ON THE DUALITY PROPERTY FOR A HERMITIAN SCALAR FIELD*

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ABSTRACT

A general hermitian scalar Wightman field is considered. On the Hilbert space of physical states "natural" domains for certain complex Lorentz transformations are constructed, and a theorem relating these transformations to the TCP symmetry is stated and proved. Under the additional assumption that the field is "locally" essentially self-adjoint, duality is considered for the algebras generated by spectral projections of smeared fields. For a class of unbounded regions duality is proved, and for certain bounded regions "local" extensions of the algebras are constructed which satisfy duality. The relationship of the arguments presented to the Tomita-Takesaki theory of modular Hilbert algebras is discussed. A separate analysis for the free field is also given.

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1. Introduction

In the theory of local observables and quantum field theory the duality condition states that the commutant of the von Neumann algebra A(R) locally associated with a region R (in a suitably selected family of regions of space-time) is precisely equal to the von Neumann algebra $A(R^c)$ locally associated with the causally complementary region R^c. [1)] A system of local algebras satisfying this condition is maximal in the sense that it has no proper local extension. Further consequences of duality have been discussed by Licht, [2)] Doplicher, Haag and Roberts, [3)] and Guenin and Misra. [4)] Araki [5)] and others [6-9)] have proved duality for so-called diamond regions for local algebras generated by a free hermitian scalar field. In a recent paper Landau[10)] has found counter-examples to duality for diamonds in the case of certain generalized free fields, but it has also been shown that there exist extended algebras which do satisfy the condition. In this dissertation we will investigate duality for a general hermitian scalar field, not necessarily free.

Our considerations will be within the framework of quantum field theory as formulated by Wightman and others.[11-13] In Section 2 we will discuss this assumption and the notation we will follow.

In Section 3 we state a variation of the theorem of Reeh and Schlieder.[14] The remainder of the section will be devoted to certain complex Lorentz transformations and a connection between these and the anti-unitary inversion transformation TCP. In particular we will be interested in the transformation

$$V(\underset{\sim}{e}_3,t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh(t) & \sinh(t) \\ 0 & 0 & \sinh(t) & \cosh(t) \end{pmatrix} \quad (1)$$

which maps the "wedge" $W_R = \{x|\ x^3 > |x^4|\ \}$ of Minkowski space onto itself for real t. On the Hilbert space $\mathcal{H}$ of physical states there is a corresponding unitary operator $U(V(\underset{\sim}{e}_3,t),\ 0)$, and a self-adjoint operator K_3 such that

$$U(V(\underset{\sim}{e}_3, t), 0) = \exp(-itK_3) \quad (2)$$

Let $P_o(W_R)$ be the polynomial algebra generated by field operators averaged with test functions with support in W_R, and let Ω be the unique Poincare invariant vacuum. We shall show that as a consequence of the "spectral condition" for the field every vector of the form $X\Omega$, $X \in P_o(W_R)$, is in the domain of the normal operator $\exp(-izK_3)$ for the complex variable z in the closed strip $0 \leq \mathrm{Im}(z) \leq \pi$, and the vector-valued function $\exp(-izK_3)X\Omega$ is strongly continuous in z on the above closed strip, and an analytic function of z on the interior of the strip. Furthermore, we will show that for any such vector

$$\exp(\pi K_3)X\Omega = JX^*\Omega \quad (3)$$

where J is the antiunitary involution defined by

$$J = U(R(\underset{\sim}{e}_3,\pi),0)\theta_o$$

where $R(\underset{\sim}{e}_3,\pi)$ is the rotation by angle π about the 3-axis and θ_o is the TCP operator. Other questions concerning the domain of $\exp(-izK_3)$ will be discussed.

In Section 4, under the assumption that the field is "locally" essentially self-adjoint, properties of the von Neumann algebras generated by the spectral projections of the self-adjoint extensions of the field are considered. Particularly, the von Neumann algebras $A(W_R)$ and $A(W_L)$ generated by field operators averaged with test functions in W_R and $W_L = -W_R$, respectively, are analyzed and it is shown that

$$\exp(\pi K_3)X\Omega = JX^*\Omega \qquad (4)$$

and

$$\exp(-\pi K_3)Y\Omega = JY^*\Omega \qquad (5)$$

for all $X \in A(W_L)'$ and $Y \in A(W_R)'$. From (4) and (5) the duality condition

$$A(W_R)' = A(W_L) \qquad (6)$$

follows. The algebras generated by smeared fields for certain bounded regions are discussed, and local extensions are constructed which satisfy duality.

In Section 5 we consider the relation of our analysis to the Tomita-Takesaki theory of modular Hilbert algebras.[15] The equivalence of $\exp(2\pi K_3)$ and the Tomita modular operator Δ for $A(W_R)$ is demonstrated.

In Section 6 we give a separate discussion of duality for wedge algebras generated by a free scalar field which is based on the well-known vacuum expectation values of the bounded operators $\exp(i\phi[f])$.

2. Assumptions and Notation

Space-time will be parametrized by the Cartesian coordinates $x = (x^1, x^2, x^3, x^4)$. The Lorentz invariant scalar product is defined as $x \cdot y = x^4y^4 - x^1y^1 - x^2y^2 - x^3y^3$. The elements $\Lambda = \Lambda(M, y)$ of the proper Poincare group $\bar{L}_o$ are parametrized by the Lorentz matrix M and a real four-vector y, such that $\Lambda(M,y)x = Mx + y$.

We denote by $D(R^n)$ the set of all complex-valued infinitely differentiable functions of compact support on n-dimentional Euclidean space R^n, and we denote by $S(R^n)$ the space of test functions on R^n on which tempered distributions are defined.

Any f in $S(R^{4n})$ or $D(R^{4n})$ will be considered as a function of n four-vectors $(x_1,\ldots,x_n)$ and will be denoted by $f(x_1,\ldots,x_n)$. $S(R^n)$ is endowed with a topology defined by a countable set of norms

Let r and s stand for sets of integers $(r_1,\ldots,r_n)$ and $(s_1,\ldots,s_n)$, respectively. Let x^r stand for $x_1^{r_1}\ldots x_n^{r_n}$ and D^s stand for $\partial^{s_1+\ldots+s_n}/\partial x_1^{s_1}\ldots\partial x_n^{s_n}$. We define the norms on $S(R^n)$ by

$$||f(x_1,\ldots,x_n)||_{r,s} = \sup_x |x^r D^s f| \qquad (7)$$

Convergence in $S(R^n)$ is defined by

$$\mathcal{S}\text{-}\lim_{n\to\infty} f_n = 0 \tag{8}$$

if $\lim_{n\to\infty} ||f_n||_{r,s} = 0$, for all r and s

We denote by (X,D) an unbounded operator with domain of definition D. The adjoint of (X,D) is denoted (X,D)* If (X,D) is closable, we denote its closure by (X,D)**. This notation is never employed for bounded operators which are regarded as defined on the entire Hilbert space.

For the sake of simplicity we limit the discussion to a single hermitian scalar Wightman field. The physical states are described by unit rays in a separable Hilbert space $\mathcal{H}$ which carries a strongly continuous unitary representation $U(\Lambda) = U(M, y)$ of the Poincare group $\bar{L}_o$. For any $\psi, \xi \in \mathcal{H}$, the scalar product, antilinear in ψ and linear in ξ , will be denoted by (ψ, ξ). The subgroup of translations $U(I, y)$ has a common spectral resolution

$$U(I, y) = \int_{(\infty)} e^{iy\cdot p} \mu(d^4p) \tag{9}$$

and the support of the spectral measure μ is contained in the closed forward light-cone $\bar{V}_+$ in momentum space. This is the "spectral condition." There exists a vacuum state Ω uniquely characterized by its invariance under all translations, and such that $U(\Lambda)\Omega = \Omega$, for all $\Lambda \in \bar{L}_o$.

