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HEAT TRANSFER IN THERMAL RADIATION ABSORBING AND SCATTERING MEDIA

by
Raymond Viskanta

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HEAT TRANSFER IN THERMAL RADIATION ABSORBING AND SCATTERING MEDIA

by

R. Viskanta

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ABSTRACT

The problem of heat transfer from media that absorb and scatter thermal radiation has been studied analytically. The fundamental quantities and definitions of the theory of thermal radiation are presented in a form useful for application to the radiant heat transfer problems. The aim was to formulate the various concepts with maximum generality. The basic equation of radiant heat transfer, which governs the radiation field in a media that absorbs, emits and scatters thermal radiation, has been derived. The mathematical analogy between thermal radiation and neutron transport is pointed out, and a few illustrations of the applicability of the solutions obtained for neutron transport problems to the radiative transfer problems are given.

The derivation of the integral equations for radiant heat exchange in a general enclosure composed of a system of surfaces separated by an absorbing and scattering media is presented. The enclosure walls under consideration can reflect specularly and the scattering from the medium is not considered to be isotropic. The equation for the conservation of energy, including contributions due to thermal radiation, was derived by evaluating the energy transported into an imaginary closed surface fixed in space and then by applying Gauss's divergence theorem. The formulations developed

are then used to gain insight into the problem by considering a few simple physical situations and obtaining numerical results for the grey case only.

The Rosseland approximation for the radiant flux vector is employed in the study of Couette flow. It is found that for large optical thicknesses the temperature distributions calculated agree well with those predicted by the exact formulation.

Numerical solutions of the boundary layer equations for the flow of a radiating media along a wedge were obtained. The effect of radiation is to decrease the temperature gradient for both the hot and the cool walls; however, the heat transfer is affected only little. The validity of the diffusion approximation for radiation in boundary layer problems is limited, and should be used with caution only in situations where the mean free path of radiation is much smaller than the thermal boundary layer thickness.

The transport of radiant energy between two parallel plates separated by an absorbing and scattering media is studied. The temperature distributions were obtained by solving the nonhomogeneous Milne integral equation of the first kind. It was found that the polynomial approximation for the black body emissive power is satisfactory for all values of the optical thickness.

The transport of energy by simultaneous conduction and radiation in a one-dimensional system has been considered. A nonlinear integral equation governing the temperature distribution in an absorbing media was solved. The results showed that the temperature distribution was strongly dependent on the optical thickness of the slab and on the dimensionless parameter, N, which determines the relative role of energy transfer by conduction to that by radiation. The presence of radiation generally increases the heat transfer by conduction.

I. INTRODUCTION

Little theoretical or experimental engineering work has been done on heat transfer in media which absorb and scatter thermal radiation. Some studies which are primarily limited to the problems occurring in boiler furnaces and combustion chambers have been made. On the other hand, during the past sixty years, astrophysicists have given considerable attention to problems connected with radiation transfer in planetary atmospheres, the sun, nebulae, and galaxies.[19, 68, 103, 4] Recent interest in radiation transfer has been stimulated by its similarity with neutron transport, energy transfer from high-temperature gases and plasmas, meteorological problems, atomic explosions and fusion reactions. The recent developments in hypersonic flight, missile reentry, rocket combustion chambers, gas-cooled nuclear reactors, and power plants for interplanetary flight have further emphasized the need to better understand the transfer of energy by radiation through absorbing and scattering media.

Heat transfer in an enclosure containing an absorbing and scattering medium, whether the medium is stationary or in motion, is one of the most complex problems occurring in engineering practice. In this case, a determination of the energy fluxes requires the solution of a system of coupled conservation equations, namely, the differential equations of motion, the integrodifferential equation of energy and an integral equation which expresses the radiosity at any point on the enclosing surface. The complexity introduced by the radiative contribution to the energy flux is in part due to the dependence of the flux on the geometrical configurations of the system, which is further complicated by the interreflections caused by the presence of walls - the essential element of any engineering system. There are no available general solutions for heat transfer problems in media which absorb and scatter thermal radiation, and only a few simplified attacks have been made.(93, 58, 3, 30) Until recently, the temperature distribution in a radiating medium has been calculated on the assumption that the medium was nonradiating. The basis for this assumption was that radiative energy exchanges do not affect temperature and velocity distributions in a flow stream, as do the usual dynamic and convective processes. The assumption is tenable when radiant energy transfer is small compared to other transport phenomena. When radiant energy transfer is of the same order of magnitude as other transport processes, a temperature distribution cannot be derived without consideration of the radiative term.

The present work has been undertaken with the hope that it will contribute to some extent toward better understanding of heat transfer in thermal radiation absorbing and scattering media, as well as in stimulating further interest in this technically important area. Thus, the purpose of this study was twofold: (1) formulation of the general heat transfer equations for thermal radiation absorbing and scattering media in the presence of the usual dynamic and convective processes, as well as (2) solution of specific heat transfer problems.

To this end, a thorough survey of literature on radiative transfer through absorbing and scattering media was made. This included surveying many contributions in fields usually unrelated to engineering heat transfer, such as astrophysics, meteorology, illumination, communication, and neutron transport. Certain works - particularly some published in the USSR - are not available in this country. Although the survey is exhaustive, it therefore may not be complete.

The radiative properties of the medium were not dealt with; but the foundations of the theory of heat transfer in thermal radiation absorbing and scattering media were examined and the general equations derived explicitly.

Concepts and results of many uncorrelated investigations of radiative transfer in the various fields, particularly from astrophysics, were used. The equations derived are detailed and assumptions explicitly stated so that all steps can be readily followed. The general equation of transfer is formulated and various special cases are discussed. The integral equations for a general enclosure are expressed in two different forms. Finally, the general energy equation for a medium in motion in the absence of electric and magnetic fields, as well as when concentration gradients are absent, is derived.

To fulfill the second purpose of this work, it was first necessary to gain insight into the problem. This was done by considering reasonably simple physical situations. In this way the essential features of the formulations were retained and the distractions of complex geometrical relationships were avoided. For this reason, one-dimensional systems are considered and the effect of radiant energy transfer on heat transfer is investigated, where the Rosseland approximation for the radiant flux is used; the transport of radiant energy between two parallel plates is studied; and simultaneous conduction and radiation between two parallel plates is considered.

Oppenheim[72] in discussing the engineering radiation problem made distinction between three methods of attack: accounting, network and calculus. If energy transfer by convection or conduction, or both, cannot be neglected compared to the energy transfer by radiation, the temperature gradients are required for the calculation of heat transfer rate. The accounting and the network methods are not suitable for the calculation of temperature distribution and temperature gradients in thermal radiation absorbing and scattering media. For this reason, the calculus method of attack is used throughout this work.

2 DEFINITIONS AND CONCEPTS

2.1 Introduction

The present section is intended to define some fundamental quantities and to present certain of the results of the theory of thermal radiation in a form useful for application to the problems of radiant heat transfer. In formulating the various concepts, maximum generality is aimed at. The basic quantity is formed by the definition of the intensity of radiation. No proofs are given. The reader is referred to Planck's treatise[78] on thermal radiation for the most complete account of the physics and thermodynamics of radiation. The usual treatment in books on radiation is supplemented, and the validity of Kirchhoff's law to systems not in thermodynamic equilibrium is discussed by Milne.[68]

2.2 The Monochromatic Intensity of Radiation

The analysis of the radiation field requires a consideration of the radiant energy, Δq_r, in a specified wavelength interval between λ and $\lambda + \Delta\lambda$ which is emitted from, reflected from, and/or transported across an element of area ΔA and confined to an element of solid angle $\Delta\Omega$ in direction $\vec{\Omega}$ making an angle θ with the outward normal $\vec{n}$ to ΔA, during the time interval between t and $t + \Delta t$ (see Fig. 2.1). This radiant energy is expressed in terms of monochromatic intensity of radiation, I_λ, and is defined as radiant energy passing through the surface (or emitted by the surface and/or reflected from the surface) per unit solid angle, per unit of time, per unit of wavelength and per unit area perpendicular to the solid angle. Mathematically, the monochromatic intensity of radiation is defined by the following limit:

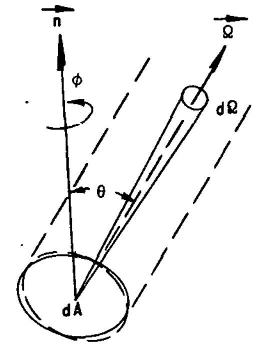

FIG. 2.1
DEFINITION OF THE INTENSITY OF RADIATION.

$$I_\lambda = \lim_{\Delta A, \Delta t, \Delta\Omega, \Delta\lambda \to 0} \left| \frac{\Delta q_r}{\cos\,\theta\,\Delta A\,\Delta t\,\Delta\Omega\Delta\lambda} \right| = \frac{dq_r}{\cos\theta\, dA dt d\Omega d\lambda} \tag{2.1}$$

The appearance of $\cos\theta$ in (2.1) is due to the fact that we are considering a pencil of rays, not in the direction of the normal $\vec{n}$, but in the direction $\vec{\Omega}$. The quantity of energy traveling across dA is determined, not by dA itself, but by its projection on a plane perpendicular to the direction $\vec{\Omega}$.

In the medium which absorbs, emits, and scatters radiation, it follows from the definition that I_λ may be expected to be a function of the position coordinates, of the direction $\vec{\Omega}$, of the time t and of the wavelength λ. Thus for a general radiation field, we can write

$$I_\lambda \equiv I_\lambda(x, y, z, \Omega_x, \Omega_y, \Omega_z, t) \equiv I_\lambda(\vec{r}, \vec{\Omega}, t).$$

2.3 The Variation of Monochromatic Intensity with the Refractive Index

Now we have to consider how the monochromatic intensity of radiation varies with the refractive index, n, of the medium by considering Fig. 2.2. The integral $\int_C nds$ taken along a curve C is known as the optical length of the curve. The radiation is propagated with the velocity of the light in the medium, $v = c/n$, along the ray:

$$nds = c/v\, ds = cdt,$$

where dt is the time needed for radiation to travel a distance ds along the ray.

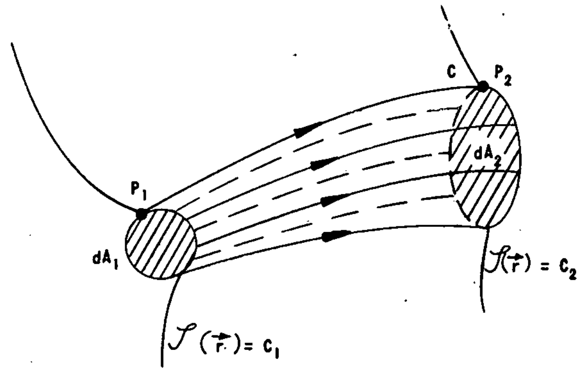

FIG. 2.2
BUNDLE OF RAYS PASSING THROUGH A MEDIUM HAVING A VARIABLE INDEX OF REFRACTION.

The principle of Fermat, or the principle of the least time, according to Born and Wolf[7] states that the "optical length

$$\int_{P_1}^{P_2} nds = c \int_{P_1}^{P_2} dt \tag{2.2}$$

of any ray between any two points P_1 and P_2 is shorter than the optical length of any other curve which joins these points and which lies in a certain regular neighborhood of it." In other words, Fermat's principle asserts that, given a starting and end point(s) for the path and given the velocities in the first and second media, the incident radiation travels along a path by which it reaches the end point of the second medium in the shortest possible time.

Denoting by square brackets the optical length of the ray which joins points P_1 and P_2, we have

$$[P_1P_2] = \int_{P_1}^{P_2} nds = c \int_{P_1}^{P_2} dt = \mathcal{L}(P_1) - \mathcal{L}(P_2) \quad , \tag{2.3}$$

where $\mathcal{L}(\vec{r})$ is the eikonal and $[\text{grad}\, \mathcal{L}(\vec{r})] = n^2$ is known as the eikonal equation; it is the basic equation of geometrical optics.[7]

The surfaces $\mathcal{L}(\vec{r})$ = constant are called geometrical wave surfaces or geometrical wave fronts.

Finally, by applying the law of conservation for energy to an arbitrary pencil of rays, as shown in Fig. 2.2, it can be shown[7] that the variation of intensity along each ray is expressed in terms of the function $\mathcal{L}$. Thus, the ratio of intensities at any two points of a ray is

$$\frac{I_{\lambda,2}}{I_{\lambda,1}} = \frac{n_2}{n_1} e^{-\int_{\mathcal{L}_1}^{\mathcal{L}_2} \frac{\nabla^2 \mathcal{L}}{n^2} d\mathcal{L}} = \frac{n_2}{n_1} e^{-\int_{s_1}^{s_2} \frac{\nabla^2 \mathcal{L}}{n} ds} \tag{2.4}$$

2.4 The Monochromatic Energy Density

The monochromatic energy density of radiation at a given point, u_λ, is the amount of energy per unit wavelength in transit in a unit volume, in the neighborhood of the point. Mathematically, the monochromatic radiant energy density is defined by the following limit:

$$u_\lambda = \lim_{\Delta V, \Delta\lambda \to 0} \left| \frac{\Delta q_r}{\Delta V \Delta\lambda} \right| = \frac{dq_r}{dV d\lambda} \tag{2.5}$$

The dependence of u_λ on I_λ can be obtained by considering Fig. 2.3. Consider a small volume, ΔV, enclosed by a convex surface, Σ, in such a way that the distance from it to any point of the surface Σ is very large compared with the dimensions of the element ΔV itself. Consider a beam of rays entering the volume bounded by Σ and passing through elements of area $d\Sigma$ and dA. Let θ denote the angle which the normal to dA makes with the line joining dA with $d\Sigma$. Further, let the element $d\Sigma$ subtend a solid angle $d\Omega$ at dA. Then the quantity of radiant energy passing through the area $d\Sigma$ which also flows across dA in time dt in the solid angle $d\Omega$ and in the wavelength interval

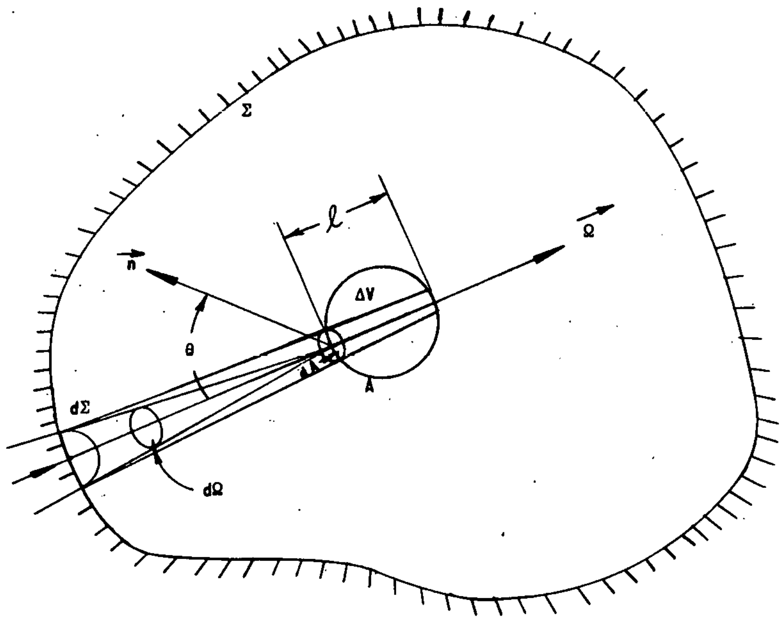

FIG. 2.3
GEOMETRICAL DATA FOR THE CALCULATION OF RADIANT ENERGY DENSITY.

between λ and $\lambda + d\lambda$ is

$$I_\lambda \cos\theta\, dA\, d\Omega\, d\lambda\, dt,$$

where I_λ is the monochromatic intensity of radiation in a medium having refractive index n. The radiation stays in ΔV as long as it takes to traverse the length ℓ, that is $dt = \ell/(c/n)$, where c is the velocity of light in vacuum. Thus, the amount of radiant energy in course of transit through ΔV by the pencil of radiation considered is

$$u_\lambda\, \Delta V = \frac{n}{c} \int_{\Omega=4\pi} \left[\int_{\Delta V} I_\lambda \cos\theta\, dA\, ds \right] d\Omega \ . \tag{2.6}$$

The product $\cos\theta\, dA \cdot \ell$ is the volume of the cylinder dV, whose length is ℓ and base area $\cos\theta dA$. The monochromatic energy density becomes

$$u_\lambda = \frac{n}{c\,\Delta V} \int_{\Omega=4\pi} \left[\int_{\Delta V} I_\lambda\, dV \right] d\Omega \ . \tag{2.7}$$

For the special case of isotropic radiation and constant index of refraction equation (2.7) reduces to

$$u_\lambda = \frac{n}{c} \int_{\Omega = 4\pi} I_\lambda \, d\Omega = \frac{4\pi n I \lambda}{c} \quad . \tag{2.8}$$

2.5 The Absorption, Scattering and Emission Coefficients

The interaction between radiation and matter is usually expressed in terms of an absorption coefficient, a scattering coefficient and an emission coefficient, all of which are defined in this section.

In defining these fundamental quantities, "the Eulerian" instead of "the Lagrangian point of view" is used.[52] In the Lagrangian point of view, the movement of individual particles (photons) is followed. In the Eulerian point of view, local variations in the radiation field itself are considered. The radiative transfer theory makes exclusive use of the Eulerian point of view. The evolution of single particles is not followed and no reference is made to the history of each individual particle.

The temperature distribution in any region of a medium is determined by the interaction between radiation and matter. The processes of thermal conduction and convection play a part in establishing the temperature distribution as well. In the case of pure scattering, radiation has no relation to the temperature. As soon, however, as absorption and emission play a role, we should have information as to how radiation is related to the temperature of the matter. At present, we are interested only in those absorption processes which lead to the conversion of radiant energy into thermal energy, and, conversely, of thermal into radiant energy.

It is implied with regard to processes of this kind that there is no direct connection between the absorbed and emitted quanta. Each absorbed quantum of wavelength λ is entirely lost, and thermal energy thereby gained by the medium is emitted in other wavelengths after some time. Absorption in which energy is converted into thermal energy (with possible subsequent re-emission in other wavelengths) is called true absorption. It can be further separated into true continuous absorption and true selective absorption or line absorption. These two types of absorption, as well as line broadening due to collisions and statistical broadening of lines, the Doppler and pressure effects, are discussed by Ambartsumyan.[4]

A pencil of radiation traversing matter is usually weakened by absorption as it is propagated. Consider a monochromatic pencil of radiation of intensity I_λ. As a result of passing through the medium of thickness ds, the decrease in intensity will be dI_λ. The coefficient of absorption or decrement in intensity of radiation, κ_λ, is thus defined as

$$dI_\lambda = -\kappa_\lambda I_\lambda ds. \qquad (2.9)$$

This definition is valid for both the continuous and the line absorption. In some astrophysics books[13,4] the decrement in intensity of radiation is defined in terms of mass absorption coefficient, $\kappa_{m,\lambda}$ as

$$\kappa_\lambda = \rho\kappa_{m,\lambda} .$$

The absorption coefficient is a property of a substance. It depends on pressure, temperature and the chemical and physical condition of the substance. Physical theory, confirmed by experiments, shows that the absorbing power of any material depends on the physical conditions in which the material is placed. The absorbing power and its dependence on the physical conditions are different for different chemical elements. Hence, the resulting absorption coefficient, determined by the chemical composition of the whole medium, will depend markedly on the relative content of various elements in the medium under consideration.

A pencil of radiation of monochromatic intensity I_λ is also weakened by the loss of radiation which is not absorbed by the media but merely redistributed, that is, scattered in direction. A material is characterized by a scattering coefficient σ_λ if from a pencil of radiation incident on an element of volume of cross section dA and height ds, the amount of energy scattered from it in all directions is

$$\sigma_\lambda I_\lambda \cos \theta dA\, ds d\Omega d\lambda dt \quad . \tag{2.10}$$

This definition of the scattering coefficient is equivalent to definitions of references (13) and (4), in which the mass scattering coefficient, $\sigma_{m,\lambda}$, is defined as

$$\sigma_\lambda = \rho\sigma_{m,\lambda} \quad .$$

To formulate more quantitatively the concept of scattering, Hopf[35] introduced the scattering function $\gamma(\vec{r}, \vec{\Omega}' \longrightarrow \vec{\Omega})$ such that

$$\sigma_\lambda I_\lambda \gamma(\vec{r}, \vec{\Omega}' \rightarrow \vec{\Omega}) \frac{d\Omega'}{4\pi} dv d\Omega d\lambda dt \tag{2.11}$$

gives the energy which is scattered into an element of solid angle $d\Omega'$. Referring to Fig. 2.4, we see that $\vec{\Omega}'$ is a unit vector in the direction of pencil of rays before collision; $\vec{\Omega}$ is the unit vector after collision; Θ is the angle between $\vec{\Omega}'$ and $\vec{\Omega}$; θ is the polar angle and ϕ is the azimuth. The loss of radiant energy from the pencil of rays due to scattering in all directions is

$$\sigma_\lambda dV d\Omega d\lambda dt \int_{\Omega'=4\pi} \frac{\gamma(\vec{r}, \vec{\Omega}' \longrightarrow \vec{\Omega})}{4\pi} I_\lambda(\vec{r}, \vec{\Omega}', t)\, d\Omega' \quad . \tag{2.12}$$

This agrees with (2.10) if

$$\int_{\Omega'=4\pi} \gamma(\vec{r}, \vec{\Omega}' \longrightarrow \vec{\Omega}) \frac{d\Omega'}{4\pi} = 1,$$

i.e., if γ is dependent of direction $\vec{\Omega}$.

The process where the wavelength of the re-emitted quantum is exactly the same as that of the absorbed quantum, the two quanta differing only in direction, is called coherent scattering. In writing (2.12)

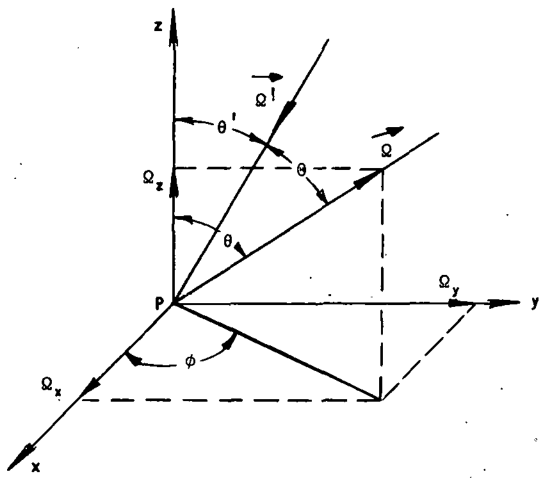

FIG. 2.4
COORDINATE SYSTEM FIXED ON STATIONARY VOLUME OF THE RADIATING MEDIUM.

we have obviously assumed that we are concerned with purely coherent scattering, since the absorbed energy is re-emitted in the same wavelength. For noncoherent scattering of radiation, the expression (2.12) has to be modified.(4) Since the effect of noncoherence in radiant heat transfer problems is not known, it will not be considered in this work. Further discussion on both coherent and noncoherent scattering can be found in books on astrophysics.(4,103)

Let a volume element, dV, of the medium emit radiant energy, dq_r, in all directions. Then in time dt and in the wavelength interval between λ and $\lambda + d\lambda$ this element will emit within the solid angle $d\Omega$ an amount of energy

$$dq_r = \epsilon_\lambda(\vec{r}, \vec{\Omega}, t)\, dV d\Omega d\lambda dt \qquad (2.13)$$

The emission coefficient $\epsilon_\lambda(\vec{r}, \vec{\Omega}, t)$* is defined as the energy emitted by a unit volume of medium per unit solid angle, per unit wavelength and per unit of time. In general, it depends on the wavelength, the composition of the emitting media and on direction. The position vector $\vec{r}$ is included in $\epsilon_\lambda(\vec{r}, \vec{\Omega}, t)$ to indicate that the emission coefficient depends on the location of the element of volume.

*The emission coefficient, $\epsilon_\lambda(\vec{r}, \vec{\Omega}, t)$, is not really a coefficient. This nomenclature is used in all books on astrophysics, however.

The monochromatic energy emitted from the volume element dV in the time dt, in the wavelength interval $d\lambda$, in all directions is

$$dq_r = dVdtd\lambda \int_{\Omega=4\pi} \epsilon_\lambda(\vec{r},\vec{\Omega},t)\, d\Omega \quad . \tag{2.14}$$

For the case of isotropic emission, that is $\epsilon_\lambda(\vec{r},\vec{\Omega},t) = \epsilon_\lambda(\vec{r},t)$, expression (2.14) may be written as

$$dq_r = 4\pi\epsilon_\lambda(\vec{r},t)dVd\lambda dt \quad . \tag{2.15}$$

For a more complete discussion of interaction of photons with matter and the distinction between the photons incident and emergent in a given direction, reference is made to Kourganoff.[52]

2.6 Thermodynamic, Local and Radiative Equilibrium

A system which is not experiencing any change with time is said to be in thermodynamic equilibrium. This means that three types of equilibrium: thermal, chemical and mechanical, must exist simultaneously. First, there must be thermal equilibrium so that the temperature is the same throughout the whole system or only a part of the system, and this temperature is the same as that of the surroundings. Second, if the system consists of more than one substance, there must be chemical equilibrium, so that the system does not undergo a spontaneous change of internal structure. Finally, the system must be in mechanical equilibrium, that is, there must be no macroscopic movement within the system itself and also between the system and its surroundings. The properties of a system in thermodynamic equilibrium provide a useful basis for consideration on nonequilibrium phenomena which occur in all forms of energy transport.

The thermodynamic analysis of radiation clearly shows that the value $\epsilon_\lambda / n^2 \kappa_\lambda$, which is constant throughout any enclosure, is the same for any two enclosures at the same temperature and is a universal function of temperature. The intensity of λ radiation in the medium is equal to the value $\epsilon_\lambda / n^2 \kappa_\lambda$. The radiation in an isothermal enclosure at a temperature T is called "black body radiation" at a temperature T. Thus, the coefficient of emission, ϵ_λ, of any matter in an enclosure at temperature T is given by the Kirchhoff law

$$\epsilon_\lambda = n^2 \kappa_\lambda I_{bb,\lambda}(T) \quad . \tag{2.16}$$

Here $I_{bb,\lambda}(T)$ is the monochromatic intensity of black body radiation given by Planck's law:

$$I_{bb,\lambda} = \frac{2\, c^2 h}{\lambda^5 \left(\exp \frac{ch}{\lambda k T} - 1\right)} \quad , \tag{2.17}$$

where k and h are Boltzmann's and Planck's constants, respectively. It is sometimes useful to express $I_{bb,\lambda}$ as a function of a frequency instead of a wavelength. We note that

$$\nu \lambda = c$$

and

$$I_{bb,\nu} d\nu = -I_{bb,\lambda} d\lambda ;$$

hence

$$I_{bb,\nu} = \left(\lambda^2 / c\right) I_{bb,\lambda} .$$

Introducing this result in (2.17), we find that

$$I_{bb,\nu} = \frac{2 h \nu^3}{c^2 \left(\exp \frac{h\nu}{kT} - 1\right)} \tag{2.18}$$

If the system we are considering is not an isothermal enclosure, we still introduce the concept of thermodynamic equilibrium so that we can define the temperature unambiguously. We assume that we can define at any small region of the medium of the system in consideration a local temperature T, such that the emission and absorption coefficients are the same as in thermodynamic equilibrium and that Kirchhoff's law is valid.[68] This is clearly a simplifying assumption.

Thus, in a nonisothermal enclosure, the temperature may vary from point to point, but each point may be characterized by a definite temperature T so that an element of matter at each point is behaving as if in local thermodynamic equilibrium at temperature T. It is to be noted that the hypothesis of local thermodynamic equilibrium is distinct from the equilibrium case where the temperature is constant throughout the region of the medium considered.

If the radiant energy absorbed per unit time by the volume ΔV is equal to the radiant energy emitted per unit time by the same volume, then the system in consideration is in radiative equilibrium. Radiative equilibrium prevails, for photon radiation, in any isothermal system that is shielded from external radiation. In such an equilibrium state, the entire system contains uniform energy density of photons, moving indiscriminately in all directions at the speed of light. The distribution of energy density and direction of the photons is likewise uniform for photons within any given wavelength interval. This kind of radiation is called isotropic. Such radiation produces no net energy flux, due to the complete balance of oppositely directed photons at all points.

2.7 Radiation from Surface and Volume

It is necessary to distinguish between radiation from a surface and from a volume. The radiation from a surface element dA is taken over the hemisphere, solid angle $\Omega = 2\pi$, while the radiation from an element of volume dV of a radiating medium in all directions is taken over the sphere, solid angle $\Omega = 4\pi$(see Fig. 2.5). A summary of analogous quantities and definitions is given in Table 2.1.

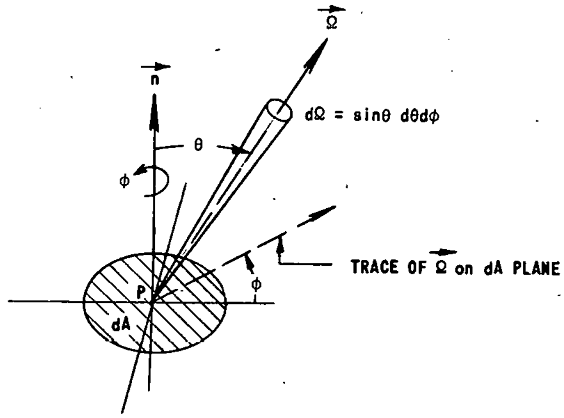

FIG. 2.5
GEOMETRICAL DATA FOR THE EVALUATION OF SURFACE AND VOLUME RADIATION.

Table 2.1

Summary of Quantities and Definitions for Radiation from a Surface and a Volume

Opaque Solid		Transparent Media	
α_λ	Monochromatic absorbtivity	κ_λ	Monochromatic absorption coefficient
ρ_λ	Monochromatic reflectivity	σ_λ	Monochromatic scattering coefficient
$\Gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega})$	Reflecting function	$\gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega})$	Scattering function
$E_\lambda(\vec{r},t)$	Monochromatic emissive power	$\mathcal{E}_\lambda(\vec{r},t)$	Monochromatic emission
$E'_\lambda(\vec{r},t)$	Monochromatic irradiation	$\mathcal{E}'_\lambda(\vec{r},t)$	Monochromatic incident radiation
$E_{a,\lambda}(\vec{r},t)$	Monochromatic radiation absorbed by a unit area	$\mathcal{E}_{a,\lambda}(\vec{r},t)$	Monochromatic radiation absorbed by a unit volume
$E_{r,\lambda}(\vec{r},t)$	Monochromatic radiation reflected from unit area	$\mathcal{E}_{s,\lambda}(\vec{r},t)$	Monochromatic radiation scattered from a unit volume
$R_\lambda(\vec{r},t)$	Radiosity-monochromatic energy leaving a unit area	$\mathcal{E}_{e,\lambda}(\vec{r},t)$	Effective emission-monochromatic radiant energy leaving a unit volume
$E_{n,\lambda}(\vec{r},t)$	Net radiant energy flux	$\mathcal{E}_{n,\lambda}(\vec{r},t)$	Net emission - net monochromatic radiant energy emitted from a unit volume

The reflecting function $\Gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega})$ is similar to the scattering function discussed previously, and it takes into account specular (non-diffuse) reflection from surfaces. The quantity $\Gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega})\, d\Omega'$ represents the probability that a pencil of rays of direction $\vec{\Omega}'$ incident on a surface is reflected into direction $\vec{\Omega}$, making an angle Θ with the original ray.

The various definitions and identities describing the heat transfer at a surface and on a volume of an emitting and scattering medium are introduced below and the radiant flux vector defined:

Radiation from Surface

Monochromatic Emissive Power

Radiant energy emitted normally from the surface per unit area, per unit time, per unit wavelength in all directions

$$E_\lambda(\vec{r},t) = \int_{\Omega=2\pi} I_\lambda(\vec{r},\vec{\Omega},t)\cos\theta\, d\Omega. \tag{2.19}$$

Monochromatic Irradiation

Radiant energy incident normally on a surface per unit time, per unit area, per unit wavelength from all directions

$$E'_\lambda(\vec{r},t) = \int_{\Omega=2\pi} I_\lambda(\vec{r},\vec{\Omega},t)\cos\theta\, d\Omega. \tag{2.21}$$

Radiation Absorbed

Radiant energy absorbed by the surface per unit time, per unit area, per unit wavelength

$$E_{a,\lambda}(\vec{r},t) = \alpha_\lambda E'_\lambda(\vec{r},t) = \alpha_\lambda \int_{\Omega=2\pi} I_\lambda(\vec{r},\vec{\Omega},t)\cos\theta\, d\Omega. \tag{2.23}$$

Radiation from Volume

Monochromatic Emission

Radiant energy emitted from a unit volume of the medium per unit volume, per unit time, per unit wavelength in all directions

$$\mathcal{E}_\lambda(\vec{r},t) = \int_{\Omega=4\pi} \epsilon_\lambda(\vec{r},\vec{\Omega},t)\, d\Omega. \tag{2.20}$$

Monochromatic Incident Relation

Radiant energy incident on a unit volume of medium per unit time, per unit area, per unit wavelength from all directions

$$\mathcal{E}'_\lambda(\vec{r},t) = cu_\lambda \int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\, d\Omega. \tag{2.22}$$

Radiation Absorbed

Radiant energy absorbed by a unit volume of the medium per unit time, per unit volume, per unit wavelength

$$\mathcal{E}_{a,\lambda}(\vec{r},t) = \kappa_\lambda \mathcal{E}'_\lambda(\vec{r},t) = \kappa_\lambda \int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\, d\Omega. \tag{2.24}$$

Radiation from Surface

Radiation Reflected

Radiant energy reflected from a surface per unit time, per unit area, per unit wavelength in all directions

$$E_{r,\lambda}(\vec{r}, t) = \frac{\rho_\lambda}{2\pi} \int_{\Omega=2\pi}\left[\int_{\Omega'=2\pi} \Gamma(\vec{r}, \vec{\Omega}' \to \Omega) I_\lambda(\vec{r}, \vec{\Omega}', t)\, d\Omega'\right] \cos\theta\, d\Omega. \qquad (2.25)$$

Radiosity

Radiant energy leaving a surface ($\tau_\lambda = 0$) per unit time, per unit area, per unit wavelength in all directions

$$R_\lambda(\vec{r}, t) = E_\lambda(\vec{r}, t) + E_{r,\lambda}(\vec{r}, t). \qquad (2.27)$$

Net Radiant Heat Flux

Net radiant energy exchange at the surface per unit time, per unit area, per unit wavelength

$$E_{n,\lambda}(\vec{r}, t) = R_\lambda(\vec{r}, t) - E'_\lambda(\vec{r}, t) = E_\lambda(\vec{r}, t) - \alpha_\lambda(\vec{r})\, E'_\lambda(\vec{r}, t). \qquad (2.29)$$

Radiation from Volume

Radiation Scattered

Radiant energy scattered from a unit volume, per unit time, per unit volume, per unit wavelength in all directions

$$\mathcal{E}_{s,\lambda}(\vec{r}, t) = \frac{\sigma_\lambda}{4\pi} \int_{\Omega=4\pi}\left[\int_{\Omega'=4\pi} \gamma(\vec{r}, \vec{\Omega}' \to \Omega) I_\lambda(\vec{r}, \vec{\Omega}', t)\, d\Omega'\right] d\Omega. \qquad (2.26)$$

Effective Emission

Radiant energy leaving a unit volume in all directions per unit time, per unit volume, per unit wavelength in all directions

$$\mathcal{E}_{e,\lambda}(\vec{r}, t) = \mathcal{E}_\lambda(\vec{r}, t) + \mathcal{E}_{s,\lambda}(\vec{r}, t). \qquad (2.28)$$

Net Emission

Net radiant energy emitted by a unit volume per unit time, per unit volume, per unit wavelength

$$\mathcal{E}_{n,\lambda}(\vec{r}, t) = \mathcal{E}_\lambda(\vec{r}, t) - \mathcal{E}_{a,\lambda}(\vec{r}, t) = \kappa_\lambda(\vec{r})\left[\mathcal{E}_{bb,\lambda}(\vec{r}, t) - \mathcal{E}'_\lambda(\vec{r}, t)\right]. \qquad (2.30)$$

Radiation from Volume

Radiant Flux Vector

Radiant flux vector is defined as the integral of the intensity in the direction of the unit vector $\vec{\Omega}_1$ over the unit sphere (solid angle, $\Omega = 4\pi$)

$$\vec{E}_\lambda(\vec{r},t) = \int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\vec{\Omega}_1 \, d\Omega \qquad (2.31)$$

2.8 Total Quantities

Throughout this work the monochromatic quantities (per unit wavelength or in the wavelength interval between λ and $\lambda + d\lambda$) are denoted by a suffix λ. The total values of energy quantities (defined in the previous section), i.e., intensity, emissive power, emission, etc., are defined as

$$f = \int_0^\infty f_\lambda \, d\lambda, \qquad (2.32)$$

where f is the desired quantity. The average radiative properties are obtained by integrating the monochromatic values over the entire spectrum from 0 to ∞ ,

$$g = \frac{\int_0^\infty g_\lambda w_\lambda \, d\lambda}{\int_0^\infty w_\lambda \, d\lambda} \quad , \qquad (2.33)$$

where w_λ is a weight function. The weight functions corresponding to a given property are presented in Table 2.2.

Table 2.2

Weight Functions for Radiative Properties

Property, g_λ	α_λ	ρ_λ	τ_λ	κ_λ	σ_λ
Weight function, w_λ	E'_λ	E'_λ	E'_λ	$\mathcal{E}'_\lambda$	$\mathcal{E}'_\lambda$

For the total quantities the suffix λ will be omitted and no ambiguity is likely to arise from it.

2.9 The Pressure of Radiation

The existence of radiation pressure follows from Maxwell's electromagnetic theory of light as well as from quantum theory and thermodynamics. According to quantum theory, the quantum of energy $h\nu$ possesses momentum $h\nu/c$ in its direction of propagation. It follows from this that radiant energy of amount q_r traversing a medium in a specific direction carries with it a momentum q_r/c. The momentum exerted is in the same direction as the pencil of radiation.

To determine the mechanical force exerted by the radiation in any direction, consider a thin cylinder of cross-sectional area dA and length ds the axis being in the direction $\vec{\Omega}$ (see Fig. 2.6). The amount of energy incident on dA in the directions contained in the solid angle $d\Omega$ about $\vec{\Omega}$ in the wavelength interval $d\lambda$ during time dt is

$$I_\lambda \cos\theta \, dA\, d\Omega\, d\lambda\, dt \quad . \tag{2.34}$$

The amount absorbed is obtained by multiplying by $\kappa_\lambda ds/\cos\theta$. The normal component of momentum in direction $\vec{\Omega}$ is obtained by multiplying by $\cos\theta$ and dividing the radiation absorbed by c

$$(1/c)\,\kappa_\lambda I_\lambda \cos\theta \, dA\, ds\, d\Omega\, d\lambda\, dt \quad .$$

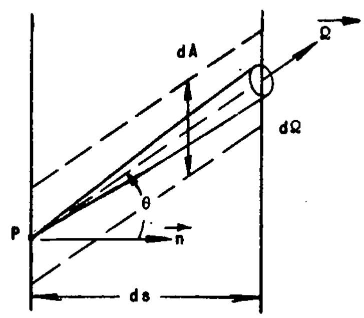

FIG. 2.6
GEOMETRICAL DATA FOR THE EVALUATION OF THE RADIATION PRESSURE ON THE SLAB OF RADIATING MEDIUM.

To obtain the normal force per unit area, we divide by the area dA and the time dt. Integrating over all direction, we find that the radiation pressure over all wavelengths of slab thickness ds is

$$P_r = \frac{\kappa_\lambda ds}{c} \int_0^\infty \left[\int_{\Omega=4\pi} I_\lambda \cos\theta \; d\Omega \right] d\lambda \quad , \tag{2.35}$$

where the integral on the right side of the above equation is the total radiation flux normal to the slab. Equation (2.35) gives the normal force per unit area of slab thickness ds.

The pressure at a point P is defined as a rate of transfer of momentum normal to an arbitrary chosen infinitesimal surface dA containing P. To obtain the pressure we divide expression (2.34) by c. The normal component of momentum across dA by the pencil of radiation in the wavelength interval λ and $\lambda + d\lambda$ is

$$(1/c) I_\lambda \cos^2 \theta dA \, d\Omega \, d\lambda \, dt \quad .$$

The total radiation pressure is the momentum transfer per unit area and is obtained by integrating the above over all directions, $\Omega = 4\pi$, all wavelengths and dividing by dA.

$$P_r = \frac{1}{c} \int_0^\infty \left[\int_{\Omega=4\pi} I_\lambda \cos^2 \theta \, d\Omega \right] d\lambda \quad . \tag{2.36}$$

A more general way of calculating the pressure due to radiation is to consider an element of surface normal to the vector $\vec{n}$. The rate of transfer of the x component of momentum per unit area per second by radiation confined in a solid angle $d\Omega$ in the direction $\vec{\Omega}$ (direction cosines Ω_x, Ω_y, Ω_z) is

$$(1/c) I \, \Omega_x^2 \, d\Omega \ .$$

The total rate of transfer of momentum in the x direction across the element per unit area per unit time is then

$$\frac{1}{c} \int_{\Omega=4\pi} I \Omega_x^2 \, d\Omega \ .$$

But this as well as (2.36) simply define the x-component of pressure exerted across the element under consideration. We write it as p_{xx}. In the same way, the y and z components of pressure across the same element are, respectively,

$$p_{xy} = \frac{1}{c} \int_{\Omega=4\pi} I \Omega_x \Omega_y \, d\Omega \ ; \quad p_{xz} = \frac{1}{c} \int_{\Omega=4\pi} I \, \Omega_x \Omega_y \, d\Omega \ .$$

By considering the stresses exerted across three perpendicular planes, each stress having three components, one[68] obtains a stress tensor whose components are

$$p_{ij} = \begin{vmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{vmatrix} = \frac{1}{c} \int_{\Omega=4\pi} I \, \Omega_i \Omega_j d\Omega \ ; \ i,j = x,y,z \ , \qquad (2.37)$$

where Ω_i and Ω_j are direction cosines. This tensor is partly analogous to the stress tensor in fluid dynamics and elasticity. We observe that it is symmetrical, $p_{ij} = p_{ji}$. If the radiative viscosity as well as the term depending on the second-order temperature gradients are not neglected, a more complete expression for the pressure tensor is obtained.[43]

The mean pressure, p_r, is defined as

$$p_r = \frac{1}{3}(p_{xx} + p_{yy} + p_{zz}) = \frac{1}{3c}\int_{\Omega=4\pi} I d\Omega = \frac{u}{3c} \tag{2.38}$$

when the radiation is isotropic. In the general case the x-component of force on a unit volume dxdydz is

$$F_x = -\left(\frac{\partial p_{xx}}{\partial x} + \frac{\partial p_{xy}}{\partial y} + \frac{\partial p_{xz}}{\partial z}\right) dxdydz \quad , \tag{2.39}$$

with similar expressions for the y and z components.

