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THE DEPOLARIZATION OF NEGATIVE MU MESONS

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THE DEPOLARIZATION OF NEGATIVE MU MESONS

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ABSTRACT

The current formulation of the theory of weak interactions predicts complete and opposite polarization for the positive and negative mu mesons created in the decay of pi mesons. The remaining polarization of the mu mesons, when they decay, may be detected by observation of the decay electrons. It is known that when positive mu mesons are stopped in certain substances they remain completely polarized; under identical circumstances the negative mu mesons are only about 13 per cent polarized when they decay. Thus it is of significance to understand this observed polarization of the negative mu mesons.

It is shown that the depolarization of the negative mu mesons may be explained by consideration of the processes attendant to the formation of mu-mesic atoms. The depolarization occurs when the mu mesons are initially captured into a highly excited bound state and in the subsequent transitions. As an essential preliminary to deducing the depolarization on capture the distribution of the mu mesons in initial states of the capturing atoms is determined. This distribution depends on the rate at which the mu mesons loose energy in the stopping process.

The depolarization in the initial capturing event is due to the spin-orbit coupling; however, the extent of the depolarization in capture is strongly conditioned by the scattering preceeding capture. It is shown that the mu mesons may be regarded as having random

direction when they are captured.

In the cascade subsequent to capture both radiative and Auger transitions are important. These are treated in an adequate manner and the final polarization of the mu mesons is derived theoretically. The circular polarization of the x-rays emitted in the last stage of the cascade is discussed.

It is found that the negative mu mesons should retain a polarization of 0.133 in the ground state of mu-mesic carbon.

The results obtained are compared with the best experimental data available. The excellent agreement and the unambiguous nature of the analysis presented indicate the validity of the basic assumption that the mu mesons interact with matter in just the same manner as an electron.

ACKNOWLEDGEMENT

The author wishes to express sincere appreciation to Dr. M. E. Rose, who suggested this problem and under whose direction it was solved.

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CHAPTER I

THE PHYSICS OF MU MESONS

In this dissertation we deal with a certain area of mu meson physics. Specifically we determine the mechanism by which negative mu mesons are depolarized when they interact with matter and give a quantitative account of the observed depolarization. In order to understand the depolarization we find it necessary to call on much of the previous work concerning the physics of electrons and mu mesons; in particular we will be concerned with the formation of mu-mesic atoms and the processes of de-excitation which occur in these atoms.

This chapter contains the necessary background for an understanding of the depolarization problem. The role of the weak interactions in providing meson polarization and a means for the analysis of this polarization is discussed and the experimental data is presented. We then use the relevant known facts concerning mu mesons to determine what processes must be given an analytical treatment so that one may obtain a quantitative understanding of the depolarization. We now turn to the discussion of the nature of mu mesons.

1. Mu Mesons as Dirac Particles

Mu mesons are known to have the properties expected of a Dirac particle. This means that the mu mesons obey the Dirac equation. Therefore the essential properties of mu mesons are exactly those of electrons. The only differences are that the mass of the mu meson is about 207 times greater than the electron mass and that the mu meson is unstable. Thus mu mesons carry an intrinsic spin 1/2, the masses of the positive and negative mu mesons are identical, and the interaction of mu mesons with the electromagnetic field is the same as for electrons except for the difference in mass. This has been verified experimentally to an accuracy sufficient to include the electrodynamic effects[1]. It has also been shown that the scattering of mu mesons by an electric field is in accordance with the Dirac theory[2]. Consequently we must treat the mu mesons as Dirac particles in all that follows. We also point out that the mu mesons are not subject to the strong interactions.

2. Mu Mesons and the Weak Interaction

Mu mesons are created by the decay of pi mesons and then after a lifetime of 2.2 microseconds decay also. These decays are discussed in the next section. The point here is that both the pi and mu decay are due to the weak interaction. The fact that the mu meson decays into an electron does not contradict the fact that the mu meson is a Dirac particle; in fact, it furnishes additional evidence for such a statement. Further evidence that the intrinsic natures of the mu

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1. R. L. Garwin, D. P. Hutchinson, S. Penman, and G. Shapiro, Phys. Rev. 118, 271 (1960). The theoretical value for the magnetic moment of the mu meson is 1.00116 $e\hbar/mc$; the experiments confirm this to within 0.007 per cent.

2. J. Rainwater, Ann. Rev. Nuclear Sci. 7, 1 (1957). This is an excellent review of meson physics and gives additional details concerning the present discussion.

meson and electron are identical comes from the decay of the pi meson into electrons[3]. The evidence is that the weak interaction of mu mesons is the same as for electrons except, of course, in that the electron has no other states to decay into and that there is never sufficient energy to produce mu mesons in nuclear beta decay.

3. The $\pi - \mu - e$ Decay Chain

The pi mesons are produced in processes involving the strong interactions. They decay in the following manner, with a mean life-time of 0.025 microseconds.

$$\pi^+ \rightarrow \mu^+ + \nu \qquad (1.1a)$$

$$\pi^- \rightarrow \mu^- + \bar{\nu} \qquad (1.1b)$$

The energy release, about 34 Mev, is the same in (1.1a) and (1.1b). The mu mesons then decay according to the scheme

$$\mu^{\pm} \rightarrow e^{\pm} + \nu + \bar{\nu} \qquad (1.2)$$

The neutrino is indicated by ν, the antineutrino by $\bar{\nu}$. The assignment of ν and $\bar{\nu}$ instead of 2ν in the mu decay is determined by the spectrum of the decay electrons[4]. The assignments of ν and $\bar{\nu}$ in the pi decay (1.1) are based on measurements of the polarization of the

---

3. E. J. Konopinski, Ann. Rev. Nuclear Sci. 9, 99 (1959). In connection with the statement above we point out that only one pi meson in many thousand decays into an electron; the decay is inhibited for kinematical reasons. These matters are discussed in detail in this review article and references to the experimental observations are given.

4. C. Bouchiat and L. Michel, Phys. Rev. 106, 170 (1957).

electrons in mu decay. These matters have been discussed at length in the literature[5]. We point out that the distinction between neutrino and antineutrino is that they have opposite helicity. The helicity is given by the expectation value of $\hat{\sigma}\cdot\hat{p}$, where p is a unit vector along the direction of propagation and $\hat{\sigma}$ is the vector composed of the Pauli spin matrices. The helicity of the neutrino is -1; the helicity of the antineutrino is $+1$.

