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Violation of Unitarity in Weinberg's Unified Theory With Bilinear Gauge Conditions*

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Abstract

The usual formalism for non-Abelian gauge theories is not completely satisfactory: Calculations show that the usual formalism for Weinberg's unified gauge theory with a bilinear gauge condition leads to violation of unitarity, contrary to general *formal proofs of unitarity*. *In contrast*, our Lagrange multiplier formalism for this theory gives rise to unitary amplitudes; their unitarity has been checked explicitly at the 1-loop and the 2-loop levels. This shows the general validity of the Lagrange multiplier formalism and its advantages over the usual formalism.

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1. Introduction

In the usual formalism for general non-Abelian gauge theories, there are general formal proofs of gauge independence and unitarity for gauge theories with suitable (linear or bilinear) gauge conditions.[1,2] These lead to a widely-held belief that conventional non-Abelian gauge theories and spontaneously broken gauge theories based on the usual formalism[3] lead to unitary amplitudes for arbitrary gauge parameters, no matter whether one chooses linear or nonlinear gauge conditions. We show that this is not true because the usual formalism leads to violation of unitarity when one chooses a class of bilinear gauge conditions.

So far, all the works in this field involve only the linear gauge condition, e.g.,

$$\partial^\mu W_\mu^\pm \pm iM\phi^\pm/\xi = 0 \quad . \qquad (1)$$

In previous papers,[4] we show that a new Lagrange multiplier formalism (LM formalism, for short) leads to the same effective Lagrangian in Weinberg's unified theory[5,6] as that obtained in the usual formalism, if one chooses linear gauge conditions. In this case, unitarity and gauge invariance have been explicitly verified.[4] Hitherto, bilinear gauge

conditions such as

$$(\partial^\mu \mp ieA^\mu)W_\mu^\pm \pm iM\phi^\pm/\xi = 0 \qquad (2)$$

have not been explored. It was taken for granted that there is no special problem for bilinear gauge conditions.

In this paper, we study bilinear gauge conditions and show that with bilinear gauge conditions in the usual formalism we have trouble.[7] Explicit calculations show that the usual formalism for Weinberg's unified theory with bilinear gauge conditions leads to violation of unitarity, contrary to general formal proof and usual belief that it satisfies unitarity. Furthermore, we show that if the theory with bilinear gauge conditions is formulated with the LM formalism, it is unitary. This has been checked at both the 1-loop and the 2-loop levels. The main difference between these two different formalisms lies in the fictitious Lagrangian (f-Lagrangian, for short) which involves fictitious scalar-fermions. The reason for this difference is discussed.

2. Usual Formalism with Bilinear Gauge Conditions

Let us consider the usual formalism for the Weinberg unified theory with *bilinear gauge conditions:*

$$(\partial^\mu \mp ieA^\mu)W^\pm_\mu \pm iMS^\pm/\xi = a^\pm \quad , \qquad (3a)$$

$$\partial^\mu A_\mu + \beta A_\mu A^\mu = a_A \quad , \qquad (3b)$$

$$\partial^\mu Z_\mu + \beta' Z_\mu Z^\mu - \frac{iM\sec\theta}{\sqrt{2}\,\eta}(S^0 - \bar{S}^0) = a_Z \quad , \qquad (3c)$$

where $a^\pm$, a_A and a_Z are some c-number functions. The Weinberg Lagrangian can be written as

$$L = L_{W1} + L_G \quad , \qquad (4)$$

$$\begin{aligned} L_{W1} = &-\tfrac{1}{2}|\partial_\mu W^+_\nu - \partial_\nu W^+_\mu + ie(W^+_\mu A_\nu - W^+_\nu A_\mu) - iG\cos^2\theta(W^+_\mu Z_\nu - W^+_\nu Z_\mu)|^2 \\ &-\tfrac{1}{4}|\partial_\mu A_\nu - \partial_\nu A_\mu - ie(W^-_\nu W^+_\mu - W^+_\nu W^-_\mu)|^2 \\ &-\tfrac{1}{4}|\partial_\mu Z_\nu - \partial_\nu Z_\mu + iG\cos^2\theta(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu)|^2 \\ &+|\partial_\mu S^+ + iG\cos\theta W^+_\mu S^0/\sqrt{2} - i(eA_\mu - G\cos 2\theta\, Z_\mu/2)S^+ + iMW^+_\mu|^2 \\ &+|\partial_\mu S^0 - iGZ_\mu S^0/2 + iG\cos\theta W^-_\mu S^+/\sqrt{2} - iM\sec\theta\, Z_\mu/\sqrt{2}|^2 \end{aligned}$$

$$-\lambda[S^+S^- + (S^0 + V/\sqrt{2})(\bar{S}^0 + V/\sqrt{2}) - V^2/2]^2 \qquad (5)$$

$$e = -G\cos\theta\sin\theta \quad , \quad S^0 = (\psi + i\chi)/\sqrt{2} \quad , \quad \bar{S}^0 = (\psi - i\chi)/\sqrt{2} \quad ,$$

$$\begin{aligned} L_G = &-\xi|(\partial^\mu - ieA^\mu)W^+_\mu + iMS^+/\xi|^2 \\ &-\frac{1}{2\alpha}(\partial_\mu A^\mu + \beta A_\mu A^\mu)^2 \\ &-\eta(\partial^\mu Z_\mu + \beta' Z_\mu Z^\mu + M\sec\theta\,\chi/\eta)^2 \quad , \qquad (6) \end{aligned}$$

$$\xi > 0 \quad , \quad \eta > 0 \quad .$$

For simplicity, we shall not include leptons because their presence does not lead to any new physical problem and, in fact, does not change the resultant f-Lagrangian we obtained in both the usual formalism and the LM formalism.