For black gas radiation, equation (2.38) becomes

$$p_r = \frac{4\pi I_{bb}}{3c} = \frac{4E_{bb}}{3c} = \frac{4\sigma T^4}{3c} = 3.66 \times 10^{-20}\, T^4 \text{ psi} \quad , \tag{2.40}$$

where T is in degrees Kelvin. The radiation pressure becomes 3.66 psi at 10^5°K. Thus the pressure of radiation could become of kinetic importance in fusion reactions. The minute, but definite, effects from radiation sources at more readily attainable temperatures have been studied both experimentally and analytically. In a recent paper Jones and Richards(44) considered experimentally some more complicated phenomena due to propagation of radiation in refracting liquid media. Their experiments show that not only does radiation exert a pressure on a mirror, but that it also gains or loses momentum when it crosses refractory surfaces.

2.10 Thermophysical Properties of Radiation

When radiation flows in an absorbing medium, there is a progressive reduction in intensity or in amplitude which is equivalent to the existence of a mean free path for the associated photons. By photon mean free path is meant a path λ_p traveled over, on the average, by the photons in that time interval which elapses between the moments of their emission and absorption by the atoms of the substance. It follows from this definition that in a mean free path of the photon, which is a statistical quantity, the radiation energy does not interact with the medium. If the mean free path of the photon is small in comparison to the dimensions of the space in which the radiation energy is being propagated, then one can often apply diffusion concepts. Radiative transfer in highly absorbing media thus acquires some of the properties of conductive transfer.

If the mean free path is sufficiently small, we can associate a diffusivity as well as viscosity with the photons. Jeans[42], by a crude calculation based on the analogy with the kinetic theory of gases, showed that the "viscosity" arising from radiation can be expressed by

$$\mu_r = \frac{1}{3} \rho C \lambda_p \quad , \tag{2.41}$$

where C is the velocity of the photons and λ_p is the mean free path. The "thermal conductivity" is given by

$$k_r = \mu_r c_v \quad , \tag{2.42}$$

where c_v is the radiative "specific heat" for constant volume. Corresponding to the temperature gradient $\partial T/\partial x$, the heat flux is given by

$$q''_r = -k \frac{\partial T}{\partial x} = -\frac{1}{3} \rho C \lambda_p c_v \frac{\partial T}{\partial x} \quad . \tag{2.43}$$

When the carriers of heat are not molecules or atoms as in gases, but photons, the specific heat at constant volume per unit mass is given by

$$c_v = \left. \frac{\partial U}{\partial T} \right|_V \quad .$$

For black gas radiation (with index of refraction n=1) this becomes

$$c_v = \frac{1}{\rho} \left. \frac{\partial (4\sigma T^4/c)}{\partial T} \right|_V \quad .$$

Since the velocity of the carriers C is equal to the velocity of light, c, we have

$$c_v = 16\sigma T^3/3\rho c \quad . \tag{2.44}$$

To find the mean free path of the photons we note that the stream is reduced in strength by e^{-x/λ_p} after traversing a distance x, while the beam of radiation is reduced by $e^{-\kappa x}$. Thus, the mean free path for the quanta is taken to be $\lambda_p = 1/\kappa$. Making these substitutions in (2.43), we find that

$$q_r'' = -\frac{16\sigma T^3}{3\kappa} \frac{\partial T}{\partial x} \quad . \tag{2.45}$$

Since the photons are the only carriers for thermal radiation, the equation also defines the "thermal conductivity" of thermal radiation, which is given by

$$k_r = -16\sigma T^3/3\kappa = 16\lambda_p \sigma T^3/3 \quad . \tag{2.46}$$

The "density" of radiation in equilibrium with a body of temperature T is obtained from Einstein's law $m/V = E/Vc^2$ and is given by

$$u/c^2 = 4\sigma T^4/c^3 \quad . \tag{2.47}$$

Now, substituting the values C = c and $\lambda_p = 1/\kappa$ as well as the above result into (2.42), we obtain for radiative "viscosity" the value

$$\mu_r = \frac{1}{3}\left(\frac{4\sigma T^4}{c^2}\right)\left(\frac{c}{\kappa}\right) = \left(\frac{4\sigma T^4}{3c^2\kappa}\right). \tag{2.48}$$

By making a momentum balance on a pencil of photons passing through an element of area, Eddington(19) obtained the same value of radiative viscosity. A more exact calculation by Jeans(43) showed that the radiative viscosity is given by

$$\mu_r = 8cT^4/15c^2\kappa \quad . \tag{2.49}$$

Hazlehurst and Sargent(31) have considered radiation as a photon gas. The relativistic terms of the order (v/c) and $(v/c)^2$ have been explicitly calculated. The radiative viscosity has been found to be twice that predicted by Jeans, but identical with that obtained previously by Thomas.[102]

The expression for radiative "thermal conductivity" may be combined with the expressions for the radiative "viscosity" and "specific heat" at constant pressure to give the dimensionless radiative "Prandtl number." The value of this parameter is found to be very small at ordinary temperatures.

3 EQUATION OF TRANSFER

3.1 Introduction

The radiative transfer problem is a quantitative study, on a phenomenological level, of the transfer of radiant energy through the media that absorbs, scatters and emits radiant energy. The problem was formulated by Hopf[35] over twenty years ago, and the foundations still remain unchanged. A new approach to the formulation of the problem was presented by Preisendorfer.[82] He introduced a set of physically motivated axioms phrased in the language of measure theory from which, as a special case, the prominent features of radiative transfer were rigorously deduced.

Equations of transfer of less general form for a medium at rest are given in the astrophysics books of Milne,[68] Rosseland,[86] Chandrasekhar,[13] Ambartsumyan[4] and others. In all of these references either the scattering or the change of intensity with time or both were not considered. Thomas[102] derived an equation of transfer for a medium in motion by using the Lorentz transformation and obtained the equation in a form including all relativistic terms in the ratio of the velocity of motion to the velocity of light. More recently, Synge,[101] by using a different approach, arrived at a similar equation.

In this chapter the writer derives the basic equation of radiant heat transfer, the equation of transfer, and by so doing, generalizes all previous results in nonrelativistic terms. This equation governs the radiation field in an isotropic medium at rest which absorbs, emits and scatters thermal radiation. In deriving this equation the Eulerian point of view is taken. The detailed mechanism of the interaction processes involving atoms and the field of radiation is not considered here. Only the macroscopic problem consisting of the study of the transformation suffered by the field of radiation passing through a medium is examined. Thus, it is unnecessary to retain the formulation of the quantum theory of radiation. It is also sufficient just to consider a parallel beam of radiation and to follow its depletion or growth as it moves along.

Thermodynamic states in which temperature varies from point to point in space and time are considered. However, this presupposes the existence of a definite temperature at each point at all time. The temperature can be uniquely defined, as mentioned in Section 2.6, only for a system in thermodynamic equilibrium. It is therefore assumed that the medium is in local thermodynamic equilibrium.

3.2 Derivation of the Monochromatic Equation of Transfer

Consider a small cylindrical element, Fig. 3.1, of cross section

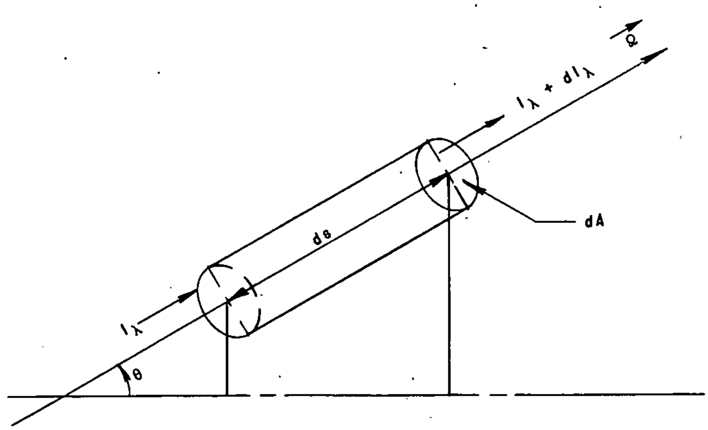

FIG. 3.1
TRANSFER OF THERMAL RADIATION.

dA and length ds in an absorbing, emitting and scattering medium. Radiant energy in the wavelength interval between λ and $\lambda + d\lambda$, confined to a pencil of rays of solid angle $d\Omega$ about direction $\vec{\Omega}$, in the time interval dt, will cross the two faces normally. This monochromatic intensity $I_\lambda(\vec{r},\vec{\Omega},t)$ will decrease on account of absorption and scattering from $d\Omega$; it will increase because of contributions from the emission and scattering into the volume element.

From the definition of intensity, it now follows that the net increase in the radiant energy is given by

$$\frac{d\,I_\lambda(\vec{r},\vec{\Omega},t)}{dt}\, dA\,d\Omega\,d\lambda\,dt \quad .$$

We can expand $dI_\lambda(\vec{r},\vec{\Omega},t)/dt$ in a Taylor series and keep only the linear terms; thus, we have

$$\frac{dI_\lambda}{dt} = \frac{\partial I_\lambda}{\partial t} + c\;\vec{\Omega}\cdot \text{grad}\, I_\lambda \quad .$$

Using the vector identity

$$\text{div}\,(\vec{\Omega} I_\lambda) = I_\lambda\,\text{div}\,\vec{\Omega} + \vec{\Omega}\cdot\text{grad}\, I_\lambda = \vec{\Omega}\cdot\text{grad}\, I_\lambda \quad ,$$

we can write

$$\frac{dI_\lambda}{dt} = \frac{\partial I_\lambda}{\partial t} + c\;\text{div}\,(\vec{\Omega} I_\lambda) \quad ,$$

so that

$$\frac{dI_\lambda(\vec{r},\vec{\Omega},t)}{dt}\, dA\,d\Omega\,d\lambda\,dt = \left\{\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\,\partial t} + \text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\right\} dA\,d\Omega\,d\lambda\,dt.$$

The quantity $\vec{\Omega}\cdot\text{grad}\, I_\lambda = \text{div}\,(\vec{\Omega} I_\lambda)$ is the directional derivative in the direction $\vec{\Omega}$. Thus, if the coordinate s is laid off in the direction $\vec{\Omega}$, then

$$\vec{\Omega}\cdot\text{grad}\, I_\lambda = \text{div}\,(\vec{\Omega} I_\lambda) = \frac{\partial I_\lambda}{\partial s} \quad .$$

The distance traveled by the pencil of rays is $ds = c\,dt$, and therefore we can finally rewrite the above expression as

$$\frac{dI_\lambda(\vec{r},\vec{\Omega},t)}{dt}\, dA\,d\Omega\,d\lambda\,dt = \left\{\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\,\partial t} + c\;\text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\right\} dA\,ds\,d\lambda\,dt.$$

The components $\vec{r}$ and $\vec{\Omega}$ are independent variables, so that, when differentiating with respect to one variable, the other must be regarded as constant. Since differentiation with respect to $\vec{\Omega}$ will never occur, it is not necessary to give an index $\vec{r}$ or $\vec{\Omega}$ to the operators "grad" and "div" to specify the variable ($\vec{r}$ or $\vec{\Omega}$) with respect to which the differentiation is to be undertaken.

The radiation scattered out of and absorbed in the pencil of rays in time dt is given by

$$(\kappa_\lambda + \sigma_\lambda)\, I_\lambda(\vec{r},\vec{\Omega},t)\; dA\,ds\,d\Omega\,d\lambda \qquad ,$$

according to the definition of the absorption and scattering coefficients.

The amount of radiant energy scattered from direction $\vec{\Omega}'$ into $\vec{\Omega}$ into the volume element, dV = dA ds, during time dt is

$$\left[\frac{\sigma_\lambda}{4\pi}\int_{\Omega' = 4\pi} \gamma(\vec{r},\vec{\Omega}'\rightarrow\vec{\Omega})\, I_\lambda(\vec{r},\vec{\Omega}',t)\, d\Omega'\right] dA\,ds\,d\Omega\,d\lambda \qquad ,$$

where $\gamma(\vec{r},\vec{\Omega}'\rightarrow\vec{\Omega})$ is the scattering function which gives the probability that the pencil of rays will be scattered into direction $\vec{\Omega}$ from $\vec{\Omega}'$.

The radiant energy emitted from the volume element dV during the time interval dt is given by

$$\epsilon_\lambda(\vec{r},\vec{\Omega},t)\; dA\,ds\,d\Omega\,d\lambda \qquad ,$$

according to the definition of the emission coefficient.

Counting up the gains and losses of radiant energy in the pencil of rays $d\Omega$ during its traversal of distance ds and dividing by dA ds $d\Omega d\lambda$, we obtain the equation

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\,\partial t} + \mathrm{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] = \epsilon_\lambda(\vec{r},\vec{\Omega},t) - (\kappa_\lambda + \sigma_\lambda) I_\lambda(\vec{r},\vec{\Omega},t)$$

$$+ \frac{\sigma_\lambda}{4\pi}\int_{\Omega' = 4\pi} \gamma(\vec{r},\vec{\Omega}'\rightarrow\vec{\Omega})\, I_\lambda(\vec{r},\vec{\Omega}',t)\, d\Omega' \quad . \qquad (3.1)$$

This integro-differential equation is called the equation of transfer or the transport equation. It is valid for coherent scattering only; for noncoherent scattering processes it has to be modified.[4]

The processes of selective attenuation of radiation, i.e., those taking place at discrete frequencies, require special attention. These processes include: (1) true selective absorption caused by discrete transitions of electrons, and (2) selective scattering. The equation of transfer can be generalized,[4,103] to account for the processes of true selective absorption and scattering. We can also allow for the possibility that a certain amount of thermal emission, δ_λ, can be associated with the scattering coefficient, σ_λ. Equation (3.1) can then be written[13] as

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\partial t} + \text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] = n^2 (\kappa_\lambda + \delta_\lambda \sigma_\lambda) I_{bb,\lambda}(\vec{r},t)$$

$$- (\kappa_\lambda + \sigma_\lambda) I_\lambda(\vec{r},\vec{\Omega},t) + \frac{(1-\delta_\lambda)\sigma_\lambda}{4\pi} \int_{\Omega'=4\pi} \gamma(\vec{r},\vec{\Omega}' \to \vec{\Omega})\, I_\lambda(\vec{r},\vec{\Omega}',t)\,d\Omega' \quad . \tag{3.2}$$

If the processes of true selective absorption are disregarded ($\delta_\lambda = 0$), this equation reduces to (3.1).

We define the monochromatic effective emission coefficient, $\epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t)$, as radiant energy leaving a unit volume of the medium per unit volume, per unit solid angle, per unit wavelength and per unit of time as

$$\epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t) = \epsilon_\lambda(\vec{r},\vec{\Omega},t) + \frac{\sigma_\lambda}{4\pi} \int_{\Omega'=4\pi} \gamma(\vec{r},\vec{\Omega}' \to \vec{\Omega})\, I_\lambda(\vec{r},\vec{\Omega}',t)\,d\Omega' \quad . \tag{3.3}$$

Thus, $\epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t)$ represents the sum of the emitted and scattered radiation. On substitution of (3.3), equation (3.1) becomes

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\partial t} + \text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] = - (\kappa_\lambda + \sigma_\lambda) I_\lambda(\vec{r},\vec{\Omega},t)$$

$$+ \epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t) \quad . \tag{3.4}$$

3.3 Equation of Transfer for Nonscattering and Diathermal Medium

In case of a purely absorbing and emitting medium with no scattering ($\sigma_\lambda = 0$, $\epsilon_{e,\lambda} = \epsilon_\lambda$), the equation of transfer (3.4) reduces to

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\partial t} + \text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] = -\kappa_\lambda\, I_\lambda(\vec{r},\vec{\Omega},t) + \epsilon_\lambda(\vec{r},\vec{\Omega},t) \quad . \qquad (3.5)$$

In a diathermal medium ($\kappa_\lambda = \sigma_\lambda = \epsilon_\lambda = \epsilon_{e,\lambda} = 0$) the equation of transfer (3.4) becomes

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\partial t} + \text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] = 0 \quad .$$

From this it follows that

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\partial t} = -\,\text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] \quad . \qquad (3.6)$$

3.4 Equation of Transfer for Steady State

In a macroscopic sense, it will sometimes (usually) occur that the intensity is independent of time. Thus, $\partial I_\lambda(\vec{r},\vec{\Omega},t)/c\,\partial t = 0$, and the equation of transfer, (3.4), reduces to

$$\text{div}\,[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega})] = -\beta_\lambda\, I_\lambda(\vec{r},\vec{\Omega}) + \epsilon_{e,\lambda}(\vec{r},\vec{\Omega}) \quad , \qquad (3.7)$$

where

$$\beta_\lambda = \kappa_\lambda + \sigma_\lambda \qquad (3.8)$$

is the extinction coefficient. Even when the intensity varies rapidly with time, it is often justifiable to neglect the time derivative with respect to other terms in equation (3.7).

It is possible to reduce the time-dependent equation of transfer to a stationary one. For instance, consider the case

$$I_\lambda(\vec{r},\vec{\Omega},t) = 0; \qquad t \leq 0 \qquad (3.9)$$

Introduce the Laplace transform, defined by

$$I_\lambda(\vec{r},\vec{\Omega},s) = \int_0^\infty e^{-st} I_\lambda(\vec{r},\vec{\Omega},t)dt \quad .$$

Then, multiplying (3.1) by e^{-st} and integrating from zero to infinity yields, on using (3.9),

$$\frac{s}{c} I_\lambda(\vec{r},\vec{\Omega},s) + \mathrm{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},s)] = \int_0^\infty \epsilon_\lambda(\vec{r},\vec{\Omega},t) e^{-st}\, dt - \beta_\lambda I_\lambda(\vec{r},\vec{\Omega},s)$$
$$+ \frac{\sigma_\lambda}{4\pi}\int_{\Omega'=4\pi} \gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega}) I_\lambda(\vec{r},\vec{\Omega}',s) d\Omega' \quad .$$

The first integral on the right-hand side cannot be simplified. However, if it should occur that the emission coefficient describes a pulse at t = 0, i.e., $\epsilon_\lambda(\vec{r},\vec{\Omega},t) = \epsilon_\lambda(\vec{r},\vec{\Omega})\delta(t)$, the above equation becomes

$$\mathrm{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},s)] = \epsilon_\lambda(\vec{r},\vec{\Omega}) - \left(\frac{s}{c} + \beta_\lambda\right) I_\lambda(\vec{r},\vec{\Omega},s)$$
$$+ \frac{\sigma_\lambda}{4\pi}\int_{\Omega'=4\pi} \gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega}) I_\lambda(\vec{r},\vec{\Omega}',s) d\Omega' \quad . \tag{3.10}$$

With the definition

$$\beta_\lambda' = \frac{s}{c} + \beta_\lambda \quad , \tag{3.11}$$

equation (3.10) may be brought into the form

$$\mathrm{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},s)] = \epsilon_\lambda(\vec{r},\vec{\Omega}) - \beta_\lambda' I_\lambda(\vec{r},\vec{\Omega},s)$$
$$+ \frac{\sigma_\lambda}{4\pi}\int_{\Omega'=4\pi} \gamma(\vec{r},\vec{\Omega}' \rightarrow \vec{\Omega}) I_\lambda(\vec{r},\vec{\Omega}',s) d\Omega' \quad . \tag{3.12}$$

This equation is the same as (3.7) with a modified β_λ.

3.5 Comparision of the Equation of Transfer with the Continuity Equation

The equation (3.4) is similar to the equation of conservation of mass:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{w}) = Q \quad , \tag{3.13}$$

where ρ is the fluid density, $\vec{w}$ the velocity vector, and Q the source or sink. If we define the right side of equation (3.4) as

$$Q_R = -\beta_\lambda I_\lambda(\vec{r},\vec{\Omega},t) + \epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t) \quad ,$$

then equation (3.4) becomes

$$\frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{c\,\partial t} + \mathrm{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)] = Q_R \quad . \tag{3.14}$$

This equation states the conservation of radiation intensity.

3.6 Equation of Conservation of Radiant Energy

Integrating equation (3.4) over all solid angles ($\Omega = 4\pi$), we obtain

$$\frac{1}{c}\int_{\Omega=4\pi} \frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{\partial t}\, d\Omega + \int_{\Omega=4\pi} \mathrm{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\, d\Omega = -\beta_\lambda \int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\, d\Omega + \int_{\Omega=4\pi} \epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t)\, d\Omega \tag{3.15}$$

In the special case of diffuse radiation from or to the unit volume of the medium, we have

$$\frac{1}{c}\int_{\Omega=4\pi} \frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{\partial t}\, d\Omega = \frac{1}{c}\frac{\partial}{\partial t}\int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\, d\Omega = \frac{\partial u_\lambda(\vec{r},\vec{\Omega},t)}{\partial t} \quad .$$

The assumption of diffuse radiation is introduced only in this term, which is ordinarily negligible in comparison with other terms in equation (3.15).

Using the definition of the radiant energy flux vector (2.31), we have

$$\int_{\Omega=4\pi} \mathrm{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\, d\Omega = \mathrm{div}\ E_\lambda(\vec{r},t) \quad .$$

The right-hand side of equation (3.15) represents the net radiant energy emitted or absorbed by a unit volume, per unit time, in the wavelength intergal between λ and $\lambda + d\lambda$, defined by equation (2.30). Then the equation expressing the conservation of monochromatic radiant energy becomes

$$\frac{\partial u_\lambda(\vec{r},t)}{\partial t} + \text{div}\, \vec{E}_\lambda(\vec{r},t) = \mathcal{E}_{n,\lambda}(\vec{r},t) \quad . \tag{3.16}$$

3.7 The Radiant Energy Flux Tensor

Integration of the equation of transfer (3.4) vectorially over all solid angles ($\Omega = 4\pi$) yields

$$\frac{1}{c}\int_{\Omega=4\pi} \frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{\partial t}\vec{\Omega}_1 d\Omega + \int_{\Omega=4\pi} \text{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\vec{\Omega}_1 d\Omega = -\beta_\lambda \int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\vec{\Omega}_1 d\Omega + \int_{\Omega=4\pi} \epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t)\vec{\Omega}_1 d\Omega$$

It is to be noted that this equation is different from (3.15). Every term of equation (3.15) has been multiplied by a unit vector $\vec{\Omega}_1$.

From the definition (2.31) and the fact that for isotropic scattering and emission $\epsilon_{e,\lambda}(\vec{r},\vec{\Omega},t)$ is independent of direction,

$$\int \vec{\Omega}_1 d\Omega = 0 \quad ,$$

we have that

$$\frac{1}{c}\frac{\partial \vec{E}_\lambda(\vec{r},t)}{\partial t} + \int_{\Omega=4\pi} \text{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\vec{\Omega}_1 d\Omega = -\beta_\lambda \vec{E}_\lambda(\vec{r},t) \quad . \tag{3.17}$$

The integral on the left-hand side equation (3.17) is a tensor of second order. It was defined by Rosseland[86] as the monochromatic radiant energy tensor:

$$\int_{\Omega=4\pi} \text{div}[\vec{\Omega} I_\lambda(\vec{r},\vec{\Omega},t)]\vec{\Omega}_1 d\Omega = \frac{\partial}{\partial x}\int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\Omega_x \vec{\Omega}_1 d\Omega + \frac{\partial}{\partial y}\int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\Omega_y \vec{\Omega}_1 d\Omega + \frac{\partial}{\partial z}\int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\Omega_z \vec{\Omega}_1 d\Omega \tag{3.18}$$

where Ω_x, Ω_y, Ω_z are the direction cosines. The components of the tensor P_λ are defined as

$$P_{\lambda,ij}(\vec{r},t) = \int_{\Omega=4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\Omega_i \Omega_j d\Omega \;\; ; \;\; i,j = x,y,z \quad . \tag{3.19}$$

On substituting (3.19) into equation (3.17), we obtain

$$\frac{1}{c}\frac{\partial \vec{E}_\lambda(\vec{r},t)}{\partial t} + \text{div}\, P_\lambda(\vec{r},t) = -\beta_\lambda \vec{E}_\lambda(\vec{r},t) \quad . \tag{3.20}$$

The radiant energy flux tensor is symmetrical: $P_{\lambda,ij} = P_{\lambda,ji}$. The invariant of this tensor is the sum of the diagonal components[86],

$$P_{\lambda,ii} = P_{\lambda,xx} + P_{\lambda,yy} + P_{\lambda,zz}$$

$$= \int_{\Omega = 4\pi} I_\lambda(\vec{r},\vec{\Omega},t)\,[\Omega_x^2 + \Omega_y^2 + \Omega_z^2]d\Omega = \mathcal{E}'_\lambda(\vec{r},t) \quad . \tag{3.21}$$

3.8 Significance of the Equation of Transfer

The mathematical foundations of the present day radiative transfer theory rest on the integro-differential equation (3.1) and its minor variants. This type of equation also arises in several branches of physical science and mathematics, namely: classical dynamics of gases, neutron transport, probability theory and others. The Boltzmann integro-differential equation in kinetic theory of gases describes the dynamics of molecular interactions.[14] Nuclear physicsts view equation (3.1) as the linearized Boltzmann neutron transport equation,[106] and the role of the photon is replaced by a neutron. Mathematicians concerned with the probability theory consider (3.1) and related variants as a representation of certain Markhoff processes.[23]

The significance of equation (3.1) in the problems of heat transfer in thermal radiation absorbing and scattering media can be compared to the importance of the Fourier-Biot equation in the mathematical theory of heat conduction. The various forms of the equation of transfer presented in this chapter are used in the following chapters. Equation (3.5) is utilized in deriving the integral equations for an enclosure containing thermal radiation absorbing, emitting, and scattering media. Equation (3.16) is used in deriving the energy equation. The Rosseland definition of the radiant energy flux tensor, equation (3.20), is employed in deducing the approximation for the radiant flux vector in the case of intense absorption.

4 PARTIAL SOLUTIONS OF THE EQUATION OF TRANSFER

4.1 Introduction

The equation of transfer (3.1) and its simpler forms are the usual starting points for heat transfer problems occurring in media absorbing and scattering thermal radiation. It is therefore of interest to look into the established solutions of this equation. Considerations of radiative transfer were first introduced in astrophysical problems in connection with the formation of absorption lines in solar spectra and have already attained a high degree of organization.[19,68,103,4] More recently, the theory of radiative transfer has been applied to neutron transport.* A sketch of the history and an extended bibliography on radiative transfer problems may be found in references 68, 13, 52, and 103. In this work only a few pertinent references dealing with the transport of radiation and neutrons will be cited.

Although the equation of transfer(3.1) has been much studied, it is very difficult to solve even for the simplest cases. The range of problems amenable to exact solution is quite small and for most cases an approximate treatment is the best that can be given. The equation of transfer can be solved rigorously for the one-dimensional case by the method based on the theory of complex variables, the basic ideas

*It is shown in Chapter 7 that the problem of transfer of thermal radiation in an absorbing, scattering and emitting medium is mathematically analogous to the problem of neutron transport in a capturing, scattering and fissioning medium. Therefore, a solution of a transport problem is a solution of a similar problem in transfer of thermal radiation.

of which are due to Wiener and Hopf.[108] A penetrating analysis of the nature of this problem and its solution was made by Lehner and Wing.[56,57] Dealing with the one-velocity neutron transport equation, they proved the existence and uniqueness of the solution for both the infinite slab and the sphere. For the slab extending from -a to +a they were able to discuss completely the structure of the solution. A new method for the solution of the neutron transport problems was suggested by Case.[12] This approach is analogous to the classical separation of variables method for partial differential equations.

Of the various approximations which have been used for total radiative intensity, it seems that the best compromise between consistent success and ease of numerical calculation is still that introduced by Eddington[19] and Milne [68], the latter using a somewhat different but mathematically more elegant method. Both methods convert the integrodifferential equation into an approximate second-order differential equation and provide a considerable simplification with, generally, a loss of accuracy.

Two methods are able to give results of arbitrarily high accuracy, provided that a sufficient amount of labor is expended on their calculation. The spherical harmonics method is the most powerful of the two. It was introduced by Wick[107] and Marshak[65] and developed in detail for a general geometry by Mark[64,65] for the solution of neutron transport equation. The method of discrete coordinates proposed by Wick[107] as an approximate method for solving the transport equation was developed by Chandrasekhar[13] into a powerful theoretical tool for the investigation of astrophysical problems. This method is not so powerful as the spherical harmonics method, which it resembles in some respects, and it has been applied only to the case of plane geometry.

Attention has so far been largely confined to one-dimensional radiation transfer problems, usually in semi-infinite, plane-parallel media. To the author's knowledge, no analytical solutions have been obtained for the problems with spherical and cylindrical symmetry. The most extensive treatise on radiative transfer is by Chandrasekhar(13), in which both the approximate Wick-Chandrasekhar method and more general methods for solving radiative transfer problems are presented. The solution of transfer problems in semi-infinite plane-parallel atmospheres with laws of scattering more general than isotropic leads to systems of integral equations which are nonlinear, nonhomogeneous, and of high degree. The scattering function, $(3/4)(1 + \cos^2\Theta)$, leads to simultaneous integral equations of second degree. The more general Rayleigh law of scattering leads to fourth-degree nonlinear integral equations.

Kourganoff(52) gives a more general exposition of the various techniques used in transport theory, as well as a summary of most of the available methods for treating the monochromatic equation of transfer. For the compilation of the techniques used in solving multi-dimensional problems in neutron transport, reference is made to the book by Davison(17) and the monograph by Marchuk(63).

4.2 Propagation of Radiation in a Diathermal Medium

For a diathermal (nonabsorbing, nonscattering and nonemitting) medium in a steady state, equation (3.6) reduces to

$$\vec{\Omega} \cdot \text{grad } I_\lambda = \text{div } (\vec{\Omega} I_\lambda) = \frac{\partial I_\lambda}{\partial s}, \tag{4.1}$$

where ds is a line element along the pencil of radiation $\vec{\Omega}$. This equation obviously indicates that i_λ is constant along a line parallel to $\vec{\Omega}$.

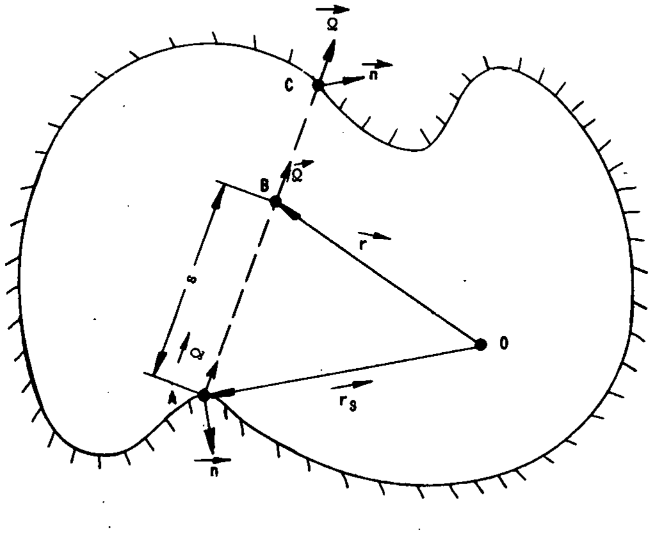

FIG. 4.1
GEOMETRICAL DATA FOR PROPOGATION OF RADIATION IN DIATHERMAL MEDIUM.

For a general region enclosed by a surface (Fig. 4.1), I_λ is the same at the points A, B and C, which lie along the line in the direction of the unit vector $\vec{\Omega}$. The intensity at point A is equal to the intensity at C, and only one of these intensities is to be specified to give a complete boundary condition. If $\vec{r_S}$ is a point on the enclosing surface which lies in the direction $\vec{\Omega}$ from $\vec{r}$ and if the intensity emitted by the surface at $\vec{r_S}$ is given, namely, $I_\lambda(\vec{r_S}, \vec{\Omega})$, (4.1) has then a solution

$$I_\lambda(\vec{r}, \vec{\Omega}) = I_\lambda(\vec{r_S}, \vec{\Omega}) = I_\lambda(\vec{r} - s\vec{\Omega}, \vec{\Omega}). \tag{4.2}$$

If I_λ is independent of $\vec{r_S}$, i.e., if

$$I_\lambda(\vec{r_S}, \vec{\Omega}) = f(\vec{\Omega}), \tag{4.3}$$

then by (4.2)

$$I_\lambda(\vec{r}, \vec{\Omega}) = f(\vec{\Omega}). \tag{4.4}$$

Thus the intensity of radiation in the region is independent of position. In particular, if the intensity on the surface is diffuse, it will also be diffuse within the enclosure.

4.3 Solution of the Equation of Transfer for a Given Direction

4.3.1 Propagation of Radiation in an Absorbing and Scattering Medium

Consider a medium which absorbs, emits and scatters thermal radiation. The absorption and scattering coefficients are assumed to depend on the position, and the direction of the pencil of rays is considered to be given. Hence for the pencil of rays in direction $\vec{\Omega}$, the directional derivative becomes

$$\vec{\Omega} \cdot \text{grad } I_\lambda = \text{div } (\vec{\Omega} I_\lambda) = \frac{dI_\lambda}{ds} ,$$

and therefore the steady-state equation of transfer (3.4) reduces to

$$\frac{dI_\lambda(s)}{ds} = -\beta_\lambda (s) \, I_\lambda(s) + \epsilon_{e,\lambda}(s) \quad . \tag{4.5}$$

An equation of this form may be solved in the following manner. Multiplying both sides by the integrating factor, $e^{\int \beta_\lambda(s)ds}$, equation (4.5) can be written in the form

$$\frac{d}{ds}\left(I_\lambda(s) \, e^{\int \beta_\lambda(s)ds}\right) = \epsilon_{e,\lambda}(s) \, e^{\int \beta_\lambda(s)ds} \quad .$$

Consequently by integration we obtain the solution

$$I_\lambda(s) \, e^{\int \beta_\lambda(s)ds} = \int \epsilon_{e,\lambda}(s') \, e^{\int \beta_\lambda(s)ds} \, ds' + C \quad .$$

Dividing by $e^{\int \beta_\lambda(s)ds}$, we get

$$I_\lambda(s) = C \, e^{-\int \beta_\lambda(s)ds} + e^{-\int \beta_\lambda(s)ds} \int \epsilon_{e,\lambda}(s') \, e^{\int \beta_\lambda(s)ds} \, ds', \tag{4.6}$$

which is the general solution of equation (4.5). The factor $e^{\int \beta_\lambda(s)ds}$ under the integral sign is a function of s' and cannot be taken outside the integration sign. The constant C can be determined from the boundary condition.

One should keep in mind that equation (4.6) does not in any real sense solve the equation of transfer in an absorbing and scattering medium. It is clear that if the effective emission coefficient, $\epsilon_{e,\lambda}(s)$, should depend on the intensity in some specific way, then one can convert the formal solution (4.6) into an integral equation for intensity. Then if the temperature distribution is known in the medium the monochromatic intensity of radiation can be calculated.

4.3.2 Propagation of Radiation in a Purely Absorbing Medium

For some engineering problems the assumption of the absence of scattering can be justified by considering the Rayleigh scattering law[13] for atoms and molecules,

$$\sigma_\lambda = 8\pi^3(n^2 - 1)^2/N\lambda^4 \quad , \tag{4.7}$$

where N is the number of molecules per cm^3. The index of refraction, n, of gases is a very weak function of density and wavelength. Thus for CO_2 at a temperature of 32°F and 1 atm pressure the index of refraction is 1.00045. The scattering coefficient at $\lambda = 2\mu = 2 \times 10^{-4}$ cm is calculated to be $\sigma_\lambda = 5.15 \times 10^{-9} cm^{-1}$.

It is seen that the scattering coefficient is quite small even for short wavelengths and can be neglected. In that case, $\sigma_\lambda = 0$, $\beta_\lambda = \kappa_\lambda$, $\epsilon_{e,\lambda} = \epsilon_\lambda$, equation (4.6) reduces to

$$I_\lambda(s) = C\, e^{-\int \kappa_\lambda(s)ds} + e^{-\int \kappa_\lambda(s)ds} \int \epsilon_\lambda(s') e^{\int \kappa_\lambda(s)ds}\, ds'. \tag{4.8}$$

With the boundary condition, $I_\lambda(s) = I_\lambda(0)$ at s = 0, equation (4.8) becomes (see Fig. 4.2)

$$I_\lambda(s) = I_\lambda(0)\, e^{-\tau_\lambda(s,0)} + \int_0^s \epsilon_\lambda(s')\, e^{-\tau_\lambda(s,s')}\, ds', \tag{4.9}$$

where $\tau_\lambda(s,s')$ is the optical thickness:

$$\tau_\lambda(s,s') = \int_{s'}^{s} \kappa_\lambda(s)\, ds \quad .$$

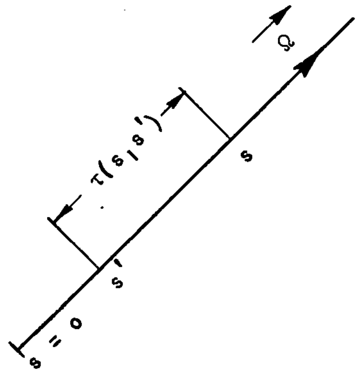

FIG. 4.2
GEOMETRICAL DATA FOR THE DEFINITION OF OPTICAL THICKNESS.

The first term on the right-hand side of (4.9) is the term which takes into account the intensity of radiation from the surface at s = 0. The physical meaning of the solution (4.9) is clear. It expresses the fact that the intensity of radiation at any point s and in any given direction $\vec{\Omega}$ results from the emission at all interior points s', reduced by the factor $e^{-\tau_\lambda(s,s')}$ to allow for the absorption by the intervening matter. Thus if the temperature distribution in the medium is known, the emission coefficient, $\epsilon_\lambda = n^2 \kappa_\lambda I_{bb,\lambda}$, can readily be determined and the intensity calculated from equation (4.9).

4.3.3 Propagation of Radiation in a Purely Scattering Medium

For a scattering medium, $\kappa_\lambda = 0$, $\beta_\lambda = \sigma_\lambda$, and $\epsilon_\lambda = 0$. Equation (4.6) then reduces to

$$I_\lambda(s) = C\, e^{-\int \sigma_\lambda(s)ds} + \frac{1}{4\pi}\, e^{-\int \sigma_\lambda(s)ds} \int \sigma_\lambda(s')\, \mathcal{E}'_\lambda(s')\, e^{\int \sigma_\lambda(s)ds} ds'. \quad (4.10)$$

The incident radiation, $\mathcal{E}'_\lambda(s)$, is defined by equation (2.22). With the boundary condition, $I_\lambda(s) = I(0)$ at s = 0, equation (4.10) becomes

$$I_\lambda(s) = I_\lambda(0)\, e^{-\tau_\lambda(s,0)} + \frac{1}{4\pi}\int_0^s \int_{\Omega'=4\pi} \sigma_\lambda(s')\; I_\lambda(s')\, e^{-\tau_\lambda(s,s')}\, ds'd\Omega', \quad (4.11)$$

where

$$\tau_\lambda(s,s') = \int_{s'}^{s} \sigma_\lambda(s)ds.$$

Thus, for a purely scattering medium, equation (4.6) reduces to a linear integral equation in $I_\lambda(s)$. The intensity of radiation at any point s, as seen from equation (4.11), results from scattering at all interior points s' as well as radiation reaching point s from the surface at s = 0.

4.4 Radiation between Two Parallel Planes

Consider a configuration consisting of a uniform, plane-parallel slab of a medium of finite thickness bounded by two planes x = -a and x = a. It is convenient to measure distances normal to the plane of stratification. Referring to Fig. 4.3, we see that x is this distance, θ is the inclination of the pencil of rays of direction $\vec{\Omega}$ to the outward normal, and ϕ is the aximuth to a suitably chosen axis.

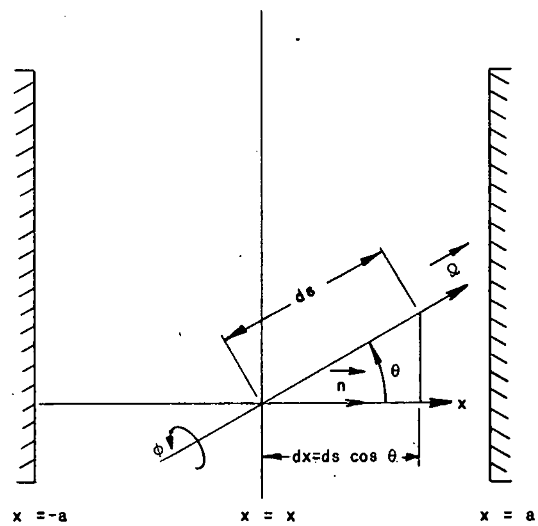

FIG. 4.3
COORDINATE SYSTEM FOR RADIATION BETWEEN TWO PARALLEL PLATES.

