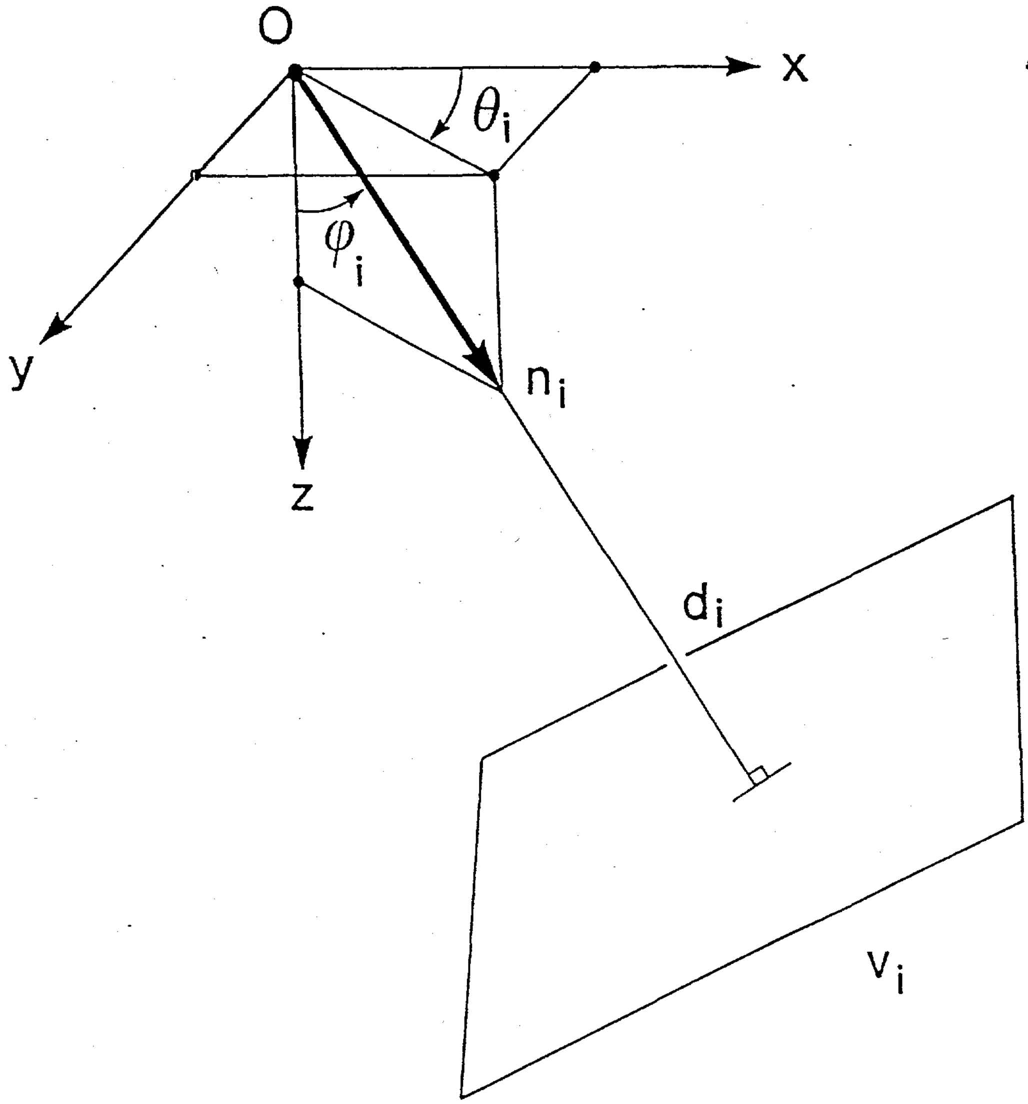
Figure 3. Noncritical refraction of the raypath at interface i in the overburden. Top: downgoing raypath. Bottom: upgoing raypath. The unit vector $\mathbf{n}_i$ is normal to the interface, and angles μ and ν describe orientations of incident and transmitted unit raypath vectors relative to this normal, respectively.