The facts of interest in the study presented here are the following. In pi decay the mu mesons are created with a definite polarization. This fact is a consequence of the parity nonconservation in weak interactions[6]. The extent of the polarization and its direction depend on the exact nature of the coupling in the decay. If the mu mesons retain any of their polarization until they decay, then the angular distribution of the decay electrons serves to analyze the polarization. These matters have been discussed theoretically by Lee and Yang[7]. The necessary experimental information was first given by the observations of Garwin, Lederman, and Weinrich[8]. These observations were of great value in the study of weak interactions and, of course, were important in the general reformulation of the theory of weak interactions following the discovery of parity nonconservation. For a treatment of these

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5. M. Gell-Mann and A. H. Rosenfeld, Ann. Rev. Nuclear Sci. 7, 407 (1957). See also Konopinski, *op. cit.*

6. *Ibid.*

7. T. D. Lee and C. N. Yang, Phys. Rev. 105, 1671 (1957).

8. R. L. Garwin, L. M. Lederman, and M. Weinrich, Phys. Rev. 105, 1415 (1957).

matters the references already cited may be consulted. It is sufficient to state that the results of the present study are in no way in disagreement with the current formulation of weak interaction theory but rather they furnish additional confirmation of the theory. We now present some quantitative information concerning the mu meson polarization just mentioned.

The angular distribution of the electrons in mu decay was first given by Lee and Yang[9]. The distribution of the decay electrons in energy and angle is given by:

$$dN = 2x^2\left[(3 - 2x) + \gamma(1 - 2x)\right] dx \, d\Omega_e \, \frac{1}{4\pi} \tag{1.3}$$

where x is the ratio of the electron momentum to its maximum momentum and γ is given by

$$\gamma = \mp (\hat{\sigma}_\mu^{\pm} \cdot p_e^{\pm}) \tag{1.4}$$

where $\hat{\sigma}_\mu$ is a unit vector along the spin direction of the mu meson and $\hat{p}_e$ is a unit vector along the electron direction. The upper signs are taken for positive mesons; the lower for negative mesons. Equation (1.3) is approximate in that the electron rest mass is ignored in its direction. Since the mu decay releases an energy of 105 Mev this approximation is well justified over almost all of the spectrum. Now the prediction of the current formulation of weak interaction theory is that the positive mu mesons are created with complete longitudinal polarization opposite to their direction of motion and the negative mu

9. Lee and Yang, loc. cit.

mesons are created with complete longitudinal polarization along their direction of motion. We discuss the experimental verification of this in the next section. Using these facts we rewrite equation (1.4) as

$$\gamma = \mp \left(\hat{p}_\mu \cdot \hat{p}_e \right) P \tag{1.5}$$

where P is the remaining polarization on decay, including the proper sign and the - or + is taken for positive or negative mu mesons respectively. Note that P is originally negative for positive muons and positive for negative muons. On integrating equation (1.3) over the electron spectrum the following result is obtained.

$$I(\theta) = 1 - \frac{\gamma}{3} \tag{1.6}$$

Thus there is an angular distribution of the decay electrons that depends on the amount of polarization of the muons when they decay. Using γ as given by (1.5) we rewrite (1.6) as

$$I(\theta) = 1 - \frac{|P|}{3} \cos \tag{1.7}$$

where θ is the angle between the electron direction and the original direction of the muon. Since there are two sign changes in going from positive to negative muons in (1.5) the angular distribution relative to the muon beam is of the same form for either positive or negative muons. We shall call the quantity, $\frac{|P|}{3}$, the asymmetry coefficient. Since (1.7) often appears in the literature as

$$I(\theta) = 1 + a \cos \tag{1.8}$$

values we give for the asymmetry coefficient will differ from some of the references by a sign.

It is possible to accept electrons of a specific energy and to measure the angular distribution at this energy. In such a case equation (1.3) would be used to analyze the polarization of the decaying muons. Since there is very little data of this type available, we do not pursue this point further. Consequently whenever we discuss the asymmetry coefficient it is always the quantity $\frac{|P|}{3}$ and thus refers to the integrated electron spectrum.

The π - μ - e decay chain is schematically represented in Figure 1. In considering the conservation of angular momentum in this diagram one should note that the pi mesons have zero spin. Since the angular distribution of the positive and negative electrons is the same one should note their respective spin directions.

4. Experimental Information Concerning Muon Polarization

In the immediately preceding section we discussed a formula that allows the polarization of mu mesons to be determined experimentally. Table I gives some representative values of the observed asymmetry coefficients. It is seen that the asymmetry coefficients that are observed vary from 0.33 to essentially zero. We discuss first the asymmetry coefficients for the positive muons.

Only two values of the asymmetry coefficient are given by the positive muon. The asymmetry coefficient given for positive muons stopping in emulsion is evidence for the mu muons being created completely polarized, since it was deduced by studying mu mesons produced