We note that the terms coming from the fourth line in (5), i.e.,

$$-iMW^-_\mu(\partial^\mu S^+ - ieA^\mu S^+) + iMW^+_\mu(\partial^\mu S^- + ieA^\mu S^-) \quad ,$$

are exactly cancelled by the terms due to L_G in (6), i.e.,

$$-\xi(iMS^+/\xi)(\partial^\mu + ieA^\mu)W^-_\mu - \xi(-iMS^-/\xi)(\partial^\mu - ieA^\mu)W^+_\mu \quad ,$$

where integrations by parts in the Lagrangian $\int d^4x L$ are used. Thus, there is neither a WSγ vertex nor a W_μ-S transition propagator in the Weinberg theory with the bilinear gauge conditions (3). We may remark that one has a nonvanishing WSγ vertex in the theory with linear gauge conditions.

In the usual formalism,[3] the f-Lagrangian (i.e., the gauge compensating terms) is obtained by considering the change of gauge conditions (3) under the infinitesimal transformation

$$W_\mu^\pm \to W_\mu^\pm \mp iW_\mu^\pm \Lambda_3 \pm iW_\mu^3 \Lambda^\pm + \frac{1}{g}\partial_\mu \Lambda^\pm \quad ,$$

$$W_\mu^3 \to W_\mu^3 + i(W_\mu^+ \Lambda^- - W_\mu^- \Lambda^+) + \frac{1}{g}\partial_\mu \Lambda_3 \quad ,$$

$$B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu (2\Lambda_0) \quad ,$$

$$S^+ \to S^+ - \frac{i}{2}S^+ \Lambda_{3+} - \frac{i}{\sqrt{2}} S^0 \Lambda^+ - \frac{iM}{g}\Lambda^+ \quad ,$$

$$S^0 \to S^0 + \frac{i}{2}S^0 \Lambda_{3-} - \frac{i}{\sqrt{2}} S^+ \Lambda^- + \frac{iM_Z}{\sqrt{2}G}\Lambda_{3-} \quad , \qquad (7)$$

where $g = G\cos\theta$, $g' = G\sin\theta$, $\Lambda_{3-} = \Lambda_3 - 2\Lambda_0$, $\Lambda_{3+} = \Lambda_3 + 2\Lambda_0$, $W_\mu^\pm = (W_{1\mu} \mp iW_{2\mu})/\sqrt{2}$, $\Lambda^\pm = (\Lambda_1 \mp i\Lambda_2)/\sqrt{2}$, and W_μ^3 and B_μ are related to A_μ and Z_μ by $A_\mu = \sin\theta\, W_\mu^3 + \cos\theta\, B_\mu$, and $Z_\mu = \cos\theta\, W_\mu^3 - \sin\theta\, B_\mu$. The transformation (7) is the infinitesimal local gauge transformation for the SU(2) x U(1) group associated with the Weinberg Lagrangian. We find that the change

of the bilinear gauge conditions (3) under the transformation (7) are

$$\delta[\partial^\mu W_\mu^\pm \mp ieA^\mu W_\mu^\pm \pm iMS^\pm/\xi]$$

$$= \frac{1}{G\cos\theta}\{(\Box + M^2/\xi)\Lambda^\pm + \frac{G\cos\theta\, M}{2\xi}\Lambda^\pm(\psi \pm i\chi) \pm iG\cos^2\theta\, \partial_\mu(Z^\mu\Lambda^\pm)$$

$$\mp ie\partial_\mu(A^\mu\Lambda^\pm) \pm ie\cos\theta\, \partial^\mu(W_\mu^\pm\Lambda_A) \mp iG\cos^3\theta\, \partial^\mu(W_\mu^\pm\Lambda_Z)$$

$$- \frac{eM\cos\theta}{\xi} S^\pm\Lambda_A + \frac{GM\cos\theta\cos 2\theta}{2\xi} S^\pm\Lambda_Z$$

$$-eG\cos\theta\, A^\mu W_\mu^\pm(\cos^2\theta\, \Lambda_Z + \cos\theta\sin\theta\, \Lambda_A)$$

$$+ eG\cos\theta\, A^\mu(\sin\theta A_\mu + \cos\theta\, Z_\mu)\Lambda^\pm \mp ieA^\mu\partial_\mu\Lambda^\pm$$

$$+ eG\cos\theta\, W_\mu^\mp(W_\mu^\pm\Lambda^\mp - W_\mu^\mp\Lambda^\pm) \mp ie\cos\theta\, W_\mu^\pm\partial^\mu\Lambda_A\} \equiv \frac{R^\pm}{G\cos\theta} \quad , \qquad (8)$$

$$\delta[\partial_\mu A^\mu + \beta A_\mu A^\mu]$$

$$= \frac{1}{G}\{\Box\Lambda_A + \frac{ie}{\cos\theta}\partial^\mu(W_\mu^-\Lambda^+ - W_\mu^+\Lambda^-) - 2\beta A^\mu\partial_\mu\Lambda_A$$

$$+ 2\beta iGA^\mu(W_\mu^+\Lambda^- - W_\mu^-\Lambda^+)\} \equiv R_A/G \quad , \qquad (9)$$

$$[\partial^\mu Z_\mu + \beta' Z_\mu Z^\mu - iM_Z/(\sqrt{2}\eta)(S^0 - \bar{S}^0)]$$

$$= \frac{1}{G}\{(\Box + M_Z^2/\eta)\Lambda_Z + 2\beta' iG\cos\theta \; Z^\mu(W_\mu^+\Lambda^- - W_\mu^-\Lambda^+)$$

$$- 2\beta' Z^\mu \partial_\mu \Lambda_Z + iG\cos\theta \; \partial^\mu(W_\mu^+\Lambda^- - W_\mu^-\Lambda^+)$$