Figure 4. Reflection of a raypath at interface k of the earth model. The unit vector $\mathbf{n}_k$ is normal to the interface, and angles μ and ν describe orientations of incident and reflected unit raypath vectors relative to this normal, respectively. If a mode conversion occurs, then the downgoing velocity v^d_{k-1} differs from the upgoing velocity v^u_{k-1}.

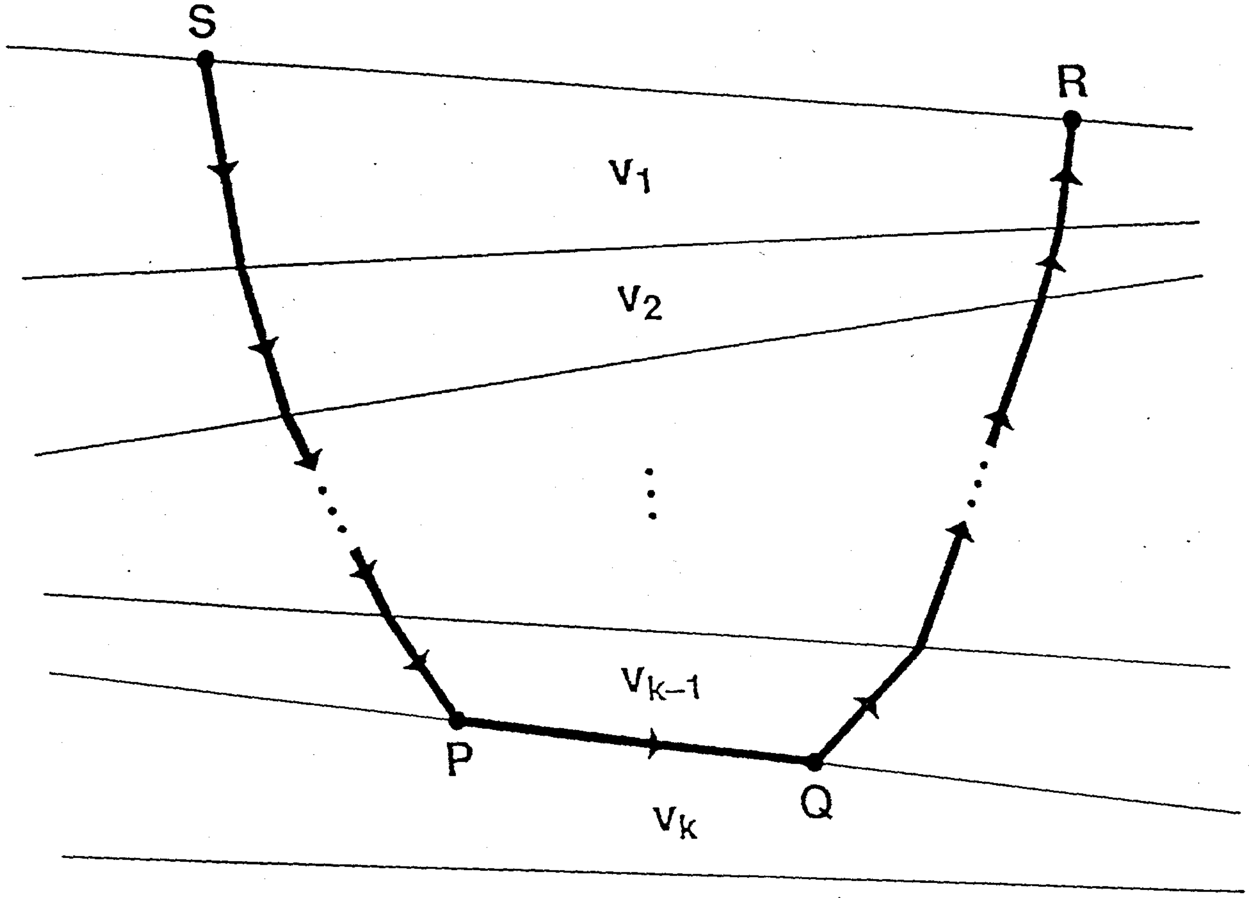
Figure 5. Schematic 3D representation of a raypath critically reflected/refracted at point P on interface k. S and R denote a surface source and receiver, respectively. Angle χ is measured from an arbitrary reference line on plane interface k, and angle Ψ is the source-receiver azimuth. Downgoing and upgoing unit raypath vectors are depicted.

