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Symmetry and Exact Dyon Solutions for Classical Yang-Mills Field Equations*

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Abstract

We show that the conserved magnetic and electric charges in non-Abelian theories have nothing to do with the Higgs scalars and/or the symmetry structure of the Lagrangian. They are a consequence of the <u>local</u> isospin gauge symmetry. We present several exact static dyon solutions to the non-linear classical field equations in both massless and massive Yang-Mills theories, which possess both electric and magnetic charges.

1. Introduction

Yang and Mills emphasized that local isospin gauge symmetry leads to gauge fields B_μ^a and completely determines the interaction dynamics of the gauge fields.[1] Several particular solutions to the classical Yang-Mills field equations have been discussed.[2-5] Recently, it was pointed out that because of the local isospin gauge symmetry, one can introduce a unit isovector, e.g., $v^a = r^a/r$, to connect the gauge field B_μ^a and the Abelian electromagnetic field tensor $\bar{F}_{\mu\nu}$ associated with the magnetic monopole in such a way that (a) $\bar{F}_{\mu\nu}$ is isospin gauge invariant and (b) $\bar{F}_{\mu\nu}$ reduces to the usual electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ if $v^a = (0,0,1)$ and B_3^μ is identified with the photon field A^μ.[6] We stress that this connection is determined by symmetry, not by dynamics. In this way, local isospin gauge symmetry also leads to a magnetic monopole with a magnetic charge $g = 1/e$ when $v^a = r^a/r$ and $B_o^a = 0$ in the Yang-Mills theory.

In this paper, we consider the static solutions with $B_o^a \neq 0$ and we obtain the dyon solution, having both magnetic and electric charges. We also find an exact static solution B_μ^a for which the total energy of the system is finite, but the solutions B_μ^a are non-pointlike and the dyon carries an imaginary electric charge. It is shown that the conserved magnetic and electric charges in non-Abelian theories have nothing to do with the Higgs scalars and/or the symmetry

structure of the Lagrangian. Rather, they are a consequence of the local isospin gauge symmetry. A massive Yang-Mills theory is also considered; it is found that in this case the dyon solution exists too.

2. Special Static Spherically Symmetric Solutions

The classical field equations for the Yang-Mills field B_μ^a are[1]

$$B_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + e\epsilon^{abc} B_\mu^b B_\nu^c \ , \qquad B_3^\mu \equiv A^\mu \ , \tag{1}$$

$$\partial B_{\mu\nu}^a + e\epsilon^{abc} B_b^\nu B_{\mu\nu}^c = 0 \ , \tag{2}$$

$$\partial^\mu B_\mu^a = 0 \ . \tag{3}$$

We look for the static spherically symmetric solution of the form,[3]

$$B_i^a = \epsilon_{iab} r^b B(r)/r \ , \qquad i,a,b = 1,2,3 \ , \tag{4}$$

$$B_o^a = r^a G(r)/r \ , \tag{5}$$

The constraint equation (3) is automatically satisfied. Equation (2) can be written as

$$\partial_i\partial_i B - 2B/r^2 - 3eB^2/r - e^2B^3 + \frac{eG^2}{r}(1 + erB) = 0 \ , \qquad (6)$$

$$\partial_i\partial_i G - 2G/r^2 - 4eBG/r - 2e^2B^2G = 0 \ . \qquad (7)$$

Introducing the variable $C(r) = 1 + erB(r)$, we have

$$r^2 \frac{d^2C}{dr^2} = C(C^2 - 1) - e^2r^2G^2C \ , \qquad (8)$$

$$\frac{d}{dr}(r^2 \frac{dG}{dr}) = 2GC^2 \ . \qquad (9)$$

The trivial solution $C = 0$ to (8) leads to the nontrivial special solution,

$$B(r) = -1/(er) \ ,$$
$$G(r) = G_o/r + G_1 \ , \quad G_o = e \ , \qquad (10)$$

where G_o and G_1 are constants of integration and $G_o = e$ because of the requirement $\mathcal{F}_{\mu\nu} = F_{\mu\nu}$ when $v^a = (0,0,1)$.