$$- \frac{GM_Z}{2\xi}(\Lambda^- S^+ + \Lambda^+ S^-) + \frac{GM_Z}{2\eta}\psi\Lambda_Z\} \equiv R_Z/G \quad , \qquad (10)$$

where $M_Z = M\sec\theta$, $\Lambda_Z \equiv \Lambda_3 - 2\Lambda_0$, $\Lambda_A \equiv \Lambda_3 g'/g + 2\Lambda_0 g/g'$. According to the usual formalism,[3] the results (8)-(10) lead to the f-Lagrangian:

$$L(\phi,\bar{\phi}) = \{-\bar{\Lambda}^+ R^+ - \bar{\Lambda}^- R^- - \bar{\Lambda}_A R_A - \bar{\Lambda}_Z R_Z\}_{\Lambda^\pm, \Lambda_A, \Lambda_Z \to \phi^\pm, \phi_A, \phi_Z} \quad . \qquad (11)$$

where $\phi \equiv \{\phi^\pm, \phi_A, \phi_Z\}$ and $\bar{\phi} \equiv \{\bar{\phi}^\pm, \bar{\phi}_A, \bar{\phi}_Z\}$ are complex fictitious scalar-fermion fields. Thus, in the usual formalism we have the following effective Lagrangian for Weinberg's theory with the bilinear gauge conditions (3):

$$L^u_{eff} = L_{W1} + L_G + L(\phi,\bar{\phi}) \quad , \qquad (12)$$

where the unphysical scalar fields $W_S^\pm \propto \partial^\mu W_\mu^\pm$, $S^\pm$, χ, ϕ and $\bar{\phi}$ are, by definition, not allowed to appear in *the external* states of a physical process.

3. Unitarity Violation from L^u_{eff}

We show that the Weinberg theory defined by the Lagrangian L^u_{eff} *in (12) violates unitarity. Let us calculate the* imagina y part Im A of the amplitude of the fourth order scattering process of the physical particles W^+ and the photon γ,

$$W^+(p)\gamma(q) \to W^+(p')\gamma(q') \quad . \qquad (13)$$

It is a one-loop process and the unphysical particles W_S^+, ϕ and $\bar{\phi}$ could be produced in the intermediate states [where W_S^+ is the negative metric spin 0 particle associated with the 4-vector field $W_\mu^+(x)$]. To be specific, $W_S^+\gamma$, $\bar{\phi}^-\phi_A$, $\phi^+\bar{\phi}_A$, W^+Z, $\bar{\phi}^-\phi_Z$, etc., could contribute in the intermediate states of the process (13). The particles W_S^+, ϕ^+, $\bar{\phi}^-$ have the same *mass* $M\xi^{-1/2}$; *the particles* γ, ϕ_A, $\bar{\phi}_A$ *are massless; while* Z, ϕ_Z, $\bar{\phi}_Z$ are massive. On the basis of phase space and unitarity considerations, we know that the net contribution due to the *unphysical intermediate states* $W_S^+\gamma$, $\bar{\phi}^-\phi_A$, $\phi^+\bar{\phi}_A$ to Im A, i.e.,

$$\text{Im } A^u = \text{Im } A(W_S^+\gamma) + \text{Im } A'(\phi,\bar{\phi}) \quad , \qquad (14)$$

must vanish for arbitrary ξ, because the unphysical amplitude Im A^u cannot be cancelled by any other unphysical amplitude in the theory (if L^u_{eff} is unitary). However, Im A^u does not vanish (see Appendix):

$$\text{Im } A^u = (2\pi)^{-2} e^4 \int d^4t\, d^4k\, \theta(t_0)\theta(k_0)\delta(t^2-M^2)\delta(k^2)\delta^4(p+q+k-t)$$

$$\times \{ -\bar{e}\cdot\bar{\varepsilon}[2e\cdot p\ \varepsilon\cdot k - 2\varepsilon\cdot q\ e\cdot k + 2e\cdot\varepsilon\ q\cdot k]/(2t\cdot k)$$

$$- e\cdot\varepsilon[2\bar{e}\cdot\bar{p}\ \bar{\varepsilon}\cdot k + 2\bar{q}\cdot k\ \bar{e}\cdot\bar{\varepsilon} - 2\bar{q}\cdot\bar{\varepsilon}\ \bar{e}\cdot k]/(2t\cdot k)$$

$$+ e\cdot(p+k)\varepsilon\cdot k\ \bar{e}\cdot\bar{\varepsilon}/(p\cdot k) + \bar{e}\cdot(\bar{p}+k)\bar{\varepsilon}\cdot k\ e\cdot\varepsilon/(\bar{p}\cdot k) \quad , \tag{15}$$

where Im $A(W^+_S\gamma) = 0$ and the non-zero contribution is due to Im $A'(\phi,\bar{\phi})$. Thus we conclude that the Lagrangian L^u_{eff} violates unitarity, contrary to the claim of formal proofs.[1] We note that this conclusion holds even if $\beta = \beta' = 0$ in the bilinear gauge conditions (3).