An idealization in which all spatial variations are one dimensional and with azimuthal symmetry of all functions about a given direction may be introduced for the purposes of simplification. I_λ then does not depend upon y and z; it is a function of x and $\mu(= \cos\theta)$. The line element ds along the direction $\vec{\Omega}$ is simply

$$ds = dx/\mu .$$

The directional derivative becomes

$$\vec{\Omega} \cdot \text{grad } I_\lambda = \text{div}(\vec{\Omega} I_\lambda) = \frac{\partial I_\lambda}{\partial s} = \frac{\partial I_\lambda}{\partial x}\frac{\partial x}{\partial s} = \mu \frac{\partial I_\lambda}{\partial x} \quad . \qquad (4.12)$$

Isotropy of scattering helps to simplify the problem further. With these assumptions, the fundamental equation of transfer (3.1), describing the steady-state condition with the help of (4.12) for $|x| \leq \alpha$ and $|\mu| \leq 1$, reduces to

$$\mu \frac{\partial I\ (x,\mu)}{\partial x} = \epsilon_\lambda(x) - (\kappa_\lambda + \sigma_\lambda)\ I\ (x,\mu) + \frac{1}{4\pi}\sigma_\lambda(x)\ \mathcal{E}'_\lambda(x), \tag{4.13}$$

where

$$\mathcal{E}'_\lambda(x) = \int\int_{\Omega'=4\pi} I_\lambda(x, \Omega')d\Omega' = \int_0^{2\pi}\int_0^{\pi} I_\lambda(x, \cos\ \theta')\sin\theta' d\theta' d\phi' = 2\pi\int_{-1}^{1} I_\lambda(x,\mu')d\mu' \tag{4.14}$$

is the radiation incident on a unit volume of the medium. With certain boundary conditions equation (4.13) defines $I_\lambda(x,\mu)$ everywhere within a single homogeneous medium. The properties of these equations are such that the boundary conditions may be specified in terms of arbitrary assignment of radiation intensity distributions on the two surfaces. Thus for a homogeneous isotropic medium between the planes $x = -a$ and $x = a$ one may specify the intensities on the boundaries:

at $x = -a$, $\mu < 0$; $I_\lambda(x,\mu) = I_\lambda(-a,\mu)$;

$x = a$, $\mu > 0$; $I_\lambda(x,\mu) = I_\lambda(a,\mu)$,

Equation (4.13) can be written as

$$\frac{\partial I\ (x,\mu)}{\partial x} = -\frac{\beta_\lambda(x)}{\mu} I_\lambda(x,\mu) + \frac{\epsilon_\lambda(x)}{\mu} + \frac{\sigma_\lambda(x)\ \mathcal{E}'_\lambda(x)}{4\pi\mu}. \tag{4.15}$$

Using the integrating factor $e^{\int\frac{\beta_\lambda(x)dx}{\mu}}$,

the solution of (4.15) is obtained in a manner similar to that of (4.5), and we have

$$I_\lambda(x,\mu) = Ce^{-\int\frac{\beta_\lambda(x)dx}{\mu}} + e^{-\int\frac{\beta_\lambda(x)dx}{\mu}}\int\left[\epsilon_\lambda(x') + \frac{1}{4\pi}\sigma_\lambda(x')\mathcal{E}'_\lambda(x')\right]e^{\int\frac{\beta_\lambda(x)dx}{\mu}}\frac{dx'}{\mu}. \tag{4.16}$$

Introducing the boundary conditions, we obtain

$$I_\lambda(x,\mu) = \begin{cases} I_\lambda(a,\mu)e^{-\int_x^a \frac{\beta_\lambda(x)dx}{\mu}} + \int_x^a \left[\epsilon_\lambda(x') + \frac{1}{4\pi}\sigma_\lambda(x')\mathcal{E}_\lambda(x')\right] e^{\int_x^{x'} \frac{\beta_\lambda(x)dx}{\mu}} \frac{dx'}{\mu} & (4.17) \\ \text{for } 0 \leq \mu \leq 1 & \\ I_\lambda(-a,\mu)e^{-\int_{-a}^{x} \frac{\beta_\lambda(x)dx}{\mu}} + \int_{-a}^{x} \left[\epsilon_\lambda(x') + \frac{1}{4\pi}\sigma_\lambda(x')\mathcal{E}_\lambda'(x')\right] e^{\int_x^{x'} \frac{\beta_\lambda(x)dx}{\mu}} \frac{dx'}{\mu} & (4.18) \\ \text{for } -1 \leq \mu \leq 0. & \end{cases}$$

The temperature distribution in the medium must be known before the intensities can be calculated, and even then an integral equation for intensity or incident radiation must be solved. Equations (4.17) and (4.18) will be used in deriving the integral equations for incident radiation and subsequently for temperature distribution.

5 THE INTEGRAL EQUATIONS

5.1 Introduction

The local radiant heat flux at the bounding surface of an enclosure receives contributions from every point in space as well as from other parts of the bounding surface or surfaces and is given by an integral equation. Hilbert(34) was probably the first to apply the theory of integral equations to the study of this general problem of radiant heat exchange in an absorbing and scattering medium without a bounding surface. For instance, he proved Kirchhoff's law for the case of thermodynamic equilibrium. Poljak(80) derived the general integral equations which describe radiation in a closed system of gray radiating surfaces in the absence of a radiation-absorbing and scattering medium, by assuming that the emissivity and temperature are constant over each surface. A solution of a heat transfer problem in a radiating medium is a combination of the two separate problems mentioned above.

In this chapter the writer derives the integral equations for an enclosure made of opaque walls and containing an absorbing and scattering medium, and, by so doing, generalizes previous results(34,80,79) to include also the effects of non-diffuse reflection from the surfaces and nonisotropic scattering from the medium. The derivation of the integral equations for monochromatic radiant heat exchange in a general closed system composed of i surfaces separated by an absorbing and scattering medium is quite straightforward and elementary. It is based, for both surface and volume radiation, on the equation of transfer, the expressions for irradiation, E', and incident radiation, $\mathcal{E}'_\lambda$, as well as appropriate boundary conditions. Other quantities describing radiant heat exchange readily follow from the definitions (2.27), (2.28), (2.29), (2.30). The following assumptions are made:

(1) Steady state of radiant heat transfer exists.

(2) The enclosing surfaces are dense and opaque, the monochromatic transmissivity of the enclosing surface is zero; that is, the enclosing surfaces are opaque.

(3) The monochromatic reflectivities of the enclosing surfaces $\rho_\lambda(A_1)$, $\rho_\lambda(A_2)$... $\rho_\lambda(A_i)$ are functions of wavelength and position.

(4) The medium has constant (independent of density) index of refraction, n.

5.2 The Integral Equations for Irradiation and Incident Radiation

The irradiation at any point S on the surface of the enclosure is defined by equation (2.21). Then, with reference to Fig. 5.1, the irradiation at point S is due to energy radiated from the enclosing surfaces and due to energy emitted from the medium. The monochromatic intensity of irradiation from a given direction at point S is obtained by applying the boundary condition to the solution (4.6) of the equation of transfer. Introducing the limits of integration, one obtains

$$I_\lambda(S) = \frac{dR'_\lambda(S')}{d\Omega} e^{-\tau_\lambda(S',S)} + \int_{r_{P'}}^{r_S} \epsilon_{e,\lambda}(P') e^{-\tau_\lambda(P',S)} ds', \tag{5.1}$$

where

$$\tau_\lambda(S', S) = \int_{r_{S'}}^{r_S} \beta_\lambda(s) ds \tag{5.2}$$

and

$$\tau_\lambda(P', S) = \int_{r_{P'}}^{r_S} \beta_\lambda(s)\, ds \tag{5.3}$$

are the optical thicknesses between points S and S', and S and P', respectively. In equation (5.1), s' is the dummy variable and $dR_\lambda(S')/d\Omega$ is the monochromatic intensity of radiation leaving the surface (point S') in the direction of point S. Thus, the monochromatic intensity of irradiation is due to radiant energy leaving the surface element dA at point S' in the direction of point S and arriving at S, plus the energy emitted by the unit volume dV at point P' in the direction of S and arriving at S. If the reflection from the surface is specular, in addition to the reflecting function, $\Gamma\ (\vec{r}, \vec{\Omega}' \rightarrow \vec{\Omega})$, the history of the pencil of rays, i.e., the direction of the pencil of radiation incident on the surface, must be known. However, for diffuse reflection $dR_\lambda(S')/d\Omega$ reduces to $R_\lambda(S')/\pi$.

Substituting equation (5.1) in (2.21), we find

$$E'_\lambda(S) = \int_{\Omega = 2\pi} \left(\frac{dR_\lambda(S')}{d\Omega}\right) e^{-\tau_\lambda(S', S)} \cos\theta d\Omega$$

$$+ \int_{\Omega = 2\pi} \int_{r_{P'}}^{r_{S'}} \epsilon_{e,\lambda}(P')\, e^{-\tau_\lambda(P', S)} \cos\theta ds' d\Omega. \tag{5.4}$$

The integrals over solid angles, appearing in equation (5.4), are transformed to surface and volume integrals by the use of the following relations:

$$\cos\theta\, d\Omega = \frac{\cos\theta'_{S'} \cos\theta'_S\, dA}{|\vec{r}_S - \vec{r}_{S'}|^2} \quad ,$$

and

$$\cos\theta ds' d\Omega = \frac{\cos\theta_S dV}{|\vec{r}_S - \vec{r}_{P'}|^2} \quad .$$

Introducing these two identities in equation (5.4), we finally get

$$E'_\lambda(S) = \int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega} \right) K_{AA}(S, S')\, dA + \int_V \epsilon_{e,\lambda}(P')\, K_{AV}(S, P') dV, \tag{5.5}$$

where

$$K_{AA}(S, S') = e^{-\tau_\lambda(S', S)} \frac{\cos\theta_{S'} \cos\theta'_S}{\left| \vec{r}_S - \vec{r}_{S'} \right|^2}$$

and

$$K_{AV}(S, P') = e^{-\tau_\lambda(P', S)} \frac{\cos\theta_S}{\left| \vec{r}_S - \vec{r}_{P'} \right|^2} .$$

The radiant energy incident on the unit volume of the medium at any point P is defined by equation (2.22). The monochromatic intensity of irradiation at point P in a given direction is obtained by applying the boundary condition to the solution (4.6) of the equation of transfer. Introducing the limits of integration (see Fig. 5.1), we obtain

$$I_\lambda(P) = \left(\frac{dR_\lambda(S')}{d\Omega} \right) e^{-\tau_\lambda(S', P)} + \int_{r_{P'}}^{r_P} \epsilon_{e,\lambda}(P')\, e^{-\tau_\lambda(P', P)} ds' , \tag{5.6}$$

where

$$\tau_\lambda(S', P) = \int_{r_{S'}}^{r_P} \beta_\lambda(s)\, ds \tag{5.7}$$

and

$$\tau_\lambda(P', P) = \int_{r_{P'}}^{r_P} \beta_\lambda(s)\, ds \tag{5.8}$$

are the optical thicknesses between points S' and P , and P' and P, respectively. Thus, the monochromatic intensity of radiation incident on the unit volume at point P is due to the radiant energy leaving the surface element dA at point S' in the direction of point P and arriving at P plus the radiant energy emitted by the volume element dV at point P' in the direction of P and arriving at P. Substituting equation (5.6) in the

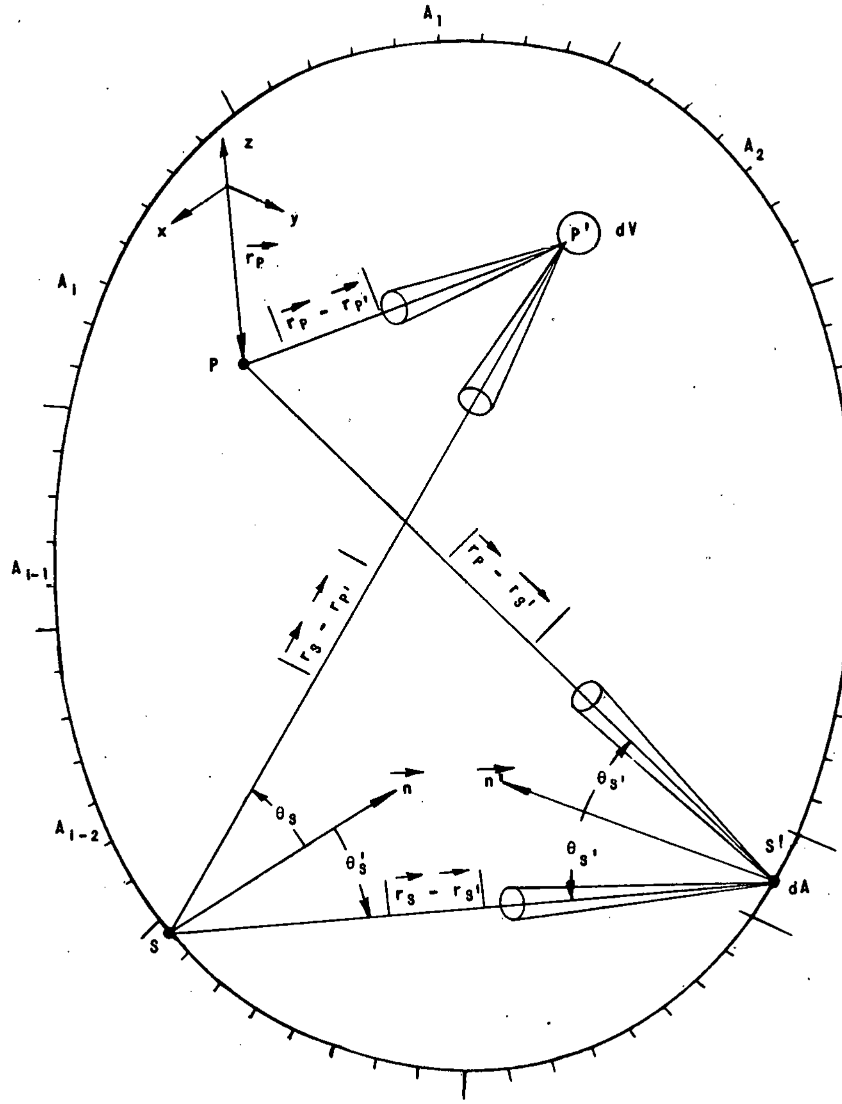

FIG. 5.1
GEOMETRICAL DATA FOR RADIATION IN A GENERAL ENCLOSURE.

definition (2.22) of the incident radiation, we find

$$\mathcal{E}'_\lambda(P) = \int_{\Omega=4\pi} \left(\frac{dR_\lambda(S')}{d\Omega}\right) e^{-\tau_\lambda(S', P)} d\Omega$$

$$+ \int_{\Omega=4\pi} \int_{r_{P'}}^{r_P} \epsilon_{e,\lambda}(P')\, e^{-\tau_\lambda(P', P)} ds'\, d\Omega. \tag{5.9}$$

We transform the integrals over solid angles appearing in the above equation to surface and volume integrals with the help of the following relations:

$$d\Omega = \frac{\cos\theta_{S'}\, dA}{\left|\vec{r}_P - \vec{r}_{S'}\right|^2}$$

and

$$d\Omega ds = \frac{dV}{\left|\vec{r}_P - \vec{r}_{P'}\right|^2} \quad .$$

With these substitutions equation (5.9) becomes

$$\mathcal{E}'_\lambda(P) = \int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega}\right) K_{VA}(P, S')\, dA$$

$$+ \int_V \epsilon_{e,\lambda}(P')\, K_{VV}(P, P')\, dV, \tag{5.10}$$

where

$$K_{VA}(P, S') = \frac{e^{-\tau_\lambda(S', P)} \cos\theta_{S'}}{\left|\vec{r}_P - \vec{r}_{S'}\right|^2} \quad ,$$

and

$$K_{VV}(P, P') = \frac{e^{-\tau_\lambda(P', P)}}{\left|\vec{r}_P - \vec{r}_{P'}\right|^2} \quad .$$

In some problems, integral equations (5.4) and (5.9) might be in a more convenient form than equations (5.5) and (5.10). If one assumes that the enclosing surfaces are diffuse, $\Gamma(\vec{r}, \vec{\Omega}' \to \vec{\Omega}) = 1$, the scattering is isotropic, $\gamma(\vec{r}, \vec{\Omega}' \to \vec{\Omega}) = 1$, and the refractive index is equal to unity, on using equations (5.5) and (5.10) one can easily arrive at the equations for radiosity and effective emission derived by Polack.[79]

5.3 Integral Equations for Net Emissive Power and Net Emission

The derivation of integral equations for net emissive power and net emission is quite straightforward. Thus, for net emissive power at point S' on the enclosure walls, from definition (2.29) we have

$$E_{n,\lambda}(S) = E_\lambda(S) - \alpha_\lambda(S)\, E'_\lambda(S).$$

Substituting for the irradiation from equation (5.5), we obtain

$$E_{n,\lambda}(S) = E_\lambda(S) - \alpha_\lambda(S) \int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega}\right) K_{AA}(S, S')\, dA$$
$$-\alpha_\lambda(S) \int_V \epsilon_{e,\lambda}(P')\, K_{AV}(S, P')\, dV. \qquad (5.11)$$

The net emission at point P is expressed by equation (2.30) as

$$\mathcal{E}_{n,\lambda}(P) = \mathcal{E}_\lambda(P) - \kappa_\lambda(P)\, \mathcal{E}'_\lambda(P) \quad .$$

Substituting for $\mathcal{E}'_\lambda$ from equation (5.10) in the above, one finds

$$\mathcal{E}_{n,\lambda}(P) = \mathcal{E}_\lambda(P) - \kappa_\lambda(P) \int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega}\right) K_{VA}(P, S')\, dA$$
$$-\kappa_\lambda(P) \int_V \epsilon_{e,\lambda}(P')\, K_{VV}(P, P')\, dV. \qquad (5.12)$$

Equations for net emissive power and net emission in a less general form were presented in references (2) and (99), but no derivations were given.

It is to be noted that R_λ and $\epsilon_{e,\lambda}$ can be expressed [see equations (2.27) and (3.3)] in terms of other variables which are more appropriate in some particular cases. These relations can then be substituted in the integral equations (5.5) and (5.10), or (5.11) and (5.12).

5.4 Integral Equations for Diathermal Medium and for Medium without an Enclosure

The system of integral equations (5.11) and (5.12) reduces to a single, much simpler equation either for $E_{n,\lambda}$ or for $\mathcal{E}_{n,\lambda}$ in two particular cases: (a) when the enclosure surfaces are separated by a diathermal medium, and (b) when the medium is not enclosed at all.

In the first case we have $\beta_\lambda = \kappa_\lambda = \sigma_\lambda = \epsilon_{e,\lambda} = 0$. Equation (5.11) reduces to

$$E_{n,\lambda}(S) = E_\lambda(S) - \alpha_\lambda(S) \int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega} \right) K_{AA}(S, S')\, dA, \qquad (5.13)$$

which gives the net emissive power at point S on the surface.

In the second case, since there is no enclosure, the radiosity at the surface is $R_\lambda(S') = 0$, and equation (5.12) reduces to

$$\mathcal{E}_{n,\lambda}(P) = \mathcal{E}_\lambda(P) - \kappa_\lambda(P) \int_V \epsilon_{e,\lambda}(P')\, K_{VV}(P, P')\, dV. \qquad (5.14)$$

The integral equation (5.14) is analogous to the equation derived by Hilbert for the case of an absorbing and scattering medium. It was used by Hilbert as the basis of the proof of Kirchhoff's law for radiation from a volume of radiating medium.

5.5 Resulting Equations for Thermodynamic Equilibrium

Examining the integral equations for a system of surfaces completely enclosing a radiating medium at thermodynamic equilibrium, from the second law of thermodynamics we know the following to be valid:

$$E_{n,\lambda} = \mathcal{E}_{n,\lambda} = 0$$

$$\mathcal{E}_{e,\lambda} = \beta_\lambda \; \mathcal{E}'_\lambda = 4\beta_\lambda E_{bb,\lambda}$$

$$E'_\lambda = R_\lambda = E_{bb,\lambda} = \pi I_{bb,\lambda} \quad .$$

With these values equations (5.11) and (5.12) for the case of diffuse enclosure walls reduce to

$$1 = \int_{A_1 \ldots A_i} K_{AA}(S, S')\, dA + \int_V \beta_\lambda(P')\, K_{AV}(S, P')\, dV, \tag{5.15}$$

and

$$4 = \int_{A_1 \ldots A_i} K_{VA}(P, S')\, dA + \int_V \beta_\lambda(P')\, K_{VV}(P, P')\, dV. \tag{5.16}$$

Using equation (5.15), either the area or the volume term can be eliminated from equation (5.11), similarly making use of equation (5.16) the area or the volume term can be eliminated from equation (5.12).

6 MATHEMATICAL FORMULATION OF THE PROBLEM

6.1 Introduction

Heat transfer from a radiating and moving medium is one of the most complex problems occurring in engineering practice. In general, the medium consists of the gaseous substances CO_2, H_2O, SO_2, and NH_3 as well as H, CH, C_nH_m and others. Furthermore, the gas or gas mixtures may include admixtures of solid particles or ash. In addition, each gas possesses its characteristic radiation properties. Thermal radiation affects heat transfer both directly and indirectly. Radiation can be absorbed directly by enclosing surfaces and cause heat transfer. Indirectly, it can be partially absorbed in the medium and alter the temperature distribution, thereby influencing conductive and convective heat transfer.

The objective of research in fluid flow and heat transfer is the prediction of the state of flow or fluid and of heat. In a single-phase fluid, in the absence of electric and magnetic fields as well as diffusion, its state is specified by the velocity vector and two thermodynamic properties (usually temperature and pressure). A complete description of the flow is a statement of the values taken by these three quantities at every point within the fluid, and for all time subsequent to some initial time. The basic problem is then this: predict the state of flow and the temperature when a fluid flows in a duct of specified shape and length, with the pressure drop and either the wall temperature or the wall heat flux being specified. Regardless of whether the flow is laminar or turbulent, the flow may be described by a coupled system of differential, integro-differential and integral equations as well as appropriate boundary conditions.

6.2 Equations of Continuity and Motion

Now let us consider the effects of radiation on the general hydrodynamical equations.

The existence of thermal radiation does not affect the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{w}) = 0 \quad , \tag{6.1}$$

since radiation has no mass.

The hydrodynamics of a fluid moving in a field of radiation is identical, as regards the dynamical equations with classical hydrodynamics, provided terms are introduced to allow for the stresses caused by the radiation. These, in general, form a stress-tensor, partly analogous to the stress-tensor in ordinary viscous motion. The radiative stress-tensor differs from the stress-tensor of viscous motion in that it does not reduce exactly to a simple hydrostatic pressure when the velocity gradients are put equal to zero. The tangential components of the stress survive.[43,68,31] These terms are, however, negligible for ordinary radiant heat transfer problems when compared to the principal components of stress. If we neglect the "radiative viscosity" (which is very small at ordinary temperatures, see Eqn. 2.49) and neglect at the same time all terms depending on the second-order temperature gradients, the radiative stresses reduce to a hydrostatic pressure (2.38). In deriving the equations of motion all we need to do is add p_r to the ordinary fluid pressure p.

Magee and Hirshfelder[62] have shown that at a temperature of 10^6 °K the radiation and hydrodynamic pressures are 2,523 and greater than 100,000 atm, respectively. Thus even at these extremely high temperatures the radiation pressure is 39.6 times smaller than the fluid pressure. Therefore, in this work the radiation pressure is neglected. The equation of motion for compressible flow becomes[88]

$$\rho \frac{D\vec{w}}{Dt} = \vec{F} - \text{grad } p + \text{div}(\mu \text{ grad} \vec{w}) + \frac{\mu}{3} \text{grad}(\text{div} \vec{w}) \quad , \qquad (6.2)$$

where $\vec{F}$ is the body force.

6.3 Equation of Energy

6.3.1 Introduction

The interaction between thermal radiation and a fluid in motion has been treated in astrophysical problems from various points of view and with various objectives in mind by Jeans,[43] Eddington,[19] Rosseland,[85] Milne,[69] and Thomas.[102] These authors have usually neglected energy transfer by conduction, work by viscous dissipation, energy generation and small terms arising from the finite velocity of light. Thomas was the only one who derived the energy equation in a fluid in motion by using the Lorentz transformation. He obtained the energy equation in a form including all orders in the ratio of the velocity of motion to the velocity of light. In this thesis the energy equation is obtained by evaluating the energy flowing into a closed surface fixed in space and then applying Gauss's divergence theorem. This method differs from Milne's chiefly in that it evaluates the energy exchanges of matter inside a fixed surface instead of a surface moving with the matter. The equation of energy derived by the method of Rosseland agrees with the equation derived by Milne when in Rosseland's equation the hydrostatic pressure p is replaced by $p + p_r$.

In this work the Rosseland derivation is followed, except that a macroscopic instead of the microscopic view is taken. The equation is derived for the case where electric and magnetic fields are absent; however, as shown by Chu,(15) the electric and magnetic energies can readily be included in the equation of energy. It is further assumed that the energy of gravity is negligible and that there is neither molecular nor thermal diffusion. The diffusive energy transport can also be readily included, as shown by Lees(54) and by Fay and Riddell.(22)

6.3.2 Derivation of the Energy Equation

We will proceed to derive the equation which represents the change in energy in a given volume per unit of time. We consider the heat transferred by conduction, convection and radiation across the surface A, bounding the volume V, in which the amount of work done per unit volume and time of the fluid is W, and the heat generated is q'''. We can write the heat flux vector as

$$\vec{q''} = -k \text{ grad } T + \rho e \vec{w} + \vec{E} \quad , \tag{6.3}$$

where -k grad T represents the heat flux vector due to thermal conduction, $\rho e \vec{w}$ is the heat flux vector due to convection, and $\vec{E} = \int_0^\infty \vec{E}_\lambda d\lambda$ [see Eqn. (2.31) for the definition of $\vec{E}_\lambda$] is the energy flux due to thermal radiation. The energy per unit mass, e, is the sum of the internal energy plus the kinetic energy,

$$e = \frac{w^2}{2} + h - pv \quad , \tag{6.4}$$

where h is the enthalpy of the fluid, $h = \int_{T_0}^{T} c_p \, dT$.

We apply the law of energy conservation to volume V. The rate of change of energy in V is due to the addition of heat across the boundary of V, due to the work done by any forces acting and due to the heat generation inside V. Accordingly, the conservation of energy equation for the fluid inside the boundary A is

$$\oint_V \frac{\partial(u + \rho e)}{\partial t} dV + \oint_A \vec{q''} \cdot d\vec{A} = \oint_V (W + q''')dV \quad ,$$

where

$$u = \int_0^\infty u_\lambda d_\lambda$$

is the total radiant energy density. Using the Gauss divergence theorem,

$$\oint_A \vec{f} \cdot d\vec{A} = \oint_V \text{div}\, \vec{f}\, dV \quad ,$$

to convert the surface integrals to volume integrals, and remembering that the volume V is arbitrary and thus may be infinitesimal, we obtain

$$\frac{\partial(u + \rho e)}{\partial t} + \text{div}\, \vec{q''} = W + q''' \quad , \tag{6.5}$$

or

$$\frac{\partial u}{\partial t} + \frac{\partial(\rho e)}{\partial t} + \text{div}(-k \text{ grad } T) + \text{div}\,(\rho e \vec{w}) + \text{div}\, \vec{E} = W + q''' \quad .$$

The work per unit volume and time, W, is the sum of the compression work plus the friction work. It is shown by Schlichting[88] that W can be expressed as

$$W = \frac{Dp}{Dt} - \rho \frac{D(pv)}{Dt} + \mu\Phi \quad , \tag{6.6}$$

where Φ is the dissipation function:

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \quad . \tag{6.7}$$

Using the vector identity

$$\text{div}(\rho e \vec{w}) = e \,\text{div}(\rho \vec{w}) + \rho \vec{w} \cdot \text{grad } e$$

and the continuity equation (6.1), we can write

$$\frac{\partial(\rho e)}{\partial t} + \text{div}(\rho e \vec{w}) = \rho\left[\frac{\partial e}{\partial t} + \vec{w} \cdot \text{grad } e\right] = \rho \frac{De}{Dt} \quad . \tag{6.8}$$

Introducing the definition (6.4) of the internal energy, we can express the substantial derivative of e as

$$\rho \frac{De}{Dt} = \frac{\rho}{2} \frac{Dw^2}{Dt} + \rho \frac{Dh}{Dt} - \rho \frac{D(pv)}{Dt} \quad . \tag{6.9}$$

Finally, substituting (6.6), (6.8) and (6.9) in the energy equation (6.5) and simplifying, we obtain

$$\frac{\partial u}{\partial t} + \rho \frac{Dh}{Dt} + \text{div } \vec{E} = \text{div}(k \text{ grad } T) + \frac{Dp}{Dt} + q''' + \mu\Phi \quad . \tag{6.10}$$

Integrating equation (3.16) over all wavelengths and substituting in equation (6.10), we obtain an alternative expression for the conservation of energy equation,

$$\rho \frac{Dh}{Dt} = \text{div}(k \text{ grad } T) - \mathcal{E}_n + \frac{Dp}{Dt} + q''' + \mu\Phi \quad , \tag{6.11}$$

where the total net emission, $\mathcal{E}_n$, is obtained by integrating equation (5.12) over all wavelengths,

$$\mathcal{E}_n(P) = \int_0^\infty \left\{ \mathcal{E}_\lambda(P) - \kappa_\lambda(P) \left[\int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega}\right) K_{AV}(P,S')dA + \int_V \epsilon_{e,\lambda}(P')K_{VV}(P,P')dV \right] \right\} d\lambda \tag{6.12}$$

It should be kept in mind that, when deriving equation (3.16), an assumption of diffuse emission was introduced in the term, $\frac{\partial u_\lambda(\vec{r},t)}{\partial t}$. In addition, equation (6.12) is valid only for the steady-state problem, since in its derivation the equation of transfer without the transient term, $\frac{1}{c} \frac{\partial I_\lambda(\vec{r},\vec{\Omega},t)}{\partial t}$, was used. However, since this term is ordinarily small, it may be neglected. To this approximation the solutions obtained above may be considered valid also for transient problems.

6.3.3 Equation of Energy for Diffuse Surfaces and Non-scattering Medium

For the case where scattering is negligible compared to absorption, the equation of energy simplifies considerably. If we can assume local thermodynamic equilibrium we can define at each point in the medium a local temperature T, the emission coefficient is then given according to Kirchhoff's law (2.17). Thus, the effective emission coefficient (3.3) reduces to

$$\epsilon_{e,\lambda} = \epsilon_{\lambda} = n^2 \kappa_{\lambda} I_{bb,\lambda} \quad .$$

In addition, if the walls of the enclosure are diffuse the equation for total net emission (6.12) simplifies to

$$\mathcal{E}_n(P) = \int_0^{\infty} \kappa_{\lambda}(P) \left\{ 4n^2 E_{bb,\lambda}(P) - \frac{1}{\pi} \left[\int_{A_1 \ldots A_i} R_{\lambda}(S') K_{VA}(P,S') dA + \int_V n^2 \kappa_{\lambda}(P') E_{bb,\lambda}(P') K_{VV}(P,P') dV \right] \right\} d\lambda \quad . \qquad (6.13)$$

A similar equation has recently been given by Surinov,[99] but no derivation was presented.

6.4 Expression for Heat Flux at the Wall

The heat flux at the wall of an enclosure containing a medium that absorbs and scatters thermal radiation, in the absence of diffusion and chemical recombination at the surface, is due to conduction and radiation;

$$q_w'' = -k \frac{\partial T}{\partial n} \bigg|_w + E_n \quad , \qquad (6.14)$$

where E_n is the total net radiant heat flux at the enclosure boundary in consideration. Equation (6.14) can be written, according to previous definitions, as

$$q''_w = -k \frac{\partial T}{\partial n}\bigg|_w + \int_0^\infty \left(E_\lambda - \alpha_\lambda E'_\lambda\right) d\lambda \quad . \tag{6.15}$$

Substituting equation (5.5) for the monochromatic irradiation, we obtain

$$q''_w = -k \frac{\partial T}{\partial n}\bigg|_w + \int_0^\infty \Bigg\{ E_\lambda(S) - \alpha_\lambda(S) \Bigg[\int_{A_1 \ldots A_i} \left(\frac{dR_\lambda(S')}{d\Omega}\right) K_{AA}(S,S')dA$$

$$+ \int_V \epsilon_{e,\lambda}(P')K_{AV}(S,P')dV \Bigg]\Bigg\} d\lambda \quad . \tag{6.16}$$

Thus, we see that even the evaluation of the wall heat flux is not a simple problem, since we have to solve equation (6.16) simultaneously with (5.5) because the radiosity, R_λ, is a function of irradiation, E'_λ.

For the case of no scattering and black surfaces, the heat flux at the wall reduces to

$$q''_w = -k \frac{\partial T}{\partial n}\bigg|_w + E_{bb}(S) - \frac{1}{\pi}\int_0^\infty \Bigg[E_{bb,\lambda}(S')K_{AA}(S,S')dA$$

$$+ \int_V n^2 \kappa_\lambda(P')E_{bb,\lambda}(P')K_{AV}(S,P')dV \Bigg] d\lambda \quad . \tag{6.17}$$

We see from equation (6.17) that, given the temperature distribution in the medium and on the enclosing surfaces, the heat flux at any point on the enclosure wall can be calculated directly. However, if the geometry of the enclosure is irregular the kernels $K_{AA}(S,S')$ and $K_{AV}(S,P')$ are quite complex, and the evaluation of the heat flux, even for this simple case, is very involved.

7 ANALOGY BETWEEN THERMAL RADIATION AND NEUTRON TRANSPORT

7.1 Introduction

In recent years there has been an immense amount of work done on the design and development of nuclear reactors. Associated with these endeavors, problems arising in biological shielding design and neutron diffusion have been investigated extensively and a large amount of information has been accumulated in this field. It was hoped that some of these techniques and results would be applicable to the solutions of problems of thermal radiation in the presence of absorbing and scattering media. It is the purpose of this chapter then, to show the mathematical analogy between thermal radiation and neutron transport, as well as to direct attention to its usefulness, by indicating some solutions previously obtained for neutron transport problems that can be borrowed for radiative transfer problems.

As far as gamma radiation is concerned, the analogy between thermal and gamma (nuclear) radiations is complete. Thermal and gamma radiations belong to the family of electromagnetic waves, the only difference being that gamma radiation has a higher frequency (shorter wavelength). Both types of radiation obey the same physical laws and equations. The work in gamma radiation associated with biological shield designs has been summarized by Rockwell[84] and Goldstein.[27] The two books give extensive bibliographies to original work. Therefore, we turn our attention to neutron transport to see what, if any, analogy exists.

7.2 Mathematical Analogy

The analogy between thermal radiation and neutron transport in absorbing and scattering media is mathematical and not physical. It is to be noted that this analogy exists only when there are no other modes of energy transport present in the thermal radiation problem, for example, heat conduction. In radiative transport we have transport of photons (quanta of energy), while in neutron transport we have transport of material particles. The main physical difference between the two phenomena is the fact that the photon has no rest mass. The velocity of propagation of the photon is the velocity of light. A neutron has a finite mass, and its velocity is between zero and, in the limit, the velocity of light. Definitions of some analogous quantities for thermal radiation and neutron transport are given in Table 7.1.

The neutrons constitute a distribution in phase space (in the terminology of Gibbs and Boltzmann); their ensemble can be fully described only by a density in a six-dimensional phase space (x, y, z, u, v, w). The fundamental variable in transport theory is the phase space distribution, $f(\vec{r}, \vec{\Omega}, E, t)$, which is the number of neutrons per unit volume and unit solid angle moving in the direction $\vec{\Omega}$. The quantities E, $\vec{r}$, t denote the energy, the space coordinates, and the time, respectively.

The general Boltzmann neutron transport equation is given by Davison[17] and by Weinberg and Wigner.[106] A great simplification of the Boltzmann neutron transport equation comes about if the neutrons are all assumed to have the same speed, that is, if all scattering occurs without change of energy. This is the usual assumption made in dealing with thermal neutrons. Since the energy is constant, the phase space distribution, $f(\vec{r}, \vec{\Omega}, E, t)$, is then a distribution in direction; that is, $f(\vec{r}, \vec{\Omega}, E, t)$ can be written as $f(\vec{r}, \vec{\Omega}, t)$ and the scattering

Table 7.1

COMPARISON OF PHENOMENA AND DEFINITIONS FOR
THERMAL RADIATION AND NEUTRON TRANSPORT

	Thermal Radiation	Neutron Transport
1	Carrier has no charge	Carrier has no charge
2	Between collisions the photon travels in straight lines at constant velocity (velocity of light)	Between collisions the neutron travels in straight lines at constant velocity
3	Thermal radiation can be absorbed, scattered and emitted	Neutrons can be absorbed, scattered and emitted
4	The scattering can be specular or isotropic	Scattering can be directional or isotropic
5	$I_\lambda(\vec{r}, \vec{\Omega}, t)$ - monochromatic intensity or radiation	$f(\vec{r}, E, \vec{\Omega}, t)$ - angular distribution of monoenergetic neutrons
6	$\mathcal{E}'_\lambda(\vec{r}, t)$ - monochromatic incident radiation $\mathcal{E}'_\lambda(\vec{r}, t) = \iint_{\Omega = 4\pi} I_\lambda(\vec{r}, \vec{\Omega}, t)\, d\Omega$	$\Phi(\vec{r}, E, \vec{\Omega}, t)$ - monoenergetic neutron flux $\Phi(\vec{r}, E, t) = \iint_{\Omega = 4\pi} f(\vec{r}, E, \vec{\Omega}, t)\, d\Omega$
7	$\mathcal{E}'(\vec{r}, t)$ - total incident radiation $\mathcal{E}'(\vec{r}, t) = \int_0^\infty d\lambda \iint_{\Omega = 4\pi} I_\lambda(\vec{r}, \vec{\Omega}, t)\, d\Omega$	$\Phi(\vec{r}, t)$ - total neutron flux $\Phi(\vec{r}, t) = \int_0^\infty dE \iint_{\Omega = 4\pi} f(\vec{r}, E, \vec{\Omega}, t)\, d\Omega$
8	$\vec{E}_\lambda(\vec{r}, t)$ - monochromatic energy flux vector $\vec{E}_\lambda(\vec{r}, t) = \iint_{\Omega = 4\pi} I_\lambda(\vec{r}, \vec{\Omega}, t)\, \vec{\Omega}\, d\Omega$	$\vec{j}(\vec{r}, E, t)$ - monoenergetic neutron current $\vec{j}(\vec{r}, E, t) = \iint_{\Omega = 4\pi} f(\vec{r}, E, \vec{\Omega}, t)\, \vec{\Omega}\, d\Omega$
9	$\epsilon(\vec{r}, \vec{\Omega}, t)$ - emission coefficient	$S(\vec{r}, \vec{\Omega}, t)$ - angular source density

cross section, $\Sigma_s(\vec{r}, E' \to E, \vec{\Omega}' \to \vec{\Omega})$, can be replaced by $\Sigma_s(\vec{r}, \vec{\Omega}' \to \vec{\Omega})$. Thus in the monoenergetic (one-velocity) case the Boltzmann neutron transport equation becomes(106)

$$\frac{1}{v}\frac{\partial f(\vec{r}, \vec{\Omega}, t)}{\partial t} + \vec{\Omega} \cdot \text{grad } f(\vec{r}, \vec{\Omega}, t) = -\Sigma(\vec{r})\, f(\vec{r}, \vec{\Omega}, t) + S(\vec{r}, \vec{\Omega}, t) + \int_{\Omega' = 4\pi} \Sigma_s(\vec{r}, \vec{\Omega}' \to \vec{\Omega})\, f(\vec{r}, \vec{\Omega}', t)\, d\Omega', \qquad (7.1)$$

where $\Sigma(\vec{r})$ is the total cross section. This equation is identical in form, except for the constants, to the equation of transfer (3.1).

The more common elementary neutron diffusion equation,(106)

$$\frac{3D}{v}\frac{\partial^2 \Phi}{\partial t^2} + \frac{1}{v}(1 + 3D\Sigma_a)\frac{\partial \Phi}{\partial t} = D\nabla^2\Phi - \Sigma_a \Phi + S, \qquad (7.2)$$

where v is the neutron velocity, D is the diffusion coefficient and Σ_a is the absorption cross section, is valid for a homogeneous system and sources constant with time. In the diffusion approximation the mean free path of the neutrons is vanishingly small but the diffusion coefficient is finite. Equation (7.2) is one of a general class of equations in mathematical physics which are first approximations of more complicated transport integral equation. It has been shown by Weinberg and Wigner that for the mononergetic case the neutron diffusion theory (except for the neglect of the term $d\vec{j}/dt$) is the P_1 approximation of the spherical harmonic method. The analogous diffusion approximation arising in thermal radiation problems will be discussed in Section 9.2.

7.3 Some Analogous Radiative Transfer Problems

In the previous section we have established the mathematical analogy between thermal radiation and neutron transport. We need to go only a step further and draw attention to a few solutions of neutron transport problems.

The general solution of the system of integral equations (5.5) and (5.10), which are more basic than equation (5.11) and (5.12), is very difficult. It is convenient, as a first approach to this problem, to isolate the different parameters appearing in equations (5.5) and (5.10) by choosing a favorable configuration for a system. The geometrical complexities of most systems are avoided by considering an infinite plane parallel medium with no enclosure.

Thus the integral equations (5.5) and (5.10) for isotropically scattering infinite medium with no enclosure reduce to

$$\mathcal{E}'_\lambda(\vec{r}) = \int_V \epsilon_{e,\lambda}(\vec{r}') \, e^{-\tau_\lambda(\vec{r},\vec{r}')} \frac{dV}{|\vec{r} - \vec{r}'|^2} \quad ,$$

since $dV = A d\vec{r}'$, where $\vec{r}'$ is understood to be a dummy position coordinate and not a vector. Taking A as unity and substituting for the effective emission coefficient, $\epsilon_{e,\lambda}(\vec{r})$, from equation (3.3), we obtain

$$\mathcal{E}'_\lambda(\vec{r}) = \int \left[\epsilon_\lambda(\vec{r}') + \frac{\sigma_\lambda(\vec{r}')\mathcal{E}'_\lambda(\vec{r}')}{4\pi} \right] e^{-\tau_\lambda(\vec{r},\vec{r}')} \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|^2} \quad . \tag{7.3}$$

The case in which σ_λ is independent of position will be dealt with. By measuring distances in the units of the mean free path, $\lambda_P = 1/\sigma_\lambda$ (this is the same as saying that $\sigma_\lambda = 1$), equation (7.3) can be simplified to

$$\mathcal{E}'_\lambda(\vec{r}) = \frac{1}{4\pi} \int \left[\frac{4\pi\epsilon_\lambda(\vec{r}')}{\sigma_\lambda} + \mathcal{E}'_\lambda(\vec{r}') \right] e^{-|\vec{r} - \vec{r}'|} \frac{d\vec{r}'}{|\vec{r} - \vec{r}'|^2} \tag{7.4}$$

It should be pointed out, though, that the general case of variable σ_λ introduces considerable complication. Absolute coordinates instead of only relative coordinates between the field and the point of emission enter the problem. In the rest of this section σ_λ is going to be considered as grey, that is, the scattering coefficient, σ_λ, is independent of wavelength.

In the absence of emission, $\epsilon(\vec{r}) = 0$, the integral equation (7.4) reduces to

$$\mathcal{E}'(\vec{r}) = \frac{1}{4\pi}\int \mathcal{E}'(\vec{r})\, \frac{e^{-|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|^2}\, d\vec{r}' \quad . \tag{7.5}$$

The solution of this equation has been obtained and the functions have been tabulated by Case et al.[10]

The integral equation (7.4) in the presence of an isotropic point source, i.e., the emission coefficient, $\epsilon_0(\vec{r}') \doteq \delta(\vec{r} - \vec{r}_0) = 4\pi\epsilon(\vec{r}')/\sigma$, can be written as

$$\mathcal{E}'(\vec{r}) = \int \left[\epsilon_0(\vec{r}') + \mathcal{E}'(\vec{r}')\right] e^{-|\vec{r} - \vec{r}'|} \frac{d\vec{r}'}{4\pi|\vec{r} - \vec{r}'|^2} \tag{7.6}$$

A detailed solution of this equation has also been worked out by Case et al.[10] Physically, for example, this problem can represent the radiation incident at any point r mean free paths distant from the center of an atomic explosion if the surrounding air does not absorb, but only scatters thermal radiation.