We try to find a solution $\mathcal{B}$ which is regular at $r = 0$. *By a stroke of luck, we find the following solution to* (8) and (9), (Note that here C and G can be either positive or negative.)

$$C(r) = \alpha r/\sin\alpha r \ ,$$
$$G(r) = i(\alpha r \cos\alpha r - \sin\alpha r)/(er \sin\alpha r) \qquad (11)$$

where α is an arbitrary constant. Unfortunately, these solutions are undefined as $r \to \infty$ if α is real. *However, if α is imaginary, i.e., $\alpha = i\beta$ with β real, we have an exact solution* to equations (8) and (9),

$$C(r) = \frac{\beta r}{\sinh\beta r} \quad \text{or} \quad B(r) = \frac{\beta r - \sinh\beta r}{er \sinh\beta r} \ ,$$
$$G(r) = \frac{i}{er \sinh\beta r}(\beta r \cosh\beta r - \sinh\beta r) \ , \qquad (12)$$

which can be easily verified. These solutions are regular at both $r = 0$ and $r = \infty$.[7] We note that such a finite solution for arbitrary r is possible because B_o^a (or $G(r)$) is non-vanishing. If B_o^a is set equal to zero, Equation (8) becomes $r^2d^2C/dr^2 = C(C^2 - 1) \approx C^3$ in the region $C^2 \gg 1$, say $r \approx r_f$, where we have the asymptotic solution $C \approx \sqrt{2}r_f/(r_f - r)$, $r \approx r_f$. In this case, *the nontrivial solution $C(r)$ diverges at some finite* $r = r_f$.[5]

On the other hand, if one looks for the static solution of the form,

$$B_i^a = r^i f^a(r)/r \ , \quad B_o^a = g^a(r) \ , \qquad (13)$$

one has *the Ikeda-Miyachi solution*[2]

$$f^a(r) = A^a/r^2 \;, \quad A^a = \text{integration constants} \;,$$
$$g^a(r) = \left(1 - \frac{k}{r}\right)\left[a'^a + b'^a \cos\left(\frac{eA}{r}\right) + c'^a \sin\left(\frac{eA}{r}\right)\right] \;, \quad k = \text{const.} \tag{14}$$

where $A \equiv |\vec{A}|$, $a'^a = \text{const.}\; A^a$, $|\vec{b}'| = |\vec{c}'|$ and $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ form a right-handed orthogonal system. It is interesting to note that if $\vec{A} \times \vec{g} \neq 0$ the vector $\vec{g}$ appears to rotate around the $\vec{A}$ axis, with the angle between $\vec{A}$ and $\vec{g}$ constant everywhere, when we travel from one point to another point in space.

3. Local Gauge Symmetry and the Dyon

What is the "physical" meaning of these classical solutions? One may regard the 6-vector $B^a_{\mu\nu}$ as composed by isovector "electric" and isovector "magnetic" fields:[3]

$$E^a_j = B^a_{jo} \;, \quad H^a_k = \tfrac{1}{2}\epsilon_{kij}B^a_{ij} \;, \tag{15}$$

which do not correspond to a physical reality in nature. It would be more interesting and meaningful if $B^a_{\mu\nu}$ can be related to the usual electromagnetic fields. Indeed, this is possible because of the local gauge symmetry.[6]

Based on local isospin gauge symmetry, we can introduce a unit isovector $v^a v^a = 1$, e.g., $v^a = r^a/r$, and construct a generalized electromagnetic field tensor $\bar{F}_{\mu\nu}$ more closely related to $F_{\mu\nu}$, so that we may actually split the 6-vector $\bar{F}_{\mu\nu}$ into the usual electric and magnetic fields associated with interesting objects such as the magnetic monopoles or the dyons. Their existence has been speculated in relation to the fundamental problem of charge quantization and discussed by many authors.[8] To accomplish this in the Yang-Mills theory with $B^\mu_3 \equiv A^\mu$, it is natural that $\bar{F}_{\mu\nu}$ satisfies the requirements: (i) $\bar{F}_{\mu\nu}$ is gauge invariant and (ii) $\bar{F}_{\mu\nu}$ can be reduced to the usual $F_{\mu\nu}$ when $v^a = (0,0,1)$. We define[6]