4. Effective Lagrangian in the LM Formalism

In the LM formalism,[4] we introduce the Lagrange multiplier fields $X_i \equiv \{X^\pm, X_Z, X_A\}$ and rewrite the Lagrangian (4) as

$$L_X = L_{W_1} + M\{X_A\partial_\mu A^\mu + X_Z[\partial_\mu Z^\mu - iM\sec\theta(S^0 - \bar{S}^0)/(\sqrt{2}\eta)$$

$$+ X^-[(\partial^\mu - ieA^\mu)W^+_\mu + iMS^+/\xi] + X^+[(\partial^\mu + ieA^\mu)W^-_\mu - iMS^-/\xi]\}$$

$$+ M^2\{\alpha X_A^2/2 + X_Z^2/(2\eta) + X^+X^-/\xi\} \quad , \tag{16}$$

where we have set $\beta = \beta' = 0$ for simplicity. It leads to the constraints

$$(\partial^\mu \mp ieA^\mu)W^\pm_\mu \pm iMS^\pm/\xi + MX^\pm/\xi = 0 \quad , \tag{17}$$

$$\partial^\mu A_\mu + \alpha M X_A = 0 \quad , \tag{18}$$

$$\partial^\mu Z_\mu - iM_Z(S^0 - \bar{S}^0)/(\sqrt{2}\eta) + MX_Z/\eta = 0 \quad , \tag{19}$$

and the field equations for $W^\pm_\mu$, Z_μ, A_μ, $S^\pm$, S^0 and $\bar{S}^0$. In order to obtain the f-Lagrangian in the LM formalism, we must first derive the equations of motion for the Lagrange multiplier fields χ_i. We find that

$$(\Box + M^2/\xi)X^{\pm} \mp 2ieA^\mu\partial_\mu X^{\pm} \mp iG\cos^2\theta Z^\mu\partial_\mu X^{\pm} \mp iG\cos^2\theta W^{\pm}_\mu\partial^\mu X_Z$$

$$+ \frac{MG\cos\theta}{2\xi}X^{\pm}(\psi \mp i\chi) - e^2W^{\mp\mu}(W^{\pm}_\mu X^{\mp} - W^{\mp}_\mu X^{\pm})$$

$$- \frac{M_Z G\cos\theta}{2\eta}X_Z S^{\pm} - (e^2A^\mu - eG\cos^2\theta Z^\mu)X^{\pm}A_\mu - T^{\pm} = 0 \;, \quad (20)$$

$$\Box X_A = 0 \;, \quad (21)$$

$$(\Box + M_Z^2/\eta)X_Z + iG\cos^2\theta\,\partial^\mu(W^+_\mu X^- - W^-_\mu X^+)$$

$$- \frac{GM}{2\xi}(X^-S^+ + X^+S^-) + \frac{GM_Z}{2\eta}\psi X_Z \equiv T_Z = 0 \;, \quad (22)$$

where $M_Z = M\sec\theta$. These equations are obtained by taking the divergences of the field equations for $W^{\pm}_\mu$, Z_μ, A_μ and using the equations derived from (16). The calculations are lengthy but straightforward. Since X_A obeys the free field equation, for the construction of the f-Lagrangian in the LM formalism we may ignore it and set $X_A = \partial_\mu A^\mu$ to be zero in (20). According to the LM formalism, the f-Lagrangian is[4,7]

$$L(D,\bar D) = [-\bar X^+T^+ - \bar X^-T^- - \bar X_Z T_Z]_{X^\pm, X_Z \to D^\pm, D_Z}$$

$$= -\bar D^+(\Box + M^2/\xi)D^+ + 2ieA^\mu\bar D^+\partial_\mu D^+ - iG\cos^2\theta\, Z^\mu\bar D^+\partial_\mu D^+$$

$$- \frac{MG\cos\theta}{2\xi}\bar D^+D^+(\psi - i\chi) + iG\cos^2\theta\, W^{+\mu}\bar D^+\partial_\mu D_Z + \frac{M_Z G\cos\theta}{2\eta}\bar D^+D_Z S^+$$

$$+ (e^2A^\mu - eG\cos^2\theta\, Z^\mu)\bar D^+D^+A_\mu + e^2W^{+\mu}\bar D^+(D^-W^+_\mu - W^-_\mu D^+)$$

$$- \bar D^-(\Box + M^2/\xi)D^- - 2ieA^\mu\bar D^-\partial_\mu D^- + iG\cos^2\theta\, Z^\mu\bar D^-\partial_\mu D^-$$

$$- \frac{MG\cos\theta}{2\xi}\bar D^-D^-(\psi + i\chi) - iG\cos^2\theta\, W^{-\mu}\bar D^-\partial_\mu D_Z + \frac{M_Z G\cos\theta}{2\eta}\bar D^-D_Z S^-$$

$$+ (e^2A^\mu - eG\cos^2\theta\, Z^\mu)\bar D^-D^-A_\mu + e^2W^{-\mu}\bar D^-(D^+W^-_\mu - W^+_\mu D^-)$$

$$- \bar D_Z(\Box + M_Z^2/\eta)D_Z - iG\cos^2\theta\,\bar D_Z\partial^\mu(W^+_\mu D^- - W^-_\mu D^+)$$

$$+ \frac{GM}{2\xi}\bar D_Z(D^-S^+ + D^+S^-) - \frac{GM_Z}{2\eta}\psi\bar D_Z D_Z \;, \quad (23)$$

where $D \equiv (D^{\pm}, D_Z)$ is a complex fictitious scalar-fermion. The theory is defined by the following path integral for the amplitude A_W:[7]

$$A_W = \int\exp\{i\int d^4x[L_X + L(D,\bar D) + L_S]\}d[X_i, F, D, \bar D]$$

$$= \int\exp\{i\int d^4x[L_{W_1} + L_G + L(D,\bar D) + L_S]\}d[F,D,\bar D] \;, \quad (24)$$

$$F \equiv (W^{\pm}_\mu, A_\mu, Z_\mu, \psi, S^{\pm}, \chi),$$

where L_S is the external source terms for the physical vector bosons $W^{\pm}$, Z, γ and scalar particle ψ. We have integrated over the Lagrange multiplier field $X_i \equiv (X^{\pm}, X_Z, X_A)$ and ignore an irrelevant constant factor in (24). The physical reason for the unitary amplitude (24) has been discussed before.[4] The result (24) show that the effective Lagrangian of the theory in the LM formalism is

$$L^{LM} = L_{W1} + L_G + L(D,\bar{D}) \quad , \qquad (25)$$

which is different from L^{u}_{eff} in (12) derived in the usual gauge formalism. We shall check that L^{LM} is unitary and gauge invariant at both 1-loop and 2-loop levels.