The hermitian scalar field ϕ is defined by the linear mapping of $f \in S(R^{4n})$, $n \geq 1$, to an operator $(\phi\{f\},D_1)$ acting on $\mathcal{H}$. The common domain D_1 consists of the linear span of the vacuum Ω and vectors of the form $\phi\{g\}\Omega$, for $g \in S(R^{4m})$, $m \geq 1$. For any $\xi \in D_1$, $\phi\{f\}\xi$ is a vector-valued tempered distribution in f, and thus if $\mathcal{S}\text{-}\lim_{n\to\infty} f_n = 0$, then $\lim_{n\to\infty} ||\phi\{f_n\}\xi|| = 0$. The field is hermitian in the sense that for any $\xi \in D_1$,

$$(\phi\{f\},D_1)^*\xi = \phi\{f^\dagger\}\xi \qquad (10)$$

where $f^\dagger(x_1,\ldots,x_n) = f^*(x_n,\ldots,x_1)$. For $f \in S(R^4)$ we employ the special notation $\phi[f] = \phi\{f\}$ and note that for $\xi \in D_1$ and $f \in S(R^{4m})$ and $g \in S(R^{4n})$

$$\phi\{f\}\phi\{g\}\xi = \phi\{h\}\xi \qquad (11)$$

where $h(x_1,\ldots,x_m,x_{m+1},\ldots,x_{m+n}) = f(x_1,\ldots,x_m) \times g(x_{m+1},\ldots,x_{m+n})$. In the literature $\phi\{f\}$ is usually expressed as

$$\phi\{f\} = \int_{(\infty)} d^4(x_1)\ldots d^4(x_n)\, f(x_1,\ldots,x_n)\,\phi(x_1)\ldots\phi(x_n)$$

Under the representation $U(\Lambda)$ of the Poincare group, the field transforms by

$$U(\Lambda)(\phi\{f\}\ ,\ D_1)U^{-1}(\Lambda) = (\phi\{\Lambda f\}\ ,\ D_1) \qquad (12)$$

where $\Lambda f = f(\Lambda^{-1}x_1,\ldots,\Lambda^{-1}x_n)$. Locality is expressed by the condition

$$\phi\{f\}\phi\{g\}\xi = \phi\{g\}\phi\{f\}\xi \qquad (13)$$

for $\xi \in D_1$, $g \in S(R^{4n})$, $f \in S(R^{4m})$, and the support of g in any x_i space-like separated from the support of f in any x_j.

Define the subset V_n of R^{4n} as

$$V_n = \{(p_1,\ldots,p_n) \mid \sum_{r=k}^{n} p_r \in \bar{V}_+,\ k = 1,\ldots,n\} \qquad (14)$$

For f, g $\in S(R^{4n})$ we have as a consequence of the spectral condition that

$$\phi\{f\}\xi = \phi\{g\}\xi, \quad \xi \in D_1 \qquad (15)$$

if $\tilde{f}(p_1,\ldots,p_n) = \tilde{g}(p_1,\ldots,p_n)$ for $(p_1,\ldots,p_n) \in V_n$ where $\tilde{f}$ is the Fourier transform defined by

$$\tilde{f}(p_1,\ldots,p_n) =$$

$$\int_{\infty} d^4(x_1)\ldots d^4(x_n) f(x_1,\ldots,x_n)\exp(i\sum_{r=1}^{n} x_r\cdot p_r) \qquad (16)$$

3. Complex Lorentz Transformations

We define the "right wedge" W_R and the "left wedge" W_L as the open subsets of Minkowski space $\mathcal{M}$

$$W_R = \{x \mid x^3 > |x^4| \} \tag{17}$$

$$W_L = \{x \mid -x^3 > |x^4| \} \tag{18}$$

Associated with these wedge regions are the algebras $P_o(W_R)$ and $P_o(W_L)$ generated by the identity and the set $\{(\phi[f], D_1)\}$, where f is any function in $S(R^4)$ with support in W_R and W_L, respectively. Certain subsets of these algebras will be of particular importance in our discussion. Let R_1 be a bounded, open, nonempty subset of W_R, and let $x_o \in W_R$ be such that $(x - x_o) \in W_L$ for all $x \in \bar{R}_1$. For any integer $n > 1$, define the set R_n by

$$R_n = \{x + (n-1)x_o \mid x \in R_1\} \tag{19}$$

R_n is a subset of W_R for all n, and if $n > k$, then $(x' - x") \in W_R$ for all $x' \in R_n$ and $x" \in R_k$. In particular R_n is space-like separated from R_k if $n \neq k$. Define the subset Q_R of $P_o(W_R)$ as the linear span of the identity and all operators (q, D_1) of the form

$$q = \phi[f_1]\phi[f_2]\ldots\phi[f_n]$$

where $n \geq 1$ and supp $f_i \subset R_i$. Similarly define $Q_L \subset P_o(W_L)$ except that R_k is replaced by $\hat{R}_k = \{(x^1,x^2,-x^3,-x^4) \mid (x^1,x^2,x^3,x^4) = x \in R_k\}$. We have the following trivial variation of the theorem of Reeh and Schlieder.[14]

Lemma 1: Let Q_R and Q_L be defined as above. Then the linear manifolds

$$D^R = Q_R\Omega \quad \text{and} \quad D^L = Q_L\Omega$$

are each dense in the Hilbert space $\mathcal{H}$.

Proof: $\{(\xi_1,\ldots,\xi_n) \mid \xi_1 = x_1;\ \xi_i = x_i - x_{i-1},\ i>1;\ x_i \in R_i\}$ is a real environment for analytic functions in C^{4n}. With this fact a slight modification of the proof of Theorem 4-2 in the monograph of Streater and Wightman[11] yields the result.

Next we consider the Lorentz velocity transformation along the 3-axis given by the matrix $V(\underset{\sim}{e}_3, t)$ in equation (1). The abelian subgroup $\{V(\underset{\sim}{e}_3,t) \mid t \text{ real}\}$ of the Poincare group maps W_R onto W_R and W_L onto W_L. On the Hilbert space $\mathcal{H}$ of physical states there is a strongly continuous unitary representation $\{U(V(e_3,t),\ 0) \mid t \text{ real}\}$ of this subgroup. By Stone's theorem there exists a self-adjoint operator $(K_3,\ D_K)$ such that

$$U(V(\underset{\sim}{e}_3,t),0) = \exp(-itK_3) \qquad (20)$$

In the following we will study the normal operators

$$\exp(-i\tau K_3) = \int_{-\infty}^{\infty} \exp(-i\tau s)\mu_K(ds) \qquad (21)$$

where μ_K is the spectral measure in the spectral decomposition of (K_3, D_K) and $\tau \in C^1$. For convenience we denote $\exp(-i\tau K_3)$ by $V(\tau)$. The domain of the closed operator $V(\tau)$ depends only on $\mathrm{Im}(\tau)$ and will be denoted by $D_V(\mathrm{Im}(\tau))$. If $\psi \in D_V(\lambda)$, λ real, then the vector-valued function $V(\tau)\psi$ of τ is well-defined, strongly continuous and bounded on the closed strip $0 \leq \frac{\mathrm{Im}(\tau)}{\lambda} \leq 1$, and is an analytic function of τ on the interior of this strip.

Common cores exist for $V(\tau)$ and for later reference we state as a lemma some well-known facts about a particular family of cores.