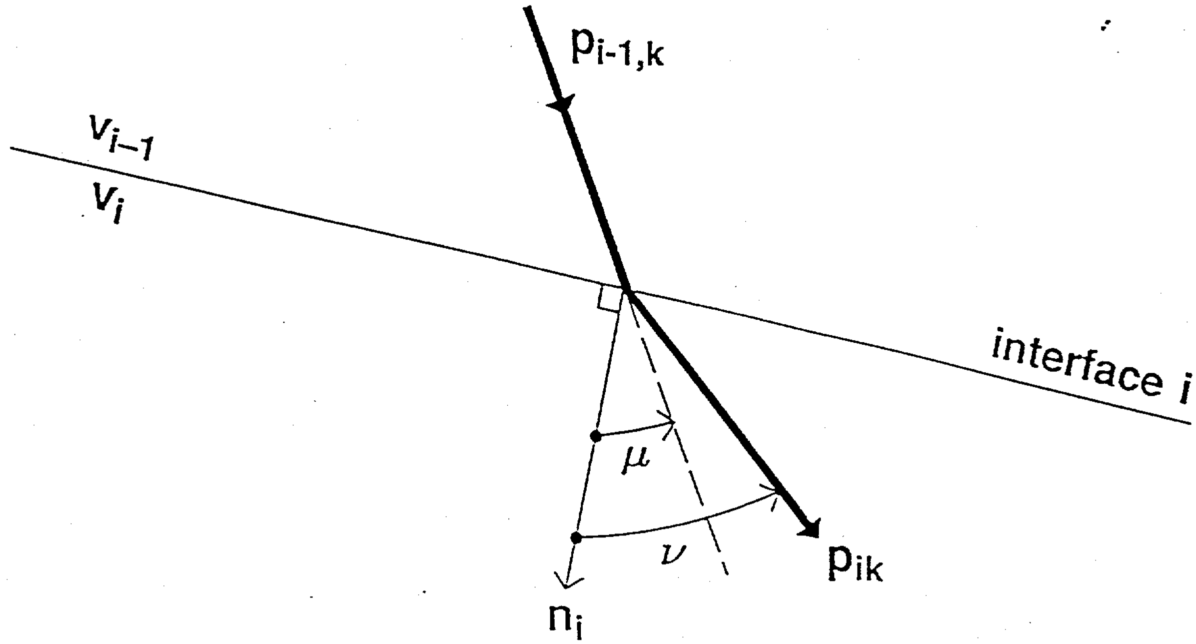
Figure 6. Direct and head wave arrival times recorded by a set of four reversed profiles over a shallow three-layer earth model. From top to bottom, profiles are oriented in a N-S, NE-SW, E-W, and SE-NW direction, respectively. All profiles are centered at the coordinate origin. In the top panel, curves labeled 1, 2, and 3 refer to arrivals from interfaces 1 (the surface), 2, and 3, respectively. The sequence of curves is the same for the remaining three panels.

Figure 7. Head wave arrival times recorded in a VSP configuration. From top to bottom, a surface source is offset horizontally from the well by 500 m to the south, west, and east, respectively. Each panel is labeled with the source-to-receiver azimuth angle Ψ. Curves labeled 2, 3, and 4 refer to critical refractions from interfaces 2, 3, and 4, respectively.