For other illustrations of the applicability of the results obtained for neutron transport problems to the problems of transfer of thermal radiation reference is made to (10), (12), (17), (63) and (106).

There also exists a possibility that some of the numerical codes available for the solution of neutron transport problems by digital computers can be used, with only minor modifications, for the solution of radiative transfer problems. For instance, Carlson's S_n method[106] can be used for the solution of the one-dimensional equation of transfer (4.13).

8 LITERATURE SURVEY

8.1 Introduction

The aim of this chapter is to present a review of the literature on heat transfer in absorbing and scattering media which are either gases, mixture of gases, solids, or gases and solid particles which form luminous flames. Emphasis is placed on papers describing more recent achievements. Radiant energy transfer problems in meteorology will not be considered here. Most of the earlier literature will be briefly referred to by way of textbooks and previous survey articles.

The study of radiant heat transfer in absorbing media has been a subject of very little theoretical and experimental investigation, and has been mostly limited to the problems occurring in boiler furnaces and combustion chambers. Recent developments in hypersonic flight, re-entry, rocket combustion chambers, power plants for interplanetary flight, and gas-cooled nuclear reactors have focussed more attention on heat transfer by radiation through absorbing media.

The radiant properties necessary for heat transfer calculations have been summarized by Jakob,(40) Pepperhoff,(76) and a recent survey of high-temperature radiative properties of gases has been given by Logan.(59) Theoretical and experimental studies of radiative properties of gases are given by Penner.(75)

The literature on heat transfer from radiating stationary media without conduction, convection and other energy transfer mechanisms are summarized in books on heat transfer by Jakob,(40) McAdams,(61) Schack,(87) and others. The paper by Wohlenberg et al.(109) contains a bibliography of 60 entries on radiant heat transfer in boiler furnaces and related topics. The papers reviewed in the above-mentioned references will not be discussed here. However, an effort will be made to review briefly all the papers dealing with heat transfer from both stationary and moving radiating media in which other modes of heat transfer (i.e., convection) are also included. In particular, the papers of the Russian engineers which have not been referred to by Jakob(40) and McAdams(61) will be reviewed.

8.2 Heat Transfer in an Enclosure

Many common problems of radiant heat transfer require the evaluation of energy transport between various parts of an enclosure, such as a combustion chamber, a rocket motor nozzle, etc. For instance, in a furnace the primary mechanism of energy transfer is the radiant exchange between the combustion gases, the enclosure walls and the absorbing area. Radiant interchange between surfaces forming a part of an enclosure and separated by an absorbing and scattering medium involves considerations of three kinds: (1) the configuration of the surfaces, (2) the radiation characteristics of the surfaces and the medium, and (3) the temperature distribution in the medium. In this section the papers concerned with heat transfer (energy transfer by radiation is predominant) in an enclosure are briefly reviewed.

A general multisurface system of source-sink and grey absorbing and emitting gas at constant temperature has been considered by Hottel(61) using a finite difference method. The problem treated is very complex and no claim is made that the solution is rigorous. A finite difference method for predicting heat exchange in enclosures where allowance is made for temperature variation in the medium is reported by Hottel and Cohen.[36] The system is divided into surface zones and gas zones, the number of zones being dependent on the desired accuracy of the result. The direct exchange factors are calculated for gas-gas, gas-surface, and surface-surface zone interchange. From these factors the net exchange factor for any zone pair can be determined, taking into account the interaction with all other zones. The resultant factors are then introduced into a set of energy balances, one for each zone. The simultaneous solution of these permit determination of the gas and surface temperatures and the distribution of heat flux over the surface.

The solution of a problem of heat exchange by radiation using the electrical network method was proposed by Oppenheim[71] and extended by Bevans and Dunkle.[6] An example to illustrate the use of the network method was given by Dunkle and Bevans,[6] who solved a multinode network. Adrianov[2] applied the electrical network analogy to the solution of the two integral equations which describe the process of radiant heat exchange in a closed system of surfaces with an absorbing and scattering medium present in an enclosure. The author has shown that the two integral equations can be sufficiently approximated by a system of linear algebraic equations. This scheme is equivalent to the solution of electric networks as first suggested by Oppenheim.[71] The problem is extended by Adrianov to the case in which the gas present in the enclosure is not at a constant temperature; however, the problem is still restricted to diffuse radiation from enclosing surfaces. An electrical analog is given in the paper for the solution of the integral equations of radiant heat transfer with the help of an electrical integrator, but no specific problems are solved.

The most practical solutions have been given by Hottel and Egbert.[37,38] Data of various investigators on the emission and absorption of radiation by carbon dioxide and water vapor were reviewed. A series of new experiments on water vapor were carried out[38] to measure total radiation from gas columns. Recommendations for a procedure to calculate heat transfer by gas radiation was given. A simplified procedure was presented to allow for the effect of gas shape on radiant heat exchange by introducing a mean value of the beam length. The procedure of mean beam length has been extended by Fax[21] for nonluminous radiation to tube banks.

The literature on configuration (some authors call it shape, angle, view, geometric) factors for radiation through nonabsorbing media, and for few simple geometries for radiation through absorbing media has been summarized by Jakob.[40] More recently, configuration factors for radiation through an absorbing medium were given by Oppenheim and Bevans.[73]

A simplified practical method of calculating heat transfer in oil-fired, pulverized coal and gas furnaces has been suggested by Mullkin.[70] A valuable additional contribution to the knowledge of heat transfer by radiation in pulverized coal furnaces has been made by Wohlenberg and Wise.[110] Introducing the concept of the radiant mean position of burning coal particles and ash, by distinguishing the coal particle while burning from the resulting ash particle, they were able to study the energy distribution and the direct transfer of energy between burning particles, ash particles, gases, refractory walls and cold walls.

Heat transfer in boiler furnaces has been studied by Poljak and Shorin,[81] Konakov,[49,50] and Mikhailov.[67] The energy equations were given in references (50) and (81) in both differential and integral form, but were not solved exactly. The energy equation was reduced to an algebraic one by introducing average quantities. Simplified solutions of temperature were obtained in a dimensionless form. In reference (81) the predicted dimensionless temperatures were compared with the correlations of other Russian investigators, and in reference (50) the calculated results were compared with the experimental data for some furnaces. Radiant heat transfer in furnaces was investigated by Yhland.[111] Simple similarity theory was applied and dimensionless quantities were deduced from the equation of energy conservation.

The role played by the radiant heat transfer on the process of combustion has been studied analytically by Shorin.[92,95] The influence of radiant energy on the activation of molecules and the contribution of the "radiant conductivity" are discussed. The energy equations are given, but are not solved. The problem of heat transfer by radiation in a combustion chamber from hot combustion gases has been studied by Shorin and Pravoverov,[94] and by Konakov et al.[51] Konakov has postulated that in combustion chambers with cooled walls a region is always present where there is thermodynamic equilibrium. The temperature of this equilibrium layer is determined by the intersection of the curves of molecular and radiation temperature change. The theoretical considerations about the existence of an equilibrium layer have been experimentally verified, and its location determined. The experiments carried out show that in combustion chambers an equilibrium layer is in close proximity to the heat-absorbing surface at a distance measured in millimeters and that the temperature of the equilibrium layer determines the heat exchange by radiation in combustion chambers.

8.3 Diffusion Approximation for Radiation

For the case of intense absorption, one can apply diffusion concepts in solving the radiant transfer problems. This type of diffusion approximation was first suggested by Jeans,[42] later extended by Rosseland,[86] and has been successfully applied in a few quite different problems.

The thermal radiation effects in atomic explosions, in particular the effect of radiant energy transfer on the shock wave, has been studied by Magee and Hirschfelder[62] using the diffusion approximation. Brickwedde[8] studied radiant energy propagation by the diffusion of photons at temperatures of atomic explosions in the range from 10^7 to 10^8 °K. He considered photons traveling in broken paths through the fireball in the emission-absorption-remission cycle. The photons therefore flowed to the surface of the fireball by a comparatively slow diffusion process described by the diffusion equation. For the heat flux by radiation the author wrote

$$\overrightarrow{q''_r} = - \frac{c\lambda_p}{3} \text{ grad } u = - \frac{4\sigma\lambda_p}{3} \text{ grad } T^4 = - \frac{4\sigma}{3\kappa} \text{ grad } T^4. \qquad (8.1)$$

Introducing an approximate value of the mean free path obtained by the help of Wien's displacement law, he showed that the radiant heat flux becomes

$$\overrightarrow{q''_r} = \text{const. } T^5 \text{ grad } T \quad . \qquad (8.2)$$

We note that the heat flux by molecular conduction is proportional to grad T, while the energy flux by radiation is proportional to grad T^6 or T^5 times the grad T. This shows the dominance of "thermal properties of radiation" over the usual thermal properties at the temperatures of thermonuclear explosions.

The diffusion approximation was used by Lubny-Gertsyk[58] to calculate the emissivity of a gas enclosed by surfaces. Shorin[93] used the same approximation to calculate the temperature distribution in combustion chambers.

Kolchenogova and Shorin[48] have studied both analytically and experimentally the diffusion of radiant energy in an absorbing media. The authors started from the definition of radiant heat flux vector,

$$\overrightarrow{q''_r} = - D \text{ grad } u = - (4\sigma/m_n^2 \kappa) \text{ grad } T^4 \quad , \tag{8.3}$$

where D is the diffusion coefficient of radiant energy defined by

$$D = c\lambda_p/m_n^2 = c/m_n^2 \kappa \quad ,$$

and m_n is defined by equation (9.8). Solutions of equation (8.3) were obtained for the case of a finite slab, cylinder, cylindrical and spherical annuli. The experimental results for a strongly absorbing gas were in good agreement with the predictions based on equation (8.3). This verifies the validity of the diffusion approximation.

The radiative conductivity has the effect of adding another term to the ordinary thermal conductivity. This contribution to molecular conductivity may be quite important even at ordinary temperatures. Heat transfer by simultaneous conduction and radiation in glass has been summarized by Pepperhoff.[76] Kellet[46] has treated mathematically the problem of steady flow of heat through glass. He found that even though infrared energy is absorbed within a short distance by almost any glass, the irradiation of energy by the glass itself is sufficient at glass-melting temperatures to cause considerable heat flow. As might be expected, the contribution of radiant conduction increases rapidly with rise in temperature, and at 1200°C the radiant heat flow in ordinary soda glass is about 50 times the ordinary conductive flow. A method for calculating temperature distributions in sheets of glass that are being heated or cooled was presented by Gardon.[26]

Van der Held[32] has discussed the contribution of radiant energy to the conduction of heat in insulating materials, particularly in connection with the experimental measurements of the thermal conductivities of these materials. He was concerned with systems in which the absorption of radiation was not so intense that the mean free paths of the photons could not be taken as negligibly small quantities. The Biot-Fourier equation was modified to include a term taking into account the effect of radiant heat transfer:

$$\rho f_p \frac{\partial T}{\partial t} = k \nabla^2 T + 4 \int_0^\infty n^2 \kappa_\lambda (E'_\lambda - E_{bb,\lambda}) \, d\lambda \quad . \tag{8.4}$$

The term E'_λ includes contributions due to the radiation from the heat source, the heated absorbing medium, and also scattered and reflected irradiation from the walls. He showed that as long as the distance d from the boundary of the system has a value so that $\kappa_\lambda d > 5$, then the effective irradiation E'_λ is equal to value $E_{bb,\lambda}$ estimated at the local temperature, and that $(E'_\lambda - E_{bb,\lambda})$ becomes proportional to the difference in the fourth power of two temperatures, the local temperature and the source temperature. Verschoor and Greembler[104] have carried out thermal conductivity measurements with four different gases in insulating materials. The measured values of thermal conductivity at low pressures confirmed the theoretical considerations of heat transfer by radiation.

8.4 Heat Transfer in Moving Radiating Medium

Only the papers which deal with radiating media at rest have been reviewed. The underlying hypothesis of the papers reviewed previously is that, unlike dynamic and convective effects, radiant energy exchanges do not affect the velocity and the temperature distribution in the flow. Such an assumption is easily acceptable when radiant heat transfer is small compared to the usual transport processes. However, this method fails in the presence of a large radiant heat flux, since its presence cannot be reconciled with the postulated temperature distribution that was derived without a radiative term. The determination of the true temperature distribution requires the solution of a coupled system of differential, integro-differential and integral equations.

Shorin[93] seems to be the first to consider a one-dimensional problem of heat transfer by radiation in the presence of both stationary and moving absorbing media. He started from the simplified equation of transfer, the definition of the radiant flux vector and the equation representing the energy balance. The equation of the energy at any point of the combustion chamber in the steady state and neglecting the energy dissipation, the work of pressure forces and gravity was written in the form

$$4\,\kappa E_{bb} - \int_{\Omega=4\pi} \kappa I \cos\,\theta\, d\Omega = -\,\mathrm{div}(\rho c_p \vec{w}) + \mathrm{div}(k\ \mathrm{grad}\ T) + q'''. \quad (8.5)$$

The boundary condition pertaining to the cooling surface was also derived. For the one-dimensional case with no heat sources between the two surfaces and neglecting the heat transfer by molecular conduction, equation (8.5) reduced to

$$\frac{4}{3\kappa}\frac{d^2E_{bb}}{dx^2} + \frac{uc_p}{3\kappa^2}\frac{d^3T}{dx^3} - uc_p\frac{dT}{dx} = 0 \quad . \tag{8.6}$$

To remove the nonlinearity of this equation with regard to temperature, the author introduced the following transformation:

$$\frac{dE_{bb}}{dT} = \frac{4E_{bb}}{T} \quad ;$$

then

$$uc_p\frac{dT}{dx} = \frac{uc_pT}{4E_{bb}}\frac{dE_{bb}}{dx} \quad .$$

Neglecting the derivative $\frac{d^3T}{dx^3}$ in comparison with $\frac{dT}{dx}$, equation (8.6) reduced to

$$\frac{d^2E_{bb}}{dx^2} - \kappa\Pi\frac{dE_{bb}}{dx} = 0 \quad . \tag{8.7}$$

The parameter $\Pi = 3uc_pT/16E_{bb}$ represents the ratio of energy transferred per unit area, per unit of time by convection to the energy of radiation transmitted through the unit area, per unit of time. According to the author, the assumption $\frac{d^3T}{dx^3} << \frac{dT}{dx}$ in the case of non-radiative equilibrium $(4\kappa E_{bb} - \kappa cu \neq 0)$ is equivalent to the assumption that

$$q''_{rad} = -\frac{4}{3\kappa}\frac{dE_{bb}}{dx} \quad , \tag{8.8}$$

an expression which signifies local thermodynamic equilibrium.

Solution of equation (8.8) is given for the radiation between two parallel planes at constant temperatures T_1 and T_2 and having emissivities ϵ_1 and ϵ_2 for various values of the heat transfer parameter Π. The solution of the problem was then applied to the calculation of heat transfer by radiation in boiler furnaces, with a single surface being cooled. The relation obtained was then compared to an empirical formula for heat transfer in furnaces.

Pukhov[83] investigated heat transfer from combustion gases of a cylindrical shape moving at constant velocity. The source of heat generation was the base plane of the cylinder, where the gases enter the system. The gas temperature changes both in the direction of the motion and along the cylindrical cavity in the radial direction. From simple approximate calculations and the definition of the effective temperature, an expression for the effective temperature is determined. The relation obtained, according to the author, takes into account the influence upon the effective temperature, not only of the temperatures of the heating and the cooling plane, but also of the properties of the radiating gas, and the shape and size of the volume which these gases occupy. However, the author himself makes the statement that the expression for the effective radiation temperature cannot be considered as final, and it is an attempt to solve the given problems.

Lubny-Gertsyk[58] has studied several problems of heat transfer between a radiating gas, both stationary and moving, and the fixed solid walls. In particular, an approximate method of calculation for a selectively emitting suspended dust medium is presented as well as a method for calculation of radiation of an emitting medium moving through an infinite plane screen. The information about some of the calculations leading to the given curves are lacking in the paper.

The paper by Adrianov and Shorin[3] is one of the first which is concerned with coupling of convective and radiative heat fluxes. The problem considered by the authors is a steady-state cooling of a radiating gas flowing in a circular pipe and between infinite parallel plates having a prescribed wall temperature. Convection and radiation are the two modes of heat transfer assumed to be present. The scattering of radiation is neglected and it is postulated that the radiant energy emitted by a unit volume of gas and absorbed by another unit volume is negligible compared to the radiant energy emitted by the gas and absorbed by the duct surfaces. The solution of a coupled system of equations of motion and the integro-differential equation of energy conservation is eliminated by assuming one-dimensional flow and various velocity profiles in the ducts. The energy equation is solved, and temperature distributions are determined for the following velocity profiles: for the circular pipe - uniform and parabolic; for the flow between two infinite parallel plates - uniform, $u/u_m = 3(1 - y/h)^2$ and $u/u_m = 3(y/h)^3$, where u is the velocity and any point y, u_m is the maximum velocity and h is the distance between the plates.

Several new dimensionless numbers, which enter radiative heat transfer problems in a natural way, are introduced. In particular, a use is made of dimensionless parameter which governs the coupling between convective and radiative energy fluxes. The results of calculations are presented in graphical form as functions and parameters of dimensionless numbers. The results show that there exists a value of optical thickness at which the heat transfer is maximum. Some of the calculated results are compared with the predictions of other Russian investigators.

The Rosseland approximation for the radiant heat flux vector was applied by Kadanoff[45] to the problem of calculating temperature distributions within bodies in steady-state ablation. Numerical results are presented for the temperature distribution with various parameters characterizing the body, i.e., reflectivity, absorption and scattering coefficients, etc. An analytical solution for temperature distribution is given in the limit of zero ablation. The author points out the concept of apparent thermal conductivity "works quite well whenever the temperature and the optical properties of the medium do not change very much within one photon mean free path." There are, however, some errors in the equations used.

The problem of a chemically reacting and radiating gas in Couette flow was studied by R. and M. Goulard.[30] With the assumption of thermodynamic equilibrium and no viscous dissipation, the energy equation reduced to

$$\text{div}\,(k\ \text{grad}\ T + \vec{E}) = 0 \quad , \tag{8.9}$$

where k is total conductivity, including the chemical part, and $\vec{E}$ is the radiant energy flux vector. Since the energy equation in Couette flow is one-dimensional, after integration of (8.9) one obtains[30]

$$k\,\frac{\partial T}{\partial y} + q_r'' = \text{constant}, \tag{8.10}$$

where

$$q_r'' = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} I_\nu\,(\theta,\phi)\cos\theta\,\sin\theta\,d\theta\,d\phi \quad .$$

Since equation (8.10) is nonlinear in T and quite complex, no closed-form solutions for the temperature are obtained. The numerical results indicate that the effect of the "long-range" process of radiation is to smooth out the temperature profiles and relieve sharp temperature gradients at the cool wall. As a result, the application of this exact method yields lower values of the total heat flux than previously calculated by approximate methods.

9. HEAT TRANSFER BY SIMULTANEOUS CONDUCTION, CONVECTION AND RADIATION

9.1 Introduction

One of the most serious mathematical difficulties is that, in solving the equation of transfer, it is necessary to take account of the variation of intensity of radiation with direction, i.e., as a function of the angle θ at each point in the medium, and on the surface of the enclosure in consideration. If high accuracy is not required in the final results, it is possible to introduce some simplifying assumptions by averaging $I_\lambda(\vec{r}, \vec{\Omega})$ over all directions. Many times, in order to simplify the problem, it is assumed that the intensity is isotropic or diffuse. Some of the schemes of the possible intensity dependence on the direction are shown in Fig. 9.1.

All of the cases idealized in the sketches approximate the dependence of intensity on direction for real surfaces and surface conditions, as well as emission of radiation from an elementary volume. The cases a and e correspond to diffuse emission from a surface and a unit volume, respectively; this is the case for black radiation. The case b corresponds to Lambert's cosine law, which is more or less valid for real bodies. All other cases give a more complicated dependence of the intensity on the direction. Knowing the dependence of intensity on direction at the surface or from an elementary volume, one can more easily calculate heat transfer to the surface or to the elementary unit volume of the medium. However, even with the assumption of diffuse radiation the net emission from a unit volume, $\mathcal{E}_n$, is a

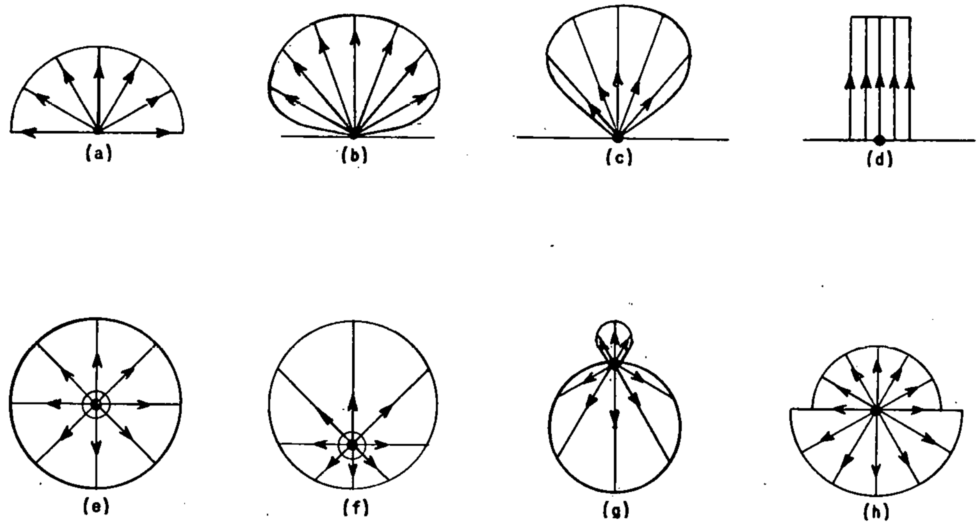

FIG. 9.1
IDEALIZED INTENSITY DISTRIBUTIONS WITH DIRECTION.

complicated integral over the entire radiating medium and the enclosing surfaces. The solution of a very simple equation of energy (11.1) is quite difficult. Thus, in view of the present state of mathematical techniques available for the solution of integral and integro-differential equations, an exact solution of (11.1) does not seem possible. There are, however, two simple forms for $\mathcal{E}_n$; one is valid when the media is optically thick and the other when it is optically thin. Both forms will be presented in this chapter.

As an illustration of these types of analyses, the approximation valid for optically thick media is employed to simplify the energy equation. The equations of motion and energy are solved for fluid in laminar flow. Two different problems are considered: Couette flow and flow along a wedge.

9.2 The Diffusion or Rosseland Approximation of Radiation

As a first simplification to the problem of radiative transfer of energy in absorbing and scattering media, the Rosseland approximation for the radiant energy flux vector is presented in this section. This approximation is valid for intense absorption and scattering; however, it fails completely at the boundary of the system. The Rosseland approximation for the radiant flux vector has been applied to such different problems as calculation of temperature distribution in boiler furnaces,(93) combustion chambers,(51) the effect of radiation on blast waves caused by nuclear explosions,(62) the transmission of radiant energy through glass,(76) the radiative energy transport within an ablating body,(45) and the transport of radiation through gases of low density.(48)

The stress tensor in hydrodynamics can be separated into normal and tangential components. Similarly, the energy tensor (3.19) was resolved by Rosseland(86) into normal and tangential components. For this purpose we resolve I_λ into two parts: I'_λ and I''_λ, where I'_λ is defined as the average value of I_λ according to the equation

$$4\pi I'_\lambda = \int_{\Omega=4\pi} I_\lambda d\Omega = cu_\lambda = \mathcal{E}'_\lambda . \qquad (9.1)$$

On the other hand, the monochromatic intensity I''_λ depends on the direction. Correspondingly, we resolve a tensor $P_{\lambda,ij}$ related to this intensity into two components $P'_{\lambda,ij}$ and $P''_{\lambda,ij}$, which are determined by the equations

$$P'_{\lambda,ij} = \int_{\Omega=4\pi} I_\lambda \Omega_i \Omega_j d\Omega = \frac{4\pi}{3} I'_\lambda \delta_{ij} = \frac{cu_\lambda}{3}\delta_{ij}, \qquad (9.2)$$

where the Kronecker delta function δ_{ij} is defined as $\delta_{ij} = \begin{Bmatrix} 1, & i=j \\ 0, & i\neq j \end{Bmatrix}$,

and

$$P''_{\lambda,ij} = \int_{\Omega=4\pi} I_\lambda \Omega_i \Omega_j \, d\Omega . \qquad (9.3)$$

The tensor $P'_{\lambda,ij}$ corresponds to normal stress, while $P''_{\lambda,ij}$ corresponds to shear stresses in a continuous medium.

From the definition of I'_{λ} it follows that when the system is in thermodynamic equilibrium the tensor $P'_{\lambda,ij}$ will increase in proportion to the monochromatic intensity of the black body radiation, or to its integral over all wavelengths, which is proportional to fourth power of temperature, according to the Stefan-Boltzmann law. On the other hand, the tensor $P''_{\lambda,ij}$ will behave quite differently. From the very definition of $P''_{\lambda,ij}$ we see that it depends on the asymmetry of the monochromatic intensity of radiation. The tensor $P''_{\lambda,ij}$ will depend on the change of I''_{λ} with direction. As a result of this, the relative magnitude of $P''_{\lambda,ij}/P'_{\lambda,ij}$ can be estimated and given as a function of $P'_{\lambda,ij}$.(93)

The divergence of the radiant energy flux tensor can be expressed as a gradient of a certain scalar potential function. This is possible only for thermodynamic equilibrium, that is, when $P''_{\lambda,ij}/P'_{\lambda,ij} \longrightarrow 0$.(86) Therefore, with this approximation we have

$$P_{\lambda,ij} = \frac{4\pi}{3} I_{bb,\lambda}\, \delta_{ij} = \frac{4}{3} E_{bb,\lambda}\, \delta_{ij} = \frac{1}{3} \mathcal{E}'_{\lambda}\, \delta_{ij}. \tag{9.4}$$

Thus for intense absorption and scattering as well as when the system is near to thermodynamic equilibrium, the nondiagonal components of the energy tensor become small, and one has(86)

$$\operatorname{div} P_{\lambda,ij} = \frac{1}{3} \operatorname{grad} \mathcal{E}'_{\lambda}. \tag{9.5}$$

Substituting (9.5) in equation (3.20), we get for a steady-state system the monochromatic radiant energy flux vector

$$\vec{E}_{\lambda} = -\frac{1}{3\beta_{\lambda}} \operatorname{grad} \mathcal{E}'_{\lambda}. \tag{9.6}$$

If the extinction coefficient depends on wavelength, the mean free path of the photon, λ_p, or the average extinction coefficient, may be defined by

$$\lambda_p = \frac{1}{\bar{\beta}} = \frac{\int_0^\infty \frac{\partial E_{bb,\lambda}}{\partial T} \frac{d\lambda}{\beta_\lambda}}{\int_0^\infty \frac{\partial E_{bb,\lambda}}{\partial T} d\lambda} = \frac{\int_0^\infty \frac{\partial E_{bb,\lambda}}{\partial T} \frac{d\lambda}{\beta_\lambda}}{\frac{\partial}{\partial T}(\sigma T^4)} = \frac{1}{4\sigma T^3}\int_0^\infty \frac{\partial E_{bb,\lambda}}{\partial T} \frac{d\lambda}{\beta_\lambda}. \quad (9.7)$$

The Rosseland definitions of the radiant heat flux vector and the mean free path, λ_p, correspond to simple kinetic theory arguments. Photons travel with the velocity c. At any time $\frac{1}{6}$ of the photons are moving in the x direction and $\frac{1}{6}$ in the opposite direction. If the mean free path is $\lambda_p = 1/\bar{\beta}$, the flux of radiation is given by equation (9.6). A similar approximation in the neutron transport theory is known as the first-order diffusion approximation.[106] Filippov[24] and Goody[29] have derived the same expression for the radiant flux vector by a different method. An improved relation for the radiant flux vector, which takes into account the change of intensity with direction, was given by Shorin[93] and will be presented in the next paragraph.

One can derive an improved relation for the radiant flux with the help of a function introduced by Kuznetsov:[53]

$$m_n = \frac{\int_{\Omega=2\pi} I \, d\Omega}{\int_{\Omega=2\pi} I \cos\theta \, d\Omega} \quad . \quad (9.8)$$

The radiant energy flux in the normal direction, q_r'', defined as

$$q_r'' = \vec{E} \cdot \vec{n} \quad , \quad (9.9)$$

which, with the help of (9.8), can be written as

$$q_r'' = \frac{1}{m_n}\left[\int_{\Omega=2\pi} I^+ \, d\Omega - \int_{\Omega=-2\pi} I^- \, d\Omega\right] \quad . \quad (9.10)$$

Using the definition of the absorption coefficient and the fact that $ds = dn/\cos\theta$, Shorin[93] has shown that the radiant flux in the normal direction is given by the following expression

$$q_r'' = -\frac{1}{m_n^2 \kappa}\frac{d(cu)}{dn} = -\frac{1}{m_n^2 \kappa}\frac{d\mathcal{E}'}{dn} \quad . \quad (9.11)$$

This equation differs from the Rosseland approximation for the radiant flux(9.6) only by a constant. It was shown by Shorin[93] that for the case of uniform distribution of intensity, as indicated in sketch (9.1e), $m_n = 2$.

One should again emphasize the limitations of the Rosseland approximation. It is valid for both extreme cases $\sigma_\lambda = 0$ and $\kappa_\lambda = 0$. The approximation is restricted to media with large optical thicknesses ($\tau \gg 1$) as well as to a system close to thermodynamic equilibrium, so that equation (9.4) is valid. This is equivalent to saying that the terms proportional to R_λ vanish in equation (5.9), that is, the radiation incident at any point in the medium is given by the second term of equation (5.9) no matter what the radiosity is. In addition, the temperature and the radiative properties do not change much within one photon mean free path, or, expressing this condition mathematically, we must have

$$\left|\frac{1}{\beta_\lambda} \text{grad} (\log \mathcal{E}'_\lambda)\right| \ll 1 \quad .$$

From the preceding discussion it is clear that the diffusional approximation of radiation breaks down completely in the vicinity of a surface, since it does not take into account the radiation leaving from a surface.

9.3 Energy Equation with the Diffusional or Rosseland Approximation for Radiation

The energy equation, including a radiative term, has been derived in Chapter 6 and is reproduced here:

$$\rho \frac{Dh}{Dt} + \frac{\partial u}{\partial t} = \text{div}(k \text{ grad } T) - \text{div } \vec{E} + q''' + \frac{Dp}{Dt} + \mu \Phi \quad .$$

Substituting the diffusion approximation (9.6) for the radiant energy flux vector, $\vec{E}$, in the energy equation, we obtain

$$\rho \frac{Dh}{Dt} = \text{div } (k_{eff} \text{ grad } T) + q''' + \frac{Dp}{Dt} + \mu \Phi \quad , \tag{9.12}$$

where

$$k_{eff} = k + k_r = k + \frac{16n^2 \sigma T^3}{3\beta} \qquad (9.13)$$

is the effective or apparent thermal conductivity.

The same restrictions as in the diffusion approximation apply also to the effective thermal conductivity. This concept holds quite well if the photon mean free path is much shorter than the distance over which the temperature changes significantly. Equation (9.13) holds except in a neighborhood of a surface.

9.4 Approximation for Weak Absorption

The second approximation to the problem of radiative transfer of energy in an absorbing and scattering medium is obtained for very weak absorption. Physically this means that the medium is transparent to thermal radiation. In this case the approximation for net emission, $\mathcal{E}_n$, may be obtained by the following consideration. The net emission is defined by equation (2.30) as

$$\mathcal{E}_n = \mathcal{E} - \kappa \mathcal{E}',$$

or

$$\mathcal{E}_n = \kappa \int_{\Omega=4\pi} \epsilon \, d\Omega - \kappa \int I \, d\Omega \quad . \qquad (9.14)$$

Since the black body intensity is isotropic, we can write (9.14) as

$$\mathcal{E}_n = 4n^2 \kappa E_{bb} - \kappa \int_{\Omega=4\pi} I \, d\Omega \quad . \qquad (9.15)$$

The first term on the right hand side of (9.15) represents the emission of thermal energy at the local temperature. The second term represents the absorption of radiant energy which was emitted by other elements of the fluid and by the bounding surfaces. If the mean free path of radiation, λ_p, is long, the main contribution to the second term will come from the points spaced at about this distance from the point in consideration. If $\lambda_p/h > 1$, where h is a characteristic dimension of the system, the irradiation at each point will originate either far away from the point, or from the boundaries. In this case, the irradiation will not vary much over distances comparable to the characteristic dimension of the system, and the variation of $\mathcal{E}_n$ will chiefly be due to the variation of E_{bb}. Hence, applying the Laplace operator ∇^2 to equation (9.15), we obtain

$$\nabla^2 \mathcal{E}_n \approx \nabla^2 (4n^2 \kappa E_{bb}) \quad . \tag{9.16}$$

An operator such as "grad" or "div" could have been applied to (9.15), but it is more convenient to use the Laplacian. If we assume that the absorption coefficient, κ, and the refractive index, n, are "weak" functions of position, (9.16) can be written as

$$\nabla^2 \mathcal{E}_n \approx 4n^2 \kappa \sigma \nabla^2 T^4 \quad . \tag{9.17}$$

For a system in a steady state, we have from equation (3.16) that $\mathcal{E}_n = \text{div } \vec{E}$, that is, the net emission of radiant energy from a unit volume per unit of time is equal to the rate of change of the radiant energy flux vector with position. The operation $\nabla^2 \mathcal{E}_n = \nabla^2(\nabla \cdot \vec{E})$, however, has no simple physical interpretation. Equation (9.17) will have to be solved simultaneously with the energy equation (6.11) to determine the temperature distribution in weakly absorbing media.

9.4 Couette Flow

9.4.1 Basic Equations

The complexity introduced by the radiative contribution to the energy flux is in part due to the dependence of the flux on the geometrical configuration of the problem. A very simple type of flow, the

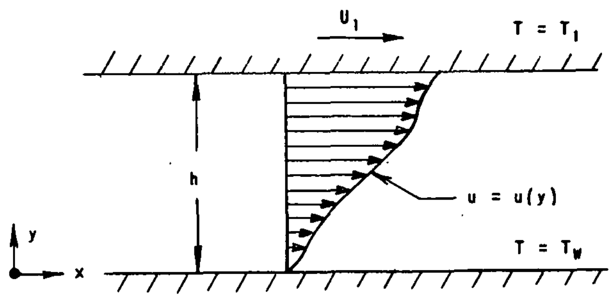

FIG. 9.2
PHYSICAL MODEL AND COORDINATE SYSTEM FOR COUETTE FLOW.

Couette flow (see Fig. 9.2), will be considered. The geometric effects are nonexistent in this problem, and it is hoped that it will contribute to the overall understanding of heat transfer in a medium absorbing thermal radiation.

The equations of motion (6.2) and energy (9.12) reduce to

$$\frac{\partial u}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0 \tag{9.18}$$

and

$$\frac{\partial}{\partial y}\left[\left(k + \frac{16n^2\sigma T^3}{3\kappa}\right) \frac{\partial T}{\partial y}\right] + \mu\left(\frac{\partial u}{\partial y}\right)^2 = 0 \tag{9.19}$$

if the following assumptions are made:

1) The flow is one-dimensional.
2) Heat conduction in the x direction is negligible compared to heat transfer in the y direction.
3) There are no pressure gradients and heat sources.
4) Scattering is negligible compared to absorption.
5) Viscosity, thermal conductivity and absorption coefficient are functions of temperature.

The heat due to viscous dissipation is important when the Eckert number,

$$N_E = \frac{\mu U_1^2}{k(T_1 - T_w)} \quad ,$$

is of the order of unit magnitude. For the problem in consideration here, we assume that $N_E << 1$. Since the viscous heat dissipation is negligible, the equations of motion (9.18) and energy (9.19) become uncoupled. Therefore, in the remainder of this section only the energy equation shall be considered.

A very simple solution for the temperature distribution is obtained when it is postulated that the temperature is constant along the wall, the boundary conditions being: $T = T_w$ at $y = 0$, and $T = T_1$ at $y = h$. Thus, to solve (9.21), let

$$\xi = y/h;$$

then

$$\frac{d}{dy} = \frac{d}{d\xi}\frac{d\xi}{dy} = \frac{1}{h}\frac{d}{d\xi} \quad .$$

Introducing the new variable and dividing by h^2, the equation of energy (9.19) reduces to

$$\frac{d}{d\xi}\left[\left(k + \frac{16 n^2 \sigma T^3}{3\kappa}\right)\frac{dT}{d\xi}\right] = 0 \quad . \qquad (9.20)$$

9.5.2 Solution of the Energy Equation

Solution of (9.20) requires the knowledge of the variations with temperature of thermal conductivity and absorption coefficient. It is assumed here that these two properties can be approximated by the relations

$$\frac{k}{k_w} = \left(\frac{T}{T_w}\right)^{a_1} \qquad (9.21)$$

and

$$\frac{\kappa}{\kappa_w} = \left(\frac{T}{T_w}\right)^{a_2} \quad . \qquad (9.22)$$

Substituting (9.21) and (9.22) into (9.20) and integrating, one obtains

$$\frac{T^{a_1+1}}{T_w^{a_1}(a_1+1)} + \frac{16 n^2 \sigma T_w^{a_2} T^{4-a_2}}{3(4-a_2)\, k_w \kappa_w} = c_1\xi + c_2 \quad . \qquad (9.23)$$

When the integration constants c_1 and c_2 are evaluated by using the boundary conditions, equation (9.23) reduces to

$$A_1 T^{a_1+1} + A_2 T^{4-a_2} = A_1\left[T_w^{a_1+1}(1-\xi) + T_1^{a_1+1}\xi\right] + A_2\left[T_w^{4-a_2}(1-\xi) + T_1^{4-a_2}\xi\right], \qquad (9.24)$$

where

$$A_1 = \frac{1}{(a_1 + 1)\ T_w^{a_1}}$$

and

$$A_2 = \frac{16n^2\sigma T_w^{a_2}}{3(4-a_2)k_w\kappa_w} \quad .$$

If the thermal conductivity and the absorption coefficient are independent of the temperature variation, $a_1 = a_2 = 0$, equation (9.24) simplifies to

$$T + B_1T^4 = [T_w\ (1 - \xi) + T_1\xi] + B_1\,[T_w^4\,(1 - \xi) + T_1^4\,\xi], \tag{9.25}$$

where

$$B_1 = 4n^2\sigma/3k\ \kappa \quad .$$

9.5.3 Discussion of Results for Couette Flow

The numerical results were obtained for the case in which the thermal conductivity and the absorption coefficient are independent of temperature, and the index of refraction is unity. For the Rosseland approximation to apply it was assumed that for all values of the absorption coefficient considered h is much less than the mean free path of radiation ($h << \lambda_p = 1/\kappa$). Predicted temperature distributions for Couette flow are plotted in Figs. 9.3 and 9.4. We consider the effect of varying κ but holding the same temperature at the boundaries. The physical nature of the results can be understood when we note that $16\ n^2\sigma T^3/3\kappa$ represents "the radiative thermal conductivity," k_r. Three separate cases must be considered: $k >> k_r$, $k << k_r$ and, that in which the thermal conductivity is of the same order of magnitude as "the radiative conductivity."

For the first case ($k \gg k_r$, $A_1 \gg A_2$), equation (9.24) can be approximated by

$$T^{a_1+1} \approx [T_w^{a_1+1}(1 - \xi) + T_1^{a_1+1}\xi] \quad . \tag{9.26}$$

This is an equation for temperature distribution for a slab with temperature-dependent thermal conductivity. The temperature profiles approach those for pure conduction as the absorption coefficient increases. From Fig. 9.4 we see that the temperature profile for the case in which $\kappa = 10^4$ ft^{-1}* is a maximum 1.5 percent higher than that for pure conduction. This difference further decreases (see Fig. 9.3) as the temperature level decreases.

When "the radiative conductivity" predominates, that is, $k \ll k_r$, $A_1 \ll A_2$, equation (9.24) reduces to

$$T^{4-a_2} \approx [T_w^{4-a_2}(1 - \xi) + T_1^{4-a_2}\xi] \quad . \tag{9.27}$$

Thus, in the limit when energy transfer by radiation predominates, the temperature distribution is independent of the value of the absorption coefficient, but depends on the variation of κ with temperature. The results for $\kappa = 10$ ft^{-1} and $\kappa = 1$ ft^{-1} are indistinguishable, and separate curves could not be drawn.

For the case when thermal conductivity is of the same order of magnitude as "the radiative conductivity," we expect an increase in T as κ decreases. This increase in temperature should be accompanied by an increase in the difference between the temperature predicted by equation (9.26) for pure conduction and by equation (9.24) for simultaneous conduction and radiation. Figure 9.4 shows that this expected increase in the temperature difference does indeed occur. A small κ indicates a large value of k_r; hence T should indeed increase with

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*This value of κ is quite high. However, much higher values of the absorption coefficients have been obtained experimentally, i.e., at a wavelength of 0.923μ. Sun and Weissler(97) obtained a value of $95{,}700 \pm 975$ ft^{-1} for CO_2 gas.

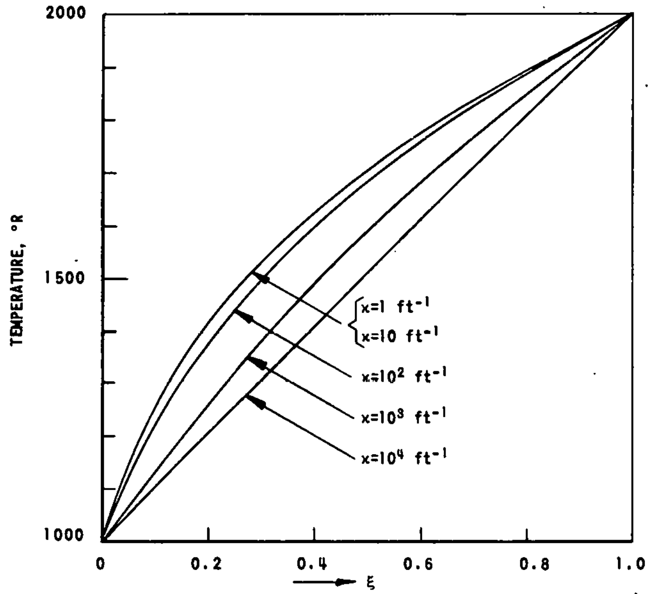

FIG. 9.3

VARIATION OF TEMPERATURE WITH THE DIMENSIONLESS DISTANCE FOR $a_1=a_2=0$ AND $k=0.054$ Btu hr^{-1} ft^{-1} R^{-1}.