5. Verification of Unitarity of L^{LM}

Let us consider the 1-loop scattering process (13). As discussed in section 3, the net contribution Im A^u due to the nine diagrams with the unphysical *intermediate* state $W_S\gamma$ to the imaginary amplitude for (13) must vanish, because there is no massless scalar-fermions in the Lagrangian L^{LM}. Indeed, we find that

$$\text{Im } A^u = \sum_{i=1}^{9} A_i(W_S^{+}\gamma) = 0 \quad , \qquad (26)$$

as required by unitarity. The amplitudes $A_i(W_S^{+}\gamma)$ are the same as those obtained in the appendix.

At the 2-loop level, we verify unitarity and gauge invariance of L^{LM} by considering the 2-loop self-energy of the physical vector boson W^{+} with one unphysical scalar particle W_S^{+} and two photons in the *intermediate states*:

$$W^{+} \to \gamma\gamma W_S^{+} \to W^{+} \quad . \qquad (27)$$

There are five such 2-loop diagrams, and their sum must vanish when the particles in the unphysical intermediate states are on their mass-shells. The imaginary amplitudes Im $A(\gamma\gamma W_S^{+})$ for (27) can be obtained by calculating the amplitudes a_i of the following decay process of $W^+(p)$ (with polarization vector $\epsilon_\mu(p)$ satisfying $p^\mu \epsilon_\mu(p) = 0$):

$$a_1: \quad W^+(p) \to \gamma(q) W_\mu^+(t) \to \gamma(q) W_S^+(p')\gamma(k) \quad , \tag{28}$$

$$a_2: \quad W^+(p) \to \gamma(k) W_\mu^+(t') \to \gamma(k) W_S^+(p')\gamma(q) \quad , \tag{29}$$

$$a_3: \quad W^+(p) \to \gamma(q) W_S^+(p')\gamma(k) \quad , \tag{30}$$

where $p = q + p' + k$ and $W_\mu^+(t)$ denotes that all four components of $W_\mu^+(x)$ are present in the intermediate states. The amplitudes a_i are

$$a_1 = \frac{ie^2}{M}\{(\xi+1)e'\cdot p' \ \epsilon\cdot e - 2e\cdot p \ \epsilon\cdot e' - 2\epsilon\cdot q \ e\cdot e' + 2\epsilon\cdot e \ e'\cdot q\} \quad , \tag{31}$$

$$a_2 = \frac{ie^2}{M}\{(\xi+1)e\cdot p' \ \epsilon\cdot e' - 2e'\cdot p \ \epsilon\cdot e - 2\epsilon\cdot k \ e'\cdot e + 2\epsilon\cdot e' \ e\cdot k\} \quad , \tag{32}$$

$$a_3 = \frac{-ie^2}{M}\{2\epsilon\cdot p' \ e\cdot e' - (1-\xi)\epsilon\cdot e \ e'\cdot p' - (1-\xi)\epsilon\cdot e' \ e\cdot p'\} \quad , \tag{33}$$

where $e_\mu \equiv e_\mu(q)$ and $e'_\mu \equiv e'_\mu(k)$ are the polarization vectors of the photons $\gamma(q)$ and $\gamma(k)$ respectively. In the calculations of a_i we have used $e_\mu q^\mu = e'_\mu k^\mu = q^2 = k^2 = 0$ because of (21) and (18), as in quantum electrodynamics. We see that

the imaginary amplitude $Im\ A(\gamma\gamma W_S^+)$ vanishes:

$$Im\ A(\gamma\gamma W_S^+) = \int |a_1 + a_2 + a_3|^2 \ d(\text{phase space}) = 0 \quad , \tag{34}$$

for arbitrary ξ, as required by gauge invariance and intensity.

We have also verified unitarity for simple 1-loop self-energy diagrams of $W^\pm$. All these non-trivial calculations at the 1-loop and the 2-loop levels indicate that the Lagrangian L^{LM} obtained in the LM formalism is unitary.

In general, we may view the amplitude A_W in (24) as arising from a system with seven degrees of freedom for each of massive charged and neutral fields, $(W_\mu^\pm, S^\pm, D^\pm, \bar{D}^\pm)$ and $(Z_\mu, \chi, D_Z, \bar{D}_Z)$: There are three normal degrees of freedom, one unphysical degree of freedom associated with the 4-vector fields, one unphysical scalar and two fictitious degrees of freedom for the scalar-fermions. The last four cancel each other in the physical scattering amplitudes (on-mass-shell). There are four degrees of freedom for the photon field A_μ and the two unphysical degrees of freedom (i.e., the longitudinal and the timelike photons) do not contribute because of (21) and (18), which are intimately related to the bilinear gauge condition (3a). This particular gauge condition makes the whole Lagrangian (4) invariant under the usual local Abelian gauge transformation, as in quantum electrodynamics. This is the basic difference between the gauge conditions (1) and (3a).

6. Remarks and Conclusion

For simplicity, let us analyze the situation when $\beta = \beta' = 0$ in (3a)-(3c). The results (21) and (18) show that the gauge condition (3b) with $\beta = 0$ is stable (i.e. can be fixed at all times). This state of affairs corresponds to the "gauge excitations" being essentially free and uncoupled. The LM formalism recognizes this and accordingly does not generate gauge compensating terms for such a stable gauge condition. However, the usual formalism[3] provides gauge compensating terms for every gauge condition no matter whether it can or cannot be fixed at all times. When gauge compensating terms are introduced for the stable gauge conditions the corresponding scalar-fermion excitations are forcibly coupled in the f-Lagrangian. One can see that the violation of unitarity in the usual formalism discussed in section 3 (with $\beta = 0$) is precisely due to the gauge compensating terms for the stable gauge condition (3b). We have already pointed out that the usual general formal proof of unitarity for non-Abelian gauge theories holds only for the case in which all gauge conditions are unstable.[2] When one of the gauge conditions is stable, the implication of the usual general formal proof of unitarity is no longer clear because a stable gauge condition is treated as if it were unstable.