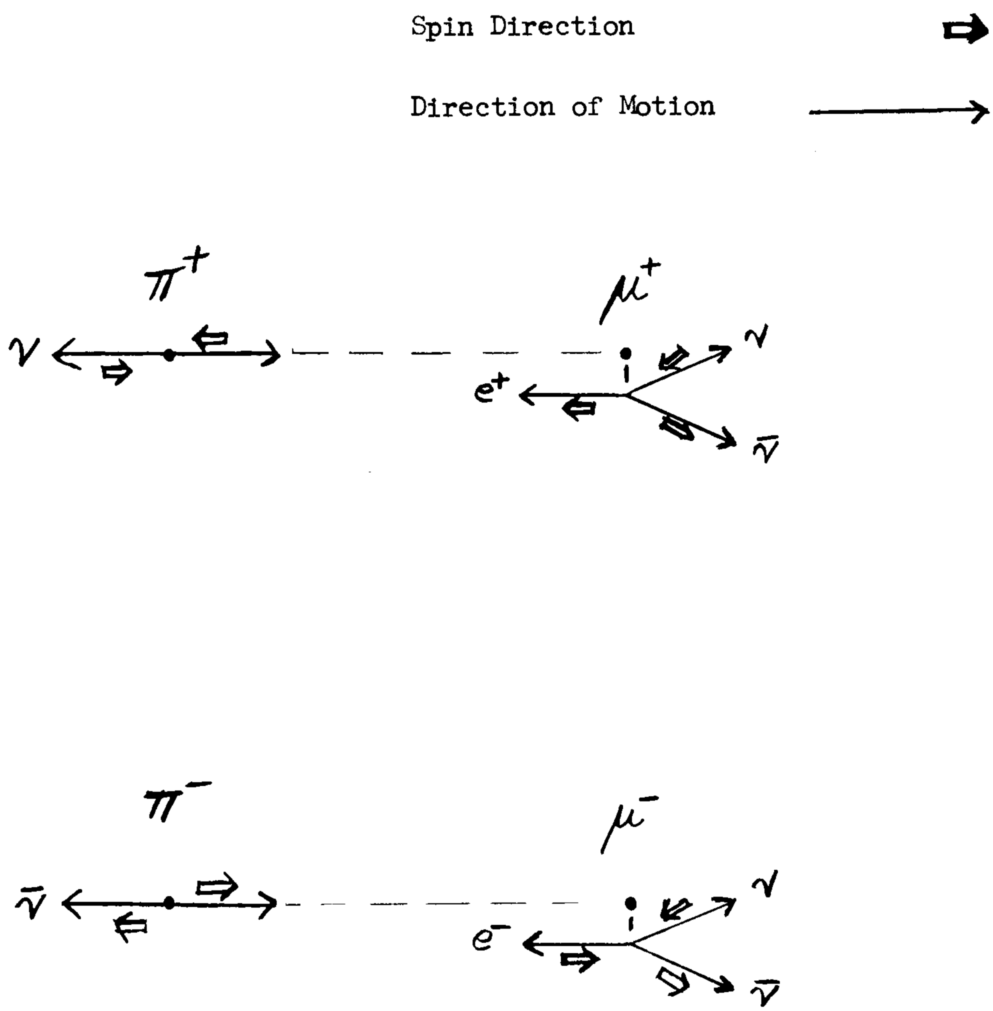

Figure 1. Schematic Representation of π - μ - e Decay Chain

TABLE I

Observed Asymmetry Coefficients for the Decay of Mu Mesons

Meson	Stopping Material	Asymmetry Coefficient
-	Hydrogen	0.01 ± 0.01
-	Carbon	0.04 ± 0.005
-	Oxygen	0.043 ± 0.005
-	Magnesium	0.058 ± 0.008
-	Sulfur	0.042 ± 0.006
-	Zinc	0.056 ± 0.012
-	Cadmium	0.055 ± 0.012
-	Lead	0.054 ± 0.013
-	Carbon	0.054 ± 0.006
-	Helium	0.024 ± 0.01
-	Magnesium	0.036 ± 0.003
-	Carbon	~ 0.05
+	Carbon	0.33 ± 0.03
+	Emulsion	0.303 ± 0.024

by the decay of pi mesons which decayed after they had stopped[10]. The study of mu meson polarization in emulsion is complicated by the fact that there is a large depolarization of positive mu mesons due to the emulsion. This was prevented in the study mentioned by applying a strong magnetic field. In view of this known complication, we present the emulsion datum but prefer to base our principle arguments on other data.

The asymmetry coefficient for positive mu mesons stopped in carbon was observed for muons produced by the decay of pi mesons in flight[11]. This value is the maximum value of the asymmetry coefficient and this means that the positive muons were created completely polarized and that they retained this polarization until they decayed. This value for the asymmetry coefficient for positive muon decay under the circumstances mentioned has been verified recently in an extremely precise experiment[12]. Thus we emphasize this point; under certain conditions positive mu mesons are experimentally observed to be completely polarized when they decay. The implications of this fact are discussed in section 6, below.

We now consider the experimental asymmetry coefficients for negative muon decay. From Table I it is seen that the maximum asymmetry coefficient for negative mu mesons does not exceed 0.06. It

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10. G. Lynch, J. Orear, and S. Rosendorff, Bull. Am. Phys. Soc., 4, 82 (1959). For a review of early emulsion work concerning the asymmetry coefficients and additional data concerning the positive mu mesons see D. H. Wilkinson, Nuovo Cimento 6, 516 (1957).

11. R. L. Garwin, L. M. Lederman, and M. Weinrich, Phys. Rev. 105, 1415 (1957).

12. R. L. Garwin, D. P. Hutchinson, S. Penman, and G. Shapiro, Phys. Rev. 118, 271 (1960).

is also seen that the asymmetry coefficient depends on the element in which the mesons are stopped. Thus we can state that the negative mu mesons suffer severe depolarization and that this depolarization depends on the stopping material.

In Table I we point out that the first eight values presented are the work of a single group[13]. It is instructive to compare the values given by this group with those obtained by other workers for magnesium and carbon[14,15]. Clearly there is room for refinement in the experimental techniques.

5. Statement of the Problem

The problem that is solved in the following chapters is implicitly stated above. Positive mu mesons are observed to remain completely polarized under certain conditions; under identical conditions negative mu mesons retain less than twenty per cent of their original polarization. The problem is to account for this difference in a quantitative fashion. To do this requires a treatment of every physical process that can contribute to the depolarization. However, one can gain sufficient understanding of many such processes by a careful comparison of the properties of positive and negative mu mesons. Therefore the

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13. A. E. Ignatenko, L. B. Egorov, B. Khalupa and D. Chultem, Soviet Phys. - JETP, 35(8), 792 (1959).

14. W. F. Baker and C. Rubbia, Phys. Rev. Letters, 3, 179 (1959).

15. R. Prepost, V. W. Hughes, S. Penman, D. McColm, and K. Ziock, Bull. Am. Phys. Soc. 5, 75 (1960).

solution of the problem may be subdivided into two steps. First one must consider the well known parts of mu meson physics and determine where there is an essential difference in the behavior of positive and negative mu mesons. This qualitative analysis will then indicate which mechanisms must be considered in detail so as to follow the process of depolarization. Therefore we turn our attention to the life history of mu mesons and consider the various events relevant to this purpose.

6. The Interaction of Mu Mesons with Matter

Since mu mesons are Dirac particles their interaction with matter will be precisely the same as for electrons except in that there are differences due to the meson's greater mass. These differences are pointed out in their proper context. In order to present the essential facts with maximum clarity we consider three different stages in the life history of mu mesons.

a. The first stage in the lifetime of the mu mesons we consider is the following. The mu meson is born in the decay of a pi meson in flight. The energy of the mu meson in the laboratory system is of the order of 100 Mev. The mu meson then looses energy by passing through various stopping materials until it has an energy of several kev. Thus as stage one we consider factors important during this period of the meson's lifetime.

Fermi and Teller have shown that this stage is complete within 10^{-9} secs after the meson is created[16]. Therefore a negligible number

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16. E. Fermi and E. Teller, Phys. Rev. 72, 399 (1947).

of the mu mesons decay during this period. The first interaction of the mu meson after it is created is with the accelerator's magnetic fringing field. If there is any depolarization due to the magnetic field, it should be the same for both the positive and negative mu mesons. There is, however, no reason to expect any depolarization because of the fringing field. Case has discussed this point in the literature and proved that such a field does not depolarize Dirac particles[17].

As discussed in section 3 above, the mu mesons are created with complete polarization, in the rest frame of the pi meson. Since the laboratory energy of the pi mesons is quite high one might expect a reduction of the polarization in the transformation from the rest frame of the pi meson to the laboratory frame. This problem has been discussed in the literature and it was shown that if the mu mesons are obtained as a well collimated beam then the loss of polarization is negligible[18]. Again, this effect would be the same for positive and negative mu mesons.

Now we consider the high energy scattering. It is known from electron theory that the sign of the charge has little to do with high energy scattering or ionization. Consequently positive and negative mu mesons will be scattered in the same fashion to a very good approximation. Further we note that the scattering is principally small

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17. K. M. Case, Phys. Rev. 106, 173 (1957).

18. J. H. P. Jenson and H. Overas, Det. Kongeleige Norske Videnskabers Selskabs for Handlinger 31, 34 (1958).

angle scattering even in inelastic events such as ionization. The depolarization expected from such scattering is very small. This problem including the effects of multiple scattering has been treated in detail in the literature[19].