$$\bar{F}_{\mu\nu} = v^a B^a_{\mu\nu} - e^{-1}\epsilon^{abc} v^a (D_\mu v^b)(D_\nu v^c) \;,$$
$$v^a v^a = 1 \;, \quad (D_\mu v^b) = \partial_\mu v^b + e\epsilon^{bcd} B^c_\mu v^d \;, \tag{16}$$

which is indeed gauge invariant and $\bar{F}_{\mu\nu} = F_{\mu\nu}$ when $v^a = (0,0,1)$. Now, the electric and the magnetic fields, E_j and H_k, are related to $\bar{F}_{\mu\nu}$ by

$$E_j = \bar{F}_{jo} \;, \quad H_k = \tfrac{1}{2}\epsilon_{kij}\bar{F}_{ij} \;. \tag{17}$$

The definition (16) is interesting because E_j and H_k in (17) can be interpreted as the fields produced by the magnetic monopole or the dyons. Since $v^a v^a = 1$, we may write (16) as

$$\bar{F}_{\mu\nu} = \partial_\mu(v^a B^a_\nu) - \partial_\nu(v^a B^a_\mu) - e^{-1}\epsilon^{abc} v^a (\partial_\mu v^b)(\partial_\nu v^c) \;. \tag{18}$$

If the theory involves a scalar field ϕ^a, one usually defines[7,9] $\bar{F}_{\mu\nu}$ in (16) with v^a replaced by $\phi^a/|\vec{\phi}|$, as proposed by 't Hooft.[8] We stress that the ratio $\phi^a/|\vec{\phi}|$ has nothing to do with dynamics, although ϕ^a is determined by dynamics. The reason is that one looks for solutions of the form $\phi^a = r^a\xi(r)$, where $\xi(r)$ is the only quantity determined dynamically. Therefore, $\phi^a/|\vec{\phi}|$ has nothing to do with $\xi(r)$ or dynamics, just like the unit isovector v^a. Note that the form $\phi^a = r^a\xi(r)$ is determined by symmetry consideration alone.

To see the sources which generate the field $\bar{F}_{\mu\nu}$, let us define the magnetic current[9] k_λ and the electric current j_μ:

$$k_\lambda = \frac{1}{2}\epsilon_{\lambda\rho\mu\nu}\partial^\rho\bar{F}^{\mu\nu} , \tag{19}$$

$$j_\mu = \partial^\nu\bar{F}_{\mu\nu} . \tag{20}$$

They are both obviously conserved currents,

$$\partial^\lambda k_\lambda = 0 , \quad \partial^\mu j_\mu = 0 , \tag{21}$$

and hence the *magnetic charge*[10] M and the electric charge Q are conserved:

$$\frac{d}{dt} M = \frac{d}{dt}\int k_o d^3r/(4\pi) = 0 ,$$
$$\frac{d}{dt} Q = \frac{d}{dt}\int j_o d^3r/(4\pi) = 0 . \tag{22}$$

When the fields B_μ^a are free from singularity lines, we have the identity,

$$\epsilon_{\lambda\rho\mu\nu}\partial^\rho[\partial_\mu(v^aB_\nu^a) - \partial_\nu(v^aB_\mu^a)] = 0 \tag{23}$$

and the magnetic current k_λ can be written as[9]

$$k_\lambda = -(1/2e)\epsilon_{\lambda\rho\mu\nu}\epsilon^{abc}\partial^\rho v^a\partial^\mu v^b\partial^\nu v^c \tag{24}$$

The relations (18)-(24) hold for arbitrary local unit isovector $v^a(\vec{r},t)$, $v^av^a = 1$. It is striking that these results are the consequence of the symmetry embedded in (18) and have nothing to do with the interaction dynamics of the system. We note that, in fact, symmetry is the most basic concept in gauge theories. The concept of symmetry leads to the gauge fields B_μ^a and completely determine the interaction dynamics of B_μ^a.[1,3] In the same sense, we may regard the dyon as the consequence of local isospin gauge symmetry.