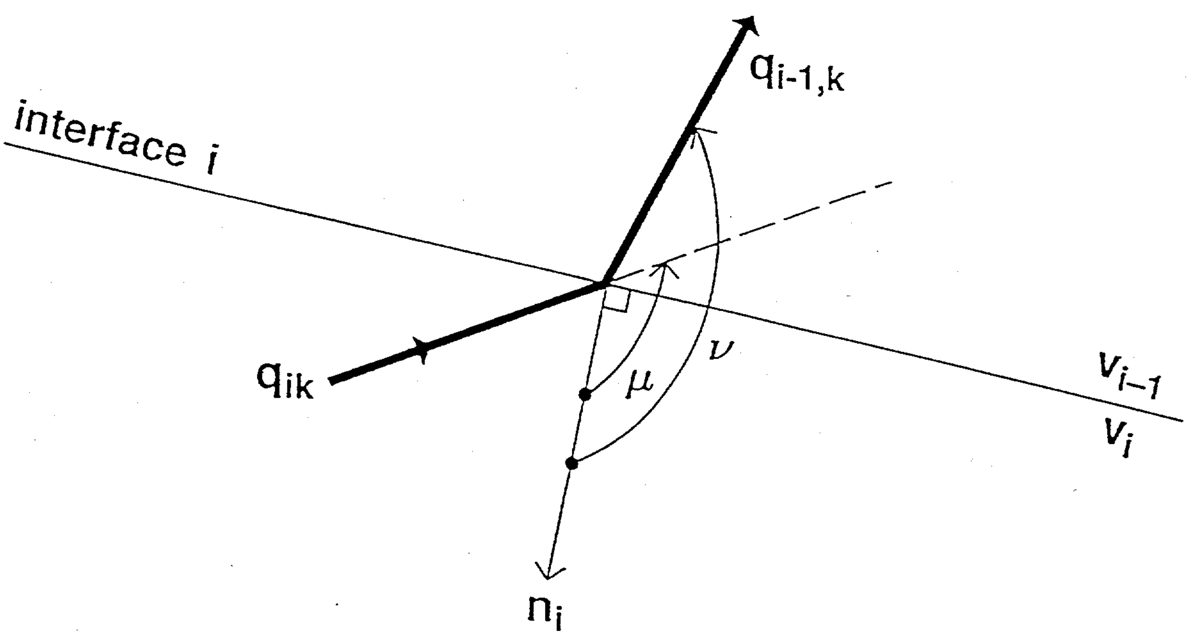


O
x
θ_i
φ_i
y
n_i
z
d_i
v_i

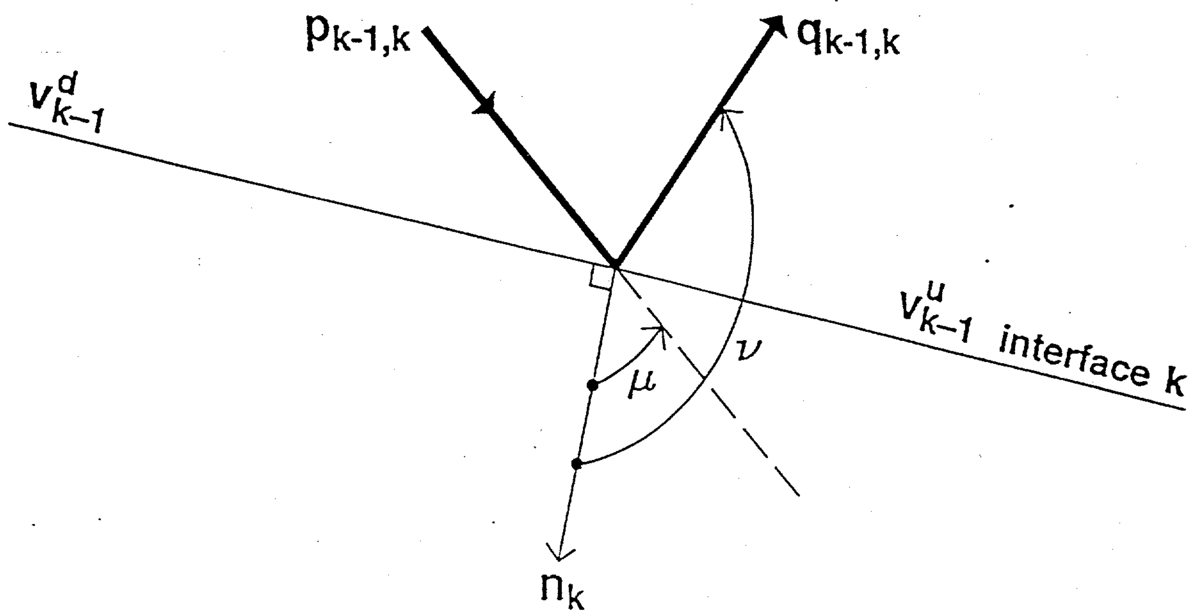

S
R