Lemma 2: a) Let $c(s) \in D(R^1)$ and let the bounded operator $c(K_3)$ be defined by

$$c(K_3) = \int_{-\infty}^{\infty} c(s)\ \mu_K(ds) \qquad (22)$$

Then $c(K_3)\psi \in D_V(\lambda)$ for all λ real and for all $\psi \in \mathcal{H}$

b) Let D be any dense linear manifold in $\mathcal{H}$ and let D_c be defined by

$$D_c = \mathrm{span}\ \{c(K_3)D \mid c(s) \in D(R^1)\} \qquad (23)$$

Then D_c is dense in $\mathcal{H}$, and a core for every operator $(V(\tau), D_V(\mathrm{Im}(\tau)))$.

c) If $c(s) \in D(R^1)$, then $c(K_3)$ is also *given* by

$$c(K_3) = \int_{-\infty}^{\infty} dt\ \hat{c}(t)V(t) \qquad (24)$$

where $\hat{c}(t)$ is the Fourier transform of $c(s)$ defined by

$$\hat{c}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} ds\ c(s)\ e^{its} \qquad (25)$$

We *furthermore* note that $\hat{c}(t)$ is an entire analytic function of t and $\hat{c}(t+i\mu)$, t and μ real, is in $S(R^1)$ as a function of t.

d) For all $\psi \in \mathcal{H}$

$$V(\tau)c(K_3)\psi = \int_{-\infty}^{\infty} dt\ c(t-\tau)V(t)\psi \qquad (26)$$

Next we consider the transformation j on Minkowski space defined by

$$jx = j(x^1,x^2,x^3,x^4) = (x^1,x^2,-x^3,-x^4) \qquad (27)$$

and note that $j = V(\underset{\sim}{e}_3, i\pi)$. Heuristically, this suggests a relation of the form

$$V(i\pi)\phi(x_1)\ldots\phi(x_n)\Omega = \phi(jx_1)\ldots\phi(jx_n)\Omega \quad (28)$$

and the remainder of this section will be devoted to giving (28) rigorous meaning.

Let z be a four-vector and consider the function

$$z(\tau) = V(\underset{\sim}{e}_3,\tau)z\ , \qquad \tau \in C^1$$

i.e., for $\mathrm{Re}(\tau) = t$ and $\mathrm{Im}(\tau) = \theta$,

$$z^1(\tau) = z^1$$

$$z^2(\tau) = z^2$$

$$z^3(\tau) = (z^3\cosh t + z^4\sinh t)\cos\theta + i(z^3\sinh t + z^4\cosh t)\sin\theta$$

$$z^4(\tau) = (z^4\cosh t + z^3\sinh t)\cos\theta + i(z^4\sinh t + z^3\cosh t)\sin\theta$$

By inspection we see that

$$\mathrm{Im}(z(\tau)) \in V_+$$

for $z \in W_R$ and $0 < \mathrm{Im}(\tau) < \pi$. Thus the function $\exp(ip\cdot z(\tau))$ will be of rapid decrease for $p \in V_+$, and by the spectral condition we might expect vectors of the form $X\Omega$, $X \in P_o(W_R)$, to be in $D_V(\lambda)$ for $0 < \lambda < \pi$. In the following lemmas we confirm this suspicion.

Lemma 3: Let $u(s)$ be an infinitely differentiable function such that $u(s) = 1$ for $s \geq 0$ and $u(s) = 0$ for $s \leq -1$. Define a function of the four-vector p by

$$E(p,z,\tau) = u(p\cdot p)u(p^4)\exp(ip\cdot z(\tau)) \quad (29)$$

Then:

a) $E(p,z,\tau) \in S(R^4)$ in p for $z \in W_R$ and

$0 < \mathrm{Im}(\tau) < \pi$.

b) $E(p,z,\tau)$ is analytic in τ in the sense of the $\mathcal{S}$-topology of test functions for $z \in W_R$ and $0 < \mathrm{Im}(\tau) < \pi$.

c) Define a function of n four-vectors p_i and n four-vectors z_i by

$$E_n(p_1,\ldots,p_n;\ z_1,\ldots,z_n;\tau\ )$$

$$= \prod_{k=1}^{n} E(\sum_{i=k}^{n} p_i, z_k, \tau) \qquad (30)$$

Then $E_n \in S(R^{4n})$ in $p_1,\ldots,p_n$ and analytic in τ in the $\mathcal{S}$-topology for $z_k \in W_R$ and $0 < \mathrm{Im}(\tau) < \pi$.

<u>Proof</u>: supp $E(p,z,\tau) \subset \{p \mid p^4 > -1\} \cap \{p \mid p^2 > -1\}$. The set $\{p \mid p^4 \leq 1\} \cap$ supp E is bounded as $|\vec{p}| \leq \sqrt{2}$. For $p^4 > 1$ and any integer $s \geq 0$,

$$|D_p^s E(p,z,\tau)| \leq \exp(\frac{-|p^4|-|\vec{p}|}{2} \mathrm{Im}(z^4(\tau)-z^3(\tau)))\times$$

$$\exp(\frac{\mathrm{Im}(z^4(\tau)+z^3(\tau))}{2}) \times |r(z(\tau),p)| \qquad (31)$$

where $r(z(\tau), p)$ is some polynomial in the components of $z(\tau)$ and p of degree s. Part a) follows immediately from this estimate and $\mathrm{Im}(z(\tau)) \in V_+$ for τ in strip $0 < \mathrm{Im}(\tau) < \pi$. For $\mathcal{S}$-analyticity we must show that for any integers $r,s > 0$ and complex h,

$$\lim_{|h|\to 0} \left|\left| \frac{dE(p,z,\tau)}{d\tau} - \frac{E(p,z,\tau+h)-E(p,z,\tau)}{h}\right|\right|_{r,s} = 0$$

We first note that $E(p,z,\tau)$ and $D_p^s E(p,z,\tau)$ are analytic in τ in the open strip $0 < \mathrm{Im}(\tau) < 1$, and $D_p^s(dE/d\tau) = d/d\tau(D_p^s E)$. Let τ_o be in the open strip, and let $\rho > 0$ be such that τ_o+h is in the open strip if $|h| < 2\rho$. We then have the estimate for integers r, $s \geq 0$

$$|(p^r D_p^s)[\frac{dE}{d\tau}(p,z,\tau_o) - \frac{E(p,z,\tau_o+\frac{h}{2})-E(p,z,\tau_o)}{h/2}]| <$$

$$2(|p|^r)\frac{h}{\rho^2} M(p,z,\tau_o), \quad |h| < \rho \qquad (32)$$

where $M(p,z,\tau_o) = \max_s |D_p^s E(p,z,\tau_o+s)|$, $|s| = \rho$. From the estimate (31) we see that $M(p,z,\tau_o)|p|^r$ is bounded in the variable p by some $M(z,\tau_o)$. Thus for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$||\frac{dE}{d\tau}(p,z,\tau_o) - \frac{E(p,z,\tau_o+h)-E(p,z,\tau_o)}{h}||_{r;s} < \varepsilon$$

for all h such that $|h| < \delta$, and b) is proved. c) is a trivial corollary of a) and b).

Lemma 4: Let R_i be as in Lemma 1, and let $f_i \varepsilon D(R^4)$ and $\mathrm{supp}\, f_i \subset R_i$. Define a function of n four-vectors $(p_1,\ldots,p_n)$ by

$$\tilde{E}(p_1,\ldots,p_n; f_1,\ldots,f_n;\tau) =$$

$$\int_{(\infty)} d^4x_1\ldots d^4x_n\, f_1(x_1)\ldots f_n(x_n) \times$$

$$E_n(p_1,\ldots,p_n; x_1,x_2-x_1,\ldots,x_n-x_{n-1};\tau) \qquad (33)$$

Then a) $\tilde{E}(p_1,\ldots,p_n;\ f_1,\ldots,f_n;\tau\ ) \in S(R^{4n})$ in $p_1,\ldots,p_n$ in the closed strip $0\leq \mathrm{Im}(\tau)\leq\pi$ and is an analytic function of τ in the $\mathcal{S}$-topology on the interior of the strip.