4. The Dyons

For the special type of solution given by (4) and (5), we have

$$\vec{E} = \vec{\nabla}G(r) \ , \tag{19}$$

$$\vec{H} = -\vec{r}/(er^3) \ , \tag{20}$$

The fields E_j and H_k related to the solution (10) are

$$\vec{E} = -e\vec{r}/r^3 \ , \quad \vec{H} = -\vec{r}/(er^3) \ , \tag{21}$$

This shows the presence of a stable dyon[11] at $\vec{r} = 0$ with an electric charge e and a magnetic charge $g = 1/e$, satisfying the Schwinger condition $eg = 1$.[8]

One may wonder why such a dyon with a point-source is possible. The answer is that the solution (10) is singular at $r = 0$ and, therefore, the solution (10) has a δ-function type source[3] S^a_μ defined by

$$S^a_\mu \equiv \partial^\nu B^a_{\mu\nu} + e\epsilon^{abc}B^\nu_b B^c_{\mu\nu}$$

$$\propto \begin{cases} \epsilon_{\mu ab}v^b\delta^3(\vec{r}) \ , & \mu = 1,2,3 \ , \quad v^b = r^b/r \ , \\ v^a\delta^3(\vec{r}) \ , & \mu = 0 \ . \end{cases}$$

However, the magnetic charge g in (21) has nothing to do directly to do with this source S^a_μ. (See Sec. 3).

The solution (12) leads to the following electric and magnetic fields:

$$\vec{E} = \frac{i}{e}\frac{\vec{r}}{r^3}\left[1 - \frac{\beta^2 r^2}{(\sinh\beta r)^2}\right] \ , \quad \vec{H} = -\vec{r}/(er^3) \tag{22}$$

Unfortunately, this corresponds to a dyon with a point magnetic charge surrounded by a cloud of imaginary electric charge without a point-like core.

The Ikeda-Miyachi solution (13) and (14) gives

$$\vec{E} = \vec{r}k(\vec{r}\cdot\vec{g})/[r^4(1-k/r)] + \vec{g}/r - \vec{r}(\vec{r}\cdot\vec{g})/r^3$$
$$+ \vec{r}(1-k/r)eA[\vec{r}\cdot\vec{b}' \sin(eA/r) - \vec{r}\cdot\vec{c}' \cos(eA/r)]/r^4 \ , \tag{23}$$

$$\vec{H} = -\vec{r}/(er^3) + \vec{A} \times \vec{r}/r^4 \ . \tag{24}$$

The magnetic flux is

$$\int \vec{H}\cdot d\vec{s} = -\frac{4\pi}{e} \tag{25}$$

because the second term in (24) does not originate from the magnetic charge. The electric field $\vec{E}$ is a rapidly oscillating function of r as $r \to 0$, and there is no simple picture for this solution.

5. Static Energy

The static energy of the system is given by the Hamiltonian of the B^a_μ field:

$$\mathcal{H} = \int d^3r[\tfrac{1}{4}B^a_{ij}B^a_{ij} - \tfrac{1}{2}B^a_{oi}B^a_{oi}]$$

$$= \frac{4\pi}{e^2}\int_0^\infty dr[(\frac{dC}{dr})^2 + \frac{(C^2-1)^2}{2r^2} - e^2G^2C^2 - \frac{e^2}{2}(\frac{dG}{dr})^2\ r^2] \qquad (26)$$

which is divergent for the solutions (10) and (11). The Ikeda-Miyachi solution (13) and (14) also leads to a divergent energy. The remarkable feature of the solution (12) is that it gives a positive and finite field energy:

$$\mathcal{H} = \frac{4\pi}{e^2}|\beta|\ , \qquad (\because\ \frac{d}{dr}[Gr^2\frac{dG}{dr}] = r^2(\frac{dG}{dr})^2 + 2G^2C^2)\ , \qquad (27)$$

where β is the constant of integration. This finiteness is intimately related to the non-pointlike nature of the solution (12).