The effects just mentioned are the only matters of importance during stage one. We conclude that there is no theoretical reason why the mu mesons, either positive or negative, should be depolarized significantly during this stage. This conclusion is verified by the fact that the positive mu mesons are found to retain their polarization in carbon. One might argue that the positive mu mesons do not retain their polarization in certain other substances, which is true; however, variations in the interactions of mu mesons with, say, carbon and emulsion are variations which occur at very low energies, not at the energies involved in stage one. Since there is no significant difference in the interactions of positive and negative mu mesons during stage one we assert that both species pass through this stage without suffering appreciable depolarization.

b. Now we define the second stage in the history of mu mesons. This is the stage in which the behavior of positive and negative mu mesons beomes totally different. Before we define the end of this stage

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19. B. Muhlschlegel and H. Koppe, Z. Physik 150, 496 (1958). In connection with depolarization in scattering see also L. Wolfenstein, Phys. Rev. 75, 1664 (1949) and G. W. Ford and C. J. Mullin, Phys. Rev. 108, 477 (1957). These studies all conclude that the depolarization due to scattering should be small.

we consider the events that are occurring at its beginning. The mu mesons have energies of several kilovolts and are being scattered. Since mu mesons of this energy have velocities small compared to the velocity of light, magnetic forces are small and the scattering is essentially spin independent. Thus neither the positive nor the negative mu mesons suffer depolarization from the scattering. There are other effects and we now define the process that we call Auger capture.

The process of Auger capture consists of a mu meson in a free state interacting with an atom to eject an atomic electron and leave the meson in a bound state. This process is possible because the meson has greater mass than the electron and therefore has lower lying energy levels in the Coulomb field. Clearly only negative mu mesons undergo Auger capture. This then is a process that involves negative mu mesons but not positive mesons; consequently there is a mechanism that can lead to differences in the asymmetry coefficients. Stage two for the mu mesons then ends for the negative mu mesons when they are captured by an atom, and for the positive mu mesons when they have been slowed down to energies roughly equivalent to the energies involved in molecular binding. From this point on the histories of the two species are totally different. The positive mu mesons would resemble a proton in their chemical interaction and therefore may form positive ions in some materials. Since all of stage two occurs in times of the order of 10^{-13} sec, the total elapsed time since the creation of the mu mesons is still of the order of 10^{-9} sec, thus a negligible fraction the mesons decay before the end of stage two.

At this point it is possible to give a qualitative outline of the processes that must be dealt with in detail. We have been unable to find any significant difference in the behavior of positive and negative mu mesons until the negative mu mesons are captured by an atom. Therefore, the depolarization of negative mu mesons is due to this capture and to subsequent events; these then are the processes that must be given an analytical treatment so that the amount of depolarization may be understood.

c. Stage three is defined only for the negative mu mesons and includes all events subsequent to the capture of the meson into an atomic bound state. At this point we only mention that the mu meson is captured into atomic states of high-excitation and then must make a series of transitions to reach the atomic ground state. These matters are discussed in detail in the following section. The mesons reach the atomic ground state in times of the order of 10^{-12} sec and, therefore, the total time between the creation of a negative mu meson and its arrival in an atomic ground state is around 10^{-9} sec. Consequently, the mu mesons either decay from the ground state of a mu-mesic atom or are captured from this state by the following reaction.

$$\mu^- + p \rightarrow n + \nu \qquad (1.9)$$

where p is a nuclear proton and n, a neutron. Except for the electromagnetic coupling, this is the only known interaction of mu mesons with nuclei. The competition of this process with the mu decay merely decreases the number of mu mesons decaying and is of no interest to the problem considered here[20]. Clearly this process will not interfere with experimental observation of the decay electrons.

Thus far we have established that the solution to the depolarization of negative mu mesons is to be found in the formation of mu mesic atoms and in the subsequent de-excitation processes that occur in these atoms. We now discuss the general properties of mu mesic atoms as a preliminary to the subsequent chapters.

7. Mu-Mesic Atoms

A mesic atom is an atom containing a meson in a bound state. Since the negative mu meson participates in electromagnetic processes just as a heavy electron, the theory of ordinary atoms applies to mu-mesic atoms with the only change being a replacement of the electron rest mass by the mu meson rest mass. Consequently, there is an adequate theoretical framework for the discussion of processes occurring after a mu meson is captured by an atom. However, as was pointed out in the preceding section, we must also understand the process of capture. We now give a qualitative discussion of this capture mechanism.

The mu meson must make a transition from a free state to an atomic bound state. The mechanism which induces this transition is the electrostatic interaction of the meson with the atomic electrons and this re-

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20. For a discussion of the mu meson proton reaction, see the previously cited review by Rainwater. This reaction is, of course, due to the weak coupling.

sults in the ejection of one of the electrons. Ir would be possible to consider the capture of the meson by a radiative process. It is shown in Chapter II that a typical cross section for capture by electron ejection is 0.1 πa_e^2 where a_e is the electron Bohr radius, whereas the known cross sections for radiative capture would be a thousand times smaller[21]. Therefore we need only consider the electron ejection mechanism which we call Auger capture, in analogy with the normal Auger effect. One notes that almost all of the atomic states are accessible to the meson and that these states will all compete in the capture of the meson. Consequently, many of the mesons will be captured into states having high excitation. We must now consider the properties of these highly excited states and understand how they liberate the energy of excitation. If effects due to screening, finite size and relativity are neglected the energy levels of a mu-mesic atom are given by:

$$E_n = -\frac{1}{2}\left(\frac{Z}{n}\right)^2 \alpha^2 m_\mu c^2 \qquad (1.10)$$

where α is the fine structure constant, m_μ the mu meson rest mass and n the principle quantum numbers. This is the same as the energy relation for normal hydrogen-like atoms if m_μ is replaced by the electron mass. Thus the energy of any given level in a mesic atom is about 207 times the energy of the corresponding electron level in a normal atom. The fact that the bound level for the mu mesons lie so much lower in energy than for the electrons explains why the mesons are captured.

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21. Hans A. Bethe and Edwin E. Salpeter, Quantum Mechanics of One and Two Electron Atoms (Academic Press, Inc., New York, 1957) Chap. 4, p. 322.

The radii of the Bohr orbit of the mu meson are given by:

$$a_{\mu}(n) = \frac{n^2}{\alpha Z} \frac{h}{m_{\mu} c} \qquad (1.11)$$

Thus for comparison, the Bohr radius of the 1s state in hydrogen is about 0.5×10^{-8} cm, whereas the radius for the 1s state of the mesic hydrogen atom is about 0.25×10^{-10} cm. Two facts are immediately obvious from the foregoing; namely, that the effect of electron screening on the meson will be small and that in heavy elements the effects of finite nuclear size will be significant. The effects of finite nuclear size on the mesic energy levels have been considered by Wheeler[22]. We do not consider these effects nor do we consider any other corrections to the hydrogen-like energy level[23]. The justification for this neglect will be presented at the pertinent point.

It is of interest to consider the transition energies involved in mu-mesic atoms. In Table II we present some transition energies based on equation (1.10). These are taken from the previously cited review by Rainwater. The experimental evidence for the existence of mu mesic atoms was the observation of such X rays corresponding to the

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22. J. A. Wheeler, Revs. Modern Phys. 21, 133 (1949).