In conclusion, the usual formalism for non-Abelian gauge theories is not really independent of different gauge conditions in the sense that it violates unitarity for a class of bilinear gauge conditions. The usual general *formal proof* of unitarity and gauge invariance does not hold in this class of bilinear gauge conditions. The advantages of the LM formalism lie in the fact that it reveals explicitly the stability of gauge conditions, if any, and that it has more general validity than the usual formalism.

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Appendix

There are nine 1-loop diagrams for the process (13) with $W_S^+\gamma$ in the intermediate states and four 1-loop diagrams with $\bar{\phi}^-\phi_A$ and $\phi^+\bar{\phi}_A$ in the intermediate states. They are given in Fig. 1. Their imaginary parts are as follows:

$$\mathrm{Im}\, A_1 = E4y\bar{y}\; e\cdot\bar{e}$$

$$\mathrm{Im}\, A_2 = E\{-4y\bar{A}\;\bar{e}\cdot e + 2y\bar{\varepsilon}\cdot\bar{e}\; e\cdot(2\bar{p}+k)$$

$$- 4y\bar{\varepsilon}\cdot e\;\bar{e}\cdot\bar{p} + 4y\bar{A}\;\bar{e}\cdot t\; e\cdot k/R\}$$

$$\mathrm{Im}\, A_3 = E\{-4y\bar{e}\cdot\bar{\varepsilon}\; e\cdot(t-\bar{q}) + 4y\bar{\varepsilon}\cdot e\;\bar{e}\cdot\bar{p} - 4y(\bar{y}-\bar{A})e\cdot\bar{e}$$

$$+ 4y\bar{y}\; e.k\;\bar{e}\cdot(\bar{p}-t)/Q + 4y\bar{A}\; e\cdot k\;\bar{e}\cdot t/Q$$

$$- 2y\; e\cdot k\;\bar{e}\cdot\bar{\varepsilon}\; R/Q\}$$

$$\mathrm{Im}\, A_4 = E\{-4\bar{y}\;\varepsilon\cdot e\;\bar{e}\cdot(t-q) + 4\bar{y}\;\varepsilon\cdot\bar{e}\; e\cdot p - 4\bar{y}(y-A)e\cdot\bar{e}$$

$$+ 4y\bar{y}\;\bar{e}\cdot k\; e\cdot(p-t)/Q + 4\bar{y}A\;\bar{e}\cdot k\; e\cdot t/Q$$

$$- 2\bar{y}\;\varepsilon\cdot e\;\bar{e}\cdot k\; T/Q\}$$

$$\mathrm{Im}\, A_5 = E\{-4\bar{y}A\;\bar{e}\cdot e + 2\bar{y}\;\varepsilon\cdot e\;\bar{e}\cdot(2p+k) - 4\bar{y}\;\varepsilon\cdot\bar{e}\; e\cdot p$$

$$+ 4A\bar{y}\; e\cdot t\;\bar{e}\cdot k/T\}$$

$$\mathrm{Im}\, A_6 = E\{-4y\bar{A}[\bar{e}\cdot t\; e\cdot k/R - e\cdot k\;\bar{e}\cdot t/Q - e\cdot\bar{e}]$$

$$+ 4A\bar{A}[\bar{e}\cdot t\; e\cdot t/R - e\cdot t\;\bar{e}\cdot t/Q - e\cdot\bar{e}]$$

$$+ 2\bar{A}[\varepsilon\cdot e(-\bar{e}\cdot t\; T/R + \bar{e}\cdot k\; T/Q + 2\bar{e}\cdot(t-q) + \bar{e}\cdot\bar{p}\; T/Q)$$

$$- 2e\cdot p\;\bar{e}\cdot\varepsilon]$$

$$+ 2y[2e\cdot\bar{\varepsilon}\;\bar{e}\cdot\bar{p} + \bar{e}\cdot\bar{\varepsilon}(-e\cdot k\; R/Q - e\cdot k + 2e\cdot\bar{q})]$$

$$+ 2A[-2\bar{e}\cdot\bar{p}\; e\cdot\bar{\varepsilon} + \bar{\varepsilon}\cdot\bar{e}\; e\cdot t\; R/Q$$

$$+ \bar{\varepsilon}\cdot\bar{e}(2\bar{p}\cdot e + e\cdot k) - e\cdot p\;\bar{\varepsilon}\cdot\bar{e}]$$

$$- \bar{\varepsilon}\cdot\bar{e}\;\varepsilon\cdot e[TR/Q + 4M^2 + 2R + T + 4q\cdot\bar{q}]$$

$$+ 4\bar{y}\;\bar{e}\cdot\bar{p}\;\varepsilon\cdot e - 4\varepsilon\cdot\bar{\varepsilon}\;\bar{e}\cdot p\; e\cdot p$$

$$- 4\bar{\varepsilon}\cdot q\;\bar{e}\cdot\bar{p}\;\varepsilon\cdot e - 4\varepsilon\cdot\bar{q}\; e\cdot p\;\bar{\varepsilon}\cdot\bar{e}\} \quad ,$$

$$\text{Im } A_7 = E\{4A\bar{y}[e\cdot t\ \bar{e}\cdot k(1/Q - 1/T) + e\cdot\bar{e}]$$