b) For $\mathrm{Re}(\tau) = t$ and $\mathrm{Im}(\tau) = \theta$

$$\mathcal{S}\text{-}\lim_{\theta\to 0+} \tilde{E}(p_1,\ldots,p_n;\ f_1,\ldots,f_n;\tau\ ) =$$

$$U(p_1,\ldots,p_n)\tilde{f}_1(V^{-1}(\underset{\sim}{e}_3,t)p_1)\ldots\tilde{f}_n(V^{-1}(\underset{\sim}{e}_3,t)p_n) \quad (34)$$

where $U(p_1,\ldots,p_n) = \prod_{k=1}^{n} [u(p_k^4+\ldots+p_n^4)u((p_k+\ldots+p_n)^2)]$

c) For $\mathrm{Re}(\tau) = t$ and $\mathrm{Im}(\tau) = \theta$

$$\mathcal{S}\text{-}\lim_{\theta\to\pi-} \tilde{E}(p_1,\ldots,p_n;\ f_1,\ldots,f_n;\tau\ ) =$$

$$U(p_1,\ldots,p_n)\ \tilde{f}_1^j\ (V^{-1}(\underset{\sim}{e}_3,t)p_1)\ldots\tilde{f}_n^j\ (V^{-1}(\underset{\sim}{e}_3,t)p_n) \quad (35)$$

where $\tilde{f}_k^j\ (p) = \tilde{f}_k(jp)$

We remark for the Fourier transform of $\tilde{E}$, $E(y_1,\ldots,y_n;\ f_1,\ldots,f_n;\quad) \in S(R^{4n})$ in $y_1,\ldots,y_n$ and thus may be used to smear the field operators.

Proof: From the support of the f_i, the variables $x_1,\ x_2-x_1,\ldots,x_n-x_{n-1}$ are each in W_R throughout the range of integration. As each $f_i \in D(R^4)$, the integration is over a compact set, and the analytic properties of E_n established . Lemma 3 carry over to the integral. The rest of the lemma is trivial.

Lemma 5: a) The vector-valued function

$\phi\{E(f_1,\ldots f_n;\tau)\}\Omega$ is a strongly continuous

function of τ in the closed strip $0\leq \mathrm{Im}(\tau)\leq\pi$ and is a strongly analytic function of τ on the interior of the strip.

Let $\mathrm{Re}(\tau) = t$ and $\mathrm{Im}(\tau) = \theta$.

b) $\underset{\theta\to 0+}{\text{s-lim}}\ \phi\{E(f_1,\ldots,f_n;\tau)\}\Omega$

$$= V(t)\phi[f_1]\ldots\phi[f_n]\Omega \tag{36}$$

c) $\underset{\theta\to\pi-}{\text{s-lim}}\ \phi\{E(f_1,\ldots,f_n;\tau)\}\Omega$

$$= V(t)\phi[f_1^j]\ldots\phi[f_n^j]\Omega \tag{37}$$

d) $\phi\{E(f_1,\ldots,f_n;\tau)\}\Omega$

$$= V(t)\phi\{E(f_1,\ldots,f_n;i\theta)\}\Omega \tag{38}$$

Proof: a)-c) follow immediately from the results of Lemma 4 and the fact that $\phi\{f\}\Omega$ is a vector-valued tempered distribution in f satisfying the spectral condition. d) follows from the fact that

$\tilde{E}(p_1,\ldots,p_n; f_1,\ldots,f_n; t+i\theta)$

$$= \tilde{E}(V^{-1}(\underset{\sim}{e}_3,t)p_1,\ldots,V^{-1}(\underset{\sim}{e}_3,t)p_n;f_1,\ldots,f_n;i\theta\ )$$

on V_n .

Let $\psi \in \mathcal{H}$ and $c(s) \in D(R^1)$.
The function $(\psi,\phi(E(f_1,\ldots,f_n;\tau))\Omega)$ is a bounded function of τ on the strip $0 \leq \mathrm{Im}(\tau) \leq \pi$ and analytic on the interior of the strip by the results of Lemma 5.
Consider the contour integral in

$$\int_{c_1+c_2+c_3+c_4} d\tau \quad \hat{c}(\tau-i\pi) \ (\psi,\phi[\ E(f_1,\ldots f_n;\tau)]\Omega) = 0$$

where the contours are indicated on figure 1.
The contributions from the contours c_1 and c_2 vanish in the limit $|\mathrm{Re}(\tau)| \to \infty$ as $\hat{c}(\tau)$ is in $S(R^1)$ in the variable $\mathrm{Re}(\tau)$.
Thus we have for $0 < \varepsilon < \pi/2$

$$\int_{\infty}^{\infty} dt\ \hat{c}(t-i(\pi-\varepsilon))\ (\psi,\ \phi[E(f_1,\ldots,f_n;t+i\varepsilon)]\Omega) =$$

$$\int_{\infty}^{\infty} dt\ \hat{c}(t+i\varepsilon)\ (\psi,\ \phi[E(f_1,\ldots,f_n;\ t+i(\pi-\varepsilon))]\Omega)$$

Taking the limit $\varepsilon \to 0$, and recalling Lemma 2 d) and Lemma 5 c) and d) we have

$$(\psi,\ V(i\pi)c(K_3)\phi[f_1]\ldots\phi[f_n]\Omega) =$$

$$(\psi,\ c(K_3)\phi[f_1^j]\ldots\phi[f_n^j]\Omega) \qquad (39)$$

Lemma 6: Let $\phi[f_1]\ldots\phi[f_n] \in Q_R$ and $c(s) \in D(R^1)$. Then

$$V(i\pi)c(K_3)\,\phi[f_1]\ldots\phi[f_n]\,\Omega =$$

$$c(K_3)\,\phi[f_1^j]\ldots\phi[f_n^j]\,\Omega \tag{40}$$

where $f_n^j(x) = f_n(jx)$.

Proof: In (39) ψ is an arbitrary vector in the Hilbert space. The result immediately follows.

Associated with the j operator is an antiunitary operator $J = U(R(\underset{\sim}{e}_3,\pi),0)\Theta_0$ where $R(\underset{\sim}{e}_3,\pi)$ is the rotation of π about the 3-axis and Θ_0 is the antiunitary TCP operator whose existence under our assumptions is guaranteed by the theorem of Jost.[17)] J has the following properties which will be of importance:

$$J^2 = 1, \quad J\Omega = \Omega, \quad JU(M,x)J = U(jMj,jx) \tag{41}$$

Furthermore, $JD_1 = D_1$ and

$$J\phi[f]J\psi = (\phi[f^j],D_1)^*\psi = \phi[f^{j*}]\psi, \quad \psi \in D_1 \tag{42}$$

For the velocity transformations V(t), in particular,

$$JV(t)J = V(t) \quad \text{for all t real} \tag{43}$$

From this relation it follows that

$$JD_K = D_K, \quad J(K_3, D_K)J = -(K_3, D_K) \tag{44}$$

$$JD_V(\lambda) = D_V(-\lambda) \quad , \quad \lambda \text{ real} \tag{45}$$

$$J(V(\tau), D_V(\mathrm{Im}(\tau))J = (V(\tau^*), D_V(-\mathrm{Im}(\tau))) \tag{46}$$

We are now prepared for the main theorem of this section.

Theorem 1: a) Let $X \in Q_R$ and $c(s) \in D(R^1)$. Then

$$V(i\pi)c(K_3)X\,\Omega = c(K_3)JX^*\Omega \tag{47}$$

b) Let $Y \in Q_L$ and let $c(s) \in D(R^1)$. Then

$$V(-i\pi)c(K_3)Y\Omega = c(K_3)JY^*\Omega \tag{48}$$

c) Let A be any operator with Ω in the domains of A and A^* and such that $(A\Omega, Y\Omega) = (Y^*\Omega, A^*\Omega)$ for all $Y \in P_o(W_L)$ Then $A\Omega$ is in $D_V(\pi)$ and

$$V(i\pi)A\,\Omega = JA^*\Omega \tag{49}$$

d) Let B be any operator with Ω in the domains of B and B^* and and such that $(B\Omega, X\Omega) = (X^*\Omega, B^*\Omega)$ for all $X \in P_o(W_R)$ Then $B\Omega$ is in $D_V(-\pi)$ and

$$V(-i\pi)B\Omega = JB^*\Omega \tag{50}$$

e) In particular result c) holds for $A \in P_o(W_R)$, and result d) holds for $B \in P_o(W_L)$.