v_1
v_2
v_{k-1}
P
Q
v_k



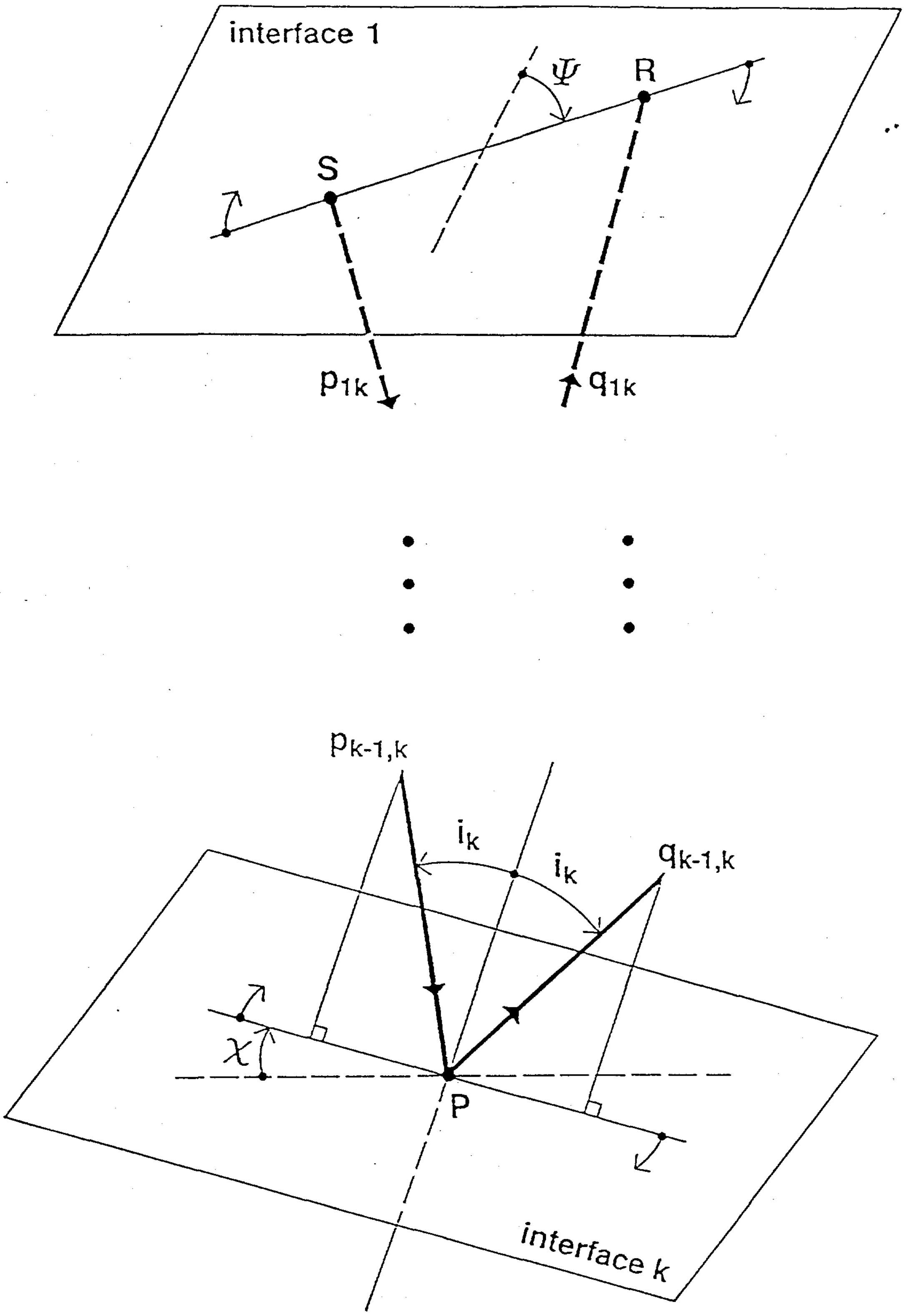
$p_{i-1,k}$
v_{i-1}
v_i
interface i
μ
ν
p_{ik}
n_i