23. There are many such corrections. Their study has led to additional confirmation of certain aspects of field theory and to information concerning nuclear radii. For a review dealing with these subjects see: M. B. Stearns, Prog. in Nuclear Phys. 6, 108 (1957). It is of interest to note that the mesic 2p 1s transition in Pb has an energy of 14 Mev according to equation (1.10). The true transition energy is 6 Mev. The difference is due to finite nuclear size. In this connection see: D. L. Hill and K. W. Ford, Phys. Rev. 94, 1617 (1954).

TABLE II

Transition Energies in Mu-mesic Atoms

Element	Transition	Energy (kev)
C	2p - 1s	76
N	3d - 2p	19
O	5g - 4f	4
Ca	5g - 4f	26
Zn	5g - 4f	57
Br	5g - 4f	78

appropriate mesonic transitions. These X rays were first observed by Chang[24]. They have since been studied in detail[25]. Our interest in this x-radiation is not so much the energy as the transition rate, since there are competing processes. The theory of radiative transitions is well known and is easily applicable to the mesonic transitions. It will suffice to state here that the transitions of interest are electric dipole and that they are of exactly the same nature as the transitions in a normal hydrogen-like atom. The process that competes with the radiative transitions is the Auger effect. We now discuss this effect and the nature of the competition.

The Auger effect in mesic atoms was first investigated by Wheeler[26]. This process is just the same as the normal Auger effect; namely, the mu meson is in an excited state and there are bound electrons in the same atom, the meson makes a transition to a state of lower excitation by ejecting one of the electrons. Such electrons have been observed in photographic emulsion[27]. There has, however, not been an extensive study of these electrons. It is possible to calculate the transition rate for the Auger process in mesic atoms in a straightforward fashion.

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24. W. Y. Chang, Revs. Modern Phys. 21, 166 (1949).

25. M. B. Stearns and M. Stearns, Phys. Rev. 105, 1573 (1957).

26. Wheeler, loc. cit. It is also shown in this paper that the process of internal pair production (possible for $Z > 26$) does not compete favorably with the other processes.

27. E. H. S. Burhop, The Auger Effect (Cambridge University Press, London, 1952) Chap. 7, p. 162.

The most informative calculations are those of Burbridge and de Borde[28]. These are not complete in that only certain types of transitions are considered, however, they do show that to a good approximation, the selection rules for the Auger process are the same as those for the electric dipole radiative transitions. This fact will be shown to be of considerable value for our purpose in Chapter V. In Table III we present some typical values for Auger and radiative transition rates. These are taken from a table due to Burhop[29]. There are two important points; namely, the Auger rates are essentially independent of Z, the nuclear charge, whereas the radiative rates are proportional to Z^4 and at moderate excitation the Auger transitions are much faster than the radiative transitions. For our purposes we take the mechanism by which mu-mesic atoms are de-excited to be the proper combination of radiative and Auger transitions. Other processes have been proposed in the literature. We now mention these and give the reasons why we do not consider such processes in detail.

It has been suggested that mesic atoms may make collisions with other atoms and that these collisions may lead to either an exchange of the meson between the two atoms or to ejection of an electron from the

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28. G. R. Burbidge and A. H. de Borde, Phys. Rev. 89, 189 (1953) and also A. H. de Borde, Proc. Phys. Soc. (London) A67, 57 (1954).

29. Burhop, *op. cit.*

TABLE III

Mu-mesic Transition Probabilities (sec^{-1})

Transition n, ℓ	Transition n', ℓ'	Type of Transition	Z = 5	Z = 20
7,6	6,5	Radiative	9.96×10^{10}	2.6×10^{13}
4,3	3,2	Auger	4×10^{13}	5.4×10^{13}
4,3	3,2	Radiative	1.86×10^{12}	4.8×10^{14}
3,2	2,1	Auger	6.0×10^{12}	9.0×10^{12}
3,2	2,1	Radiative	8.68×10^{12}	2.25×10^{15}
2,1	1,0	Auger	2.1×10^{11}	3.1×10^{11}
2,1	1,0	Radiative	8.44×10^{12}	2.15×10^{16}

second atom with the mesic atom going to a state of lower excitation[30]. In the case of hydrogen such mechanisms may be reasonable since the mesic hydrogen atom is a small neutral system and therefore could penetrate another atom. For other mesic atoms these processes appear to be somewhat unreasonable; the arguments have been presented in the literature[31]. There has also been a conjecture that collisions with other atoms could induce transitions between the fine structure levels in mesic atoms[32]. For certain light elements the level structure is such that this is plausible, however, those proposing the mechanism could exhibit no reason why the effect should be competitive with either Auger or radiative transitions. There is some experimental data which points to an inadequacy in the theory of the Auger effect in light mesic atoms[33]. This data indicates that the predicted Auger transition rate is too low for mesic atoms lighter than carbon. This point remains unsettled; however, for reasons which we present in Chapter V it is of little consequence to the problem solved herein. In any case, it is of interest to point out that the validity of the data mentioned has been questioned in a recent paper[34]. We now turn to a qualitative discussion of the depolarization suffered by the negative mu mesons in connection with the processes occurring in mesic atoms.

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30. T. B. Day and P. Morrison, Phys. Rev. 107, 912 (1957).
31. J. Bernstein and T. Y. Wu, Phys. Rev. Letters 2, 404 (1959).
32. N. A. Krall and E. Gerjuoy, Phys. Rev. Letters 3, 142 (1959).
33. M. B. Stearns and M. Stearns, Loc. cit.
34. R. A. Ferrell, Phys. Rev. Letters 4, 425 (1960).

8. The Process of Depolarization

We anticipate the results of the following chapters and state here that the observed depolarization of the negative mu mesons may be accounted for by an adequate treatment of the meson capture and the subsequent radiative and Auger transitions. The reason for the depolarization in the capture is that the orbital motion of the mu meson causes the meson to experience a magnetic field which by virtue of the mesons magnetic moment causes the spin to precess. Quantum mechanically this states that the effect of the spin-orbit coupling is to mix the spin states of the captured meson. The additional depolarization due to the various transitions depends upon the number and type of transitions and therefore it is determined by the initial state of the meson (immediately following capture). One may think of this depolarization in transitions as an additional manifestation of the effects of spin orbit coupling since depolarization occurs only in transitions in which there is a change in the nature of the coupled state. A more specific discussion of this point is given in Chapter V. We now wish to mention the effect of nuclear spin.

Since it is our motivation to account quantitatively for the observed depolarization we wish to check our results as precisely as possible against experiment. At the moment, it is not possible to do so to a thoroughly satisfying extent since the experimental errors quoted in Table I are quite large. Except for hydrogen the elements listed in Table I are composed principally of spin zero isotopes. These are the elements which show the largest asymmetry coefficients, and thus the relative uncertainty in their measured asymmetry coefficients is the smallest. For this reason we find no need to consider the effects of nuclear spin in detail; however, for completeness we give the following information. The effect of nuclear spin is to split the levels of the bound mu mesons through the hyperfine interaction. This leads to greatly increased depolarization. The depolarization due to the hyperfine coupling has been investigated for certain types of radiative cascades in atoms with nuclear spin $\frac{1}{2}$.[35] Applying the published results in an approximate fashion leads to the conclusion that the asymmetry coefficient observed for atoms with spin $\frac{1}{2}$ nuclei will be less than one third of the coefficient observed in atoms with zero nuclear spin.