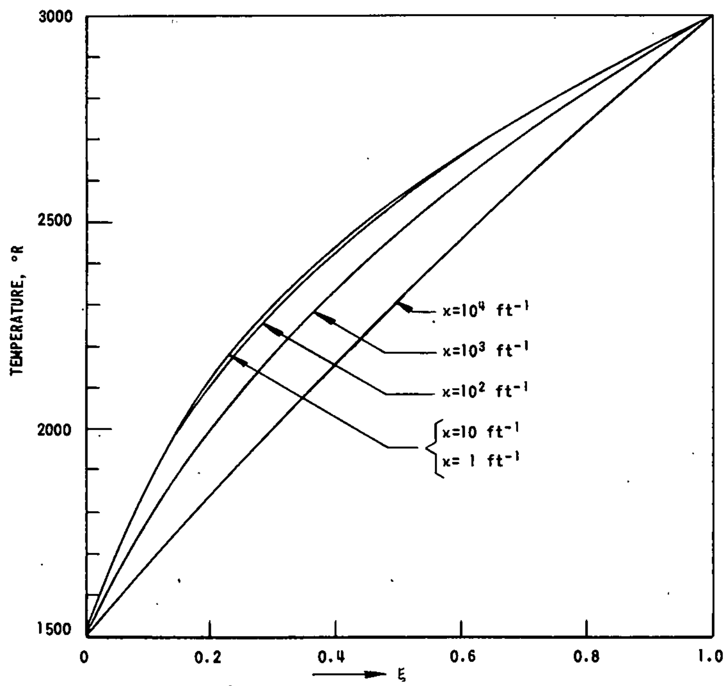

FIG. 9.4

VARIATION OF TEMPERATURE WITH THE DIMENSIONLESS DISTANCE FOR $a_1=a_2=0$ AND $k=0.054$ Btu hr^{-1} ft^{-1} R^{-1}.

decreasing absorption coefficient. For small values of κ, less than 10 ft^{-1}, the increase in temperature with the decrease in κ is negligible.

The results show that the presence of radiation increases the temperature gradient above that of pure conduction at the cool wall and decreases it at the hot wall. This trend in the temperature gradient is expected. Since the effective thermal conductivity increases with temperature, the temperature gradients must decrease to conserve the total energy flux (conduction plus radiation). The Rosseland approximation breaks down in the close vicinity of the surface and only the presence of conduction insures the continuity of the temperature near the wall. It is therefore expected that in close proximity to the walls the temperature predicted by using the diffusion approximation for the radiant flux vector will be in error; however, this error decreases as the mean free path of radiation decreases.

9.6 Flow Along a Wedge

9.6.1 Introduction

Two-dimensional laminar flow and heat transfer have been studied by many investigators. Historical sketches and extended bibliographies of analytical and experimental studies are given in the texts of Goldstein,[28] Howarth,[39], Schlichting,[88] Pai[74] and others. Factors that affect the development of a laminar boundary layer are pressure gradients, heat transfer, Mach number, and the properties of the fluid under consideration. Since mathematical complexities preclude solutions of this problem in a completely general fashion, the literature consists largely of solutions treating particular combinations of these factors.[16] In previous analyses the heat transfer by radiation has been ignored. The objective of this section is to investigate briefly the effect of the absorption coefficient on the temperature distribution and heat transfer in a medium that strongly absorbs thermal radiation.

9.6.2 Basic Boundary Layer Equations

The first step in the analysis of laminar flow and heat transfer along a wedge is to apply the conservation laws of mass, momentum and energy. The resulting partial differential equations require simplification before they become amenable to numerical solution. The coordinate system for the wedge is shown in Fig. 9.5, and the fundamental equations which express the law of conservation of mass (6.1), momentum (6.2), and energy (9.12), with the standard laminar boundary layer approximations,[88] reduce to:

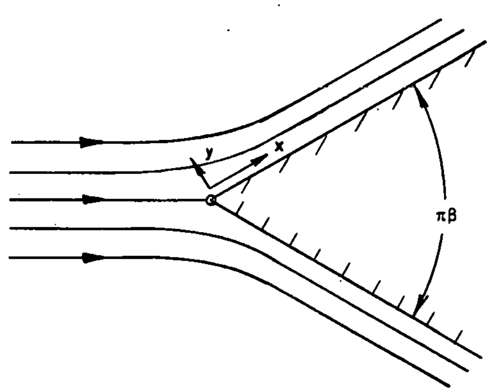

FIG. 9.5
COORDINATE SYSTEM FOR FLOW PAST A WEDGE.

Continuity:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0, \qquad (9.28)$$

Momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \quad \frac{\partial p}{\partial y} = 0, \qquad (9.29)$$

Energy:

$$\rho u \frac{\partial (c_p T)}{\partial x} + \rho v \frac{\partial (c_p T)}{\partial y} = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left[\left(k + \frac{16 n^2 \sigma T^3}{3\kappa}\right)\frac{\partial T}{\partial y}\right] + \mu\left(\frac{\partial u}{\partial y}\right)^2 . \qquad (9.30)$$

The heat generation per unit volume has been neglected.

The boundary conditions for this system of equations are

$$\left.\begin{array}{lll} u = v = 0, & T = T_W & \text{at} \quad y = 0 \\ u = U, & T = T_0 & \text{at} \quad y \rightarrow \infty \end{array}\right\} \qquad (9.31)$$

9.6.3 Similarity Transformation

"Similar" solutions are discussed in reference 88. They constitute a particularly simple class of solutions of u(x,y), and in this case the system of partial differential equations (9.28), (9.29), and (9.30) reduce to two ordinary differential equations. It is proved in reference 88 that such similar solutions exist when the velocity of the potential flow is proportional to a power series of the length coordinate measured from the stagnation point, i.e., for

$$U(x) = u_1 x^m \quad . \qquad (9.32)$$

We now introduce dimensionless coordinate η, first suggested by Faulkner and Skan,[20] so that

$$\eta = y\sqrt{\frac{(m+1)}{2}\,\frac{U}{\nu^* x}} \quad . \qquad (9.33)$$

The asterisk designates a physical property evaluated at an arbitrary temperature T^*. The continuity equation (9.28) can be integrated by introducing the stream function:

$$\psi(x,y) = \sqrt{\frac{2}{(m+1)}\,\nu^* u_1 x^{m+1}}\, f(\eta). \qquad (9.34)$$

The velocities in the conservation equations can be replaced through the definitions

$$\frac{\partial\psi}{\partial y} = \frac{\rho}{\rho^*}\, u \quad \text{and} \quad \frac{\partial\psi}{\partial x} = -\frac{\rho}{\rho^*}\, v \quad .$$

Thus the velocity components become

$$u = \frac{\rho^*}{\rho}\frac{\partial\psi}{\partial y} = \frac{\rho^*}{\rho} u_1 x^m f'(\eta) = \frac{\rho^*}{\rho} U f'(\eta) \tag{9.35}$$

and

$$v = -\frac{\rho^*}{\rho}\frac{\partial\psi}{\partial x} = -\frac{\rho^*}{\rho}\sqrt{\frac{(m+1)}{2}\nu^* u_1 x^{m-1}}\left[f(\eta) + \frac{m-1}{m+1}\eta f'(\eta)\right]. \tag{9.36}$$

From Bernoulli's equation:

$$p + \tfrac{1}{2}\rho U^2 = \text{constant},$$

we have

$$\frac{dp}{dx} = -\rho U \frac{dU}{dx} \quad . \tag{9.37}$$

Assuming that density and viscosity are functions of temperature and substituting equation (9.37) into (9.29), we obtain

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu^*\left(\frac{\rho^*}{\rho}\right)\frac{\partial}{\partial y}\left(\frac{\mu}{\mu^*}\frac{\partial u}{\partial y}\right) \quad . \tag{9.38}$$

We define the dimensionless temperature, Θ, as

$$\Theta \equiv T/T^* \quad ,$$

where T^* is an arbitrary temperature. Assuming that the density, viscosity, and thermal conductivity are functions of temperature and introducing the dimensionless temperature, as well as the pressure gradient, in the equation of energy, (9.30), we obtain

$$u\frac{\partial\Theta}{\partial x} + v\frac{\partial\Theta}{\partial y} = -\frac{uU}{c_p T^*}\frac{dU}{dx} + \alpha^*\left(\frac{\rho^*}{\rho}\right)\frac{\partial}{\partial y}\left(\frac{k_{eff}}{k^*}\frac{\partial\Theta}{\partial y}\right) + \frac{\mu}{\rho c_p T^*}\left(\frac{\partial u}{\partial y}\right)^2 \quad . \tag{9.39}$$

Introducing in equations (9.38) and (9.39) the independent dimensionless variable η, there result the equations

$$\left\{\left(\frac{\mu}{\mu^*}\right)\left[\left(\frac{\rho}{\rho^*}\right)' f' + \left(\frac{\rho}{\rho^*}\right) f''\right]\right\}' + f\left[\left(\frac{\rho}{\rho^*}\right)' f' + \left(\frac{\rho}{\rho^*}\right) f''\right] + \beta\left[\left(\frac{\rho}{\rho^*}\right) - \left(\frac{\rho^*}{\rho}\right) f'^2\right] = 0 \tag{9.40}$$

and

$$\left[\left(\frac{k_{eff}}{k^*}\right)\Theta'\right]' + N_{Pr} f \Theta' = \frac{N_{Pr} U^2}{c_p T^*}\left\{\beta f - \left(\frac{\mu}{\mu^*}\right)\left[\left(\frac{\rho}{\rho^*}\right)' f' + \left(\frac{\rho}{\rho^*}\right) f''\right]^2\right\} = 0, \tag{9.41}$$

where the pressure gradient parameter β is defined as $\beta = 2m/(m+1)$. The prime on f and Θ denotes differentiation with respect to η. The boundary conditions for this set of equations are obtained from equations (9.31) and (9.33):

$$\left.\begin{array}{llll} f = f' = 0, & \Theta = \Theta_w & \text{at} & \eta = 0 \\ f = 1, & \Theta = \Theta_0 & \text{at} & \eta \rightarrow \infty \end{array}\right\} \tag{9.42}$$

are the transformed boundary conditions.

The pressure gradient parameter β is related to the exponent m of the velocity distribution (9.32) through the relation $\beta = 2m/(m+1)$. The case $\beta < 0$ corresponds to an unfavorable pressure gradient; $\beta = 0$ (m=0) corresponds to a flat-plate flow; and $\beta = 2$ ($m=\infty$) corresponds to an infinitely favorable pressure gradient. It is shown in reference 88 that the case of stagnation point in axisymmetric flow can be transformed to the solution for $\beta = \frac{1}{2}$ ($m = \frac{1}{3}$).

Because of the nonlinearity of the system, its high order, and its classification as a "two-point boundary value problem," no standard integration methods will yield results expressible in closed form. Equations (9.40) and(9.41) were solved numerically by the forward integration method, as indicated in Appendix C.

The local heat transfer rate to the body is determined by the sum of conductive and radiative transports. The conductive energy transport is given by

$$q_c'' = -k\left(\frac{\partial T}{\partial y}\right)_w \quad . \tag{9.43}$$

In terms of the dimensionless temperature distribution, this heat flux becomes

$$q_c'' = -k\left(\frac{\partial T}{\partial \Theta}\right)\left(\frac{\partial \Theta}{\partial \eta}\right)_w\left(\frac{\partial \eta}{\partial y}\right) \quad ,$$

or

$$q_c'' = -\frac{kT^*}{x}\sqrt{\frac{m+1}{2}}\ \sqrt{N_{Re_x}}\left(\frac{\partial \Theta}{\partial \eta}\right)_w \quad . \tag{9.44}$$

Thus, the local heat flux by conduction is directly proportioned to the square root of the Reynolds number based on x as the characteristic dimension and inversely proportional to x.

Since the diffusion approximation fails at the surface, the radiative energy flux at the wall is more difficult to estimate. It is necessary to obtain the temperature distribution as a function of y and not of the similarity variable η, and then to calculate the net radiant energy flux from equation (2.29). This procedure is quite awkward and very cumbersome. The total heat flux might be approximated in a simpler manner. Instead of evaluating the heat flux at the wall, we can estimate it at a very small distance away from it, where the fluid velocity is still small and where the Rosseland approximation for the radiant flux vector is valid.

9.6.4 Discussion of Results for Flow Along a Wedge

The flow along a wedge was investigated with the purpose of studying the effect of the absorption coefficient on the temperature distribution in the flowing radiating media. The numerical calculations were performed for two distinct cases. First, in order to separate out the effect of the absorption coefficient on the temperature distribution from other variables, the physical properties were assumed to be independent of the temperature. The dimensionless temperature distributions for a Prandtl number of 1 and pressure gradient parameter β of 0 and $\frac{1}{2}$ are shown for both the hot and the cool walls in Figs. 9.6 through 9.10. Second, the case in which the viscosity is a very strong function of temperature was considered. Both the velocity and temperature profiles are therefore affected by this dependence. Pyrex glass,

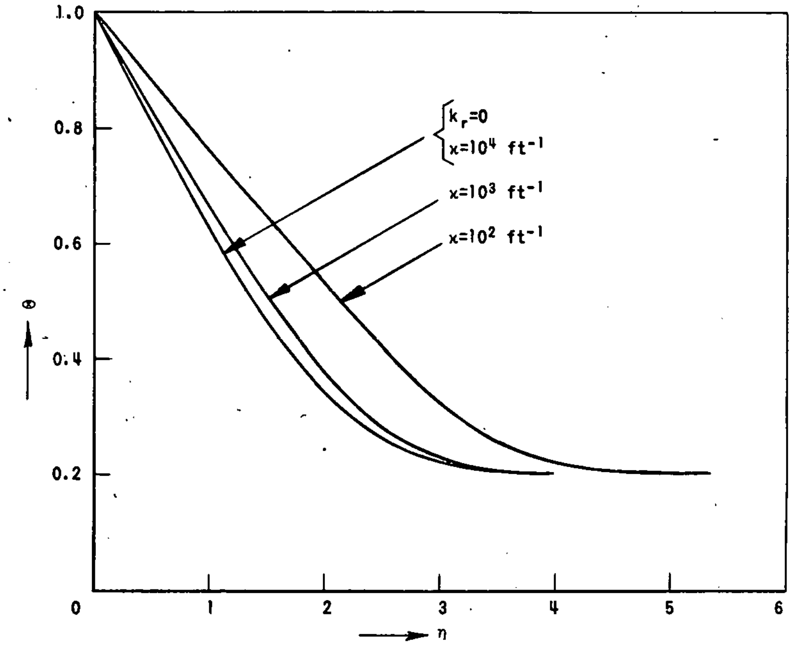

FIG. 9.6
DIMENSIONLESS TEMPERATURE PROFILES AS A FUNCTION OF THE SIMILARITY VARIABLE η FOR $\beta=0$, $N_{Pr}=1.0$, $k=0.05$ Btu hr^{-1} ft^{-1} R^{-1} AND $T^*=3000$°R.

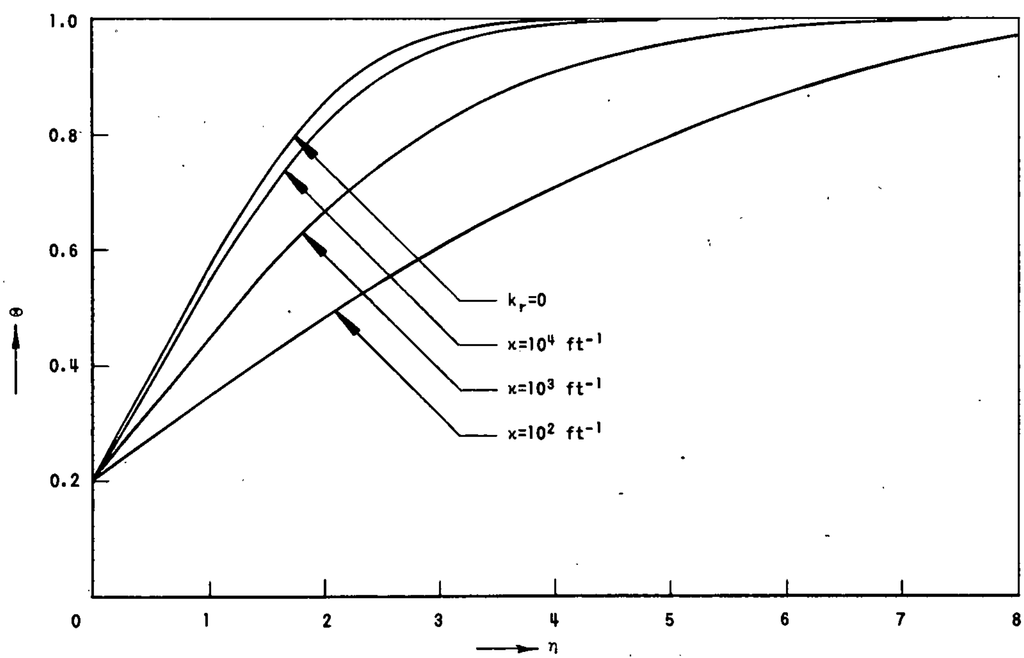

FIG. 9.7
DIMENSIONLESS TEMPERATURE PROFILES AS A FUNCTION OF THE SIMILARITY VARIABLE η, FOR $\beta=0$ $N_{Pr}=1.0$, $k=0.05$ Btu hr^{-1} ft^{-1} R^{-1} AND $T^*=3000$°R.

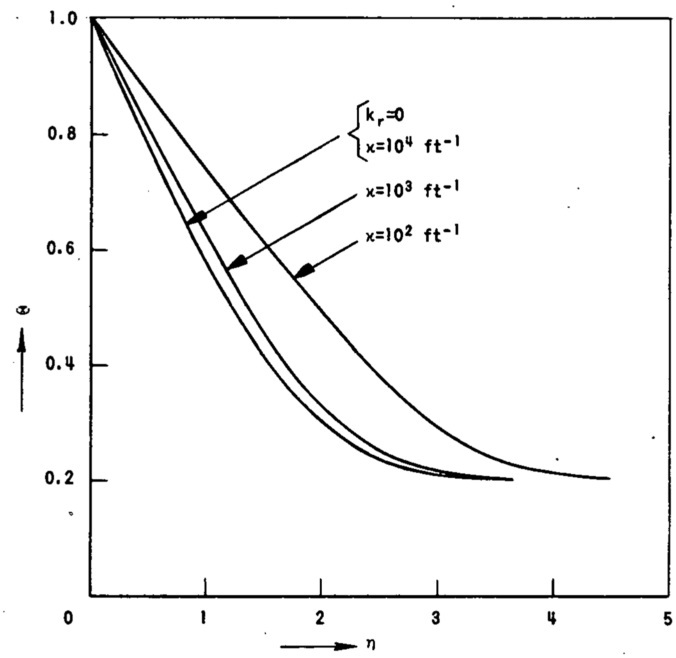

FIG. 9.8

DIMENSIONLESS TEMPERATURE PROFILES AS A FUNCTION OF THE SIMILARITY VARIABLE η FOR β=0.5, N_{Pr}=1.0, k=0.05 Btu hr^{-1} ft^{-1} R^{-1} AND T*=3000°R.

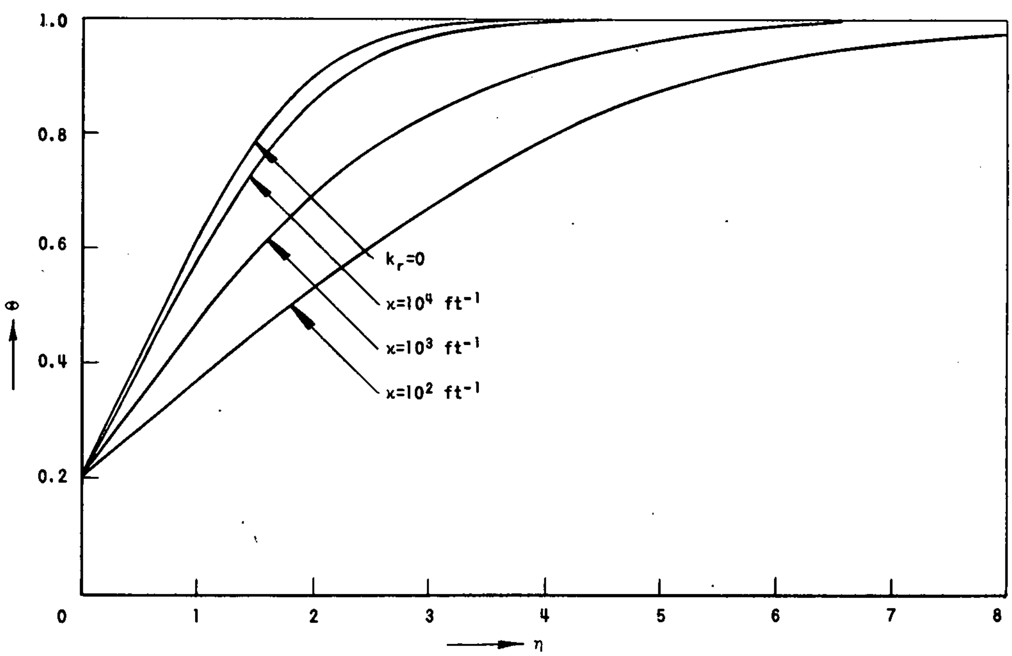

FIG. 9.9

DIMENSIONLESS TEMPERATURE PROFILES AS A FUNCTION OF THE SIMILARITY VARIABLE η FOR β=0.5, N_{Pr}=1.0, k=0.05 Btu hr^{-1} ft^{-1} R^{-1} AND T*=3000°R.

which is of physical interest in reentry problems, was chosen as an example. The results of these calculations are given in Figs. 9.11 through 9.13.

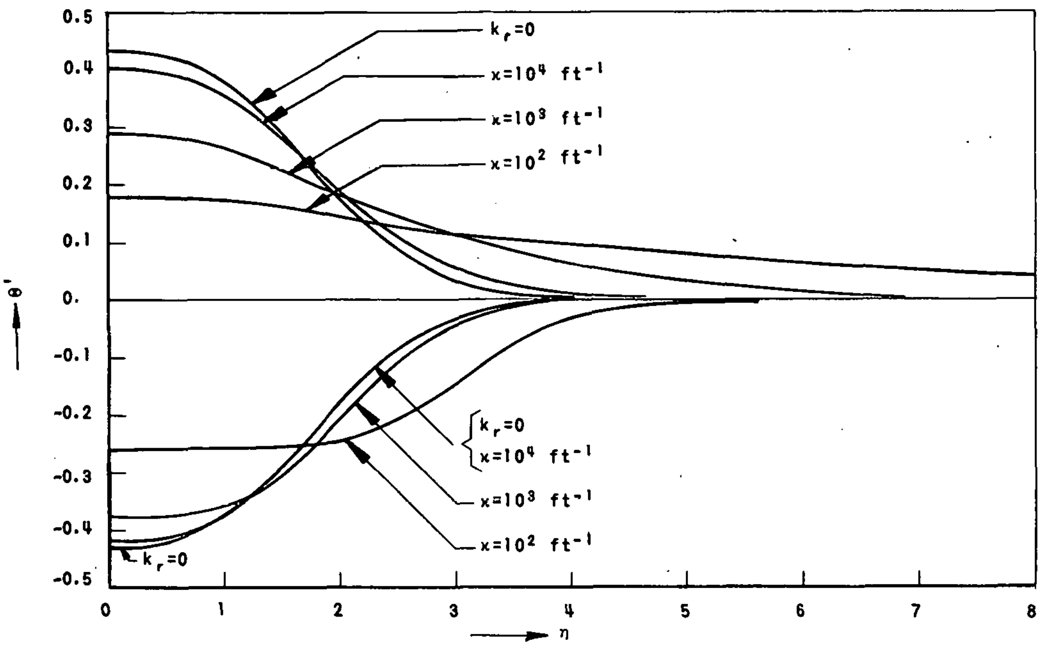

FIG. 9.10
DIMENSIONLESS TEMPERATURE GRADIENT ACROSS THE BOUNDARY LAYER FOR β=0.5, N_{Pr}=1.0, k=0.05 Btu hr^{-1} ft^{-1} R^{-1} AND T*=3000°R.

The velocity profiles for the constant-property case are well known[20] and therefore will not be considered in this work. Figures 9.6 through 9.9 show that, for N_{Pr} = 1 and a given κ, the dimensionless temperature varies monotonically across the boundary layer from the wall value to the free stream value. When the absorption coefficient is large, the temperature profiles approach those of a nonradiating media. For the hot wall when $\kappa = 10^4$ ft^{-1} ($k_{eff} \rightarrow k$), the temperature distributions are indistinguishable. The temperature gradients are decreased with the decrease in the absorption coefficient. In the case of boundary layer problems in which the thermal boundary layer thickness is small, the diffusion approximation of the radiant flux vector is not applicable, even when the absorption coefficient is large. Hence, one cannot expect to find a value of κ for which the temperature gradient is a minimum. The difference between the temperature gradients for the cool wall and the hot wall is as expected. Because at the cool wall k_r is much smaller than at the hot wall, the temperature gradients for the latter are greater.

The temperature gradients for the case of a favorable pressure gradient, $\beta = \frac{1}{2}$, are plotted in Fig. 9.10. As seen from the curves, the changes in the gradients near the surface are quite small. The total heat flux was calculated at an arbitrary value of the dimensionless similarity variable, $\eta = 0.03$, from equation (9.44) by replacing k with k_{eff}. The heat transfer results are given in Table 9.1.

Table 9.1

Heat Transfer Results for Flow Along a Wedge Expressed in Terms of the Ratio $q''x/\sqrt{N_{Re_x}}$

β	Cool wall			Hot wall		
	$k_r = 0$	$\kappa = 10^{-4}$	$\kappa = 10^{-3}$	$k_r = 0$	$\kappa = 10^{-4}$	$\kappa = 10^{-3}$
0	43.1	39.9	29.3	43.1	43.8	44.7
$\frac{1}{2}$	57.0	52.7	38.3	57.0	56.9	68.1

In order for the Rosseland approximation to be valid, it is necessary for the thermal boundary layer thickness to be at least an order of magnitude greater than the mean free path of radiation. Consequently, the heat flux was not calculated for $\kappa = 10^2 ft^{-1}$ since the mean free path is only 0.01 ft. For a flat plate ($\beta = 0$) and the hot wall, the heat flux for $\kappa = 10^3 ft^{-1}$ exceeds that for pure conduction. This is due to the fact that the increase in the effective conductivity is greater than the decrease in the temperature gradient. In the case of a favorable pressure gradient, $\beta = \frac{1}{2}$, the results are similar in trend, but the heat fluxes are about 30 percent higher.

Pyrex glass is of physical interest because it has been considered[1,5,100] as an ablating material for protection of ballistic missiles reentering the earth's atmosphere. In these studies the energy transfer by thermal radiation has been ignored. Since it is possible to change the absorption coefficient of Pyrex glass by addition of a carbonizing plastic,[1] it is of interest to study the effect of κ on the heat transfer. The values of the physical properties and the dependence of viscosity on temperature were taken from reference 100. The viscosity variation with temperature was approximated by the relation

$$\mu = 0.0672 e^{-\left(\frac{8720}{T-460}\right)^{1.612}} \quad . \tag{9.45}$$

The results are presented in Figs. 9.11 through 9.13. The dimensionless stream function, f, the velocity ratio, f', and the shear function, f", distributions are plotted in Fig. 9.10 as a function of η. The shear function f" is related to the shear stress τ through the expression

$$\tau = \mu \frac{\partial u}{\partial y} = \mu\left(\frac{\rho^*}{\rho}\right) U \sqrt{\frac{m+1}{2} \frac{U}{\nu^* x}}\, f'' \quad . \tag{9.46}$$

Because of a very strong temperature dependence, the shear is a maximum, not at the wall, but at some point away from it. For other values of κ the maximum shear occurs at different values of the similarity variable η. The shear stress at the wall was found to vary little with the absorption coefficient. Looking at Figs. 9.11 and 9.12, we can see that the thermal boundary layer thickness is about five times smaller than the momentum boundary layer thickness.

The Navier-Stokes equations of motion are not applicable for a solid material, and since Pyrex glass possesses no definite melting temperature, 2500°F was chosen as the boundary condition at the cool wall. This value is higher than the working point for this type of glass. The dimensionless temperature, Fig. 9.12, varies monotonically across the boundary layer from the wall value to the value at the edge of the boundary layer. The temperature gradients, Fig. 9.13, at the wall for all values of κ do not differ from one another by more than one percent. For $\kappa = 10^2$ ft^{-1}, the "radiative conductivity" at the wall is of the same order of magnitude as the thermal conductivity. Therefore, with the increase in η the "radiative conductivity" also increases and the temperature gradient has a slightly different trend than for other values of κ. The total heat fluxes for $k_r = 0$, $\kappa = 10^4$ and 10^3 ft^{-1} differ from one another by a few percent only.

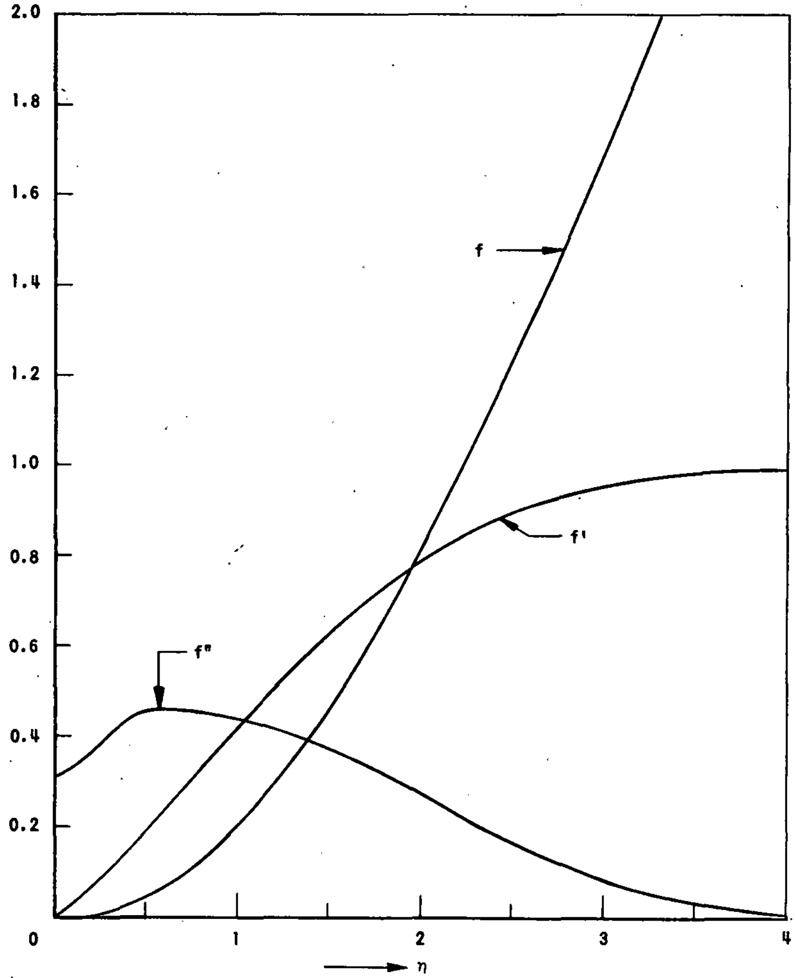

FIG. 9.11
DIMENSIONLESS STREAM FUNCTION, VELOCITY RATIO AND SHEAR FUNCTION VS. SIMILARITY VARIABLE η FOR PYREX GLASS β=0, $\kappa = 10^3$ ft^{-1} AND $T^* = 4460°R$.

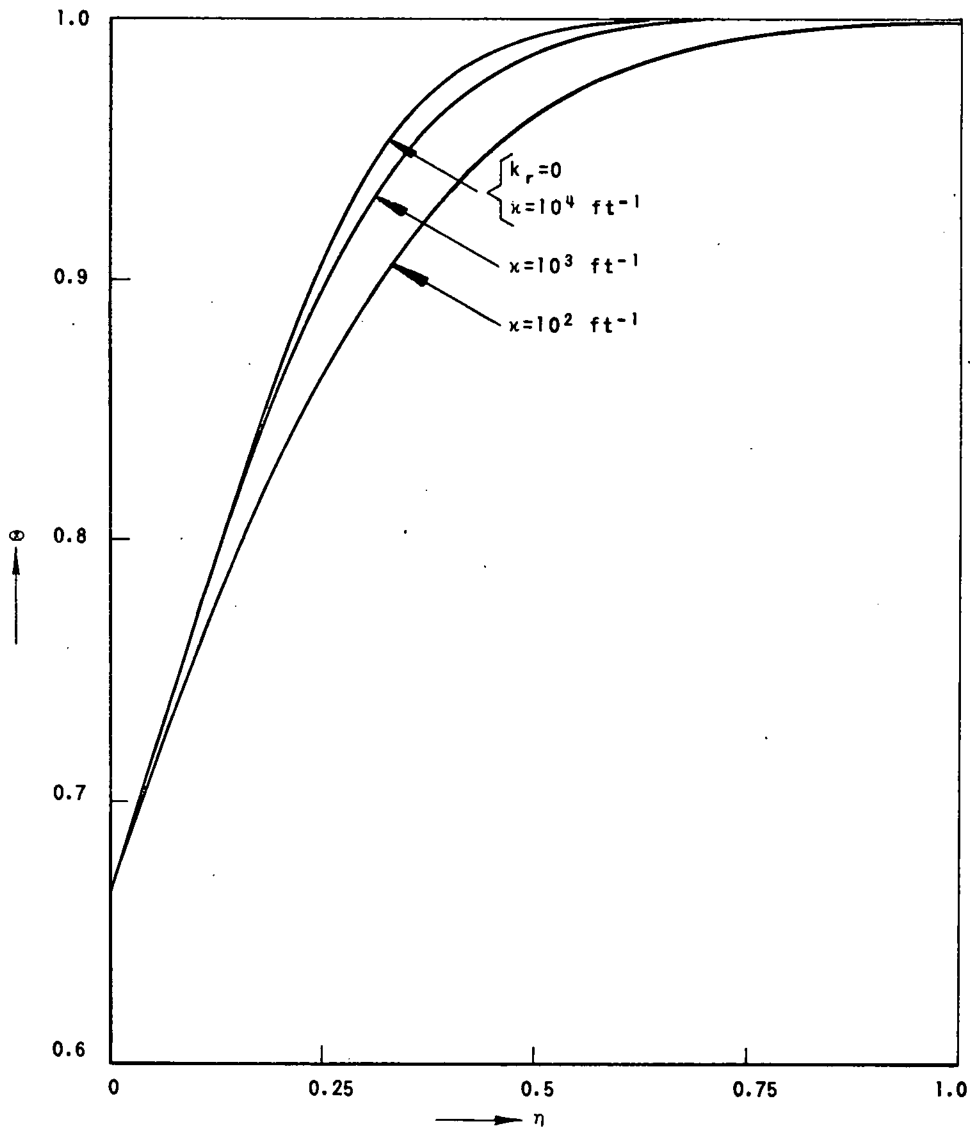

FIG. 9.12
DIMENSIONLESS TEMPERATURE AS A FUNCTION OF SIMILARITY VARIABLE η FOR PYREX GLASS β=0 AND T^*=4460°R.

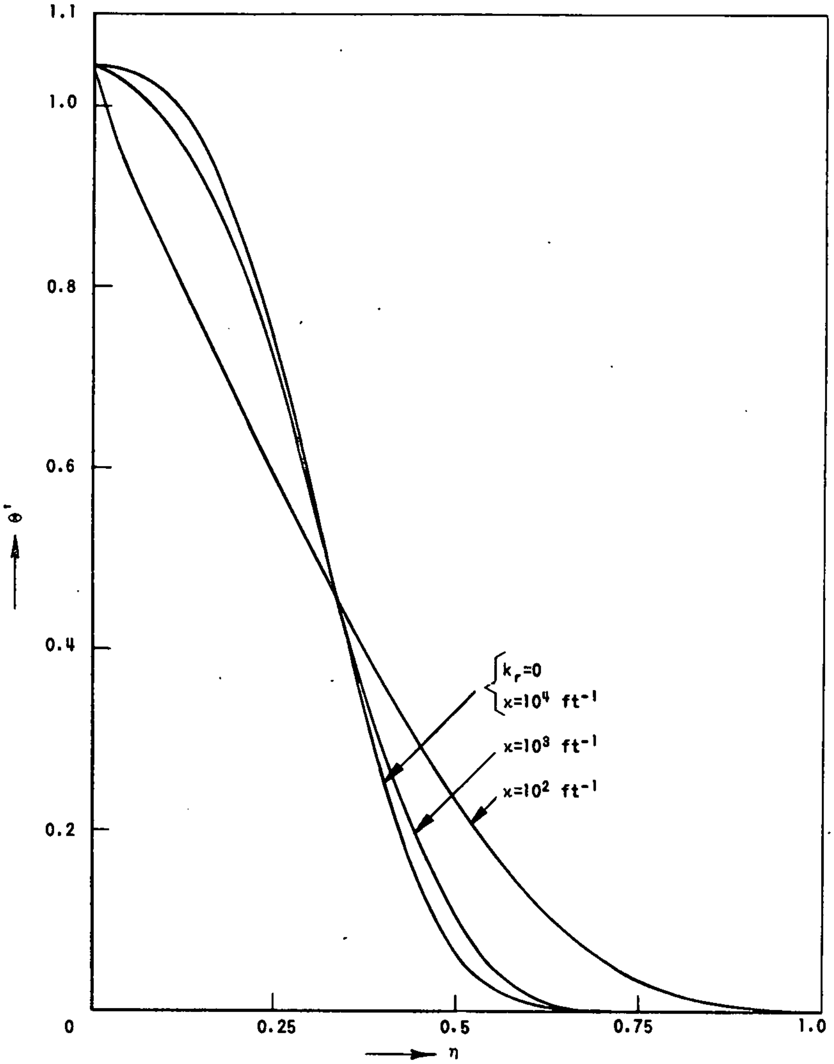

FIG. 9.13
DIMENSIONLESS TEMPERATURE GRADIENT AS A FUNCTION OF SIMILARITY VARIABLE η FOR PYREX GLASS β=0 AND T^*=4460°R.

In boundary layer problems when the thermal boundary layer thickness is small the Rosseland approximation for the radiant flux vector must be used with caution and only in cases in which the mean free path of radiation is much shorter than the thermal boundary layer thickness. When the absorption coefficient is large, the radiative conductivity is quite small, and the effect of radiation on the temperature distribution and the gradients is not appreciable. The validity of the diffusional approximation of radiation is further discussed in Chapter 11.

10 THERMAL RADIATION BETWEEN TWO INFINITE PARALLEL PLATES

10.1 Introduction

With higher temperatures being utilized in various fields of engineering, thermal radiation is becoming more important. Further, the simplified approximate approaches presented in standard heat transfer texts[61,40,87] for calculation of energy transfer and temperature distribution become unsatisfactory. It is therefore of interest to take a more basic approach and attack the problem by starting from the phenomenological equation, the equation of transfer of radiation.

The problem considered here is that of thermal radiation between two parallel planes separated by a finite distance. The problem of this type requires the simultaneous solution of the system of integral equations (5.5) and (5.10). To make the problem tractable by analytical means, it is assumed that the planes are infinite in extent in the directions parallel and normal to the plane of the Fig. 4.3. This assumption simplifies the problem in that the irradiation becomes constant on the surfaces bounding the medium. In addition we assume that the absorption and scattering coefficients are independent of position. The medium under consideration can emit, absorb and scatter thermal radiation. The formulation of the integral equation for incident radiation and net emission from a unit volume as well as derivation of the equation for radiant heat flux are presented. Finally, the integral equation governing the temperature distribution is solved approximately and heat transfer calculated.

To the writer's knowledge, no work dealing with this type of a problem has been reported in the literature. However, some related problems for the special case of slabs of infinite thickness have been dealt with extensively by a great number of astrophysicists and mathematicians some time ago. No effort will be made to give a survey or summary of the literature of these investigations, and only a few pertinent references will be cited.

Schwarzschild(91) derived and solved the equation of transfer for radiative equilibrium for the radiation in the outer atmosphere of the sun. Schuster(90) dealt with the problem of pure scattering. Hilbert(34) and Schwarzschild(91) showed that the boundary value problems (solution of the equation of transfer) of elementary theory of radiation reduce to an integral equation. Hilbert used this in his effort to prove Kirchhoff's laws. A comprehensive exposition to the theory of radiative equilibrium is summarized by Milne.(68) He established integral equations for net emission in the case of an infinite slab. The homogeneous form, (10.7), of this equation has been investigated extensively, and Milne(68) has given approximate solutions to this problem. Its solution, explicit as far as the angular distribution of emerging radiation is concerned, has been obtained by means of Fourier-Laplace integrals by Wiener and Hopf(108) and Hopf.(35) More recently, the solution of the nonhomogeneous form of Milne's first integral equation has been obtained by Busbridge.(9) The Wiener-Hopf technique for solving the homogeneous equation (10.7) has been extended by Case.(11)

The problems arising in neutron transport theory are mathematically identical to those of the tranfer of radiation. A problem of half space $z > 0$ bounded by the plane $z = 0$ filled by a noncapturing medium, which scatters neutrons isotropically without changing their velocity, has been considered by Placzek and Seidel.(77) No sources are present in the medium and no neutrons enter the plane z = 0 from outside. This problem, which represents an important case in the study of neutron transport, is completely identical with the Milne problem. More extensive bibliographies on neutron transport problems are given in the book by Davison[17] and the monograph by Case _et al._[10] The discrete coordinate, the variational, the exact and other methods used to solve the equation of transfer are described in the book by Kourganoff.[52]

10.2 Derivation of Integral Equations

10.2.1 Equation for Incident Radiation

The equation for incident radiation can be obtained from the general equation (5.10). However, this equation will be derived directly by starting from the definition of incident radiation and the intensity distribution. A general method of solving the simultaneous equations in $I_\lambda(x,\mu)$ is to solve the single integral equation in $\mathcal{E}'_\lambda(x)$ by performing the integration of equation (2.22) on the functions $I_\lambda(x,\mu)$ defined by equations (4.17) and (4.18). Thus equation (2.22) may be put in the form

$$\mathcal{E}'_\lambda(x) = 2\pi \int_0^1 I_\lambda(a,\mu)\, e^{-\frac{\beta_\lambda(a-x)}{\mu}}\, d\mu + 2\pi \int_{-1}^0 I_\lambda(-a,\mu)\, e^{-\frac{\beta_\lambda(a+x)}{\mu}}\, d\mu$$

$$+ 2\pi \int_0^1 \int_x^a \epsilon_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}}\, dx' \frac{d\mu}{\mu} - 2\pi \int_{-1}^0 \int_{-a}^x \epsilon_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}}\, dx' \frac{d\mu}{\mu}$$

$$+ \frac{\sigma_\lambda}{2} \int_0^1 \int_x^a \mathcal{E}'_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}}\, dx' \frac{d\mu}{\mu} - \frac{\sigma_\lambda}{2} \int_{-1}^0 \int_{-a}^x \mathcal{E}'_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}}\, dx' \frac{d\mu}{\mu} \qquad (10.1)$$

At this point it is advantageous to define the exponential integral function:

$$E_n(x) = \int_0^1 \mu^{n-2} e^{-\frac{x}{\mu}} d\mu = \int_1^\infty e^{-x\mu} \mu^{-n} d\mu = \int_x^\infty e^{-\mu} \mu^{-n} d\mu \quad . \qquad (10.2)$$

In the French and German literature the symbol $K_n(x)$ instead of $E_n(x)$ is used, but the general tendency now is to use the notation $E_n(x)$, in order to avoid confusion with the Bessel functions. It is hoped that no ambiguity will arise from this choice, since the symbol E is used for the emissive power.