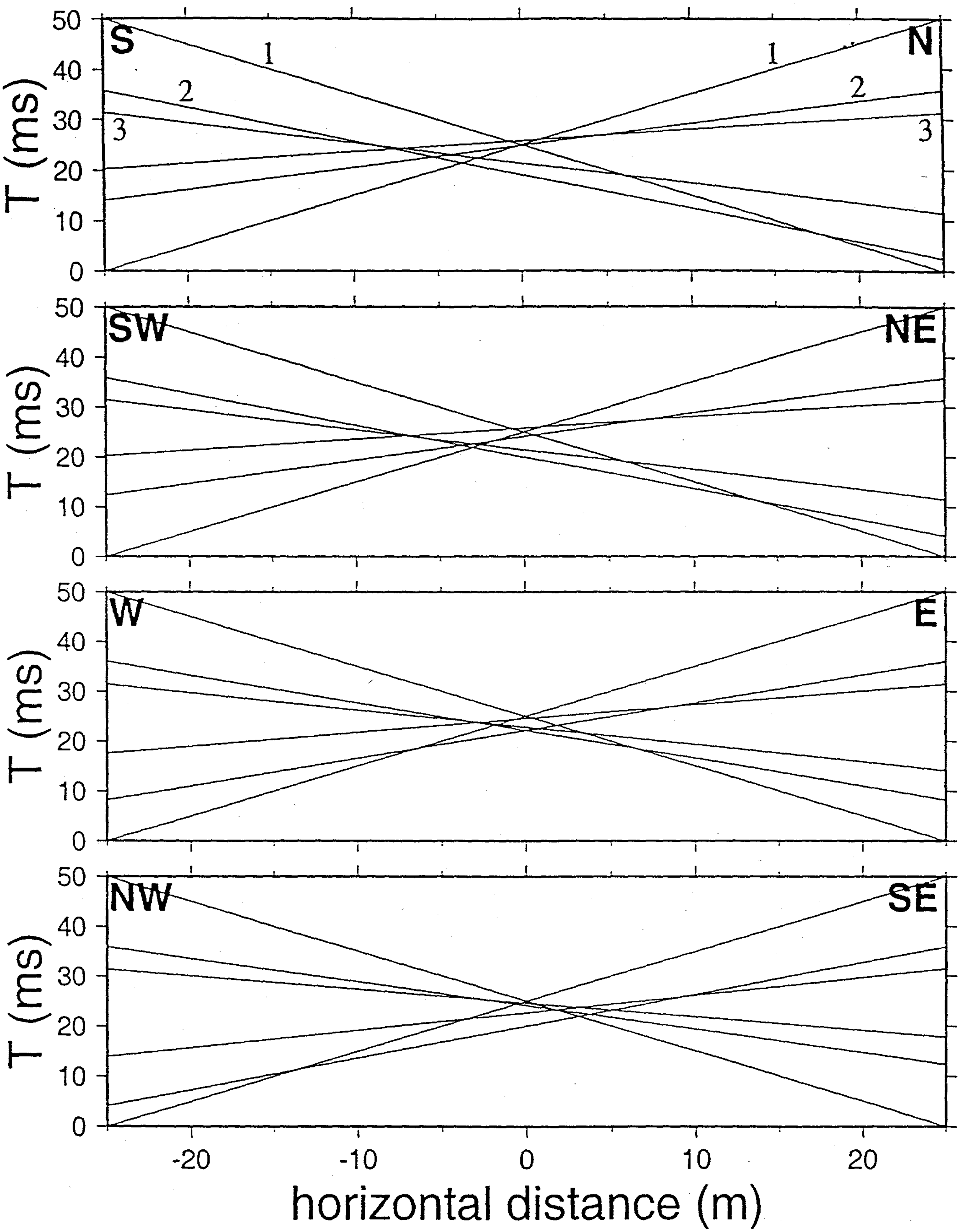


$q_{i-1,k}$
interface i
ν
μ
q_{ik}
v_{i-1}
v_i
n_i

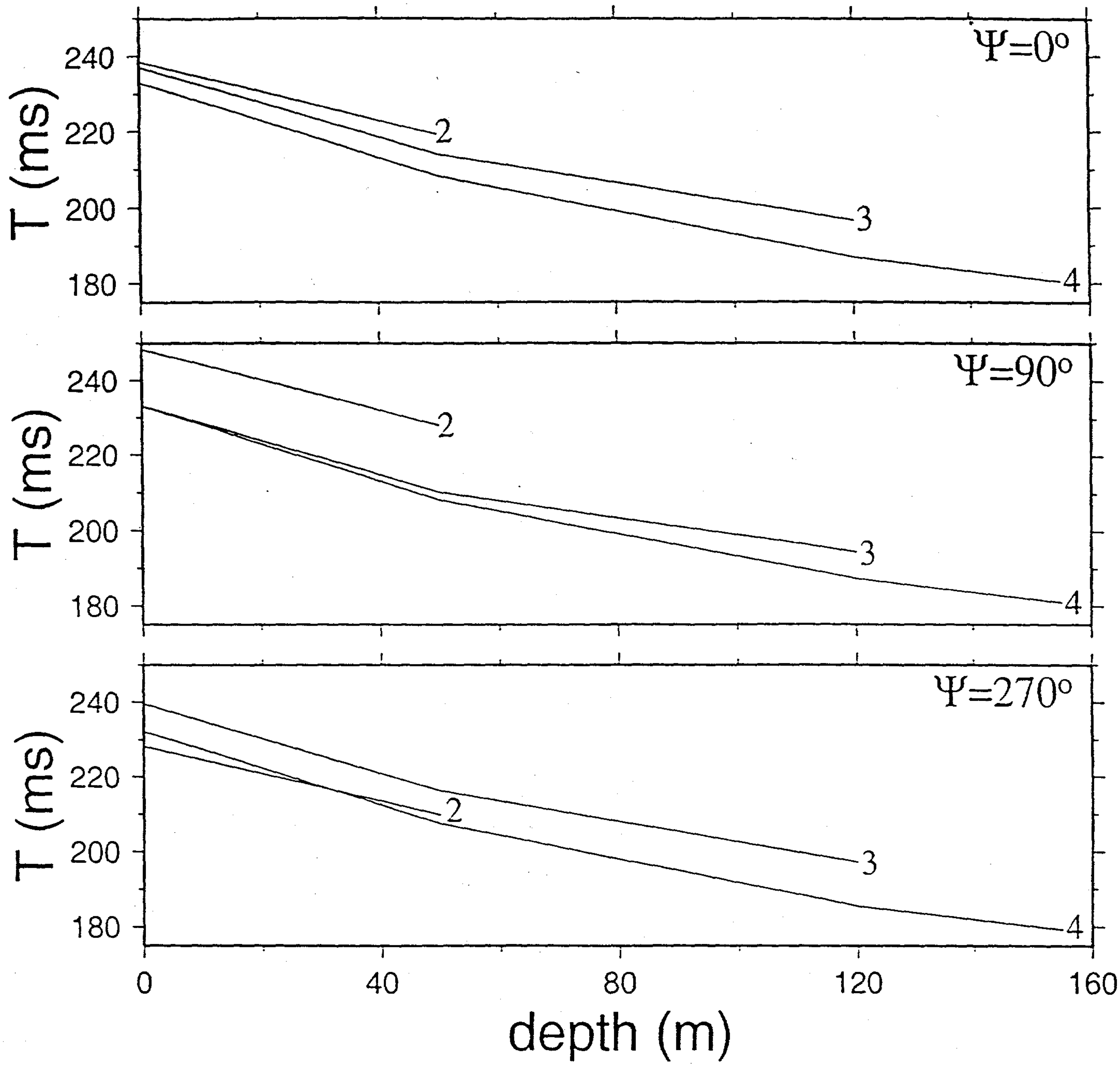


$p_{k-1,k}$
$q_{k-1,k}$
v_{k-1}^{d}
v_{k-1}^{u}
interface k
μ
ν
n_k


interface 1
Ψ
R
S
p1k
q1k
pk-1,k
ik
ik
qk-1,k
χ
P
interface k



S
N
SW
NE
W
E
NW
SE
1
2
3
T (ms)
0
10
20
30
40
50
-20
-10
0
10
20
horizontal distance (m)


Ψ=0°
Ψ=90°
Ψ=270°
T (ms)
240
220
200
180
2
3
4
0
40
80
120
160
depth (m)