It is of interest to consider briefly the asymmetry coefficient for hydrogen given in Table I. One might be tempted to use our final results and the immediately preceeding remarks to predict an asymmetry coefficient for hydrogen. This procedure would not be meaningful since there are two effects not considered. These are peculiar to the isotopes of hydrogen and occur because the experiments require hydrogen as a liquid. The mu mesic hydrogen atom can exchange its meson with a normal hydrogen atom[36]. In such a transition one might expect some depolarization. There is also the formation of mu mesic hydrogen mole-

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35. M. E. Rose, Bull. Am. Phys. Soc. 4, 80 (1959).

36. V. B. Beliaev and B. N. Zakharev, Soviet Phys. - JETP 35(8), 696 (1959).

cules which also depolarizes[37]. Thus we state that we find no inconsistency in the observed asymmetry coefficient for hydrogen. We now give an outline of the means by which we predict asymmetry coefficients in the case of spin zero nuclei.

9. Program of Analysis

The problem of accounting for the observed asymmetry coefficients is solved by proceeding in the steps outlined below. The method of analysis is applicable to any element containing only spin zero isotopes; however, throughout the following chapters we will consider carbon to be the element of principle interest. The reasons why we are especially concerned with carbon are the following: It consists almost entirely of spin zero isotopes, thus there need be no correction for hyperfine complications and it is a common stopping material for meson experiments, thus there is experimental data from several sources. The steps in the analysis are as follows:

a. The problem of computing the capture cross section for the formation of mu-mesic atoms is formulated and reduced to a problem suitable for machine computation. This is the subject of Chapter II.

b. Since the mu-mesons are captured strongly over a considerable energy range it is not sufficient to know only the capture cross sections as a function of energy. To get the distribution of the mu

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37. Ia. B. Zeldovich and S. S. Gershtein, Soviet Phys. - JETP 35(8), 451 (1959).

mesons among the states of the capturing atom it is necessary to take account of the number of mesons captured at each energy increment as they are slowed down. A slowing mechanism is introduced and the problem of capture is solved. The results obtained are discussed in reference to certain previous assumptions by others. These matters are treated in Chapter III.

c. Given the initial distribution in atomic states on capture one must calculate the depolarization due to the capture. We show how to calculate the depolarization for two extreme cases. It is shown that the amount of scattering before capture determines which, if either, of these cases has physical meaning. Using a conclusion which we show to be very well justified we find that the polarization after capture may be calculated in a fashion totally independent of assumptions concerning the atomic model. These results are derived in Chapter IV.

d. After the mu mesons are captured they undergo a cascade to the atomic ground state. The problem of depolarization in the various transitions is solved. Results are presented which show the importance of the Auger transitions in causing depolarization. Certain illustrative data concerning the radiative transitions is also given. The polarization of the emitted X rays is discussed. Thus in Chapter V we present a theoretical asymmetry coefficient.

CHAPTER II

THE CROSS SECTION FOR AUGER CAPTURE

In this chapter we calculate the cross section for an incident mu meson to be captured into an atomic state by the ejection of an atomic electron. This calculation is carried out by using first order perturbation theory. The wave functions of the free particles are taken to be plane waves. The wave functions of the bound particles are those appropriate for a hydrogen like atom.

1. The Interaction

The total hamiltonian for a meson and electron in the field of a nucleus of charge Z is:

$$H = H^{o} + H' = -\frac{\hbar^2 \nabla_1^{\,2}}{2m_\mu} - \frac{\hbar^2 \nabla_2^{\,2}}{2m_e} - \frac{Ze^2}{r_1} - \frac{Ze^2}{r_2} + \frac{e^2}{r_{12}} \qquad (2.1)$$

where subscripts, 1, refer to meson coordinates, subscripts, 2, refer to electron coordinates, r_{12} is $\left|\vec{r}_1 - \vec{r}_2\right|$. We consider two different decompositions of this hamiltonian into components H^{o} and H'. For a meson approaching from infinity the unperturbed hamiltonian, H_i^{o}, is given by:

$$H_i^{o} = -\frac{\hbar^2 \nabla_1^{\,2}}{2m_\mu} - \frac{\hbar^2 \nabla_2^{\,2}}{2m_e} - \frac{Ze^2}{r_2} \qquad (2.2)$$

If the initial wave function is defined as:

$$\Psi = \Psi_{free}(\mu)\, \Psi_{bound}(e) \qquad (2.3)$$

then clearly:

$$H^o \Psi = E^o \Psi = (E_\mu + E_e) \tag{2.4}$$

is satisfied. The perturbation is, in this decomposition,

$$H_i' = - \frac{Ze^2}{r_1} + \frac{e^2}{r_{12}} \tag{2.5}$$

If we now consider the system after interaction H_f^o must be:

$$H_f^o = - \frac{\hbar^2 \nabla_1^{\,2}}{2m_\mu} - \frac{\hbar^2 \nabla_2^{\,2}}{2m_e} - \frac{Ze^2}{r_1} \tag{2.6}$$

and,

$$H_f' = - \frac{Ze^2}{r_2} + \frac{e^2}{r_{12}} \tag{2.7}$$

The effect of applying H_f^o to the wave function of the final state,

$$\Phi = \Phi_{bound}(\mu) \, \Phi_{free}(e) \tag{2.8}$$

is:

$$H_f^o \Phi = \mathcal{E}^o \Phi = (\mathcal{E}_\mu + \mathcal{E}_e) \Phi \tag{2.9}$$

Although the requirement of conservation of energy is that:

$$\mathcal{E}_\mu + \mathcal{E}_e = E_\mu + E_e \tag{2.10}$$

the wave functions Ψ and Φ are eigenfunctions of different operators and are not orthogonal. Thus we must determine the proper formulation of the perturbation theory.