Proof: We first recall that by definition $X \in Q_R$ has domain D_1. For any $(\phi[f_1]\ldots\phi[f_n],\ D_1) \in Q_R$

$$(\phi[f_1]\ldots\phi[f_n],D_1)^*\psi =$$

$$\phi[f_n^*]\ldots\phi[f_1^*]\psi \quad =$$

$$\phi[f_1^*]\ldots\phi[f_n^*]\psi\ , \quad \psi \in D_1$$

as supp f_i is space-like separated from supp f_j, $i \neq j$. Thus

$$J(\phi[f_1]\ldots\phi[f_n],D_1)^*J\psi \quad =$$

$$\phi[f_1^j\ ]\ldots\phi[f_n^j\ ]\psi \qquad , \ \psi \in D_1$$

and a) then follows from Lemma 6. As $Y \in Q_L$ is equal to JXJ for some $X \in Q_R$, (48) is a consequence of a) and relation (46). To prove c) we first note that

$$D_c^R \ = \ \text{span}\ \{c(K_3)Q_R\Omega|\ c(s) \ \in D(R^1)\}$$

is a core for $V(i\pi)$ by Lemma 2 b). The following string of equalities yields the desired result. Let $X\epsilon Q_R$.

$$(A\Omega,\ V(i\pi)c(K_3)X\Omega) =$$

$$(A\Omega,\ c(K_3)JX^*J\Omega) =$$

$$\int_{-\infty}^{\infty} dt\ \hat{c}(t)\ (A\Omega,\ JV(t)X^*V^{-1}(t)J\Omega) =$$

$$\int_{-\infty}^{\infty} dt\ \hat{c}(t)\ (JV(t)XV^{-1}(t)J\Omega,\ A^*\Omega) =$$

(since $JV(t)X^*V^{-1}(t)J \in P_o(W_L)$)

$$\int_{-\infty}^{\infty} dt\ \hat{c}(t)\ (JV(t)X\Omega,\ A^*\Omega) =$$

$$\int_{-\infty}^{\infty} dt\ \hat{c}(t)\ (JA^*\Omega,\ V(t)X\Omega) =$$

$$(JA^*\Omega,\ c(K_3)X\Omega)$$

d) is similarly proved and e) is trivial.

4. Local von Neumann Algebras and Duality

In the theory of local observables there is a correspondence between certain regions R of space-time and von Neumann algebras A(R). R is selected from a collection $\mathcal{K}$ of regions of Minkowski space which is invariant under Poincare transformations. Let $A(R)'$ denote the commutant of A(R) and let R^c denote the causal complement of R, i.e.,

$$R^c = \{x \mid (x-y)^2 < 0 \text{ for all } y \in R\} \qquad (51)$$

A physically reasonable system of local algebras should minimally satisfy the conditions:

i) locality, i.e.,

$$A(R^c) \subset A(R)' \ , \ R \in \mathcal{K} \qquad (52)$$

ii) covariance, i.e.,

$$U(\Lambda)A(R)U^{-1}(\Lambda) = A(\Lambda R) \ , \ R \in \mathcal{K} \qquad (53)$$

where $U(\Lambda)$ is the unitary operator associated with the Poincare transformation Λ and

$$\Lambda R = \{\Lambda x \mid x \in R\}$$

In this dissertation we wish to discuss the duality condition

$$A(R^c) = A(R)' \tag{54}$$

for systems of local algebras associated with a hermitian scalar field. In view of some results of Araki[18) duality is usually conjectured only for regions R such that $R^{cc} = R$. We also note that the causal complement of R is often defined in the literature as the interior of the set R^c. This issue will be clarified in the course of the discussion.

Unfortunately, it is not known in general whether a nontrivial system of local algebras exists which is relatively local to a hermitian scalar field ϕ in the sense that

$$(X\xi, \phi[f]\zeta) = (\phi[f^*]\xi, X^*\zeta), \quad \xi,\zeta \in D_1$$

for R any open subset of Minkowski space, $X \in A(R)$, $f \in S(R^4)$ and supp $f \subset \overline{R^c}$. One condition which guarantees the existence of such systems is as follows: Special Condition: For every real $f \in S(R^4)$ the operator $(\phi[f], D_1)$ is essentially self-adjoint. Furthermore, if $r \in S(R^4)$ and real, and supp $r \subset$ (supp $f)^c$, then

$$EF = FE \tag{55}$$

for any spectral projection E associated with $(\phi[r], D_1)^{**}$ and any spectral projection F associated with $(\phi[f], D_1)^{**}$.

In the remainder of this section we shall discuss duality under the assumption of the Special Condition using the results of Section 3. In the following lemma we establish that for any nonempty open subset R of Minkowski space the von Neumann algebras A(R) generated by the spectral projection of $\{(\phi[f],D_1)^{**} | f \in S(R^4)$ and real, and supp $f \subset R\}$ are local algebras.

Lemma 8: Let ϕ be a hermitian scalar field satisfying the Special Condition. For any nonempty open subset R of Minkowski space the von Neumann algebras A(R) generated by the spectral projections of $\{(\phi[f], D_1)^{**} | f \in S(R^4)$ and real, and supp $f \subset R\}$ form a local system of algebras in the sense that:

a) For any two nonempty open subsets R_1 and R_2 of Minkowski space

$$A(R_1) \subset A(R_2)' \quad \text{if } R_1 \subset \overline{R}_2^c \tag{56}$$

b) For any nonempty open subset R of Minkowski space

$$U(\Lambda)A(R)U^{-1}(\Lambda) = A(\Lambda R), \text{ for all } \Lambda \in \overline{L}_o \tag{57}$$

Furthermore,

c) $$JA(R)J = A(jR) \tag{58}$$

where $jR = \{jx | x \in R\}$

d) $\{X\Omega \mid X \in A(R)\}$ is dense in $\mathcal{H}$

Proof: Under the Special Condition D_1 is a core for $(\phi(f), D_1)^{**}$, $f \in S(R^4)$ and real. By the maximality of self-adjoint operators, we have

$$U(\Lambda)(\phi[f],D_1)^{**}U^{-1}(\Lambda) =$$

$$(\phi[\Lambda f],D_1)^{**}$$

and

$$J(\phi[f],D_1)^{**}J = (\phi[f^j),D_1)^{**}$$

for any real $f \in S(R^4)$.

Let μ_f, $\mu_{\Lambda f}$, and $\mu_f(j)$ be the associated spectral measures. By the uniqueness of the spectral resolution we have

$$U(\Lambda)\mu_f U^{-1}(\Lambda) = \mu_{\Lambda f} \tag{59}$$

and

$$J\mu_f J = \mu_f(j) \tag{60}$$

b) and c) immediately follow from (59) and (60), and a) is trivial.

Let g_k be an arbitrary real element of $S(R^4)$ with support in R. Among the operators in A(R) are those of the form $(\exp(it_k\phi[g_k]) - 1)$, t_k real. Since vectors of the form

$$\phi[g_1]\phi[g_2]\ldots\phi[g_n] \text{ , } n > 0, \text{ supp } g_i \subset R \tag{61}$$

may be approximated arbitralily closely by vectors of the form

$$\frac{(\exp(it_1\phi[g_1])-1)}{it_1}\cdots\frac{(\exp(it_n\phi[g_n])-1)}{it_n}\,\Omega$$

and since the linear manifold generated by Ω and the vectors of (61) are dense in $\mathcal{H}$, d) holds.

We are now in a position to extend the results of Section 3 for $P_o(W_R)$ and $P_o(W_L)$ to the associated local von Neumann algebras $A(W_R)$ and $A(W_L)$.