$$+ 4A\bar{A}[e\cdot t\ \bar{e}\cdot t(1/T - 1/Q) - e\cdot\bar{e}]$$

$$+ 2\bar{A}[e\cdot\varepsilon\ \bar{e}\cdot t\ T/Q - 2e\cdot p\ \bar{e}\cdot\varepsilon + e\cdot\varepsilon(\bar{e}\cdot t - 2\bar{e}\cdot q)]$$

$$+ 2A[-e\cdot t\ \bar{e}\cdot\bar{\varepsilon}\ R/T + \bar{e}\cdot\bar{\varepsilon}(e\cdot p + e\cdot k)R/Q$$

$$- 2\bar{e}\cdot\bar{\varepsilon}(e\cdot\bar{q} - e\cdot t) - 2e\cdot\bar{\varepsilon}\ \bar{e}\cdot\bar{p}]$$

$$+ 2\bar{y}[-e\cdot\varepsilon\ \bar{e}\cdot k\ T/Q + 2e\cdot p\ \bar{e}\cdot\varepsilon + e\cdot\varepsilon(2\bar{e}\cdot q - \bar{e}\cdot k)]$$

$$- \bar{e}\cdot\bar{\varepsilon}\ e\cdot\varepsilon[+ TR/Q + R + 2T + 4M^2 + 4q\cdot\bar{q}]$$

$$+ 4y\ e\cdot p\ \bar{e}\cdot\bar{\varepsilon} - 4\varepsilon\cdot\bar{\varepsilon}\ e\cdot p\ \bar{e}\cdot p$$

$$- 4\varepsilon\cdot\bar{q}\ e\cdot p\ \bar{e}\cdot\bar{\varepsilon} - 4\bar{\varepsilon}\cdot q\ \bar{e}\cdot\bar{p}\ e\cdot\varepsilon\}$$

$$\text{Im } A_8 = E\{4A\bar{y}[2e\cdot t\ \bar{e}\cdot\bar{p}/Q - 2e\cdot t\ \bar{e}\cdot t/Q - e\cdot\bar{e}]$$

$$+ 4A\bar{A}[2e\cdot t\ \bar{e}\cdot t/Q + e\cdot\bar{e}]$$

$$+ 4A[-\bar{e}\cdot\bar{\varepsilon}\ e\cdot t(R/Q + 1) + \bar{e}\cdot\bar{\varepsilon}\ e\cdot\bar{q} + e\cdot\bar{\varepsilon}\ \bar{e}\cdot\bar{p}]$$

$$+ 4y\bar{y}[2e\cdot k\ \bar{e}\cdot t/Q - 2\bar{e}\cdot\bar{p}\ e\cdot k/Q + e\cdot\bar{e}]$$

$$- 4y\bar{A}[2e\cdot k\ \bar{e}\cdot t/Q + e\cdot\bar{e}]$$

$$- 4y[\bar{e}\cdot\bar{\varepsilon}\ e\cdot\bar{q} + e\cdot\bar{\varepsilon}\ \bar{e}\cdot\bar{p} - \bar{e}\cdot\bar{\varepsilon}\ e\cdot k(R/Q + 1)]$$

$$+ 4\bar{y}[2e\cdot\varepsilon\ \bar{e}\cdot\bar{p}\ q\cdot k/Q - 2e\cdot\varepsilon\ \bar{e}\cdot t\ q\cdot k/Q - e\cdot\varepsilon\ \bar{e}\cdot q - \varepsilon\cdot\bar{e}\ e\cdot p]$$

$$+ 4\bar{A}[2e\cdot\varepsilon\ \bar{e}\cdot t\ q\cdot k/Q + \bar{e}\cdot q\ e\cdot\varepsilon + \varepsilon\cdot\bar{e}\ e\cdot p]$$

$$+ e\cdot\varepsilon\ \bar{e}\cdot\bar{\varepsilon}[2T + 2TR/Q + 4q\cdot\bar{q} + 4M^2 + 2R]$$

$$+ 4\varepsilon\cdot\bar{\varepsilon}\ e\cdot p\ \bar{e}\cdot\bar{p} + 4e\cdot\varepsilon\ \bar{\varepsilon}\cdot q\ \bar{e}\cdot\bar{p} + 4\varepsilon\cdot\bar{q}\ \bar{e}\cdot\bar{\varepsilon}\ e\cdot p\}$$

$$\text{Im } A_9 = E\{4A\bar{A}[-e\cdot t\ \bar{e}\cdot t(1/R + 1/T) + e\cdot\bar{e}]$$

$$+ 2A[2e\cdot\bar{\varepsilon}\ \bar{e}\cdot\bar{p} + \bar{e}\cdot\bar{\varepsilon}\ e\cdot t(R/T - 1)$$

$$- 2\bar{e}\cdot\bar{\varepsilon}\ e\cdot\bar{p} + 2\bar{e}\cdot\bar{\varepsilon}\ e\cdot p]$$

$$+ 2\bar{A}[e\cdot\varepsilon\ \bar{e}\cdot t(T/R - 1) - 2e\cdot\varepsilon\ \bar{e}\cdot p$$

$$+ 2\varepsilon\cdot\bar{e}\ e\cdot p + 2e\cdot\varepsilon\ \bar{e}\cdot\bar{p}]$$

$$- 4\bar{y}\ e\cdot\varepsilon\ \bar{e}\cdot\bar{p} - 4y\ \bar{e}\cdot\bar{\varepsilon}\ e\cdot p$$

$$+ 4\bar{\varepsilon}\cdot q\ e\cdot\varepsilon\ \bar{e}\cdot\bar{p} + 4\varepsilon\cdot\bar{q}\ \bar{e}\cdot\bar{\varepsilon}\ e\cdot p + 4\varepsilon\cdot\bar{\varepsilon}\ e\cdot p\ \bar{e}\cdot\bar{p}$$