We define the exact wave function of the system, Π, to be an eigenfunction of H such that

$$H \Pi = i \, h \frac{\partial \Pi}{\partial t} \tag{2.11}$$

The wave functions Ψ and the wave functions Φ both form complete sets. Therefore we may expand Π in terms of either. When we remove the time dependence from a wave function we lower the case of the latter. Using n and m as eigenstate labels we write:

$$\Pi = \sum_n (a_n^o + a_n^! + \ldots) \psi_n e^{-i\frac{E_n^o}{\hbar}t} \tag{2.12}$$

when the expansion coefficients of the orders in t are explicitly indicated. Alternatively,

$$\Pi = \sum_m (b_m^o + b_m^! + \ldots) \phi_m e^{-i\frac{\mathcal{E}_m^o}{\hbar}t} \tag{2.13}$$

We may now write:

$$\left(H_f^o - i\hbar \frac{d}{dt}\right)\Pi = - H_f' \Pi \tag{2.14}$$

and then use either (2.12) or (2.13) as we choose:

$$\left(H_f^o - i\hbar \frac{d}{dt}\right)\sum_m (b_m^o + b_m^! + \ldots) \phi_m e^{-\frac{i\mathcal{E}_m^o}{\hbar}t} =$$

$$- H_f' \sum_n (a_n^o + a_n^! + \ldots) \psi_n e^{-\frac{iE_n^o}{\hbar}t} \tag{2.15}$$

we now multiply (2.15) by $\phi^*_{m'}$ and integrate over all coordinate space[1]. This gives in first order:

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1. Note that we use a standard notation for the matrix elements. The notation (X, AY) means explicitly $\int X^* AYdV$ where dV is the appropriate volume element.

$$b_m^1 = \frac{1}{i\hbar} (\phi_m, H_f' \psi_n) e^{-i(E_n^o - \mathcal{E}_m^o)t/\hbar} a_n^o \qquad (2.16)$$

From this point the treatment is the same as that of standard perturbation theory as given, for example, by Schiff[2]. The result for the transition probability is:

$$w = \frac{2\pi}{\hbar} \left| (\phi_m, H_f' \psi_n) \right|^2 \rho(\mathcal{E}_e) \qquad (2.17)$$

With the requirement that all quantities not observed be summed over. H_f' is defined by (2.7). The term in $\frac{1}{r_2}$ in H_f' arises because we use plane waves for the free particles; if we had used Coulomb wave functions in the free states this term would not have occurred because the hamiltonian would have been decomposed differently.

2. The Cross Section

We now apply equation (2.17) to determine the cross section. The mu mesons are taken as plane waves incident along the axis of quantitazation and as having their spin along the direction of motion. The cross sections will not depend on these specifications; they are taken here so that certain intermediate results may be used in the following chapters. The initial state of the electron is taken as the ground state of a hydrogen-like atom. It would be possible to consider electrons outside of the K shell. We do not do so because their greatly reduced binding energy means that they would contribute only at incident meson energies much lower than we need consider. The final state for

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2. Leonard I. Schiff, Quantum Mechanics (McGraw Hill Book Company, Inc., New York, 1955), 2nd ed., Chap. 8, p. 197.

the electron is a plane wave. We discuss the approximations implicit in this treatment in Section 3 below.

For the initial state

$$\psi_n = \psi(\mu)\,\psi(e) \qquad (2.18)$$

with

$$\psi(\mu) = \frac{1}{\sqrt{V}} \sum_{\ell_1} \sqrt{4\pi(2\ell_1+1)}\; i^{\ell_1}\, Y_{\ell_1 0}(\hat{r}_1)\, j_{\ell_1}(k_1 r_1)\, \chi_{\frac{1}{2}}^{\tau_1} \qquad (2.19)$$

where the Rayleigh expansion has been used for the plane wave. V is a normalization volume, $Y_{\ell_1 0}(\hat{r}_1)$ is a spherical harmonic, $j_{\ell_1}(k_1 r_1)$ is a spherical Bessel function and

$$\chi_{\frac{1}{2}}^{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad ; \quad \chi_{\frac{1}{2}}^{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad (2.20)$$

We will choose $\tau_1 = \frac{1}{2}$ at a convenient point.

$$\psi(e) = \frac{1}{\sqrt{4\pi}} R(r_2)\, \chi_{\frac{1}{2}}^{\tau_e} \qquad (2.21)$$

where $R(r_2)$ is a radial function defined below. For the final state:

$$\phi_m = \phi(\mu)\,\phi(e) \qquad (2.22)$$

with

$$\phi(e) = \frac{4\pi}{\sqrt{V}} \sum_{\ell_2, m_2} i^{\ell_2}\, Y^*_{\ell_2 m_2}(\hat{k}_2)\, Y_{\ell_2 m_2}(\hat{r}_2)\, j_{\ell_2}(k_2 r_2)\, \chi_{\frac{1}{2}}^{\tau_2} \qquad (2.23)$$

where the plane wave, $e^{i\vec{k}_2 \cdot \vec{r}_2}$, has been expanded.

$$\phi(\mu) = \sum_{\tau} C(\ell \tfrac{1}{2} j; m-\tau, \tau)\, Y_{\ell, m-\tau}(\hat{r}_1)\, R_{n,\ell}(r_1)\, \chi_{\frac{1}{2}}^{\tau} \qquad (2.24)$$

$R_{n,\ell}(r_1)$ is a radial function defined below. For the Clebsch-Gordan coefficients we follow the notation and conventions of Rose[3]. The choice of $\phi(\mu)$ is dictated by the requirement that the final state wave functions diagonalize the spin-orbit coupling.

We are interested in the cross section for absorption of a partial wave, ℓ_1, with the meson going to a state n, ℓ, j and the electron being emitted into the partial wave ℓ_2. Therefore, the quantity that we refer to as cross section will be the partial cross section for this process, unless otherwise indicated. The cross section is given by:

$$\sigma = \frac{W}{j_{inc}} = \frac{Vw}{v_1} \tag{2.25}$$

where j_{inc} is the incident meson current; V, a normalization volume, and v_1 is the velocity of the incident meson. In all that follows subscripts 1 refer to the meson; subscripts 2 refer to the electron. The transition probability is obtained according to

$$w = \frac{2\pi}{\hbar} \left| (\phi, H_f' \psi) \right|^2 \rho(E_2) \tag{2.17}$$

We first evaluate the density of final states for the ejected electron.

$$\rho(E_2) dE_2 = \frac{V d\vec{p}_2}{(2\pi\hbar)^3} = \frac{V}{(2\pi\hbar)^3} p_2^2 \, d\Omega_{k_2} \, dp_2 \tag{2.26}$$

where $d\Omega_{k_2}$ is the solid angle in the electron direction. At this

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3. M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957). All of the relations used in the following pages concerning the Clebsch-Gordan coefficients and the Racah algebra are proven in this book. A summary of the basic relations is given in Appendix I.