<u>Theorem 2</u>: a) Let $X \in A(W_R)$. Then the vector $X\Omega$ is in $D_V(\pi)$ and

$$V(i\pi)X\Omega = JX^*\Omega \tag{62}$$

b) Let $Y \in A(W_L)$. Then the vector $Y\Omega$ is in $D_V(-\pi)$ and

$$V(-i\pi)Y\Omega = JY^*\Omega \tag{63}$$

c) The sets of vectors $A(W_R)\Omega$ and $A(W_L)\Omega$ are cores for the operators $V(i\pi)$ and $V(-i\pi)$, respectively.

d) Let $Z \in A(W_L)'$. Then the vector $Z\Omega$ is in $D_V(\pi)$ and

$$V(i\pi)Z\Omega = JZ^*\Omega \tag{64}$$

e) Let $W \in A(W_R)'$. Then the vector $W\Omega$ is in $D_V(-\pi)$ and

$$V(-i\pi)W\Omega = JW^*\Omega \tag{65}$$

Proof: a) and b) are a consequence of the Special Condition and Theorem 1 c) and d). Since

$V(\underset{\sim}{e}_3,t)W_R = W_R$ and $V(\underset{\sim}{e}_3,t)W_L = W_L$, Lemma 8 b implies that $V(t)A(W_R)V^{-1}(t) = A(W_R)$ and $V(t)A(W_L)V^{-1}(t) = A(W_L)$. Thus, operators of the form

$$X_c = \int_{-\infty}^{\infty} dt\hat{c}(t)\ V(t)XV^{-1}(t)$$

are in $A(W_R)$ for $X \in A(W_R)$ and $c(s) \in D(R)$. Since

$X_c\Omega = c(K_3)X\Omega$ Lemmas 8 e) and 2 b) imply c).

Let $Z \in A(W_L)'$ and $X \in A(W_R)$. Then

$$(Z\Omega,\ V(i\pi)X_c\Omega) = (Z\Omega,\ JX_c{}^*J\Omega) =$$

$$(JX_cJ\Omega,\ Z^*\Omega) = (JZ^*\Omega,\ X_c\Omega)$$

which follows from $JX_cJ \in A(W_L)$, together with c) implies d). A similar argument yields e).

From Theorem 2 the duality condition for wedge regions, in particular

$$A(W_R)' = A(W_L)$$

will now follow.

Theorem 3: a) Let $Y \in A(W_R)'$ and $X \in A(W_L)'$. Then

$$XY = YX \tag{66}$$

b) $$A(W_R)' = A(W_L) \tag{67}$$

<u>Proof</u>: Since $X\Omega \in D_V(\pi)$ and $Y\Omega \in D_V(-\pi)$, we have

$$(Y\Omega, \ X\Omega) = (V(-i\pi)Y\Omega, V(i\pi)X\Omega) =$$

$$(JY^*\Omega, JX^*\Omega) = (X^*\Omega, Y^*\Omega)$$

Let $M, N \in A(W_R) \subset A(W_L)'$. Then

$$(M\Omega, YXN\Omega) = (Y^*\Omega, M^*XN\Omega) =$$

$$(N^*X^*M\Omega, Y\Omega) = (M\Omega, XYN\Omega)$$

as $M^*XN \in A(W_L)'$. Since $\{M\Omega \mid M \in A(W_R)\}$ is dense in $\mathcal{H}$

$$XY = YX$$

and a) is proved. Reexpressing this result as

$$A(W_L) \subset A(W_R)' \subset A(W_L)'' = A(W_L)$$

we also have part b).

We define the set $\mathcal{W}$ of "wedge regions" as

$$\mathcal{W} = \{\Lambda W_R \mid \Lambda \in \mathcal{L}_o\} \qquad (68)$$

and the associated local von Neumann algebras

$$A(\Lambda W_R) = U(\Lambda)A(W_R)U^{-1}(\Lambda)$$

As a corollary to Theorem 3 we have

$$A(W)' = A(\overline{W}^c) \quad , W \varepsilon \mathcal{W} \tag{69}$$

Next we wish to consider the duality condition for bounded regions of space-time, and in particular, for so-called double-cones. For any two points x_1 and x_2 of space-time such that $x_2 \varepsilon \ V_+(x_1)$, (where $V_+(x_1)$ is the forward light cone with x_1 as apex), we define the double-cone $C = C(x_1,x_2)$ by

$$C(x_1,x_2) = V_+(x_1) \cap V_-(x_2)$$

where $V_-(x_2)$ is the backward light cone with x_2 as apex. The double-cones so defined are thus open and non-empty. We denote by $\mathcal{C}$ the set of all double-cones. Again under the assumption of the Special Condition, there exists for each $C \varepsilon \ \mathcal{C}$ the locally associated algebra

$A(C)$. In his discussion of generalized free fields, Landau [10] constructs counter-examples to the duality condition for double-cones. However, he also exhibits local extensions of these algebras which do satisfy duality. It is in this spirit that we proceed in the more general case.

For any double-cone C we define the von Neumann algebra $B(\bar{C})$, which we regard as associated with the closed, convex set $\bar{C}$, by

$$B(\bar{C}) = \bigcap \{ A(W) \mid W \in \mathcal{W} , W \supset \bar{C}\} \qquad (70)$$

$B(\bar{C})$ is an extension of the algebra $A(C)$, and in the following theorem we demonstrate that the set $\bigcup_{C \in \mathcal{C}} \{ B(\bar{C}), A(\bar{C}^c)\}$ form a local system of algebras which satisfy the duality condition.

<u>Theorem 4</u>: Let $B(\bar{C})$ be defined as above. Then:

a) The algebras $B(\bar{C})$ are local in the sense that for any C_1, $C_2 \in \mathcal{C}$, such that $C_1 \subset \bar{C}_2^c$,

$$B(\bar{C}_1) \subset B(\bar{C}_2)' \qquad (71)$$

b) For any $C \in \mathcal{C}$ and $\Lambda \in \bar{L}_o$,

$$U(\Lambda)B(\bar{C})U^{-1}(\Lambda) = B(\Lambda\bar{C}) \qquad (72)$$

c) For any $C \in \mathcal{C}$,

$$B(\bar{C})' = A(\bar{C}^c) \qquad (73)$$

i.e., the duality condition is satisfied.

<u>Proof</u>: a) follows from the fact that for any two disjoint, space-like separated double-cones C_1 and C_2

there exists a wedge W, such that $\overline{C}_1 \subset W$ and $\overline{C}_2 \subset \overline{W}^c$. By definition $B(\overline{C}_1) \subset A(W)$ and $B(\overline{C}_2) \subset A(\overline{W}^c)$, and $A(W)' = A(\overline{W}^c)$ by (69). Thus $B(\overline{C}_1)' \supset B(\overline{C}_2)$. b) is a trivial consequence of the definitions. To prove c) we first note that

$$B(\overline{C})' = \{A(W)' \mid W \in \mathcal{W}, W \supset \overline{C}\}''$$

By duality for wedges, we have

$$B(\overline{C})' = \{A(W) \mid W \in \mathcal{W}, \overline{W}^c \supset \overline{C}\}''$$

Since $\overline{W}^c \supset \overline{C}$ implies $\overline{W} \subset \overline{C}^c$, we have

$$B(\overline{C})' = \{A(W) \mid W \in \mathcal{W}, \overline{W} \subset \overline{C}^c\}''$$

and

$$B(\overline{C})' \subset A(\overline{C}^c)$$

To prove the reverse inclusion, we turn to the definition of $A(\overline{C}^c)$. $A(\overline{C}^c)$ is generated by the spectral projections of $(\phi\{f\}, D_1)^{**}$, where $f \in S(R^4)$, f real, and supp $f \subset \overline{C}^c$. Let $X \in B(\overline{C})'' = B(\overline{C})$. Let $y \in \overline{C}^c$. Then there exists a wedge W and an open neighborhood N_y of y such that $\overline{W} \subset \overline{C}^c$ and $N_y \subset W$. Under the assumption of the Special Condition we have

$$(\phi[g]\psi\ ,\ X\xi) = (\ X^*\psi,\ \phi[g^*]\xi), \qquad \psi,\xi \in D_1 \quad (74)$$

for all $g \in S(R^4)$ and supp $g \subset N_y$. It immediately follows that (74) holds for all $g \in S(R^4)$ and supp $g \subset \bar{C}^c$. As we have assumed that D_1 is a domain of essential self-adjointness for ϕ smeared with real test functions

$$X(\phi[g],\ D_1)^{**} \subset (\phi[g],\ D_1)^{**}X$$

for all $X \in B(\bar{C})$, and $g \in S(R^4)$, g real, and supp $g \subset \bar{C}^c$. This relation implies that for any spectral projection E associated with $(\phi[g],\ D_1)^{**}$,

$$XE = EX$$

and similarly for all elements of $A(\bar{C}^c)$, which is generated by such spectral projections. Thus, we have

$$B(\bar{C}) \subset A(\bar{C}^c)'$$

and

$$B(\bar{C})' \supset A(\bar{C}^c)$$

which completes the proof.