$$+ e\cdot\varepsilon\ \bar{e}\cdot\bar{\varepsilon}(4q\cdot\bar{q} + 4M^2 + T + R)\} \quad ,$$

$$\mathrm{Im}\ A_1' = +M^2E\{\bar{e}\cdot\bar{\varepsilon}/Q[2e\cdot p\ \varepsilon\cdot k - 2\varepsilon\cdot q\ e\cdot k + 2e\cdot\varepsilon\ q\cdot k]\} \quad ,$$

$$\mathrm{Im}\ A_2' = +M^2E\{e\cdot\varepsilon/Q[2\bar{e}\cdot\bar{p}\ \bar{\varepsilon}\cdot k + 2\bar{q}\cdot k\ \bar{e}\cdot\bar{\varepsilon} - 2\bar{q}\cdot\bar{\varepsilon}\ \bar{e}\cdot k]\} \quad ,$$

$$\mathrm{Im}\ A_3' = M^2E\{2e\cdot(p+k)\varepsilon\cdot k\ \bar{e}\cdot\bar{\varepsilon}/T\} \quad ,$$

$$\mathrm{Im}\ A_4' = M^2E\{2\bar{e}\cdot(\bar{p}+k)\bar{\varepsilon}\cdot k\ e\cdot\varepsilon/R\} \quad ,$$

where $p+q = t-k = \bar{p}+\bar{q}$, $Q \equiv P'^2 - M^2$, $R \equiv \bar{S}^2 - M^2$, $T \equiv S^2 - M^2$, $y \equiv \varepsilon\cdot t$, $\bar{y} \equiv \bar{\varepsilon}\cdot t$, $A \equiv \varepsilon\cdot k$, $\bar{A} \equiv \bar{\varepsilon}\cdot k$, $p' = p+q$, $\bar{S} = \bar{p}+k$, $S = p+k$, and

$$E \equiv \frac{e^4}{(2\pi)^2M^2}\int d^4t\ d^4k\,\theta(t_0)\theta(k_0)\delta(t^2-M^2)\delta(k^2)\delta^4(p+q+k-t) \quad .$$

In these calculations, we have set $\xi = \alpha = 1$ for simplicity.

Figure Caption

Fig. 1. The fourth order scattering process of W^+ and γ. In our calculations, only W_S^+ contribute in the propagator with the momentum t_μ. All four components of $W_\mu^+(x)$ contribute to the propagators with momenta p_μ', S_μ and $\bar{S}_\mu$, which are not on mass-shell.

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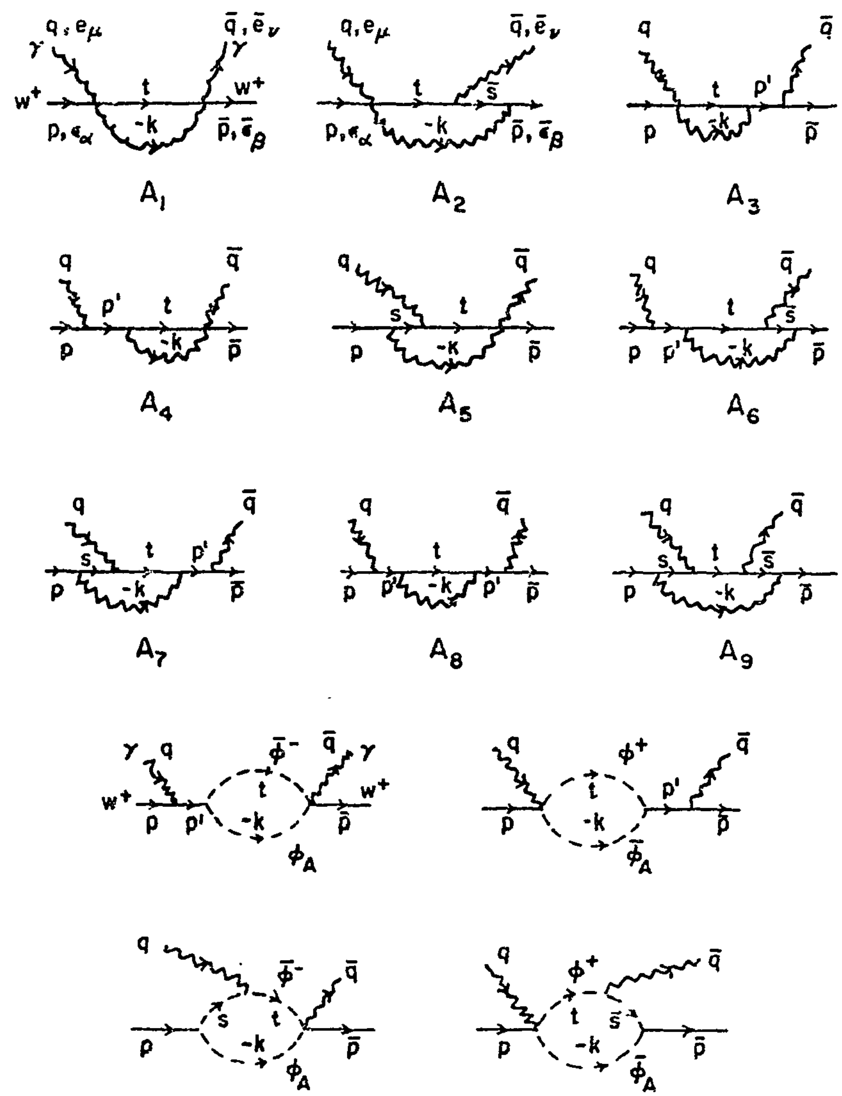

Fig. 1