Substituting equation (10.2), interchanging the order of integration and introducing the symmetrical kernel $E_1(\beta_\lambda|x - x'|)$, the integral equation (10.1) can be put in the form

$$\mathcal{E}'_\lambda(x) = f(x) + \frac{\sigma_\lambda}{2} \int_{-a}^{a} E_1(\beta_\lambda|x - x'|)\, \mathcal{E}'_\lambda(x')\, dx' \quad , \qquad (10.3)$$

where

$$f(x) = 2\pi \Big\{ I_\lambda(a)\, E_2\left[\beta_\lambda(a-x)\right] + I_\lambda(-a)\, E_2\left[\beta_\lambda(a+x)\right] + \int_{-a}^{a} E_1\left(\beta_\lambda|x-x'|\right) \epsilon_\lambda(x')dx' \Big\}$$

An interesting special case of the integral equation (10.3) arises when one considers the monochromatic radiation in a infinite medium and neglects the radiation from the surfaces, that is, we let $I_\lambda(-a) = I_\lambda(a) = 0$. For pure scattering the emission coefficient, $\epsilon_\lambda(x)$, is equal to zero. In this special case the integral equation (10.1) simplifies to

$$\mathcal{E}'_\lambda(x) = \frac{\sigma_\lambda}{2} \int_0^1 \left[\int_{-a}^{x} \mathcal{E}'_\lambda(x') e^{-\frac{\sigma_\lambda(x-x')}{\mu}} dx' + \int_x^a \mathcal{E}'_\lambda(x') e^{-\frac{\sigma_\lambda(x'-x)}{\mu}} dx' \right] \frac{d\mu}{\mu} \qquad (10.4)$$

Within the limits of integration, the exponents $\sigma_\lambda\left(\frac{x-x'}{\mu}\right)$ and $\sigma_\lambda\left(\frac{x'-x}{\mu}\right)$ are positive, so that one can write

$$\mathcal{E}'_\lambda(x) = \frac{\sigma_\lambda}{2}\int_{-a}^{a}\mathcal{E}'_\lambda(x)\,dx'\int_0^1 e^{-\frac{\sigma_\lambda|x-x'|}{\mu}}\frac{d\mu}{\mu} \quad . \tag{10.5}$$

Using the definition (10.2) of the exponential integral function, we arrive at Milne's integral equation of the first kind:

$$\mathcal{E}'_\lambda(x) = \frac{\sigma_\lambda}{2}\int_{-a}^{a} E_1\left(\sigma_\lambda|x-x'|\right)\mathcal{E}'_\lambda(x')\,dx' \quad . \tag{10.6}$$

In Milne's problem, $|a| = \infty$. In addition, if one measures the distances in the units of mean free path, $\lambda_p = 1/\sigma_\lambda$, and considers a problem in which the lower limit of integration in equation (10.6) is zero, one obtains

$$\mathcal{E}'_\lambda(x) = \frac{1}{2}\int_0^{\infty} E_1(|x-x'|)\,\mathcal{E}'_\lambda(x')\,dx' \quad . \tag{10.7}$$

This is the form of Milne's integral equation of the first kind which has been studied by many mathematicians and physicists.

10.2.2 The Equation for Net Emission of Radiation

The net emission of radiation from a unit volume is defined by equation (2.30). Thus for our system, the use of equation (10.3) leads to

$$\mathcal{E}_{n,\lambda}(x) = \mathcal{E}_\lambda(x) - \kappa_\lambda\left[f(x) + \frac{\sigma_\lambda}{2}\int_{-a}^{a} E_1\left(\beta_\lambda|x-x'|\right)\mathcal{E}'_\lambda(x')\,dx'\right] \quad . \tag{10.8}$$

Since $\epsilon_\lambda(x) = n^2\kappa_\lambda I_{bb,\lambda}(x)$ and $\mathcal{E}_\lambda(x) = 4\pi n^2\kappa_\lambda I_{bb,\lambda}(x)$ for a medium having a unit refractive index, n = 1, (10.8) becomes

$$\mathcal{E}_{n,\lambda}(x) = 4\kappa_\lambda E_{bb,\lambda}(x) - 2\kappa_\lambda \Big\{ E_\lambda(a)\, E_2\left[\beta_\lambda(a-x)\right] + E_\lambda(-a)\, E_2\left[\beta_\lambda(a+x)\right]$$

$$+ \kappa_\lambda \int_{-a}^{a} E_1\left(\beta_\lambda |x-x'|\right) E_{bb,\lambda}(x')dx' + \frac{\sigma_\lambda}{4} \int_{-a}^{a} E_1\left(\beta_\lambda |x-x'|\right) \mathcal{E}'_\lambda(x')\, dx' \Big\} \quad . \tag{10.9}$$

No difficulty is introduced by considering a medium having refractive index different from unity. It happens that for gases the index is very close to one, i.e., for CO_2, $n = 1.000449$ to 1.000450 at 1 atm pressure and 0°C.

The total net emission from a unit volume per unit of time is obtained by integrating (10.9) over all wavelengths:

$$\mathcal{E}_n(x) = \int_0^\infty \kappa_\lambda \Big\{ 4E_{bb,\lambda}(x) - 2\Big[E_\lambda(a)\, E_2\left[\beta_\lambda(a-x)\right] + E_\lambda(-a)\, E_2\left[\beta_\lambda(a+x)\right]$$

$$+ \kappa_\lambda \int_{-a}^{a} E_1\left(\beta_\lambda |x-x'|\right) E_{bb,\lambda}(x')dx' + \frac{\sigma_\lambda}{2} \int_{-a}^{a} E_1\left(\beta_\lambda |x-x'|\right) \mathcal{E}'_\lambda(x')\, dx' \Big]\Big\} d\lambda \quad . \tag{10.10}$$

10.2.3 The Equation for the Radiant Heat Flux

The monochromatic radiant energy flux perpendicular to the planes at any point in the medium is given with the help of equation (2.31) by

$$q''_{r,\lambda} = \vec{E_\lambda} \cdot \vec{n} = \int_{\Omega=4\pi} I_\lambda(\vec{r}, \vec{\Omega})\, \vec{n} \cdot \vec{\Omega_1} \; d\Omega \quad . \tag{10.11}$$

Since $d\Omega = \sin\theta d\theta d\phi$, the radiant heat flux can be expressed as

$$q''_{r,\lambda} = E^+ - E^- = \int_0^{2\pi} d\phi \int_0^{\pi/2} I_\lambda \cos\theta \sin\theta d\theta + \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} I_\lambda \cos\theta \sin\theta d\theta \quad . \tag{10.12}$$

Since $\mu = \cos\theta$ and $d\mu = -\sin\theta d\theta$, the above relation can also be written as

$$q''_{r,\lambda} = 2\pi\left[\int_0^1 I_\lambda \mu d\mu + \int_{-1}^0 I_\lambda \mu d\mu\right] \quad . \tag{10.13}$$

Equation (10.12) expresses, for an infinitely small surface element of a layer, the net rate of flow of photons across a layer per unit area, per unit wavelength interval, and per unit interval of time. With the proper choice of the direction of the normal to the layers of the medium from which θ is measured (see Fig. 4.3), $q''_{r,\lambda}$ represents the difference between the energy of wavelength λ transmitted to the left across each unit of surface area of a given layer from the layers to the right of it, E^+, and the energy E^- transmitted towards the right across each unit of area of the same layer from the layers to the left of it.

Substituting the appropriate I_λ's from (4.17) and (4.18) in equation (10.13), one gets

$$q''_{r,\lambda} = E^+_\lambda - E^-_\lambda = 2\pi\int_0^1 I_\lambda(a,\mu)\, e^{-\frac{\beta_\lambda(a-x)}{\mu}} \mu\, d\mu + 2\pi\int_{-1}^0 I_\lambda(-a,\mu)\, e^{-\frac{\beta_\lambda(a+x)}{\mu}} \mu\, d\mu$$

$$+ 2\pi\int_0^1 \int_x^a \epsilon_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}} dx'\, d\mu - 2\pi\int_{-1}^0 \int_{-a}^x \epsilon_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}} dx'\, d\mu$$

$$+ \frac{\sigma_\lambda}{2}\int_0^1 \int_x^a \mathcal{E}'_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}} dx'\, d\mu - \frac{\sigma_\lambda}{2}\int_{-1}^0 \int_{-a}^x \mathcal{E}'_\lambda(x')\, e^{-\frac{\beta_\lambda(x'-x)}{\mu}} dx'\, d\mu \ . \tag{10.14}$$

Interchanging the order of integration and introducing the exponential integral function, $E_n(x)$, equation (10.14) becomes

$$q''_{r,\lambda} = E^{+}_{\lambda} - E^{-}_{\lambda} = 2\Big\{E_{\lambda}(a)\, E_3\left[\beta_{\lambda}(a-x)\right] - E_{\lambda}(-a)\, E_3\left[\beta_{\lambda}(a+x)\right]$$

$$+ \kappa_{\lambda}\int_{x}^{a} E_2\left[\beta_{\lambda}(x'-x)\right] E_{bb,\lambda}(x')\, dx' - \kappa_{\lambda}\int_{-a}^{x} E_2\left[\beta_{\lambda}(x-x')\right] E_{bb,\lambda}(x')dx'$$

$$+ \frac{\sigma_{\lambda}}{4}\int_{x}^{a} E_2\left[\beta_{\lambda}(x'-x)\right]\mathcal{E}'_{\lambda}(x')\, dx' - \frac{\sigma_{\lambda}}{4}\int_{a}^{x} E_2\left[\beta_{\lambda}(x-x')\right]\mathcal{E}'_{\lambda}(x')dx'\Big\} \quad (10.15)$$

The total radiant heat flux is obtained by integrating (10.15) over all wavelengths:

$$q''_{r} = E^{+} - E^{-} = 2\int_{0}^{\infty}\Big\{E_{\lambda}(a)\, E_3\left[\beta_{\lambda}(a-x)\right] - E_{\lambda}(-a)\, E_3\left[\beta_{\lambda}(a+x)\right]$$

$$+ \kappa_{\lambda}\int_{x}^{a} E_2\left[\beta_{\lambda}(x'-x)\right] E_{bb,\lambda}(x')dx' - \kappa_{\lambda}\int_{-a}^{x} E_2\left[\beta_{\lambda}(x-x')\right] E_{bb,\lambda}(x')dx'$$

$$+ \frac{\sigma_{\lambda}}{4}\int_{x}^{a} E_2\left[\beta_{\lambda}(x'-x)\right]\mathcal{E}'_{\lambda}(x')\, dx' - \frac{\sigma_{\lambda}}{4}\int_{-a}^{x} E_2\left[\beta_{\lambda}(x-x')\right]\mathcal{E}'_{\lambda}(x')dx'\Big\}\, d\lambda \quad . \quad (10.16)$$

10.3 Methods of Solution of Fredholm Integral E uations of the Second Kind

The solution of the linear Fredholm integral equation of the second kind with parameter λ

$$\phi(x) = f(x) + \lambda\int_{a}^{b} K(x,\xi)\,\phi(\xi)\, d\xi \quad , \quad (10.17)$$

where a and b are constants, can be obtained by five different methods:(89)

(1) The first method, that of successive substitutions, due to Neumann and Liouville gives the unknown function, $\phi(x)$, as an integral series in λ.

(2) The second method, due to Fredholm, gives the unknown function, $\phi(x)$, as a ratio of two integral series in λ, each series has an infinite radius of convergence.

(3) The third method developed by Hilbert and Schmidt gives the unknown function, $\phi(x)$, in terms of a set of fundamental functions.

(4) The fourth method is that of integral transforms: Fourier, Laplace, Mellin, and others. This method is sometimes advantageous because it can give a closed form solution expressed as contour integrals. However, when the kernel $K(x, \xi)$ or $f(x)$ are complicated functions, a series solution might be simpler.

(5) Integral equation (10.17) can also be studied either by iterative methods or by variational methods. Both are extremely flexible. The weak point about the variational methods is that they can only give approximate solutions, though the order of approximation can be extremely high.

The solution of equation (10.17) in terms of Neumann integral series is given by Lovitt[60] if the following are satisfied.

(1) The kernel $K(x,\xi)$ is real and continuous in the region R, for which $a \leq x \leq b$, $a \leq \xi \leq b$; $|K(x, \xi)| \leq M$, where M is the upper bound and $K(x, \xi) \not\equiv 0$.

(2) The function $f(x) \not\equiv 0$, is real and continous in I; $a \leq x \leq b$.

(3) The absolute value of the constant parameter λ is

$$|\lambda| \leq 1/M|b-a| \quad .$$

The equation (10.17) has then one and only one continuous solution given by absolutely convergent integral series:

$$\phi(x) = f(x) + \lambda \int_a^b K(x,\xi)\, f(\xi)\, d\xi$$

$$+ \sum_{n=2}^{\infty} \lambda^n \int_a^b K(x,\xi_1) \int_a^b K(\xi_1, \xi_2) \cdots \int_a^b K(\xi_{n-1},\xi_n)\, f(\xi_n)\, d\xi_n \cdots d\xi_1 .$$

10.4 Solution of the Equation for Incident Radiation

Very little data are available on such radiative properties as absorption and scattering coefficients. In particular, the dependence of these properties on wavelength, temperature, and pressure has been investigated only in certain wavelength regions only for low pressures and temperatures. Since the data are so scanty, to make the problem more tractable analytically, the medium is considered to be grey. In the rest of this chapter the surfaces shall be considered to be separated by a distance a, i.e., $0 \leq x \leq a$. In addition, measuring the distances in the units of the mean free path, $\lambda_p = 1/\beta$, or introducing the optical thickness, $\tau = \beta x$, as the independent variable, equation(10.3) reduces to

$$\mathcal{E}'(\tau) = f(\tau) + \frac{\sigma}{2\beta} \int_0^{\tau_0} E_1(|\tau-\tau'|)\, \mathcal{E}'(\tau')\, d\tau', \quad 0 \leq \tau \leq \tau_0, \quad 0 \leq \frac{\sigma}{\beta} \leq 1, \qquad (10.19)$$

where

$$f(\tau) = 2\left[E(0)\, E_2(\tau) + E(\tau_0)\, E_2(\tau_0-\tau) + \frac{\kappa}{\beta} \int_0^{\tau_0} E_1(|\tau-\tau'|)\, E_{bb}(\tau')\, d\tau'\right]$$

and

$$\tau_0 = \beta a \quad .$$

Because of the properties of $E_1(|\tau-\tau'|)$, the general solution of equation (10.19), which is a Fredholm integral equation of the second kind, can be written in the form of an infinite Neumann series:

$$\mathcal{E}'(\tau) = f(\tau) + \left(\frac{\sigma}{2\beta}\right) \int_0^{\tau_0} E_1(|\tau-\xi|)\, f(\xi)\, d\xi$$

$$+ \left(\frac{\sigma}{2\beta}\right)^2 \int_0^{\tau_0} E_1(|\tau-\xi_1|) \int_0^{\tau_0} E_1(|\xi_1-\xi_2|)\, f(\xi_2)\, d\xi_2\, d\xi_1$$

$$+ \left(\frac{\sigma}{2\beta}\right)^3 \int_0^{\tau_0} E_1(|\tau-\xi_1|) \int_0^{\tau_0} E_1(|\xi_1-\xi_2|) \int_0^{\tau_0} E_1(|\xi_2-\xi_3|)\, f(\xi_3)\, d\xi_3\, d\xi_2\, d\xi_1 + \dots \,. \qquad (10.20)$$

The logarithmic infinity of the kernel function at zero argument $(|\tau-\xi|) = 0$, causes no difficulty, since

$$\frac{1}{2}\int_0^{\tau_0} E_1(|\tau-\xi|)\, d\xi \leq 1 \quad ,$$

for $\tau_0 \leq \infty$, and therefore the convergence of the series may be easily demonstrated(35) for either $\sigma/\beta < 1$ or $\tau_0 < \infty$. The limiting case of $\sigma/\beta = 1$ and $\tau_0 = \infty$ has been settled by Hopf.(35) One can further show that no other bounded continuous solutions exist. However, for a medium in which the absorption coefficient is much smaller than the scattering coefficient, $\sigma/\beta \approx 1$, the convergence of the series (10.20) is very slow and a large number of terms would have to be included. The reason that this happens is that the kernel $E_1\,(|x-\xi|)$ is not a well-behaved function. In the other extreme case in which the scattering coefficient is much smaller than the absorption coefficient, $\sigma/\beta \approx 0$, the equation (10.20) reduces to $\mathcal{E}'(\tau) = f(\tau)$.

Before the incident radiation can be calculated from equation (10.20), the temperature distribution in the medium must be known. In most problems this information is not available. The temperature distribution (or the black body emissive power) can be obtained from equation (10.10), which, if rewritten and rearranged, takes the form

$$E_{bb}(\tau) = g(\tau) + \frac{\kappa}{2\beta}\int_0^{\tau_0} E_1(|\tau-\tau'|)\, E_{bb}(\tau')\, d\tau' \quad , \qquad (10.21)$$

where

$$g(\tau) = \frac{1}{2}\left[\frac{\mathcal{E}_n(\tau)}{2\kappa} + E(0)\,E_2(\tau) + E(\tau_0)\,E_2(\tau_0-\tau) + \frac{\sigma}{4\beta}\int_0^{\tau_0} E_1(|\tau-\tau'|)\,\mathcal{E}'(\tau')d\tau'\right].$$

For steady state, the net flow of radiation through each volume element vanishes, that is, the radiant energy absorbed per unit time by the volume ΔV is equal to the radiant energy emitted per unit time by the same volume; thus for the net emission, $\mathcal{E}_n = 0$. This is defined as radiative equilibrium. However, this does not mean that energy is not transferred from one surface to another by radiation. It should be noted that in the general case in which other modes of energy transfer are present, the net emission from a unit volume, $\mathcal{E}_n$, could be negative as well as positive. Even for this simple one-dimensional problem, the simultaneous solution of the integral equations (10.19) and (10.21) is not easy.

10.5 Equations for Heat Flux in the Medium

If the incident radiation and temperature are known, the heat flux at any point in the medium can be given directly by equation (10.16):

$$q_r'' = E^+ - E^- = 2\Big[E(\tau_0)\,E_3(\tau_0-\tau) - E(0)\,E_3(\tau) + \frac{\kappa}{\beta}\int_{\tau}^{\tau_0} E_2(\tau'-\tau)\,E_{bb}(\tau')\,d\tau' - \frac{\kappa}{\beta}\int_0^{\tau} E_2(\tau-\tau')\,E_{bb}(\tau')\,d\tau' + \frac{\sigma}{4\beta}\int_{\tau}^{\tau_0} E_2(\tau'-\tau)\,\mathcal{E}'(\tau')\,d\tau' - \frac{\sigma}{4\beta}\int_0^{\tau} E_2(\tau-\tau')\,\mathcal{E}'(\tau')\,d\tau'\Big] \quad . \tag{10.22}$$

Equation (10.22) expresses the conservation of radiant energy flux, that is, for the system in consideration the flux is constant: $dq_r''/d\tau = 0$.

At the plane $\tau = 0$ $(x = 0)$, the heat flux can be written as

$$q_r''\Big|_{\tau=0} = 2\Big[E(\tau_0)\,E_3(\tau_0) - \frac{1}{2}E(0) + \frac{\kappa}{\beta}\int_0^{\tau_0} E_2(\tau')\,E_{bb}(\tau')\,d\tau' + \frac{\sigma}{4\beta}\int_0^{\tau_0} E_2(\tau')\,\mathcal{E}'(\tau')\,d\tau'\Big] \quad , \tag{10.23}$$

and similarly the heat flux at the plane $\tau = \tau_0$ $(x = a)$ is given by

$$q''_r\Big|_{\tau=\tau_0} = 2\left[\frac{1}{2}E(\tau_0) - E(0)\,E_3(\tau_0) - \frac{\kappa}{\beta}\int_0^{\tau_0} E_2(\tau_0-\tau')\,E_{bb}(\tau')\,d\tau'\right.$$

$$\left. - \frac{\sigma}{4\beta}\int_0^{\tau_0} E_2(\tau_0-\tau')\,\mathcal{E}'(\tau')\,d\tau'\right] \quad . \qquad (10.24)$$

Since the radiant energy flux is constant, equations (10.23) and (10.24) yield the same result for the heat flux.

10.6 Temperature Distribution for Radiative Equilibrium

The problem of radiative equilibrium has been investigated by Milne,[68] Hopf[35] and others. The equation solved by these investigators is of the same form as equation (10.7). The solution of Milne's integral equation of the first kind becomes extremely complex. Moreover, the solutions obtained for the integral equation are usually expressed as contour integrals. These are put in a tractable form for numerical calculation only after transformations.

In order to obtain the temperature distribution in the radiating medium when scattering is present, equation (10.21) must be solved simultaneously with equation (10.19). For radiative equilibrium, $\mathcal{E}_n = 0$, and hence equations (10.19) and (10.21) reduce to

$$E_{bb}(\tau) = \frac{1}{2}\left[E(0)\,E_2(\tau) + E(\tau_0)\,E_2(\tau_0-\tau) + \int_0^{\tau_0} E_1(|\tau-\tau'|)\,E_{bb}(\tau')\,d\tau'\right] \quad . \qquad (10.25)$$

Since $E_{bb}(\tau) = \sigma T^4(\tau)$, the solution of (10.25) yields the temperature distribution in the medium.

To the author's knowledge, no analytical solutions have been obtained for this type of an integral equation. Integral equations of the convolution or the Faltung type [equation (10.25) is of this type] were first studied by Doetsch[18] and Fock[25] with the help of Laplace transforms. However, for the problem of consideration here it is very difficult to obtain the transforms of the right-hand side of equation (10.25). Because of this and the inversion difficulties, the Laplace transform method of solving the integral equation had to be abandoned. Since an exact solution cannot be obtained, an effort was made to arrive at an approximate solution. Of the several methods available for solving equation (10.25), the method of successive substitutions and the variational method are the most attractive. The variational method as suggested by Sparrow[96] is more involved, since a double integral of the type

$$\int_0^{\tau_0} \int_0^{\tau_0} E_1(|\tau-\tau'|)\, E_{bb}(\tau')\, E_{bb}(\tau)\, d\tau'\, d\tau$$

has to be evaluated, and the complexity of integration increases greatly with the number of terms approximating the black body emissive power $E_{bb}(\tau)$. We therefore turn to the method of successive substitutions with undetermined parameters, which is similar to the variational method but less elegant mathematically.

When the unknown function, such as $E_{bb}(\tau)$, can be expressed by equation (10.25), in which the unknown function appears explicitly on the left-hand side of the equation and as part of the integrand on the right, it is usually possible to find the unknown function by means of successive approximations resulting from a sequence of iterations. On starting from a reasonable first approximation, a convergent sequence of functions is obtained, each being found by substituting its predecessor in place of the unknown function on the right-hand side and evaluating the resulting integral. The limit of which the sequence converges is the rigorous solution of the equation.

Various iterative methods for solving Milne's first integral equation have also been presented by Kourganoff.[52] The iteration of functions containing parameters, as discussed in reference 52, will be followed in this work. The integral equation (10.25) can be written as

$$E_{bb,j}(\tau) = \frac{1}{2}\left[E(0)\,E_2(\tau) + E(\tau_0)\,E_2(\tau_0-\tau) + \int_0^{\tau_0} E_1(|\tau-\tau'|)\,E_{bb,j-1}(\tau')\,d\tau'\right] \quad . \tag{10.26}$$

If a function $E_{bb,1}(\tau)$ is assumed and inserted under the integral sign on the right side of equation (10.26), a function $E_{bb,2}(\tau)$ is produced on the left. If the function $E_{bb,2}(\tau)$ is inserted under the integral sign of (10.26) and the procedure repeated, a sequence of functions $E_{bb,1}(\tau)$, $E_{bb,2}(\tau) \ldots E_{bb,j}(\tau)$ is obtained. This sequence converges, as $j \rightarrow \infty$, to the rigorous solution of equation (10.25).

As a first approximation to $E_{bb}(\tau)$, we can assume a function such as

$$E_{bb}(\tau) = c_0 + c_1\tau + c_2\,E_2(\tau_0-\tau) + c_3\,E_3(\tau) + c_4\,E_3(\tau_0-\tau) \quad . \tag{10.27}$$

This form is suggested by the fact that if the method of successive substitution is applied to equation (10.26), beginning with

$$E_{bb,1}(\tau) = E(0) + \frac{\tau}{\tau_0}\left[E(\tau_0) - E(0)\right] \quad ,$$

the next approximation is

$$E_{bb,2}(\tau) = E(0) - \left[E(\tau_0) - E(0)\right]\left[\frac{\tau}{\tau_0} + \frac{1}{2}\,E_2(\tau_0-\tau) + \frac{1}{2}E_3(\tau) - \frac{1}{2\tau_0}\,E_3(\tau_0-\tau)\right] \quad .$$

However, the integrals of the form

$$\int_0^{\tau_0} E_1(|\tau-\tau'|)\,E_n(\tau_0-\tau')\,d\tau' \quad .$$

when τ_0 is finite are very difficult to evaluate, and we are forced to abandon this approximation for $E_{bb}(\tau)$. Various other functions suggest themselves, but none appears quite as simple as the polynomial. It should be pointed out that the polynomial approximation does not seem to be well adapted to the problem we are considering in that it is not orthogonal. However, its simplicity outweighs this disadvantage. One other advantage is that integrals of the type

$$\int_0^{\tau_0} E_1(|\tau-\tau'|)\, \tau'^n \, d\tau'$$

yield exponential integral functions which are already computed and tabulated, i.e., in references 10 and 52. So, as an approximation to $E_{bb}(\tau)$, we assume

$$E_{bb}(\tau) = E(0) \sum_{n=0}^{n} D_n \tau^n \quad . \tag{10.28}$$

The parameters $D_0, D_1, \ldots D_n$ are to be determined. With two parameters at one's disposal, it is clearly impossible to approximate the exact function in the whole interval $0 \leq \tau \leq \tau_0$. It is natural, therefore, to suppose that better solutions would be obtained if a larger number of terms were retained in (10.28). The conservation of radiant flux, as expressed by equation (10.22), at optical depths $\tau_1, \tau_2 \ldots \tau_n$ gives us enough conditions to determine any desired number of the constants. As $n \rightarrow \infty$, the approximation (10.28) converges to the rigorous solution of equation (10.25).

The integrations necessary for the evaluation of the undetermined parameters are given in Appendix A. The black body emissive power is assumed to be approximated by five terms of the series (10.28). The solution of the system of five simultaneous algebraic equations (A.7) through (A.9) is straightforward.

10.7 Discussion of the Results

The state of radiative equilibrium, when the medium is stratified in parallel layers, is characterized by a certain net radiant energy flux, and at the same time the net emission at any point from the medium vanishes. From the definition of the net emission, equation (2.30), we have that

$$\mathcal{E}_n = 0 = \kappa \int_{\Omega=4\pi} (I_{bb}-I)\, d\,\Omega \quad ,$$

or

$$E_{bb} = \frac{1}{4} \int_{\Omega=4\pi} I\, d\,\Omega = \frac{\pi}{2} \int_{-1}^{1} I\, d\,\mu \quad , \tag{10.29}$$

where the intensity I is obtained from the solution of the equation of transfer. The composition of $E_{bb}(\tau)$ is shown in Fig. 10.1, and it is self-explanatory.

The physics of the problem considered here is well understood, and the mathematics of the formulation is straightforward; nevertheless, the solution is most difficult. The unfortunate fact is that the kernel $E_1(|\tau-\tau'|)$ is undefined at the exact point of interest. This singularity is not physical, but only mathematical.

Integral equation (10.25) has very simple solutions for two special cases. For a diathermal medium, $\beta = 0$, the solution of equation (10.25) becomes

$$E_{bb}(x) = \frac{E(0) + E(a)}{2} \quad . \tag{10.30}$$

For very large optical thicknesses, $\tau_0 \rightarrow \infty$, the solution of equation (10.25) is given by

$$E_{bb}(\tau) = E(0) + \frac{\tau}{\tau_0}\left[E(\tau_0) - E(0)\right] \quad . \tag{10.31}$$

This can be readily verified by substituting (10.31) into the integral equation (10.25). The value of this exact solution for large optical thicknesses lies in the fact that it tells us the asymptotic form of $E_{bb}(\tau)$ for large values of τ_0 when the proximity of the boundary is taken into account. A trivial solution of the integral equation (10.25) can also be mentioned. When $E(0) = E(\tau_0)$, that is, if the two surfaces are at the same temperature, the temperature at any point in the

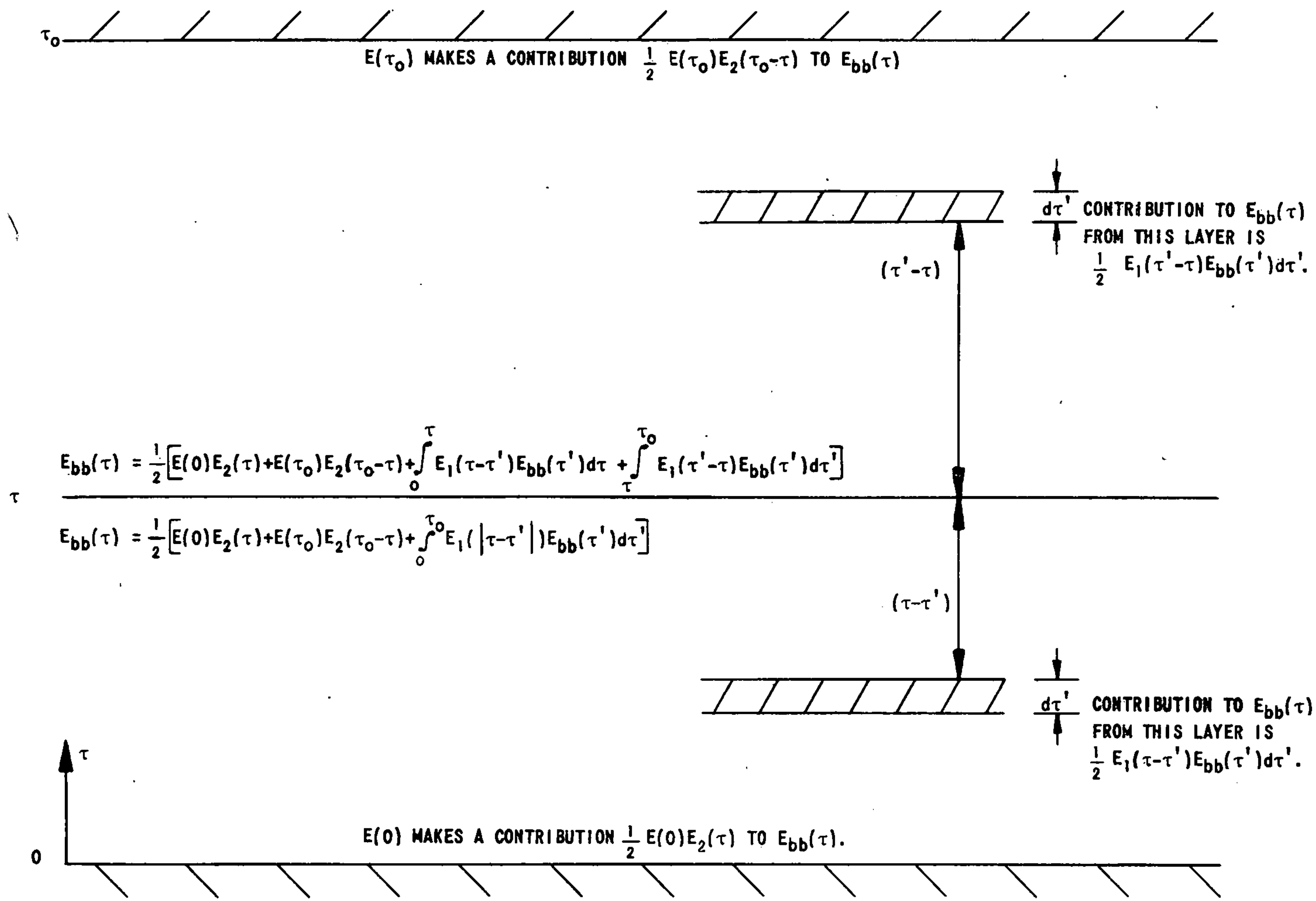

FIG. 10.1
THE COMPOSITION OF $E_{bb}(\tau)$.

medium between the two plates is constant. This is the condition of thermal equilibrium; thus we have not only radiative but thermal equilibrium as well.

The results of calculations are presented in Figs. 10.2 through 10.4 for several values of $E(\tau_0)/E(0)$. It is evident that the black body emissive power distributions are practically straight lines. The biggest departure from a straight line occurs at larger ratios of $E(\tau_0)/E(0)$ when the optical thickness is equal to one (see Fig. 10.3). The temperature distribution can readily be calculated from the relation $T(\tau) = \left[E_{bb}(\tau)/\sigma\right]^{\frac{1}{4}}$.

It is seen from the figures that in all cases a temperature step exists at the radiating surfaces, the magnitude of which depends on the optical thickness of the medium and on the values of $E(\tau_0)$ and $E(0)$ (the amount of radiant energy transmitted). The temperature step is small at large optical thicknesses ($\tau_0 = 10$), and one can readily see from equation (10.31) that no temperature step exists for $\tau_0 \rightarrow \infty$. Further, the step decreases with the decrease in the ratio $E(\tau_0)/E(0)$, and in the limit when $E(\tau_0)/E(0) \rightarrow 1$, the step vanishes.

The fact that a temperature step exists at the radiating surfaces is a little hard to understand physically. To shed more light on this point, we consider an example. Take the case in which $E(\tau_0)/E(0) = 10$ and $\tau_0 = 0.1$. Since τ_0 is small, the contribution to $E_{bb}(\tau)$ will be due mainly to $E(0)$ and $E(\tau_0)$. Thus, neglecting the emission from the medium, at $\tau = 0$ we have that

$$E_{bb}(0) = \frac{1}{2}[E(0) + E(\tau_0)\, E_2(\tau_0)] = \frac{1}{2}E(0)[1 + (10)(0.722545)] = 4.112725E(0),$$

a value which is considerably higher than what we postulated originally. On the other hand, for $E_{bb}(\tau)$ at $\tau = \tau_0$, we have

$$E_{bb}(\tau_0) = \frac{1}{2}[E(0)\, E_2(\tau_0) + E(\tau_0)] = \frac{1}{2}E(0)\,[0.722545 + 10] = 5.361275\, E(0),$$

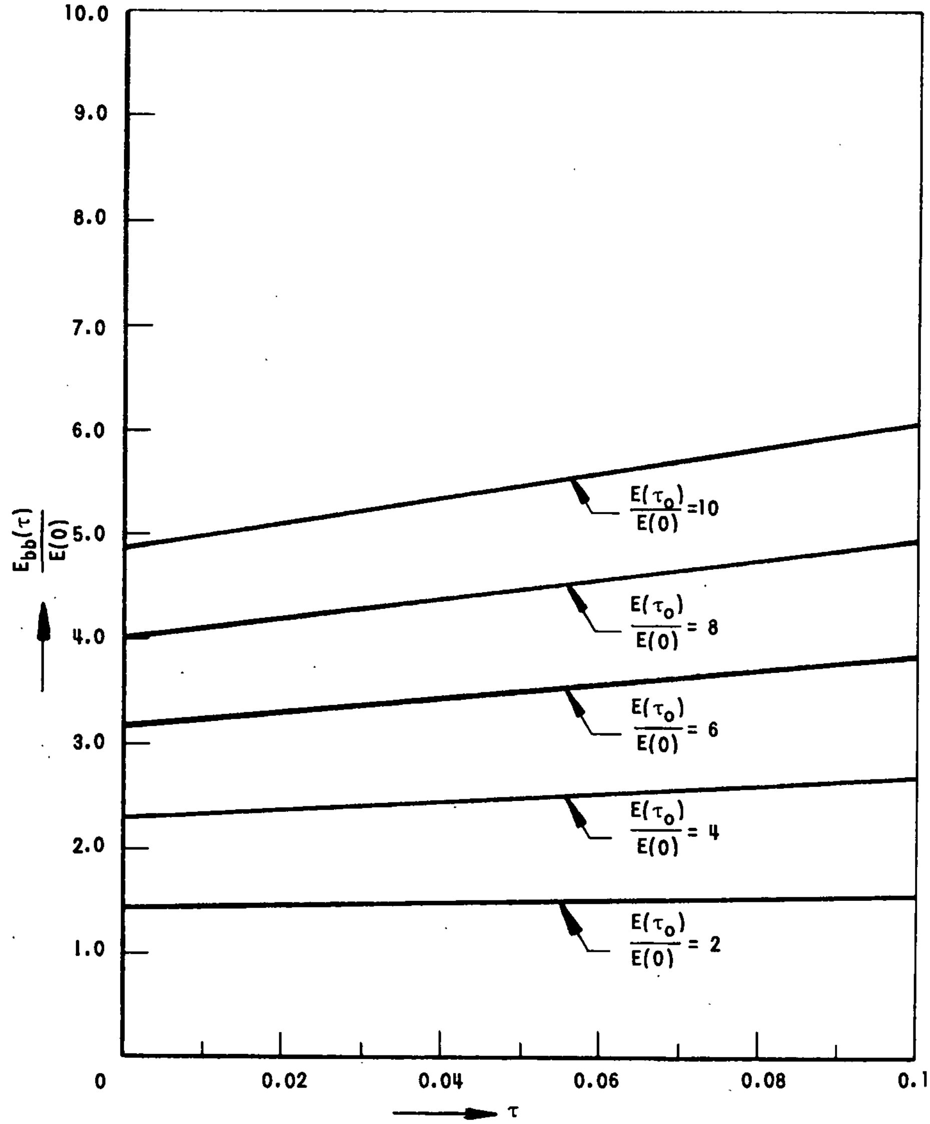

FIG. 10.2

VARIATION OF THE BLACK BODY EMISSIVE POWER WITH THE OPTICAL THICKNESS FOR τ_o=0.1.

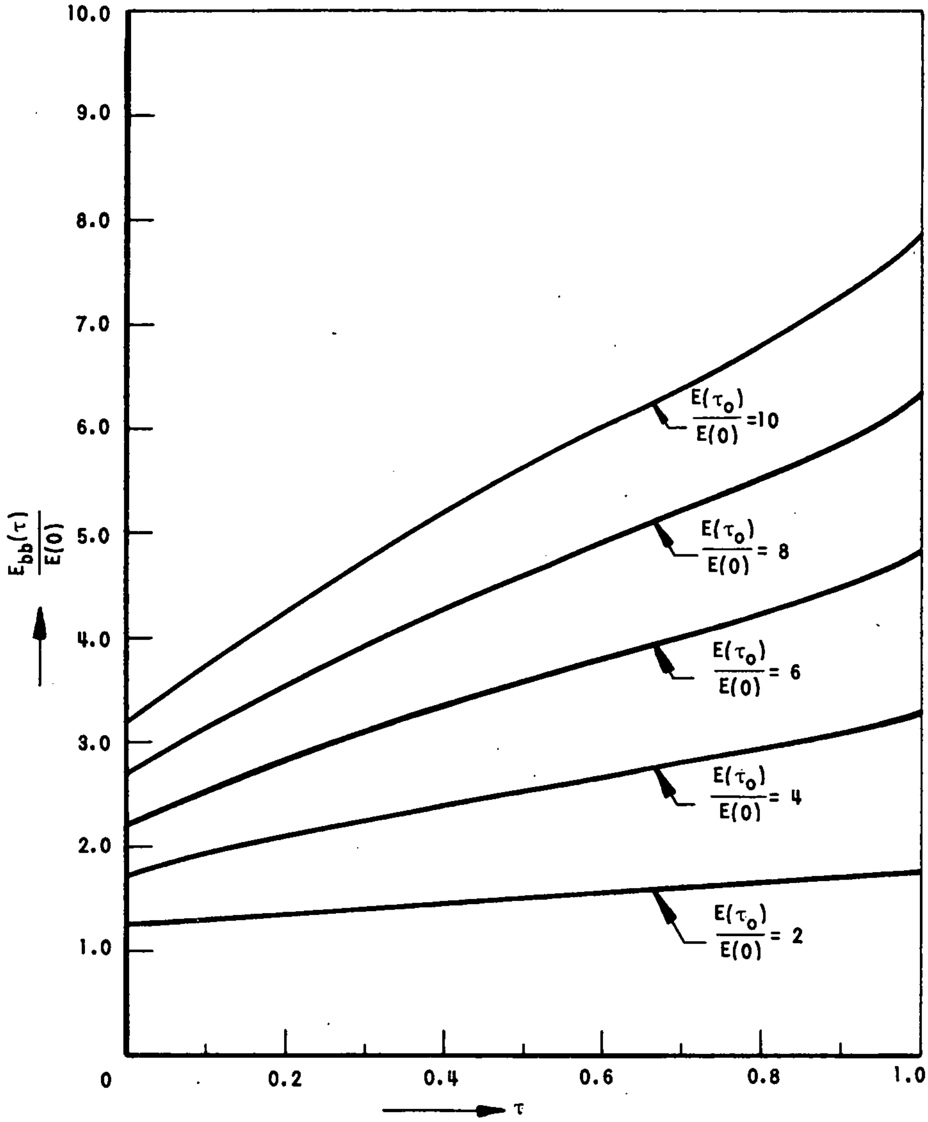

FIG. 10.3
VARIATION OF THE BLACK BODY EMISSIVE POWER WITH THE OPTICAL THICKNESS FOR τ_0=1.0.