5. Relation to Tomita-Takesaki Theory

The analysis of sections 3 and 4 is closely related to the Tomita-Takesaki theory of modular Hilbert algebras.[15,19] As the extensive results of this approach yields information concerning factors, types, and symmetries of von Neumann algebras, we wish to establish the precise nature of this relationship. The main theorem (from our point of view) is due to Tomita, and we will state the facts in the following form:

Let A be a von Neumann algebra on a separable Hilbert space with a cyclic and separating vector Ω, and let A' denote its commutant. Then there exists a unique antiunitary involution J_T, and a unique self-adjoint operator $(\Delta, D(\Delta))$, which satisfy the conditions:

a) $$J_T\Omega = \Omega \ , \quad \Omega \in D(\Delta) \ , \quad \Delta\Omega = \Omega \tag{75}$$

b) $$J_T A J_T = A' \tag{76}$$

c) $$J_T D(\Delta) = D(\Delta^{-1}), \quad J_T(\Delta, D(\Delta)) J_T = (\Delta^{-1}, D(\Delta^{-1})) \tag{77}$$

d) $$\Delta^{it} A \Delta^{-it} = A \tag{78}$$

$$\Delta^{it} A' \Delta^{-it} = A' \tag{79}$$

for all real t.

e) If $(S, A\Omega)$ is the antilinear operator defined by

$$S X\Omega = X^*\Omega \quad , \text{ for all } X \in A \tag{80}$$

then

$$(J_T \Delta^{1/2}, D(\Delta^{1/2})) = (S, A\Omega)^{**} \tag{81}$$

In the literature on the subject, Δ is called the modular operator, and the automorphism in d) is the modular automorphism. The relationship of the analysis of Section 4 and Tomita-Takesaki theory for wedge algebra $A(W_R)$ is established in the following theorem:

Theorem 5 : Let ϕ be a hermitian scalar field satisfying the Special Condition and let $A(W_R)$ be the associated von Neumann algebra of the "right wedge" W_R. Let J_T, S, and $(\Delta, D(\Delta))$ be the Tomita operators associated with $A(W_R)$. Then

$$J_T = J \tag{82}$$

$$(\Delta, D(\Delta)) = (V(2i\pi), D_V(2\pi)) \tag{83}$$

Proof: By Theorem 2 we have that $A(W_R)\Omega$ is a core for the operator $(V(i\pi), D_V(\pi))$. As J is an antiunitary involution, $A(W_R)\Omega$ is also a core for $J(V(i\pi), D_V(\pi))$, and by definition is a core for the operator $(S, A(W_R)\Omega)^{**}$

From the relation

$$SX\Omega = X^*\Omega = JV(i\pi)X\Omega \quad , \text{ for all } X \in A(W_R)$$

and the uniqueness of the polar decomposition we have $J_T = J$ and $(\Delta^{1/2}, D(\Delta^{1/2}) = (V(i\pi), D_V(\pi))$ and the theorem follows.

We remark that for $W_\Lambda = \Lambda W_R$, $\Lambda \in \vec{L}_o$, the Tomita J_T and $(\Delta, D(\Delta))$ for $A(W_\Lambda) = U(\Lambda)A(W_R)U^{-1}(\Lambda)$ are respectively $U(\Lambda)JU^{-1}(\Lambda)$ and $U(\Lambda)(V(2i\pi), D_V(2\pi))U^{-1}(\Lambda)$. Also, we note the similarity of our discussion in Section 4 to that of Haag, Hugenholtz and Winnink [20] and Kastler, Pool, and Thue Poulsen[21].

Finally, we state as a lemma a paraphrase of Theorem 13.2 of Takesaki [22] which gives another set of conditions which characterizes the modular operator Δ. This lemma will be used in the next section for a separate discussion of the free hermitian scalar field.

<u>Lemma 9</u> : Let A be a von Neumann algebra with a cyclic and separating vector Ω. Let $U(t)$, t real, be a one-parameter group of unitary operators such that $U(t)\Omega = \Omega$, and such that

$$U(t)AU^{-1}(c) = A, \quad \text{for all real } t \tag{84}$$

Furthermore, for all a, b ε A let there exist a function $F(z)$ continuous in the closed strip $0 \leq \mathrm{Im}(z) \leq 1$ and analytic in the corresponding open strip with boundary values

$$F(t) = (\Omega, aU^{-1}(t)b\Omega) \tag{85}$$

$$F(t+i) = (\Omega, bU(t)a\Omega) \tag{86}$$

for t real. Then $U(t) = \Delta^{it}$, where Δ is the self-adjoint modular operator for the von Neumann algebra A.

6. Duality for Free Field

For a free hermitian scalar field we consider the von Neumann algebra $A_o(W_R)$ generated by the unitary operators $\exp(i\phi[f])$, where $f \in S(R^4)$, f real, and supp $f \subset W_R$. Since the vacuum expectation values of these operators are now explicitly available, we present a separate proof of duality for "wedge" algebras by direct computation.

For any f, g $\in S(R^4)$ and real

$$(\Omega, \exp(i\phi[f])\exp(i\phi[g])\Omega) =$$

$$\exp(-\tfrac{1}{2}[f,f] - [f,g] - \tfrac{1}{2}[g,g]) \tag{87}$$

where, for example,

$$[f,g] = \frac{1}{2(2\pi)^3} \int_{(-)} \frac{d^3p}{\omega_p} \tilde{f}(-\vec{p},-\omega_p)\tilde{g}(\vec{p},\omega_p) \tag{88}$$

$$\omega_p = \sqrt{p^2 + m^2}$$

Let supp $f \subset W_R$ and supp $g \subset W_R$. Consider the function

$$F(t) = (\Omega, \exp(i\phi[f])V(+2\pi t)\exp(i\phi[g])\Omega)$$

$$= (\Omega, V(-\pi t)\exp(i\phi[f])V(+\pi t)V(+\pi t)\exp(i\phi[g])V(-\pi t)\Omega)$$

$$= (\Omega,\ \exp(i\phi[V(\underset{\sim}{e}_3,-\pi t)f])\exp(i\phi[V(\underset{\sim}{e}_3,+\pi t)g])\Omega) \tag{89}$$

From equation (87) we have

$$F(t) = \exp\Big(-\tfrac{1}{2}\ [V(\underset{\sim}{e}_3,-\pi t)f, V(\underset{\sim}{e}_3,-\pi t)f] - [V(\underset{\sim}{e}_3,-\pi t)f, V(\underset{\sim}{e}_3,+\pi t)g] -\tfrac{1}{2}\ [V(\underset{\sim}{e}_3,+\pi t)g, V(\underset{\sim}{e}_3,+\pi t)g]\Big) \tag{90}$$

By Lorentz invariance the first and third terms in the exponential are actually constant functions of t and the second term is explicitly

$$\frac{1}{16\pi^3}\int_{(\infty)} \frac{d^3p}{\omega_p}\ \tilde{f}(V(\underset{\sim}{e}_3,+\pi t)(-\vec{p},-\omega_p))\tilde{g}(V(\underset{\sim}{e}_3,-\pi t)(p,\omega_p)) \tag{91}$$

Consider the equation

$$\tilde{f}(V(\underset{\sim}{e}_3,\ \pi t)(-\vec{p},-\omega_p)) = \int_{(\infty)} d^4x\ \exp(-ip\cdot V(\underset{\sim}{e}_3,-\pi t)x) f(x) \tag{92}$$

where $p = (\vec{p},\omega_p)$. For $x \in W_R$, $-\text{Im}(V(\underset{\sim}{e}_3,-\pi t)x) \in V_+$ for $0 < \text{Im}(t) < 1$. Thus for $f \in S(R^4)$ and $\text{supp } f \subset W_R$, $\tilde{f}(V(\underset{\sim}{e}_3,\pi t)(-\vec{p},-\omega_p)) \in S(R^3)$ in $\vec{p}$ for $0 \le \text{Im}(t) \le 1$, and is analytic in t in the corresponding open strip.