6. Massive Yang-Mills Fields

From experimental viewpoint, if the dyon exists, it is probably massive. We wish to point out that the above discussions hold also for a theory involving massive Yang-Mills

fields $f^a_\mu(x)$. Let us consider the Lagrangian[12]

$$\mathcal{L} = -\frac{1}{4}\vec{f}_{\mu\nu}\cdot\vec{f}^{\mu\nu} + \frac{1}{2}M_f^2\vec{f}_\mu^2 + \frac{1}{2}\partial_\mu U\partial^\mu U$$

$$+ \frac{1}{2}\partial_\mu\vec{\phi}\cdot(\partial^\mu + e\vec{f}^\mu\times)\vec{\phi} - \frac{1}{2}e\vec{f}_\mu\cdot(U\partial^\mu\vec{\phi} - \vec{\phi}\partial^\mu U)$$

$$+ \frac{1}{8}e^2\vec{f}_\mu^2(\vec{\phi}^2 + U^2) + \frac{1}{2}eM_f\vec{f}_\mu^2U + M_f\vec{\phi}\partial_\mu\vec{f}^\mu$$

$$- \frac{1}{2}\xi(\partial_\mu\vec{f}^\mu + M_f\vec{\phi}/\xi)^2 \ . \qquad (28)$$

$$\vec{f}_{\mu\nu} = \partial_\mu\vec{f}_\nu - \partial_\nu\vec{f}_\mu + e\vec{f}_\mu\times\vec{f}_\nu \ .$$

which is, except the last gauge-fixing term with the parameter[13] ξ, invaraint under distorted local SU(2) gauge transformation. The Lagrangian (28) leads to field equations for $f^a(x)$, $\phi^a(x)$ and $U(x)$, and $\phi^a(x) = 0$ is a trivial solution. In analogy with (4) and (5), we set $f^a_i = \varepsilon_{iab}v^b\bar{B}(r)$, $f^a_o = v^a\bar{G}(r)$, $v^a = r^a/r$. We find that

$$\nabla^2\bar{B} - \frac{2}{r^2}\bar{B} - \frac{3e}{r}\bar{B}^2 - e^2\bar{B}^3 + \frac{e\bar{G}^2}{r}(1 - er\bar{B}) - \frac{e^2}{4}\bar{B}(U + \frac{2M_f}{e})^2 = 0 \qquad (29)$$

$$\nabla^2\bar{G} - \frac{2}{r^2}\bar{G} - \frac{4e}{r}\bar{B}\bar{G} - 2e^2\bar{B}^2\bar{G} - \frac{e^2}{4}\bar{G}(U + \frac{2M_f}{e})^2 = 0\ , \qquad (30)$$

$$\nabla^2U + \frac{e^2}{4}(\bar{G}^2 - 2\bar{B}^2)(U + 2M_f/e) = 0 \ . \qquad (31)$$

By inspection, we see that $U = -2M_f/e$ is a trivial solution to Equation (31), and Equations (29) and (30) reduces to Equation (6) and (7) respectively. Therefore, all subsequent discussions hold equally well for the Lagrangian (28) involving massive Yang-Mills fields.

7. Remarks and Conclusions

In the solutions (10) and (14), B_o^a may be zero and the nonvanishing B_i^a leads to the magnetic monopole.[6] When $B_o^a = 0$, the static particle-like solution with finite energy such as (12) has not been found.[5,6] We conjecture that the nonvanishing B_o^a is necessary for a positive and finite energy.

The electromagnetic field tensor $F_{\mu\nu}$ defined in (16) is not a direct dynamical consequence of the Lagrangian in non-Abelian gauge theory. In Ref. 9, the conserved magnetic charge in non-Abelian gauge theories is regarded as the consequence of the topological structure of three Higgs scalar field in a three-dimensional space. However, from our discussions, the conservations of the electric charge Q and the magnetic charge M, as shown in (22), has nothing to do with the symmetry structure of the Yang-Mills Lagrangian and/or the existence of Higgs scalars. Rather, they follow from the basic concept of local gauge symmetry, as discussed in Sec. 3.

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respectively. See Ref. 9.

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