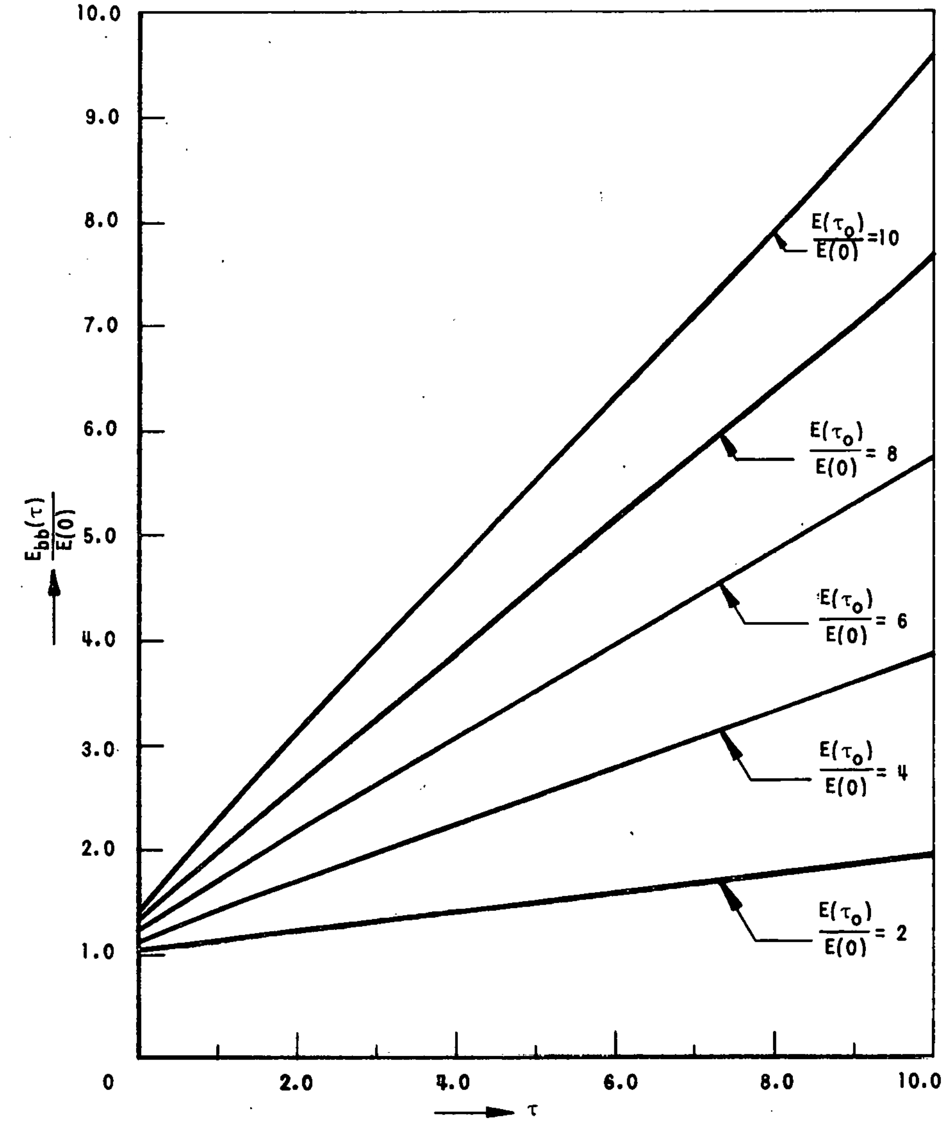

FIG. 10.4

VARIATION OF THE BLACK BODY EMISSIVE POWER WITH THE OPTICAL THICKNESS FOR τ_0=10.0.

and thus a very sharp decrease in the temperature due to the emission of radiation from the surface at $\tau = \tau_0$. A step increase in temperature at the radiating surface was also predicted by Shorin(93) for a moving radiating medium in the absence of energy transfer by conduction.

The radiant energy fluxes computed from equation (10.24) are given in Table 10.1. For instance, if the media were diathermal, the heat transfer for $E(\tau_0)/E(0) = 2$ would be $q_r''/E(0) = 1$, instead of 0.9172, 0.5281 and 0.1098 for $\tau_0 = 0.1$, 1.0 and 10.0, respectively. Thus, the heat transfer is reduced considerably by the presence of an absorbing and scattering medium. The absorption and scattering coefficients enter then into the problem only through the optical thickness, which is the pertinent variable.

Table 10.1

The Calculated Values of the Normalized Heat Flux, $q_r''/E(0)$

τ_0 \ $E(\tau_0)/E(0)$	2	4	6	8	10
0.1	0.9172	2.751	4.588	6.421	8.240
1.0	0.5281	1.586	2.642	3.682	5.023
10.0	0.1098	0.3270	0.5471	0.7820	0.9847

To the author's knowledge, there are neither numerical nor analytical solutions of the integral equation of the type (10.25). The solutions obtained in this work are probably unique; however, the validity and accuracy of the results obtained by using only five terms in the equation approximating $E_{bb}(\tau)$ could not be checked independently. The difference between results obtained by using a smaller number of terms in equation (10.28) can be readily estimated. Thus, using only two terms from equations (A.7) and (A.8), we obtain

$$E_{bb}(\tau) = D_0 + D_1\tau \quad , \tag{10.32}$$

where

$$D_0 = \frac{E(0)\left\{\frac{1}{2} + \tau_0 + E_2(\tau_0)\left[\frac{1}{2} + \tau_0 E_2(\tau_0)\right] - E_3(\tau_0)[1+E_2(\tau_0)]\right\} + E(\tau_0)[1+E_2(\tau_0)]\left[\frac{1}{2} - E_3(\tau_0)\right]}{[1 + E_2(\tau_0)][1 + \tau_0 - 2E_3(\tau_0) - \tau_0 E_2(\tau_0)]}$$

and

$$D_1 = \frac{[E(\tau_0) - E(0)][1 - E_2^2(\tau_0)]}{[1 + E_2(\tau_0)][1 + \tau_0 - 2E_3(\tau_0) - \tau_0 E_2(\tau_0)]} .$$

For the range of parameters considered, the results obtained from equation (10.32) differ only by a maximum of ± 3 per cent from those with five terms in equation (10.28). In view of this fact, the results are believed to be accurate to ± 0.5 per cent.

11 HEAT TRANSFER BY SIMULTANEOUS CONDUCTION AND RADIATION

11.1 Introduction

Energy exchange between an absorbing medium and the walls of the duct takes place by conduction, convection and radiation. However, as higher temperatures are reached, the radiant energy contribution tends to become a larger percentage of the total heat transport, and the temperature distribution cannot be calculated by neglecting the radiative energy transfer in the energy equation.

The problem is further complicated by the fact that the absorption coefficients of such common gases as CO_2 and H_2O are quite distinct, and, although the absorption spectra of these and other gases have been studied for a long time, data on the frequency distribution of absorption coefficients are incomplete and the determination of integral absorption is complex.

Then, too, the equation of energy (6.11) is most intractable in these cases. In fact, no general solutions are known. Therefore, as the first step in the analysis of this general problem, the transport of energy by simultaneous conduction and radiation is studied in this chapter. The geometrical complexity is avoided by considering a one-dimensional system. However, before we proceed to this problem we will briefly consider the more general equation of energy for flow of radiating medium between two parallel plates (see Fig. 11.1).

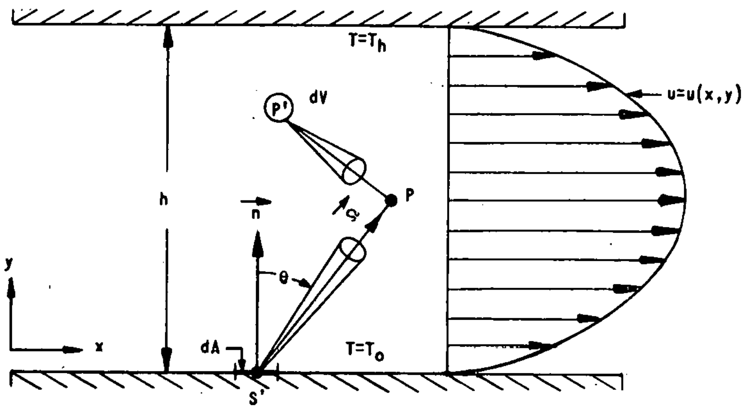

FIG. 11.1
PHYSICAL MODEL AND COORDINATE SYSTEM FOR FLOW BETWEEN TWO PARALLEL PLATES.

Now the general energy equation (6.11) is simplified if the following assumptions are made:

1. The flow is steady and in the x-direction only.
2. The physical properties are independent of temperature.
3. The viscous dissipation of energy is negligible.
4. There are no body forces and the energy generation due to pressure gradients is negligible.
5. The surfaces are diffuse and the radiosity on a surface is constant.
6. The index of refraction of the medium is unity.
7. The scattering is negligible compared to the absorption

and becomes

$$\rho c_p u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + q''' - \mathcal{E}_n \,, \qquad (11.1)$$

where

$$\mathcal{E}_n = \int_0^\infty \kappa_\lambda(P) \left[4E_{bb,\lambda}(P) - \int_{A_1, A_2} R_\lambda(S') \frac{e^{-\tau_\lambda(S', P)} \cos\theta \, dA}{\pi \left| \vec{r}_P - \vec{r}_{S'} \right|^2} \right.$$

$$\left. - \int_V \kappa_\lambda(P') E_{bb,\lambda}(P') \frac{e^{-\tau_\lambda(P', P)} \, dV}{\pi \left| \vec{r}_P - \vec{r}_{P'} \right|^2} \right] d\lambda \quad ,$$

and if the absorption coefficient is independent of wavelength the net emission can be written as

$$\mathcal{E}_n = \kappa(P) \left[4E_{bb}(P) - \int_{A_1, A_2} R(S') \frac{e^{-\tau(S', P)} \cos\theta \, dA}{\pi \left| \vec{r}_P - \vec{r}_{S'} \right|^2} \right.$$

$$\left. - \int_V \kappa(P') \, E_{bb}(P') \frac{e^{-\tau(P', P)} \, dV}{\pi \left| \vec{r}_P - \vec{r}_{P'} \right|^2} \right] .$$

The following simple boundary conditions are postulated:

$$\left. \begin{array}{ll} x = 0, & T = T_i \\ y = 0, & T = T_0 \\ y = h, & T = T_h \end{array} \right\} . \qquad (11.2)$$

A solution of the nonlinear integro-differential equation (11.1) is very difficult to obtain. Therefore, first, an attempt will be made to take advantage of the knowledge built into the integro-differential equation and the boundary conditions to obtain information without attempting to solve the equation and, second, to restrict ourselves to the solution of one specific problem.

11.2 Analysis of the Dimensionless Energy Equation

The differential method of dimensional analysis as developed rigorously by Kline[47] will be used. Therefore, we non-dimensionalize the dependent and independent variables by using

$$\zeta = \frac{x}{\ell}; \qquad \xi = \frac{y}{h}; \qquad \Theta = \frac{T}{T^*};$$

where ℓ and h are the longest dimensions in the x and y directions, respectively, and T^* is an arbitrary temperature. It is to be noted that the above definitions of the dimensionless variables are quite arbitrary. However, these definitions make the dependent nondimensional variables of the order of magnitude of unity, and the independent variables run from zero to the order of unity over the range of integration.

Then

$$\frac{\partial T}{\partial x} = \frac{\partial \Theta}{\partial \zeta}\frac{\partial \zeta}{\partial x} = \frac{T^*}{\ell}\frac{\partial \Theta}{\partial \zeta}; \quad \frac{\partial^2 T}{\partial x^2} = \frac{T^*}{\ell^2}\frac{\partial^2 \Theta}{\partial \zeta^2};$$

and

$$\frac{\partial T}{\partial y} = \frac{\partial \Theta}{\partial \xi}\frac{\partial \xi}{\partial y} = \frac{T^*}{h}\frac{\partial \Theta}{\partial \xi}; \quad \frac{\partial^2 T}{\partial y^2} = \frac{T^*}{h^2}\frac{\partial^2 \Theta}{\partial \xi^2}.$$

Substitution of the new variables in (11.1) yields

$$\frac{\rho c_p u T^*}{\ell}\frac{\partial \Theta}{\partial \zeta} = \frac{kT^*}{\ell^2}\frac{\partial^2 \Theta}{\partial \zeta^2} + \frac{kT^*}{h^2}\frac{\partial^2 \Theta}{\partial \xi^2} + q''' - \mathcal{E}_n, \tag{11.3}$$

where

$$\mathcal{E}_n = 4\kappa\sigma T^{*4}\left[\Theta^4(P) - \int_{A_1,A_2} \Theta^4(S')\frac{e^{-\tau(S',P)}\cos\theta dA}{4\pi|\vec{r}_P - \vec{r}_{S'}|^2} - \int_V \kappa(P')\,\Theta^4(P')\frac{e^{-\tau(P',P)}\,dV}{4\pi|\vec{r}_P - \vec{r}_{P'}|^2}\right].$$

Dividing both sides of (11.3) by $4\kappa\sigma T^{*4}$, which is the energy emitted by the unit volume of the medium at temperature T^* per unit time, we have

$$\frac{\rho u c_p T^*}{4\ell\kappa\sigma T^{*4}} \frac{\partial\Theta}{\partial\zeta} = \frac{kT^*}{4\ell^2\kappa\sigma T^{*4}} \frac{\partial^2\Theta}{\partial\zeta^2} + \frac{kT^*}{4h^2\kappa\sigma T^{*4}} \frac{\partial^2\Theta}{\partial\xi^2} + \frac{q'''}{4\kappa\sigma T^{*4}} - \frac{\mathcal{E}_n}{4\kappa\sigma T^{*4}} \quad . \tag{11.4}$$

This equation is nondimensional. It has been formulated in a fashion that each set of terms containing variables (that is, each derivative term in this instance) will be of the order of magnitude of unity when integrated. The parameters appearing in front of the partial derivatives can be interpreted as follows:

$$\pi_1 = \frac{\rho u c_p T^*}{4\ell\kappa\sigma T^{*4}} = \frac{\text{Energy content in the flowing fluid}}{\text{Energy radiated from the flowing fluid}}$$

$$\pi_2 = \frac{kT^*}{4\ell^2\kappa\sigma T^{*4}} = \frac{\text{Energy transfer by conduction in the x direction}}{\text{Energy radiated in the x direction from the flowing fluid}}$$

$$\pi_3 = \frac{kT^*}{4h^2\kappa\sigma T^{*4}} = \frac{\text{Energy transfer by conduction in the y direction}}{\text{Energy radiated in the y direction from the flowing fluid}}$$

$$\pi_4 = \frac{q'''}{4\kappa\sigma T^{*4}} = \frac{\text{Energy generated in the fluid}}{\text{Energy radiated from the flowing fluid}}$$

Thus the nondimensional equation establishes a relationship between the dependent variable, the independent variables, and the π's. It is possible to consider fixed values of π's, which implies studying one particular problem in the class considered and then to study relations among the variables. It is also possible to consider the variables fixed, at a given location, and then to study the variation of π's, which means comparing one problem of the class considered with a different physical problem in the same class.

For large values of T^* the dimensionless parameters π are very much less than unity. In this case the transfer of energy is by radiation only. In our particular problem $\ell \gg h$, so that $\pi_2 \ll \pi_3$; this indicates that the term with $\frac{\partial^2\Theta}{\partial\zeta^2}$ can be neglected in comparison with the term $\frac{\partial^2\Theta}{\partial\xi^2}$. If π_1 and π_3 are of the same order of magnitude, the terms containing ζ and ξ cannot be dropped and no further simplification is possible. Two additional special cases arise: a) $\pi_1 \gg \pi_3$ (in this case the energy transfer by convection is of a much greater order of magnitude than the energy transfer by molecular conduction and therefore the latter can be neglected when compared with π_1), and b) $\pi_3 \gg \pi_1$, for which the reverse is true.

11.3 Equation of Energy for Simultaneous Conduction and Radiation

Consider the transfer of energy by conduction and radiation only. Further, assume that energy transfer by conduction in the x direction is negligible compared with that in the y direction. Then, introducing the dimensionless temperature and dividing by $4\kappa\sigma T^{*4}$, the steady-state energy equation (11.1) reduces to

$$\frac{kT^*}{4\kappa\sigma T^{*4}} \quad \frac{d^2\Theta}{dy^2} = \frac{\mathcal{E}_n}{4\kappa\sigma T^{*4}} \quad . \tag{11.5}$$

Since there is no temperature variation in the x direction, the distances can be measured in the units of the mean free path of radiation. Introducing the optical thickness, $\tau = \kappa y$, as the new independent variable and utilizing the expression for net emission, $\mathcal{E}_n$, obtained in Section 10.2.2 for a media enclosed by two parallel planes, we have

$$\frac{\mathcal{E}_n}{4\kappa\sigma T^{*4}} = \frac{1}{2\sigma T^*}\left[2E_{bb}(\tau) - E(0)\,E_2(\tau) - E(\tau_0)\,E_2(\tau_0-\tau) - \int_0^{\tau_0} E_1(|\tau-\tau'|)\,E_{bb}(\tau')\,d\tau'\right], \quad (11.6)$$

where $\tau_0 = \kappa h$. Substituting (11.6) in the energy equation (11.5), one obtains

$$\frac{k\kappa^2 T^*}{4\kappa\sigma T^{*4}}\frac{d^2\Theta}{d\tau^2} = \frac{1}{2\sigma T^{*4}}\left[2E_{bb}(\tau) - E(0)\,E_2(\tau) - E(\tau_0)\,E_2(\tau_0-\tau) - \int_0^{\tau_0} E_1(|\tau-\tau'|)\,E_{bb}(\tau')\,d\tau'\right]. \quad (11.7)$$

If we assume the surfaces are black, $E = E_{bb} = \sigma T^4$, equation (11.7) can be written as

$$N\frac{d^2\Theta}{d\tau^2} = \Theta^4 - \frac{1}{2}\left[\Theta^4(0)E_2(\tau) + \Theta^4(\tau_0)E_2(\tau_0-\tau) + \int_0^{\tau_0} E_1(|\tau-\tau'|)\,\Theta^4(\tau')\,d\tau'\right], \quad (11.8)$$

where

$$N = \frac{k\kappa^2 T^*}{4\kappa\sigma T^{*4}} = \frac{k\kappa}{4\sigma T^{*3}}.$$

The magnitude of this dimenionless parameter determines the relative role of the conduction term vs. the radiative terms. For large values of N conduction predominates, while radiation is the important energy transport process for small values of N. The importance of the temperature in this parameter is obvious. The influence of other physical variables is as expected. Further, the integro-differential equation (11.8) is nonlinear and of the second order in which the unknown function, $\Theta(\tau)$, occurs under the integral sign to the fourth power.

The boundary conditions in dimensionless notation for equation (11.8) are assumed to be

$$\left.\begin{aligned} \tau &= 0\,(y = 0), \quad \Theta(\tau) = \Theta(0); \\ \tau &= \tau_0\,(y = h), \quad \Theta(\tau) = \Theta(\tau_0). \end{aligned}\right\} \qquad (11.9)$$

The differential equation for temperature distribution can also be obtained by a different consideration. Since the problem studied is a steady-state one and one-dimensional along the y axis, after integration of equation (6.5) we obtain

$$- k\frac{\partial T}{\partial y} + q_r'' = q'' ,$$

where q'' is the total energy flux (conduction + radiation), which must stay constant. The energy flux by radiation, q_r'', is given by equation (10.16), except that the independent variable here is y instead of x. The above differential equation, instead of (11.5), can also be the starting point for the determination of the temperature distribution.

11.4 Methods of Solution of the Integro-Differential Equation

To the author's knowledge there are no exact mathematical methods of solving the integro-differential equation (11.8). Three approximate methods of solving this equation will be indicated below.

1. The first method is that of linearization of the dependent variable. If the temperature difference between $\Theta(0)$ and $\Theta(\tau_0)$ is small, we can define the average temperature as

$$\overline{\Theta} = \frac{\Theta(0) + \Theta(\tau_0)}{2},$$

and then the temperature difference

$$\chi(\tau) = \Theta(\tau) - \overline{\Theta}$$

does not exceed $\frac{\Theta(0) - \Theta(\tau_0)}{2}$. Expanding $\Theta^4(\tau)$ by binomial expansion, the temperature can then be approximated by

$$\Theta^4(\tau) \approx \overline{\Theta}^4 + 4\,\overline{\Theta}^3\,\chi(\tau),$$

which is linear in $\chi(\tau)$. Using this relation, the nonlinear integro-differential equation (11.8) can be reduced to a linear one. The linear integro-differential equation can then be reduced to a linear integral equation and solved.

2. A solution of equation (11.8) can be obtained by an iterative method. To this end, solve first the pure conduction equation ($\mathcal{E}_n = 0$):

$$\frac{d^2\,\Theta}{d\,\tau^2} = 0. \tag{11.10}$$

The solution of this equation with the boundary conditions (11.9) can be written as

$$\Theta(\tau) = \Theta(0) + \frac{\tau}{\tau_0}\left[\Theta(\tau_0) - \Theta(0)\right] . \tag{11.11}$$

Substituting this result for $\Theta(\tau')$ under the integral sign of equation (11.8), one obtains a very complex nonlinear ordinary differential equation. The solution of this equation can again be substituted under the integral sign of equation (11.8) and the procedure repeated until convergence is achieved. However, the equation becomes very involved and already after the first substitution of (11.11) only a numerical solution of the differential equation is possible.

3. The third method is also that of iteration. The integro-differential equation is converted into a nonlinear integral equation. The iteration is performed on this equation. The solution of equation (11.8) will be obtained by this method and is described below.

11.5 Solution of the Integro-Differential Equation

To solve equation (11.8), Volterra[105] has suggested to integrate twice with respect to τ from 0 to τ. This gives a nonlinear Fredholm integral equation of the second kind:

$$\Theta(\tau) = G(\tau) + \frac{1}{N}\int_0^{\tau_0} \phi(\tau,\tau')\,\Theta^4(\tau')\,d\tau', \qquad (11.12)$$

where

$$G(\tau) = \frac{1}{N}\left\{-\frac{1}{2}\int_0^{\tau} d\xi \int_0^{\xi}\left[\Theta^4(0)\,E_2(\zeta) + \Theta^4(\tau_0)\,E_2(\tau_0-\zeta)\right]d\zeta + C_1\tau + C_2\right\},$$

and

$$\phi(\tau,\tau') = \int_{\tau'}^{\tau}\left[1 - \frac{1}{2}\int_{\tau'}^{\xi} E_1(\,|\zeta-\tau'|\,)\,d\zeta\right]d\xi\ .$$

Here C_1 and C_2 are integration constants to be determined from the boundary conditions of equation (11.8). The evaluations of functions $G(\tau)$ and $\phi(\tau,\tau')$ as well as of the constants C_1 and C_2 are performed in the Appendix B. In this way, equation (11.12) for the temperature distribution reduces to

$$\Theta(\tau) = G(\tau) + \frac{1}{2N}\int_0^{\tau_0} \left\{ -E_3(|\tau-\tau'|) + E_3(\tau') + \frac{\tau}{\tau_0}\left[E_3(\tau_0-\tau') - E_3(\tau')\right]\right\} \Theta^4(\tau')d\tau', \tag{11.13}$$

where

$$G(\tau) = \frac{1}{N}\left\{\frac{1}{2}\,\Theta^4(0)\left[-E_4(\tau) + \frac{\tau}{\tau_0}E_4(\tau_0) + \frac{1}{3}(1-\frac{\tau}{\tau_0})\right]\right.$$
$$+\frac{1}{2}\Theta^4(\tau_0)\left[(1-\frac{\tau}{\tau_0})E_4(\tau_0) - E_4(\tau_0-\tau) + \frac{\tau}{3\tau_0}\right]$$
$$\left. + N\left[\Theta(0) + \frac{\tau}{\tau_0}\Big(\Theta(\tau_0) - \Theta(0)\Big)\right]\right\} .$$

To the author's knowledge there are no known solutions of the equation of this type. It is to be noted that for the case when conduction predominates, the parameter N is large and equation (11.13) reduces to (11.11).

A solution of (11.13) can be obtained by iteration.[89] If $\Theta(\tau)$ is a monotonic function, then the following recursion relation:

$$\Theta_{j+1}(\tau) = G(\tau) + \frac{1}{2N}\int_0^{\tau_0}\left\{- E_3(|\tau-\tau'|) + E_3(\tau') + \frac{\tau}{\tau_0}\left[E_3(\tau_0-\tau') - E_3(\tau')\right]\right\} \Theta_j^4(\tau')\,d\tau', \tag{11.14}$$

gives a sequence $\{\Theta_j(\tau)\}$ which converges to the solution $\Theta(\tau)$ of equation (11.13). The limit to which the sequence converges when $j \to \infty$ is the rigorous solution of the equation with appropriate constants C_1 and C_2 which satisfy the boundary conditions (11.9) for given parameters τ_0 and N. The numerical solution of equation (11.14) is discussed in Appendix D.

11.6 Heat Transfer

Since the system considered here is in a steady state, the heat flux is constant and is given by equation (6.14). In our particular case this equation reduces to

$$q'' = -k \frac{\partial T}{\partial y}\bigg|_w + E - \int_{\Omega=2\pi} I \cos\theta d\Omega. \tag{11.15}$$

Since $d\Omega = \sin\theta d\theta d\phi = -2\pi d\mu$, the heat flux at the upper wall $(\tau = \tau_0)$ becomes

$$q'' = k \frac{\partial T}{\partial y}\bigg|_w + E + 2\pi \int_{-1}^{0} I \mu \, d\mu, \tag{11.16}$$

where

$$I = I(0)\, e^{-\frac{\tau_0}{\mu}} - \int_0^{\tau_0} I_{bb}(\tau')\, e^{(\tau_0-\tau')/\mu} \frac{d\tau'}{\mu}$$

and was obtained from equation (4.18). Introducing the exponential integral function (10.2), equation (11.16) can be written as

$$q'' = k \frac{\partial T}{\partial y}\bigg|_w + E - 2E(0)E_3(\tau_0) - 2\int_0^{\tau_0} E_2(\tau_0-\tau')E_{bb}(\tau')d\tau' . \tag{11.17}$$

The dimensionless temperature gradient,

$$\frac{d\Theta}{d\tau} = \frac{d\Theta}{dT}\,\frac{dT}{dy}\,\frac{dy}{d\tau} = \frac{1}{\kappa T^*}\,\frac{dT}{dy} \quad , \tag{11.18}$$

is obtained from equation (11.13) by differentiation:

$$\frac{d\Theta}{d\tau} = \frac{1}{N}\left\{ \frac{1}{2}\Theta^4(0)\left[E_3(\tau) + \frac{1}{\tau_0} E_4(\tau_0) + \frac{1}{3\tau_0}\right] \right.$$

$$+ \frac{1}{2}\Theta^4(\tau_0)\left[-\frac{1}{\tau_0} E_4(\tau_0) - E_3(\tau_0-\tau) + \frac{1}{3\tau_0}\right] + \frac{N}{\tau_0}\left[\Theta(\tau_0)-\Theta(0)\right]$$

$$\left. + \frac{1}{2}\int_0^{\tau_0} \left\{ E_2\left(|\tau-\tau'|\right) + \frac{1}{\tau_0}\left[E_3(\tau_0-\tau') - E_3(\tau')\right]\right\} \Theta^4(\tau')d\tau' \right\} . \tag{11.19}$$

Substituting equations (11.18) and (11.19) into equation (11.17), one obtains

$$q'' = \frac{k}{h}\left[T_h - T_0\right] + 2\sigma\left\{ T_0^4\left[E_3(\tau_0) + \frac{1}{\tau_0}E_4(\tau_0) - \frac{1}{3\tau_0}\right]\right.$$

$$+ T_h^4\left[-\frac{1}{\tau_0}E_4(\tau_0) - \frac{1}{2} + \frac{1}{3\tau_0}\right] + \int_0^{\tau_0}\left\{E_2(\tau_0-\tau')\right.$$

$$\left.\left. + \frac{1}{\tau_0}\left[E_3(\tau_0-\tau') - E_3(\tau')\right]\right\} T^4(\tau')d\tau' \right\} + \sigma T_h^4$$

$$- 2\sigma T_0^4 E_3(\tau_0) - 2\sigma\int_0^{\tau_0} E_2(\tau_0-\tau')\,T^4(\tau')d\tau' \quad . \tag{11.20}$$

It is obvious that the presence of thermal radiation changes the temperature distribution in the radiating media. If only thermal radiation were present, the heat flux would be given by the first term of equation (11.20). In the presence of absorbing media the energy flux by conduction is given by the first two terms. The third, fourth and fifth terms represent the heat transfer by thermal radiation. After cancellation of some of the terms, equation (11.20) reduces to

$$q'' = \frac{k}{h}\left[T_h - T_0\right] + \frac{2\sigma}{\tau_0}\left\{T_0^4\left[E_4(\tau_0) - \frac{1}{3}\right]\right.$$

$$\left. + T_h^4\left[\frac{1}{3} - E_4(\tau_0)\right] + \int_0^{\tau_0}\left[E_3(\tau_0 - \tau') - E_3(\tau')\right]T_4(\tau')d\tau'\right\}. \quad (11.21)$$

11.7 Discussion of Results

The solution of equation (11.14) by iteration is quite lengthy and presents some mathematical difficulties. When the parameter N is small, the convergence of this equation is poor, even when a very reasonable initial guess for temperature distribution is made. The temperature profile and the function $G(\tau)$ are plotted in Fig. 11.2 for the case that N = 0.01. The contribution of the integral term, appearing in equation (11.14), is predominant. Since the temperature under the integral sign is raised to the fourth power, the value of the integral is very sensitive to the initial guess.

The computed results for the temperature distributions are presented in Figs. 11.3 through 11.5. The range of the dimensionless parameter N covered in the calculations is from 10 to 0.01. The curves for values of $N > 10$ are indistinguishable from those for pure conduction ($N \to \infty$), which are straight lines. The temperature profiles for the case that the parameter N = 1 are on the average less than one percent higher than the temperature profiles for pure conduction. For large values of the parameter N, the difference between the temperature profiles for pure conduction and simultaneous conduction and radiation is small; however, as the parameter N is decreased, the difference, as seen from Fig. 11.3 and 11.4, between the temperature

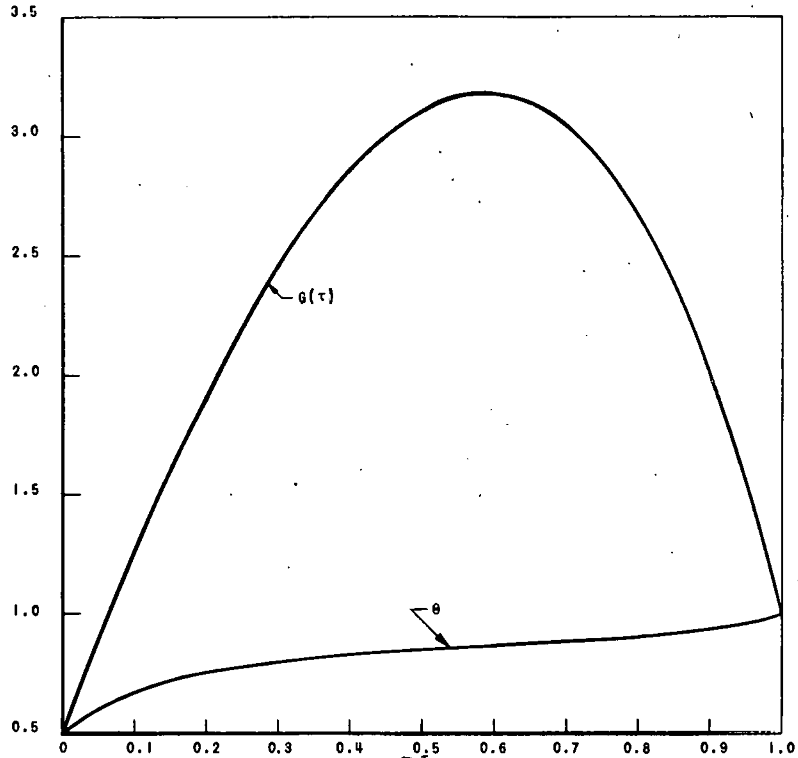

FIG. 11.2
VARIATION OF THE FUNCTION G AND THE DIMENSIONLESS TEMPERATURE θ VS. OPTICAL THICKNESS FOR N=0.01.

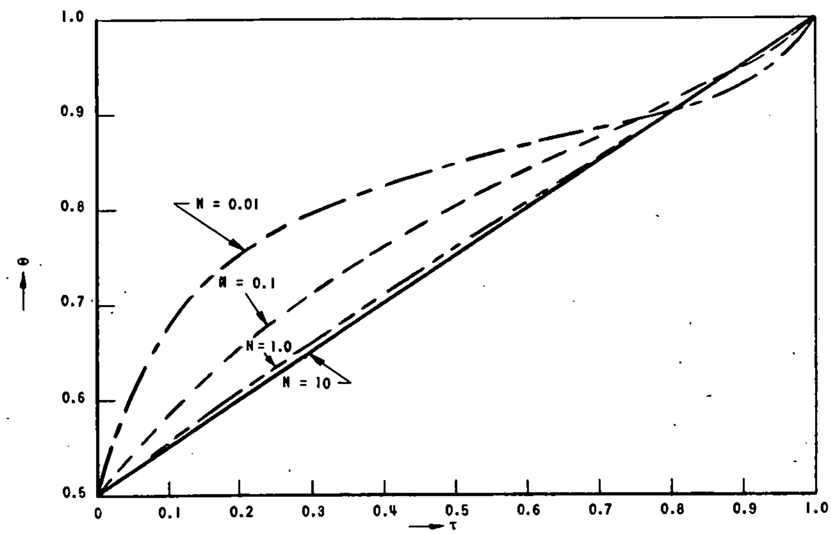

FIG. 11.3
DIMENSIONLESS TEMPERATURE DISTRIBUTION VS. OPTICAL THICKNESS, $\tau_0=1.0$.

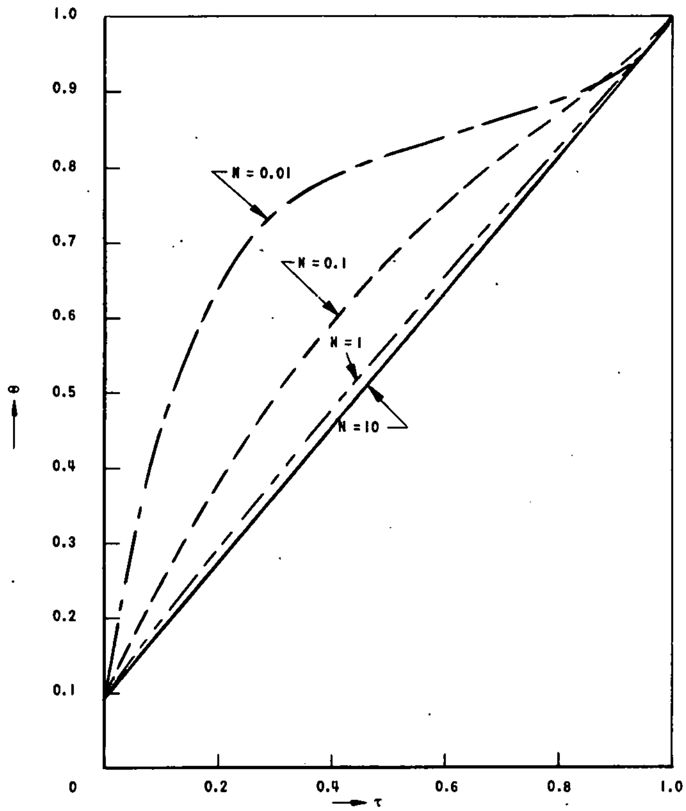

FIG. 11.4
DIMENSIONLESS TEMPERATURE DISTRIBUTION VS. OPTICAL THICKNESS, τ_o=1.0.

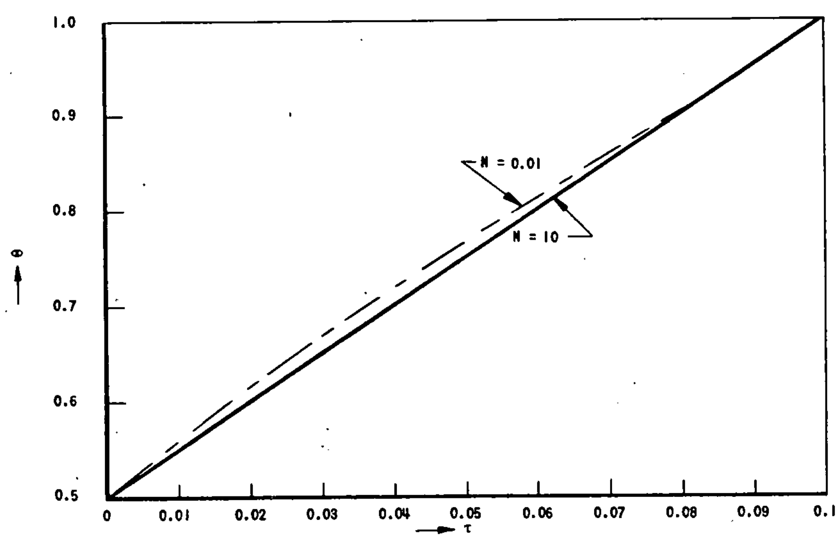

FIG. 11.5
DIMENSIONLESS TEMPERATURE DISTRIBUTION VS. OPTICAL THICKNESS, τ_o=0.1.

profiles for pure conduction and simultaneous conduction and radiation widens. The temperature distribution for N = 0.01 agrees in trend with the results of R. and M. Goulard,[30] who studied a similar problem, but in their case the thermal conductivity varied with temperature and N<<0.01. The results of the present work agree in trend with those of Walther et al., as reported by Pepperhoff,[76] who investigated the flow of heat through glass.

Temperature profiles for a medium having an optical thickness τ_0 = 0.1 are given in Fig. 11.5. The curves for N $\geq$ 1 are indistinguishable from the temperature profile for pure conduction. Even for small values of the parameter N, the difference between the profiles for pure conduction and simultaneous conduction and radiation is small, i.e., for N = 0.01 the maximum difference is only three percent. From Fig. 11.3 we see that when the optical thickness τ_0 = 1.0, the maximum difference is about 25 percent. We can therefore conclude that for radiative heat transfer problems the optical thickness, and not the spacing between the plate, is a pertinent parameter.

Since the system under consideration is in a steady state, the total energy flux (conduction plus radiation) across the medium is constant. To insure this, it is necessary for the conductive energy flux variations to be compensated by inverse variations in radiative energy flux. The temperature gradients at the cool wall are always steeper than those for pure conduction, and they increase with a decreasing value of the dimensionless parameter N. Heat transfer by conduction to a cool wall is therefore always increased if the medium is radiative. On the other hand, at the hot wall the temperature gradients can be larger or smaller than those for pure conduction, depending on the parameter N. For large values of N (N$\geq$ 1), the temperature gradients are a fraction of a percent smaller than for conduction alone. For smaller values of N the gradients increase with the decreasing N.

The results of heat transfer calculations are given in Table 11.1.

Table 11.1

Parameters and Results for Simultaneous Conduction and Radiation

τ_0	h	k	κ	N	T_0	T_h	q_c''	q_r''	q''
0.1	0.001	0.547	100	1.0	1000	2000	547,000	24,100	571,100
0.1	0.01	0.547	10	0.1	1000	2000	54,800	24,100	78,900
0.1	0.1	0.547	1	0.01	1000	2000	5,340	24,200	29,500
1.0	0.01	0.547	100	1.0	1000	2000	55,800	15,500	71,300
1.0	0.1	0.547	10	0.1	1000	2000	7,150	14,600	21,800
1.0	1.0	0.547	1	0.01	1000	2000	2,090	14,200	16,300
0.1	0.0002	0.123	500	1.0	1500	3000	923,000	122,000	1,045,000
0.1	0.002	0.123	50	0.1	1500	3000	92,500	122,200	214,700
0.1	0.02	0.123	5	0.01	1500	3000	8,540	123,500	131,000
1.0	0.002	0.123	500	1.0	1500	3000	97,500	78,300	175,800
1.0	0.02	0.123	50	0.1	1500	3000	18,100	74,300	92,400
1.0	0.2	0.123	5	0.01	1500	3000	9,600	72,300	81,900
0.1	0.000125	0.0547	800	0.1	2000	4000	888,800	394,100	1,282,900
0.1	0.00125	0.0547	80	0.01	2000	4000	96,600	395,200	491,800
1.0	0.00125	0.0547	800	0.1	2000	4000	119,200	237,000	356,200
1.0	0.0125	0.0547	80	0.01	2000	4000	37,100	231,100	268,200
1.0	0.0025	1.095	400	1.0	400	4000	1,497,000	307,000	1,804,000
1.0	0.025	1.095	40	0.1	400	4000	147,500	282,000	429,500
1.0	0.25	1.095	4	0.01	400	4000	38,300	249,600	287,900
1.0	0.0025	2.13	400	1.0	500	5000	3,742,000	742,500	4,484,500
1.0	0.025	2.13	40	0.1	500	5000	368,000	692,000	1,060,000
1.0	0.25	2.13	4	0.01	500	5000	93,300	612,000	705,300
0.1	0.001	0.054	100	0.02916	1500	3000	81,800	122,100	203,900
1.0	0.01	0.054	100	0.02916	1500	3000	17,800	72,100	89,900
10.0	0.1	0.054	100	0.02916	1500	3000	2,100	14,200	16,300
0.3	0.0328	0.532	9.14	0.02575	3012	2292	29,400	73,400	102,800
0.1	0.0025	0.267	40	1.0	580	1160	61,960	2,780	64,740
1.0	0.025	0.267	40	1.0	580	1160	6,300	1,750	8,050

The separate conductive and radiative energy fluxes were obtained from equation (11.20). It is obvious that in calculating the energy transfer by conduction by simply using the temperature gradient for the nonradiative medium [first term in equation (11.20)], an incorrect result is obtained. It was found that for small optical thicknesses, $\tau_0 = 0.1$, the effect of radiation on the heat transfer by conduction was small. For the range of the dimensionless parameter N considered, the maximum difference occurred at N = 0.01, and it was never greater than 10 percent. For larger optical thicknesses, $\tau_0 = 1.0$, the effect of radiation on heat flux by conduction was much greater. When the parameter was small, N = 0.01, the energy transfer by conduction was increased by at least a factor of two and in one case by a factor of 10.

As seen from equation (11.20), the energy radiated by the wall at $\tau = 0$ (y = 0) and reaching the wall at $\tau = \tau_0$ (y = h) is given by $2\sigma T^4 E_3(\tau_0)$. The radiant energy flux due to radiation emitted from the medium which reaches the wall at $\tau = \tau_0$ is $2\sigma \int_0^{\tau_0} E_2(\tau_0 - \tau) T^4(\tau) d\tau$. This latter flux is always greater than the first. These two effects tend to reduce the net radiant flux and, in the limit when $T_h \rightarrow T_0$, the net radiant energy flux vanishes. The results show that the radiant heat flux for a given value of τ_0 does not vary appreciably with N, and that the variation of q_r'' with τ_0 for a constant value of parameter N is more pronounced.