By a similar argument for $\tilde{g}(V(e_3,-\pi t)(\vec{p},\omega_p))$, we have that (91) is well-defined for t in the closed strip $0 \leq \mathrm{Im}(t) \leq 1$, and is continuous in t in the closed strip and analytic in t in the open strip. For $\mathrm{Im}(t) = 1$ it has the boundary value

$$\int_{(\infty)} \frac{d^3p}{\omega_p} \tilde{f}(V(e_3,+\pi s)(\vec{p},\omega_p))\tilde{g}(V(e_3,-\pi s)(-\vec{p},-\omega_p)) \quad (93)$$

where $s = \mathrm{Re}(t)$. But since (93) is just the expression for $[V(e_3,+\pi s)g, V(e_3,-\pi s)f]$, we have

<u>Lemma 10</u>: Let f, g ε $S(R^4)$, real, and supp f $\subset W_R$ and supp g $\subset W_R$. Then there exists a function $F(z)$ continuous in the closed strip $0 \leq \mathrm{Im}(z) \leq 1$, and analytic in the interior of the strip with boundary values

$$F(t) = (\Omega, \exp(i\phi[f])V(2\pi t)\exp(i\phi[g])\Omega) \quad (94)$$

$$F(t+i) = (\Omega, \exp(i\phi[g])V(-2\pi t)\exp(i\phi[f])\Omega) \quad (95)$$

for all real t.

Thus, for operators of the form $\exp(i\phi[f])$ ε $A_o(W_R)$ we have the conditions of Lemma 9 satisfied with $U(t) = V(-2\pi t)$, since $V(-2\pi t)A_o(W_R)V(2\pi t) = A_o(W_R)$ for all real t. We will now extend this result to all operators in $A_o(W_R)$.

First consider operators of the form

$$X = \sum_{n=1}^{m} a_n \exp (i\phi[f_n]) \tag{96}$$

with a_n complex, and real $f_n \in S(R^4)$ and supp $f \subset W_R$.
This set of operators is in fact a polynomial algebra since

$$\exp(i\phi[f])\exp(i\phi[g]) = (\text{constant})\exp(i\phi[f+g])$$

and we denote this set by $G_o(W_R)$.

Lemma 11: Let $X, Y \in G_o(W_R)$. Then there exists a function $F(z)$ continuous in the closed strip $0 \leq \mathrm{Im}(z) \leq 1$ and analytic in the interior of the strip with boundary values

$$F(t) = (\Omega, XV(2\pi t)Y\Omega) \tag{97}$$

$$F(t+i) = (\Omega, YV(-2\pi t)X\Omega) \tag{98}$$

for all real t.

Proof: Since X and Y are of the form (96), this lemma is a trivial consequence of Lemma 10.

Theorem 6: Let $X, Y \in A_o(W_R)$. Then:

a) There exists a function $F(z)$ continuous in the closed strip $0 \leq \mathrm{Im}(z) \leq 1$ and analytic in the interior of the strip with boundary values

$$F(t) = (\Omega, XV(2\pi t)Y\Omega) \tag{99}$$

$$F(t+i) = (\Omega, YV(-2\pi t)X\Omega) \tag{100}$$

for all real t.

b) $V(-2\pi t) = \Delta^{it}$, where Δ is the modular operator for the algebra $A_o(W_R)$.

c)
$$A_o(W_R)' = JA_o(W_R)J = A_o(W_L) \tag{101}$$

Proof: Since $A_o(W_R) = G_o(W_R)''$, for any $X, Y \in A_o(W_R)$ there exist bounded sequences of operators $X_n, Y_n \in G_o(W_R)$ such that [23]

$$\underset{n\to\infty}{s\text{-}\lim} X_n\Omega = X\Omega \qquad \underset{n\to\infty}{s\text{-}\lim} X_n^*\Omega = X^*\Omega$$

$$\underset{n\to\infty}{s\text{-}\lim} Y_n\Omega = Y\Omega \qquad \underset{n\to\infty}{s\text{-}\lim} Y_n^*\Omega = Y^*\Omega$$

Thus we have

$$(\Omega, XV(+2\pi t)Y\Omega) = \lim_{n\to\infty} (X_n^*\Omega, V(+2\pi t)Y_n\Omega)$$

$$(\Omega, YV(-2\pi t)X\Omega) = \lim_{n\to\infty} (Y_n^*\Omega, V(-2\pi t)X_n\Omega)$$

By Lemma 11 there exists a function $F_n(z)$ continuous in the closed strip $0 \leq \mathrm{Im}(z) \leq 1$, and analytic in the interior of the strip, with boundary values (t real)

$$F_n(t) = (X_n^*\Omega, V(+2\pi t)Y_n\Omega)$$

$$F_n(t+i) = (Y_n^*\Omega, V(-2\pi t)X_n\Omega)$$

$F_n(t)$ and $F_n(t+i)$ are uniformly bounded with respect to t, and converge uniformly. Therefore

$F_n(z)$ converges to a function $F(z)$ which is continuous in closed strip, and analytic in the interior of the strip with boundary values (t real)

$$F(t) = (\Omega, XV(+2\pi t)Y\Omega)$$

$$F(t+i) = (\Omega, YV(-2\pi t)X\Omega)$$

Finally noting that $V(t)A_o(W_R)V^{-1}(t) = A_o(W_R)$, for all real t, the conditions of Lemma 9 are satisfied and $V(-2\pi t) = \Delta^{it}$, and a) and b) are proved. Moreover, $\Delta = V(2\pi i)$.

By direct computation it is seen that $V(i\pi)\exp(i\phi[f])\Omega = \exp(i\phi[f^j])\Omega$, for $\exp(i\phi[f]) \in A_o(W_R)$. From the defining relation

$$J_T(\exp(i\phi[f]))\Omega = \Delta^{1/2}(\exp(i\phi[f]))^*\Omega$$

for the Tomita J_T associated with the algebra $A_o(W_R)$, we have

$$\begin{aligned} J_T(\exp(i\phi[f]))\Omega &= V(i\pi)(\exp(-i\phi[f]))\Omega \\ &= \exp(-i\phi[f^j])\Omega \\ &= J(\exp(i\phi[f]))\Omega \end{aligned}$$

Therefore, $J_T = J$, and c) follows immediately from (76).

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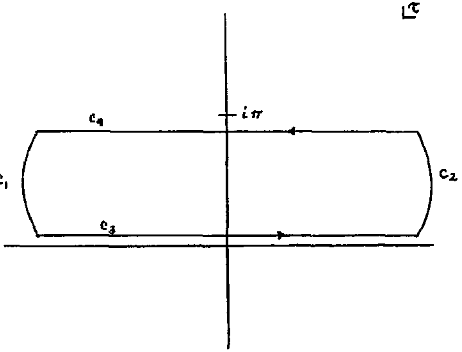

Figure 1