The temperature distributions obtained by the exact formulation are compared in Fig. 11.6 with those predicted by using the Rosseland approximation for the radiant flux vector. As expected, when the medium is optically thick (the radiation mean free path, $\lambda_p = 1/\kappa << h$), the agreement between the two sets of results is good. However, the temperature gradients at the two bounding planes, as predicted by the Rosseland approximation, are too small. This is not surprising since the approximation fails completely in the vicinity of the boundary and only the molecular conduction insures the continuity of the temperature.

In Fig. 11.7 the same comparison is made, except now the mean free path of radiation is of the same order of magnitude as the separation distance between the two plates, that is, $\lambda_p = 1/\kappa \approx h$. The temperature gradients at the cool wall are in very good agreement, but at the hot wall the temperature gradient for the Rosseland approximation is lower than that for the exact formulation. The agreement between the two temperature profiles is poorer for this case than for $\tau_0 = 10$.

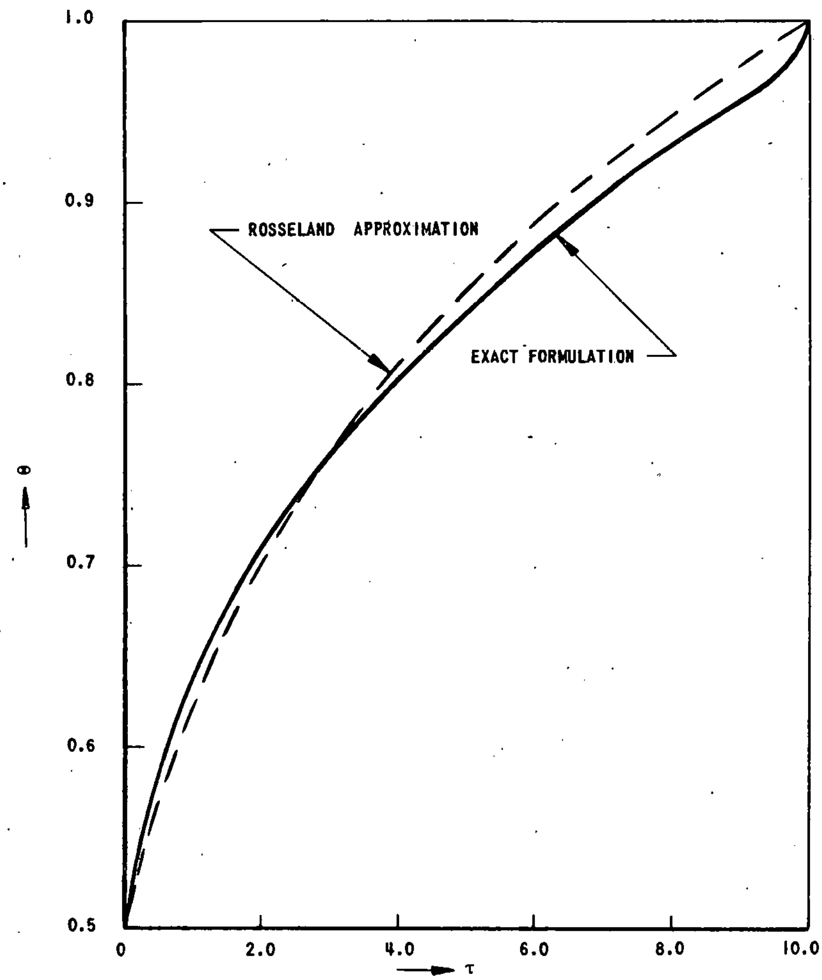

FIG. 11.6
COMPARISON OF DIMENSIONLESS TEMPERATURE DISTRIBUTIONS FOR k=0.054 BTU.HR^{-1} FT^{-1} R^{-1}, $\varkappa$=100 FT^{-1}, N=0.02916 AND τ_0=10.

The resulting temperature distributions and heat transfer for a medium of large optical thickness, τ_0 = 10, are worthy of note. The striking feature of the results is that the heat flux is about 20 times what would be expected from conduction alone, without considering the presence of a radiating medium, and about seven times larger than the heat flux obtained by considering the medium to be radiative. Even though the primary radiation falling on the medium penetrates

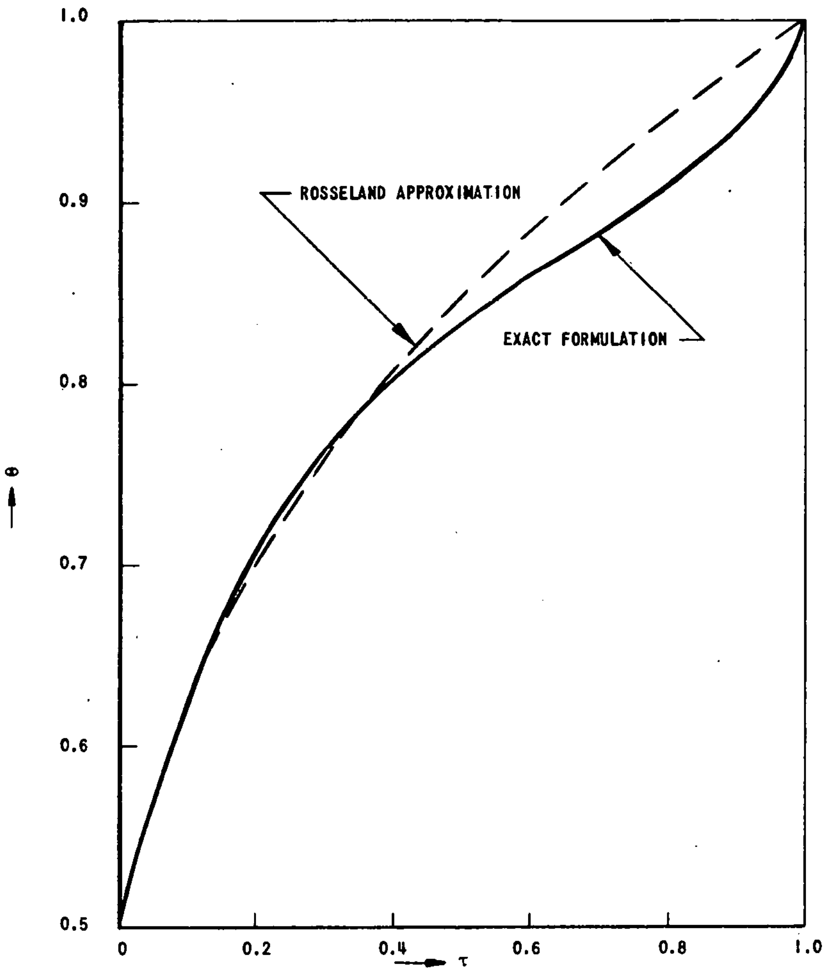

FIG. 11.7
COMPARISON OF DIMENSIONLESS TEMPERATURE DISTRIBUTIONS FOR $k=0.054$ BTU HR^{-1} FT^{-1} R^{-1}, $\kappa=100$ FT^{-1}, $N=0.02916$ AND $\tau_0=1$.

only a short distance (in 0.03 ft the intensity has fallen to $e^{-0.03 \times 100} = 5$ percent), the process of absorption and emission within the medium transports a considerable quantity of energy. The radiant heat flux of 14,200 Btu/(hr)(ft^2) calculated from the exact formulation is in substantial agreement with the value of 11,800 Btu/(hr)(ft^2) obtained by using the formula derived by Shorin:(93)

$$q_r'' = \frac{\sigma (T_1^4 - T_2^4)}{\left(\frac{1}{\epsilon_1} - \frac{1}{2}\right) + \left(\frac{1}{\epsilon_2} - \frac{1}{2}\right) + \tau_0} \qquad (11.22)$$

This simple formula was deduced for a strongly absorbing medium at rest by using the Rosseland approximation for the radiant energy flux vector, but it predicts lower radiant fluxes than equation (11.21).

It is of interest to compare the theoretical results of Kellett[46] with those of the present work. Kellett derived a differential equation which expresses the energy conservation in a slab. An approximate solution of this equation was obtained by replacing T^4 occurring in the radiative terms by a linear expression. Figure 11.8 shows the comparison between the temperature profiles obtained from

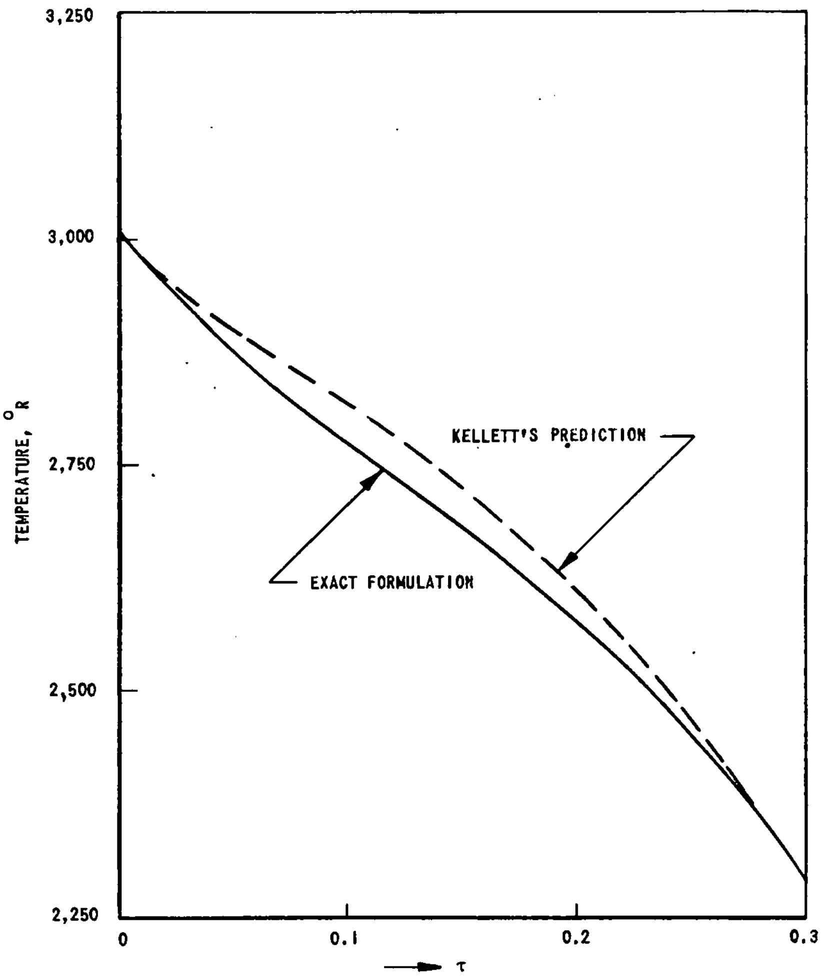

FIG. 11.8

COMPARISON OF TEMPERATURE DISTRIBUTIONS FOR $k=0.532$ BTU HR^{-1} FT^{-1} R^{-1}, $\varkappa=9.14$ FT^{-1}, $N=0.02575$ AND $\tau_0=0.3$.

the approximate formulation of reference 46 with the exact formulation of the present section. The conditions and the physical property values for which the results have been evaluated are identical. The predicted temperatures are on an average about one percent lower than those of Kellett. The temperature gradients at both walls are in very good agreement; however, the total heat flux calculated in the present study is about 10 percent higher.

In summary, the problems considered here can be divided into three classes: (1) that in which conduction predominates, (2) an intermediate case wherein conduction and radiation are of the same order of magnitude, and (3) that in which radiation predominates. In the first class of problems the parameter N is large and/or the optical thickness τ_0 is small. The energy transport by conduction predominates, and the effect of radiation on the temperature distribution and the gradients is negligible. In the intermediate class of problems and in the problems where radiation predominates, the parameter N is small and/or the optical thickness τ_0 is large. The temperature distribution and the heat transfer cannot be calculated by neglecting the radiative term from the energy equation. Generally, in these problems the presence of radiation changes the temperature distribution in such a way as to increase the heat transfer by conduction. Finally, the Rosseland approximation for the radiant flux vector is satisfactory for optically thick media.

12 SUMMARY AND CONCLUSIONS

A systematic presentation of energy transfer from thermal radiation absorbing and scattering media is given. The relations for radiation from surfaces and from media (see Section 2.7) are fundamental to the theory of radiant heat exchange. They are the basis for the derivation of a system of integral equations for irradiation and incident radiation in an enclosure containing an absorbing and scattering media.

The radiant flux vector has an integral representation and is defined as a vectorial function of a point and also of a functional operator which depends on the geometrical configuration, on the temperature field in the media as well as on the enclosure walls, and on the absorption and scattering coefficients. For the case of intense absorption and a system near thermodynamic equilibrium the radiant energy flux vector can be represented by a simple expression.

The equation of transfer which governs the intensity distribution in a radiating medium was derived and various special cases discussed. This phenomenological equation, which describes the kinetics of radiation, is analogous in its form to integro-differential equations encountered in other branches of physical science. The intensity of radiation obtained from the solution of the equation of transfer was used in the subsequent derivations of the integral equations for irradiation and incident radiation, as well as the integro-differential equation of energy conservation.

The integral equations derived in Section 5.2 are the basis for analytical methods of investigation of the problem of radiant heat exchange in a system of nondiffuse surfaces separated by an absorbing and scattering media. They can readily be expressed in terms of other variables which are more appropriate to a particular problem. The system of integral equations (5.5) and (5.10) is very complex, however, and a general solution will be very difficult, if not impossible, to obtain. Since scattering generally is not isotropic and the reflection from the surfaces is not diffuse, but depends on direction, the scattering and reflecting functions must be known, or some simplifying assumptions introduced, before a solution of this system of equations can even be attempted. In addition, to obtain a solution of radiant heat transfer problem for an enclosure, a number of other simplifying assumptions will be required. These assumptions will be inevitable because sufficient information does not exist regarding the radiative properties of materials to permit a more accurate analysis of the problem.

The conservation of energy equation, including the contribution due to thermal radiation, was derived. Since radiative transfer is an integral problem, the analytical studies must be based on integral and integro-differential equations, which have a general and rigorous character. Differential equations in particular cannot be employed to formulate the mathematics of this physical problem. Only in the simplest particular case, that of very intense absorption and scattering, as well as of a system close to thermodynamic equilibrium, can the radiant flux vector be approximated by a differential equation.

The subject matter covered in this work may be regarded as basic for the understanding of the specific problems treated here, as well as more complicated ones dealing with heat transfer from radiating media. Since very little data are available on radiative properties such as absorption and scattering coefficients and only scanty information exists in regard to the dependence of these properties on wavelength, temperature and pressure, the solutions obtained for the specific problems were for the grey case. This is only an approximation which might not correspond to physical reality, and in the future refinements will have to be made. However, the grey case is of particular interest as it provides a physically significant standard of comparison for interpreting the general case, and the author is of the opinion that the simple problem must be solved first before the non-grey case can be attempted.

It was shown that for large optical thicknesses the temperature distributions calculated by using the Rosseland approximation are in good agreement with those predicted by the exact formulation. The results for the flow along a wedge indicated that the effect of radiation is to decrease the temperature gradients at both hot and cool walls, but the heat transfer is affected only little.

The transfer of radiant energy between two parallel plates separated by an absorbing and scattering media was studied. A non-homogeneous Milne integral equation was solved by the method of undetermined parameters. The black body emissive power (temperature) distributions were determined. For the range of parameters investigated, it was found that $E_{bb}(\tau)$ can be approximated by a straight line, and the radiant heat fluxes were strongly dependent on the optical thickness of the media. The polynomial approximation used for $E_{bb}(\tau)$ was satisfactory for all values of optical thickness and ratios of $E(\tau_0)/E(0)$.

For the transfer of radiant energy by simultaneous conduction and radiation, when the two transport processes are of the same order of magnitude, or when radiation predominates, the temperature distribution and heat flux cannot be calculated by neglecting the radiative terms from the energy equation. The results showed that the temperature distribution is strongly dependent on the optical thicknesses of the media and on the dimensionless parameter, N, which determines the role of energy transfer by conduction to that by radiation. Radiation effects are relatively unimportant for small optical thicknesses; an increased value of optical thickness increases the role of radiation. The presence of radiation generally changes the temperature distribution in such a way as to increase the heat transfer by conduction, and the energy transport by radiation is a weak function of the dimensionless parameter N.

The future work in the field of heat transfer, where thermal radiation is important or predominant, should be undertaken along the following general directions:

a) Since the accuracy of the results obtained for the temperature distribution and heat transfer depend largely on the radiative properties, theoretical and experimental effort should be directed towards the evaluation of these properties.

b) It is evident that microscopic analysis of the radiative properties and other contributory effects will be excessively complicated. Moreover, a theory which starts out on such detailed premises will, by its very nature, obscure the essential factors which are operative. Therefore, in theoretical studies simple physical situations should be chosen so that the geometrical and the property evaluation complexities would not conceal the effect of radiation on temperature distribution and heat transfer.

c) The solution of the integral equations by iterative methods takes up a considerable amount of time even on very fast digital computers, e.g., IBM 704. Consequently, accurate approximate methods are needed for solving complicated integral equations or systems of integral equations occurring in radiant heat transfer problems.

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APPENDIX A

Approximate Solution of Milne's Integral Equation (10.25)

We assume that the black body emissive power can be approximated by a fourth-degree polynomial:

$$E_{bb}(\tau) = E(0)\left[D_0 + D_1\tau + D_2\tau^2 + D_3\tau^3 + D_4\tau^4\right] \quad . \tag{A.1}$$

$E_{bb}(\tau)$ and $E_{bb}(\tau')$ are the same functions, the only difference being that τ and τ' are interchanged as independent variables. Substituting (A.1) under the integral sign of equation (10.25), we have

$$\begin{aligned} E_{bb}(\tau) &= \tfrac{1}{2}[E(0)E_2(\tau) + E(\tau_0)E_2(\tau_0-\tau)] \\ &+ \frac{E(0)}{2}\int_0^{\tau_0} E_1(|\tau-\tau'|)(D_0 + D_1\tau' + D_2\tau'^2 + D_3\tau'^3 \\ &+ D_4\tau'^4)d\tau' \quad . \end{aligned} \tag{A.2}$$

Introducing the definition of the exponential integral function, $E_1(\tau)$, in equation (A.2) and integrating, we obtain

$$\begin{aligned} E_{bb}(\tau) &= \tfrac{1}{2}[E(0)E_2(\tau) + E(\tau_0)E_2(\tau_0-\tau)] + \tfrac{1}{2}E(0)\Big\{D_0[2 - E_2(\tau) - E_2(\tau_0-\tau)] \\ &+ D_1[2\tau + E_3(\tau) - \tau_0E_2(\tau_0-\tau) - E_3(\tau_0-\tau)] + D_2[2\tau^2 + \tfrac{4}{3} - 2E_4(\tau) \\ &- \tau_0^2E_2(\tau_0-\tau) - 2\tau_0E_3(\tau_0-\tau) - 2E_4(\tau_0-\tau)] + D_3[2\tau^3 + 4\tau \\ &+ 6E_5(\tau) - \tau_0^3E_2(\tau_0-\tau) - 3\tau_0^2E_3(\tau_0-\tau) - 6\tau_0E_4(\tau_0-\tau) - 6E_5(\tau_0-\tau)] \\ &+ D_4[2\tau^4 + 8\tau^2 + \tfrac{48}{5} - 24E_6(\tau) - \tau_0^4E_2(\tau_0-\tau) - 4\tau_0^3E_3(\tau_0-\tau) \\ &- 12\tau_0^2E_4(\tau_0-\tau) - 24\tau_0E_5(\tau_0-\tau) - 24E_6(\tau_0-\tau)]\Big\} \quad . \end{aligned} \tag{A.3}$$

For radiative equilibrium the radiant heat flux at any point in the medium can be written directly from equation (10.22), and we have

$$q_r'' = 2[E(\tau_0)E_3(\tau_0-\tau) - E(0)E_3(\tau) + \int_{\tau}^{\tau_0} E_2(\tau'-\tau)E_{bb}(\tau')d\tau'$$

$$- \int_0^{\tau} E_2(\tau-\tau')E_{bb}(\tau')d\tau'] \quad . \qquad (A.4)$$

Substituting equation (A.1), as well as the definition of the exponential integral function, under the integral signs and integrating, we obtain

$$q_r'' = 2[E(\tau_0)E_3(\tau_0-\tau) - E(0)E_3(\tau)] + 2E(0)\Big\{D_0[E_3(\tau) - E_3(\tau_0-\tau)]$$

$$+ D_1[\tfrac{2}{3} - E_4(\tau) - \tau_0E_3(\tau_0-\tau) - E_4(\tau_0-\tau)] + D_2[\tfrac{4\tau}{3} - \tau_0^2 E_3(\tau_0-\tau)$$

$$- 2\tau_0E_4(\tau_0-\tau) + 2E_5(\tau) - 2E_5(\tau_0-\tau)] + D_3[\tfrac{12}{5} + 2\tau^2 - \tau_0^3 E_3(\tau_0-\tau)$$

$$- 3\tau_0^2 E_4(\tau_0-\tau) - 6\tau_0E_5(\tau_0-\tau) - 6E_6(\tau) - 6E_6(\tau_0-\tau)]$$

$$+ D_4[\tfrac{48\tau}{5} + \tfrac{8\tau^3}{3} - \tau_0^4 E_3(\tau_0-\tau) - 4\tau_0^3 E_4(\tau_0-\tau) - 12\tau_0^2 E_5(\tau_0-\tau)$$

$$- 24\tau_0E_6(\tau_0-\tau) - 24E_7(\tau_0-\tau) + 24E_7(\tau)]\Big\} \quad . \qquad (A.5)$$

We do not know the radiant energy flux, but we know that for radiative equilibrium it is constant, that is, $dq_r''/d\tau = 0$. Differentiating (A.5) with respect to τ, we obtain

$$\frac{dq_r''}{d\tau} = 0 = 2[E(\tau_0)E_2(\tau_0-\tau) + E(0)E_2(\tau)] + 2E(0)\Big\{D_0[-E_2(\tau) - E_2(\tau_0-\tau)]$$

$$+ D_1[E_3(\tau) - \tau_0E_2(\tau_0-\tau) - E_3(\tau_0-\tau)] + D_2[\tfrac{4}{3} - \tau_0^2 E_2(\tau_0-\tau)$$

$$- 2\tau_0E_3(\tau_0-\tau) - 2E_4(\tau) - 2E_4(\tau_0-\tau)] + D_3[4\tau - \tau_0^3 E_2(\tau_0-\tau)$$

$$- 3\tau_0^2 E_3(\tau_0-\tau) - 6\tau_0E_4(\tau_0-\tau) + 6E_5(\tau) - 6E_5(\tau_0-\tau)] + D_4[\tfrac{48}{5} + 8\tau^2$$

$$- \tau_0^4 E_2(\tau_0-\tau) - 4\tau_0^3 E_3(\tau_0-\tau) - 12\tau_0^2 E_4(\tau_0-\tau) - 24\tau_0E_5(\tau_0-\tau)$$

$$- 24E_6(\tau_0-\tau) - 24E_6(\tau)]\Big\} \quad . \qquad (A.6)$$

We need five equations to determine D's. Two of these can be obtained by satisfying the emissive powers at the two boundaries:

at $\tau = 0$, $E_{bb}(0) = E(0)D_0$

and

at $\tau = \tau_0$, $E_{bb}(\tau_0) = E(0)(D_0 + D_1\tau_0 + D_2\tau_0^2 + D_3\tau_0^3 + D_4\tau_0^4)$.

Introducing these in equation (A.3), we get

$$D_0[-1 - E_2(\tau_0)] + D_1[\tfrac{1}{2} - \tau_0 E_2(\tau_0) - E_3(\tau_0)] + D_2[\tfrac{2}{3} - \tau_0^2 E_2(\tau_0) - 2\tau_0 E_3(\tau_0) - 2E_4(\tau_0)] + D_3[\tfrac{3}{2} - \tau_0^3 E_2(\tau_0) - 3\tau_0^2 E_3(\tau_0) - 6\tau_0 E_4(\tau_0) - 6E_5(\tau_0)] + D_4[\tfrac{24}{5} - \tau_0^4 E_2(\tau_0) - 4\tau_0^3 E_3(\tau_0) - 12\tau_0^2 E_4(\tau_0) - 24\tau_0 E_5(\tau_0) - 24E_6(\tau_0)] = -1 - \frac{E(\tau_0)}{E(0)} E_2(\tau_0) \quad , \tag{A.7}$$

and

$$D_0[-1 - E_2(\tau_0)] + D_1[-\tau_0 - \tfrac{1}{2} + E_3(\tau_0)] + D_2[-\tau_0^2 - \tau_0 + \tfrac{2}{3} - 2E_4(\tau_0)] + D_3[-\tau_0^3 - \tfrac{3}{2}\tau_0^2 + 2\tau_0 - \tfrac{3}{2} + 6E_5(\tau_0)] + D_4[-\tau_0^4 - 2\tau_0^3 + 4\tau_0^2 - 6\tau_0 + \tfrac{24}{5} - 24E_6(\tau_0)] = -E_2(\tau_0) - \frac{E(\tau_0)}{E(0)} \quad . \tag{A.8}$$

Three other equations can be obtained by satisfying the flux conservation equation (A.6) at optical depths τ_1, τ_2 and τ_3. We then have

$$D_0[-E_2(\tau_i) - E_2(\tau_0-\tau_i)] + D_1[E_3(\tau_i) - \tau_0 E_2(\tau_0-\tau_i) - E_3(\tau_0-\tau_i)] + D_2[\tfrac{4}{3} - \tau_0^2 E_2(\tau_0-\tau_i) - 2\tau_0 E_3(\tau_0-\tau_i) - 2E_4(\tau_i) - 2E_4(\tau_0-\tau_i)] + D_3[4\tau_i - \tau_0^3 E_2(\tau_0-\tau_i) - 3\tau_0^2 E_3(\tau_0-\tau_i) - 6\tau_0 E_4(\tau_0-\tau_i) + 6E_5(\tau_i) - 6E_5(\tau_0-\tau_i)] + D_4[\tfrac{48}{5} + 8\tau_i^2 - \tau_0^4 E_2(\tau_0-\tau_i) - 4\tau_0^3 E_3(\tau_0-\tau_i) - 12\tau_0^2 E_4(\tau_0-\tau_i) - 24\tau_0 E_5(\tau_0-\tau_i) - 24E_6(\tau_0-\tau_i) - 24E_6(\tau_i)] = -E_2(\tau_i) - \frac{E(\tau_0)}{E(0)} E_2(\tau_0-\tau_i) \quad , \tag{A.9}$$

where i = 1,2,3.

One should note that all five equations could have been obtained from equation (A.6). It can readily be shown that, by satisfying the flux conservation equation at optical depths $\tau = 0$ and $\tau = \tau_0$, we would have obtained equations (A.7) and (A.8), respectively.

The equation (A.9) was evaluated at three different optical depths, τ_1, τ_2 and τ_3, more or less arbitrarily. The values chosen were $\tau_1 = 0.2\tau_0$, $\tau_2 = 0.5\tau_0$ and $\tau_3 = 0.8\tau_0$, and thereby three equations were obtained. Great care was exercised in the evaluation of the coefficients appearing in the equations. In some cases for $\tau_0 = 0.1$, the exponential integral functions accurate to 10 significant figures were used in order to obtain the coefficients with an accuracy to four significant figures.

APPENDIX B

Reduction of the Nonlinear Integral Equation (11.12)

In this Appendix the functions $G(\tau)$ and $\phi(\tau,\tau')$, as well as the constants C_1 and C_2, appearing in the nonlinear integral equation (11.12) are evaluated explicitly. Substituting the definition of the exponential integral function, $E_n(\tau)$, in the expression of the function $G(\tau)$, we have

$$G(\tau) = \frac{1}{N}\left\{-\frac{1}{2}\int_0^{\tau} d\xi \int_0^{\xi}\left[\Theta^4(0)\int_1^{\infty}\frac{e^{-\zeta\mu}d\mu}{\mu^2} + \Theta^4(\tau_0)\int_1^{\infty}\frac{e^{-(\tau_0-\zeta)\mu}d\mu}{\mu^2}\right]d\zeta\right\} + C_1\tau + C_2 \qquad . \qquad \text{(B.1)}$$

Interchanging the order of integration and integrating once, we obtain

$$G(\tau) = \frac{1}{N}\left\{\frac{1}{2}\int_1^{\infty}\frac{d\mu}{\mu^3}\left[\Theta^4(0)\int_0^{\tau}\left(e^{-\xi\mu}-1\right)d\xi - \Theta^4(\tau_0)\int_0^{\tau}\left(e^{-(\tau_0-\xi)\mu}-e^{-\tau_0\mu}\right)d\xi\right]\right\} + C_1\tau + C_2 \qquad . \text{(B.2)}$$

One more integration reduces $G(\tau)$ to

$$G(\tau) = \frac{1}{2N}\left\{\Theta^4(0)\left[-E_4(\tau) + \frac{1}{2}(1-\tau)\right] + \Theta^4(\tau_0)\left[-E_4(\tau_0-\tau) + E_4(\tau_0) + \tau E_3(\tau_0)\right]\right\} + C_1\tau + C_2 \quad . \text{(B.3)}$$

Integrating the function $\phi(\tau,\tau')$ once, we obtain

$$\phi(\tau,\tau') = \int_{\tau'}^{\tau}\left[\frac{3}{2} - \frac{1}{2}E_2(|\xi-\tau'|)\right]d\xi \qquad ,$$

and finally

$$\phi(\tau, \tau') = \frac{3}{2}(\tau - \tau') + \frac{1}{4} - \frac{1}{2} E_3(|\tau - \tau'|) \quad . \tag{B.4}$$

The integral equation (11.12) can now be written as

$$\theta(\tau) = G(\tau) + \frac{1}{2N}\int_0^{\tau_0}\left[3(\tau - \tau') + \frac{1}{2} - E_3(|\tau - \tau'|)\right]\theta^4(\tau')d\tau' \quad , \tag{B.5}$$

where

$$G(\tau) = \frac{1}{2N}\left\{\Theta^4(0)\left[-E_4(\tau) + \frac{1}{2}(1 - \tau)\right]\right.$$
$$\left. + \Theta^4(\tau_0)\left[E_4(\tau_0) - E_4(\tau_0 - \tau) + \tau E_3(\tau_0)\right]\right\} + C_1\tau + C_2 \quad .$$

To evaluate the constant C_2 we apply the first of the boundary conditions (11.9) and obtain

$$C_2 = \Theta(0) - \frac{1}{12N}\Theta^4(\tau_0) - \frac{1}{2N}\int_0^{\tau_0}\left[-3\tau' - E_3(\tau') + \frac{1}{2}\right]\Theta^4(\tau')d\tau' \quad . \tag{B.6}$$

Constant C_1 is obtained by applying the second of the boundary conditions (11.9):

$$C_1 = \frac{1}{2N\tau_0}\left\{2N\Theta(\tau_0) - \Theta^4(0)\left[-E_4(\tau_0) + \frac{1}{2}(1 - \tau_0)\right]\right.$$
$$- \Theta^4(\tau_0)\left[E_4(\tau_0) - \frac{1}{3} + \tau_0 E_3(\tau_0)\right] - 2NC_2$$
$$\left. - \int_0^{\tau_0}\left[3(\tau_0 - \tau') - E_3(\tau_0 - \tau') + \frac{1}{2}\right]\Theta^4(\tau')d\tau'\right\} \quad . \tag{B.7}$$

Substitution of C_1 and C_2 in equation (B.5) and simplification yields

$$\Theta(\tau) = G(\tau) + \frac{1}{2N}\int_0^{\tau_0}\left\{-E_3(|\tau - \tau'|) + E_3(\tau')\right.$$
$$\left. + \frac{\tau}{\tau_0}\left[E_3(\tau_0 - \tau') - E_3(\tau')\right]\right\}\Theta^4(\tau')d\tau' \quad , \tag{B.8}$$

where

$$
\begin{aligned}
\dot{G}(\tau) = \frac{1}{N}\Bigg\{ & \frac{1}{2}\,\Theta^4(0)\left[\frac{\tau}{\tau_0}\,E_4(\tau_0) - E_4(\tau) + \frac{1}{3}\left(1 - \frac{\tau}{\tau_0}\right)\right] \\
& + \frac{1}{2}\,\Theta^4(\tau_0)\left[\left(1 - \frac{\tau}{\tau_0}\right)E_4(\tau_0) - E_4(\tau_0 - \tau) + \frac{\tau}{3\tau_0}\right] \\
& + N\left[\Theta(0) + \frac{\tau}{\tau_0}\Big(\Theta(\tau_0) - \Theta(0)\Big)\right]\Bigg\} \quad .
\end{aligned}
$$

APPENDIX C

Numerical Solution of Equations (9.40) and (9.41)

The differential equations derived in Section 9.6.3 were solved numerically[33] by the Runge-Kutta and Milne methods on a digital computer. The computer program used was originally coded to solve a system of first-order ordinary differential equations with the initial conditions being specified.

In general, integration methods which keep track of truncation error, such as that of Milne,[33] are preferable to those which do not, for obvious reasons. For the Milne method, the manner in which the precision is computed is important, since its proper choice may reduce machine time more than by a factor of two. One disadvantage of this method is that a starting procedure is required before the method can be applied. The first three points (not counting the initial point) were therefore computed by the Runge-Kutta method; then a switch to the Milne method was made. When using the Milne method, the routine required the examination at each point of the sum of the absolute values of maximum deviations for the last three points. When this sum was exceeded by a certain specified quantity, the interval was doubled, and the interval was decreased by a factor of two if the criteria were violated.

Equations (9.40) and (9.41) were reduced to a system to five first-order ordinary differential equations. Unfortunately, the conditions to be satisfied by the equations were specified on both ends of the integration interval. Thus, since only $f(0)$, $f'(0)$ and $\Theta(0)$ were known, the method of solution was to pick values of $f''(0)$, as well as $\Theta'(0)$, and integrate the equations directly, recording the resultant asymptotic values of f' and Θ for large values of η. After two such integrations, a linear interpolation produced better values of $f''(0)$ and $\Theta'(0)$. After the third integration the second-order Newton interpolation formula with divided differences was used, and the procedure was repeated until the required conditions at large η were met - i.e., $f'(\infty) \rightarrow 1$ and $\Theta(\infty) \rightarrow 1$. This interpolation was made an integral part of the numerical program so that by starting with two initial guesses for $f''(0)$ and $\Theta'(0)$ the program would run to completion.

In the numerical solution the right-hand side of equation (9.41) was neglected. The Rosseland approximation, which breaks down completely at the wall, was assumed to hold in the interval $0.03 \leq \eta \leq \infty$.

Both the Runge-Kutta and Milne methods yield fourth-order precision, i.e., have truncation errors of order $(\Delta\eta)^5$. Any desired accuracy within reason may be obtained by choosing appropriately small values of $\Delta\eta$. A $\Delta\eta$ of 0.01 was used initially for all solutions. Generally, after the first three points were calculated this increment was doubled.

The boundary conditions at $\eta \rightarrow \infty$ were considered met when f' and Θ were satisfied simultaneously to within ± 0.5 percent of the values at the boundary. The computer was programmed to print out the values of f, f', f'', Θ and Θ' at specified intervals of η.

APPENDIX D

Numerical Solution of Equation (11.14)

The method of successive approximations used in solving the nonlinear integral equation (11.14) is as follows: A function $\Theta_j(\tau)$ is assumed and inserted into the right side of equation (11.14). This produces a new function, $\Theta_{j+1}(\tau)$, on the left. The procedure was repeated until the convergence criteria were satisfied. It was found that for large values of $N(N > 0.1)$ with an assumed function $\Theta_j(\tau)$ of the form (11.11), the convergence was achieved with less than six iterations. For values at $N = 1$ convergence was obtained after three iterations. When the value of N for which solution was sought was sufficiently small ($N < 0.075$), the successive iterations showed a tendency to oscillate and then to diverge if a linear approximation for $\Theta_j(\tau)$ was substituted. In the present case, the property of oscillation cannot be developed analytically; however, it has been found by trial that if, $\frac{\Theta_j(\tau) + \Theta_{j+1}(\tau)}{2}$ is used in place of $\Theta_{j+1}(\tau)$ to obtain $\Theta_{j+2}(\tau)$, the oscillation in the first few iterations is reduced and convergence takes place.

The successive approximation calculations were carried out by means of an IBM 650 digital computer. The exponential integral functions, $E_n(\tau)$, are well behaved for $n > 1$, and the infinite series expressions for these functions were taken from reference 10. The integration interval $0 \leq \tau \leq \tau_0$ was divided into i equally spaced sub-intervals. The number of steps selected depended on the value of the optical thickness, τ_0. The inequality

$$\frac{\Theta_{j+1}(\tau)-\Theta_{j}(\tau)}{\Theta_{j}(\tau)} < |0.0005|$$

had to be satisfied before the convergence criteria were considered as met. Therefore, the accuracy of the solutions obtained by the method of successive substitutions is believed to be ±0.1 percent.

APPENDIX E

List of Symbols

Symbol	Definition	Units
A	Area	ft^2
A_1	Parameter defined in equation (9.24)	-
A_2	Parameter defined in equation (9.24)	-
a	Slab thickness	ft
a_1	Exponent in equation (9.21) for variation of thermal conductivity with temperature	none
a_2	Exponent in equation (9.22) for variation of absorption coefficient with temperature	none
B_1	Parameter defined in equation (9.25)	R^{-3}
c	Velocity of light	ft/hr
c_p	Specific heat at constant pressure	$Btu/(lb_m)(R)$
D_n	Parameters defined by equation (10.28)	none
E	Emissive power defined by equation (2.19)	$Btu/(hr)(ft^2)$
E'	Irradiation defined by equation (2.21)	$Btu/(hr)(ft^2)$
$\vec{E}$	Radiant energy flux vector defined by equation (2.31)	$Btu/(hr)(ft^2)$
E_n	Exponential integral function defined by equation (10.2)	none
E_n	Net radiant energy flux defined by equation (2.29)	$Btu/(hr)(ft^2)$
$\mathcal{E}$	Emission from a unit volume defined by equation (2.20)	$Btu/(hr)(ft^3)$

Symbol	Definition	Units
$\mathcal{E}'$	Incident radiation on a unit volume defined by equation (2.22)	Btu/(hr)(ft^2)
$\mathcal{E}_n$	Net emission from a unit volume defined by equation (2.30)	Btu/(hr)(ft^3)
e	Internal energy defined by equation (6.4)	Btu/lb$_m$
f	Function defined in equation (10.19)	none
f	Dimensionless stream function defined by equation (9.33)	none
g	Function defined in equation (10.21)	Btu/(hr)(ft^2)
G	Function defined in equation (11.13)	none
h	Enthalpy	Btu/lb$_m$
h	Vertical distance between two parallel surfaces	ft
I	Intensity of radiation defined by equation (2.1)	Btu/(hr)(ft^2)
k	Thermal conductivity	Btu/(hr)(ft)(R)
k_{eff}	Effective thermal conductivity defined by equation (9.13)	Btu/(hr)(ft)(R)
m	Exponent in equation (9.32)	none
m_n	Function defined by equation (9.8)	none
N	Dimensionless parameter in equation (11.8)	none
n	Index of refraction	none
$\vec{n}$	Unit vector normal to the surface	none
N_{Pr}	Prandtl number, $N_{Pr} = \frac{\mu c_p}{k}$	none
p	Pressure	lb$_f$/ft^2
P	Radiant energy flux tensor defined by equation (3.19)	Btu/(hr)(ft^2)
q"	Heat flux	Btu/(hr)(ft^2)
q"'	Heat generation	Btu/(hr)(ft^2)

Symbol	Definition	Units
R	Radiosity defined by equation (2.27)	Btu/(hr)(ft^2)
$\vec{r}$	Position radius vector	ft
s	Position coordinate in a given direction	ft
u	Velocity in the x direction	ft/hr
u	Radiant energy density defined by equation (2.7)	Btu/ft^3
U	Velocity outside the boundary layer	ft/hr
t	Time	hr
T	Temperature	R
v	Velocity in the y direction	ft/hr
v	Specific volume	ft^3/lb_m
V	Volume	ft^3
w	Velocity in the z direction	ft/hr
$\vec{w}$	Fluid velocity vector	ft/hr
W	Work done by the fluid	Btu/(hr)(ft^3)
x	Position coordinate	ft
y	Position coordinate	ft
z	Position coordinate	ft

Greek Symbols

α	Thermal diffusivity, $\alpha = k/\rho c_p$	ft^2/hr
α	Radiation absorptivity of the surface	none
β	Radiation extinction coefficient defined by equation (3.8)	ft^{-1}
β	Pressure gradient parameter defined as $\beta = 2m/m + 1$	none
Γ	Reflecting function defined by equation (2.25)	none
γ	Scattering function defined by equation (2.26)	none
ϵ	Radiation emissivity of the surface	none

Symbol	Greek Symbols	Units
ϵ	Emission coefficient of the medium	Btu/(hr)(ft^3)
ϵ_e	Effective emission coefficient of the medium defined by equation (3.3)	Btu/(hr)(ft^3)
ζ	Dimensionless independent variable, x/ℓ	none
ζ	Dummy integration variable in equation (11.12)	none
η	Dimensionless independent variable defined by equation (9.33)	none
θ	Angle between the outward normal $\vec{n}$ and the direction of the pencil at rays $\vec{\Omega}$	none
Θ	Angle between the direction rays $\vec{\Omega}'$ and $\vec{\Omega}$	none
Θ	Dimensionless temperature defined as $\Theta = T/T^*$	none
κ	Absorption coefficient of the medium defined by equation (2.9)	ft^{-1}
λ	Wavelength	microns
λ_p	Mean free path of radiation	ft
μ	Dynamic viscosity	(lb)(hr)/ft^2
μ	Cos θ	none
ν	Frequency	hr^{-1}
ν	Kinematic viscosity	ft^2/hr
ξ	Dimensionless independent variable, y/h	none
ξ	Dummy integration variable	none
ρ	Density	lb_m/ft^3
ρ	Radiation reflectivity of the surface	none
σ	Stefan-Boltzmann constant, 1.714×10^{-9}	Btu/(hr)(ft^2)(R^4)
σ	Scattering coefficient of the medium defined by equation (2.10)	ft^{-1}
τ	Optical thickness (depth) of the medium defined as $\tau(s,s') = \int_{s'}^{s} \beta(s)ds$	none

Symbol	Greek Symbols	Units
ϕ	Azimuthal angle	none
Φ	Dissipation function defined by equation (6.7)	hr^{-2}
ψ	Stream function defined by equation (9.34)	ft^2/hr
Ω	Solid angle	none
$\vec{\Omega}$	Unit vector in the direction of the pencil of radiation	none
$\vec{\Omega}_1$	Unit vector	none
Ω_x	Direction cosine in the x direction	none
Ω_y	Direction cosine in the y direction	none
Ω_z	Direction cosine in the z direction	none

Subscripts

bb	Black body
c	Refers to conduction
r	Radiant
λ	Refers to monochromatic (a given wavelength or per unit wavelength)