point it is evident that all normalization volumes must cancel in W and we therefore drop the Vs. Using $\frac{dp}{dE} = \frac{E}{c^2 p}$ and taking the energy of the electron as $m_e c^2$ yields:

$$\rho(E) = \frac{m_e^2 v_2}{(2\pi\hbar)^3} \, d\Omega_{k_2} \tag{2.27}$$

Thus the cross section becomes

$$\sigma = \frac{m_e^2 \, d\Omega_{k_2}}{(2\pi)^2 \hbar^4} \left(\frac{v_2}{v_1}\right) \left|(\phi, H_f' \psi)\right|^2 \tag{2.28}$$

and

$$H_f' = e^2 \left(- \frac{Z}{r_2} + \frac{1}{r_{12}}\right) \tag{2.7}$$

taking the e^2 out of H_f' we have

$$\sigma = \left(\frac{v_2}{v_1}\right) \frac{1}{(2\pi a_e)^2} \left|\left(\phi, \left(\frac{1}{r_{12}} - \frac{Z}{r_2}\right)\psi\right)\right|^2 d\Omega_{k_2} \tag{2.29}$$

where $a_e = \frac{\hbar^2}{me^2}$ is the electron Bohr radius. To carry out the angular interaction we use the standard expansion[4]

$$\frac{1}{r_{12}} = \sum_{\lambda, M_\lambda} \frac{4\pi}{2\lambda+1} Y_\lambda^{M_\lambda *}(\hat{r}_1) \, Y_\lambda^{M_\lambda}(\hat{r}_2) \, f_\lambda(r_1, r_2) \tag{2.30}$$

$$\text{where } f_\lambda = \frac{r_<^\lambda}{r_>^{\lambda+1}} \tag{2.31}$$

We now define functions F_λ as follows:

$$F_0 = \frac{1}{r_>} - \frac{Z}{r_2} \quad ; \quad F_\lambda = f_\lambda \quad , \quad \text{if } \lambda \neq 0 \tag{2.32}$$

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4. See, for example, Schiff, op. cit., p. 175.

We now evaluate $\left(\phi, \left(-\frac{Z}{r_2} + \frac{1}{r_{12}}\right)\psi\right)$ which we denote by H'_{fi}.

$$H'_{fi} = (4\pi)^2 \sum_{\lambda, M_\lambda, m_2, \tau} (-i)^{\ell_2} (i)^{\ell_1} \frac{(2\ell_1+1)^{\frac{1}{2}}}{2\lambda+1} \left(Y_{\ell, m-\tau}(\hat{r}_1), Y^*_{\lambda M_\lambda}(\hat{r}_1) \, Y_{\ell_1 0}(\hat{r}_1)\right)$$

$$\times \left(Y_{\ell_2 m_2}(\hat{r}_2), Y_{\lambda M_\lambda}(\hat{r}_2)\right) C(\ell \tfrac{1}{2} j; m-\tau, \tau)\left(\chi^{\tau}_{\frac{1}{2}}, \chi^{\tau_1}_{\frac{1}{2}}\right)$$

$$\left(\chi^{\tau_2}_{\frac{1}{2}}, \chi^{\tau_e}_{\frac{1}{2}}\right) \mathcal{J}(\ell_1, \ell_2, \ell, \lambda) \, Y_{\ell_2 m_2}(\hat{k}_2) \qquad (2.33)$$

Where $\mathcal{J}(\ell_1, \ell_2, \ell, \lambda)$ is the radial integral defined by:

$$\mathcal{J}(\ell_1, \ell_2, \ell, \lambda) = \iint r_1^2 \, dr_1 \, r_2^2 \, dr_2 \, j_{\ell_1}(k_1 r_1) \, R_{n,\ell}(r_1) \, F_\lambda(r_1, r_2)$$

$$\times \; R(r_2) \, j_{\ell_2}(k_2 r_2) \qquad (2.34)$$

We now have the following:

$$\left(\chi^{\tau_2}_{\frac{1}{2}}, \chi^{\tau_e}_{\frac{1}{2}}\right) = \delta_{\tau_2 \tau_e} \qquad (2.35a)$$

$$\left(\chi^{\tau}_{\frac{1}{2}}, \chi^{\tau_1}_{\frac{1}{2}}\right) = \delta_{\tau \tau_1} \qquad (2.35b)$$

From the orthonormality of the spherical harmonics we have:

$$\left(Y_{\ell_2 m_2}(\hat{r}_2), Y_{\lambda m_\lambda}(\hat{r}_2)\right) = \delta_{\ell_2 \lambda} \, \delta_{m_2 M_\lambda} \qquad (2.36)$$

We also have[5]

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5. Rose, op. cit. p. 62. The general form of the relation is given in Appendix I.

$$\left((Y_{\ell,m-\tau}(\hat{r}_1),\ Y^*_{\lambda M_\lambda}(r_1)\, Y_{\ell_1 0}(r_1) \right) = (-)^{M_\lambda} \left[\frac{(2\lambda+1)(2\ell_1+1)}{4\pi(2\ell+1)} \right]^{\frac{1}{2}}$$

$$\times \quad C(\ell_1 \lambda \ell; 0; M_\lambda, m-\tau)\, C(\ell_1 \lambda \ell; 00) \qquad (2.37)$$

Where use has been made of the following two relations

$$Y^*_{\ell m}(\hat{r}) = (-)^m\, Y_{\ell,-m}(\hat{r})$$

and

$$\int Y^*_{\ell_1 m_1}(\hat{r})\, Y_{\ell_2 m_2}(\hat{r})\, Y_{\ell_3 m_3}(\hat{r})\, d\Omega = \left[\frac{(2\ell_3+1)(2\ell_2+1)}{(2\ell_1+1)\, 4\pi} \right]^{\frac{1}{2}}$$

$$C(\ell_3 \ell_2 \ell_1; m_3 m_2 m_1)\, C(\ell_3 \ell_2 \ell_1; 000)$$

$C(\ell_3 \ell_2 \ell_1; 000)$ is called the parity coefficient since it has the value zero unless $\ell_3 + \ell_2 + \ell_1$ = even integer. We replace λ by ℓ_2 and M_λ by m_2 as is permitted by (2.36); but we are interested in the partial cross section for which the electron is emitted into the ℓ_2'th partial wave so we drop the sum on λ. Since τ_1 is to be fixed, we drop the sum on τ as indicated by (2.35b). We now have

$$H'_{fi} = (4\pi)^{3/2} \sum_{m_2} (-i)^{\ell_2}\, i^{\ell_1}\, (-)^{m_2} \left[\frac{(2\ell_1+1)^2}{(2\ell_2+1)(2\ell+1)} \right]^{\frac{1}{2}} C(\ell_1 \ell_2 \ell; 00)$$

$$C(\ell \tfrac{1}{2} j; m-\tau, \tau)\, C(\ell_1 \ell_2 \ell; 0, -m_2, m-\tau)\, Y_{\ell_2 m_2}(\hat{k}_2)\, \mathcal{J}(\ell_1 \ell_2 \ell) \qquad (2.38)$$

We are not concerned with the direction of the emitted electron, therefore we next find $\sum_{m,m_2} \int d\Omega_2\, |H'_{fi}|^2$ since we must sum over the quantities not observed (m). We have:

$$\sum_{m,m_2}\int\left|H'_{fi}\right|^2 d\Omega_2 = (4\pi)^3 \sum_{m,m_2} (-i)^{\ell_2}\, i^{\ell'_2}\, (-i)^{\ell'_1}\, (i)^{\ell_1}$$

$$\times\left[\frac{(2\ell_1+1)^2}{(2\ell_2+1)(2\ell+1)}\right]^{\frac{1}{2}}\left[\frac{(2\ell'_1+1)^2}{(2\ell'_2+1)(2\ell+1)}\right]^{\frac{1}{2}} C(\ell_1\,\ell_2\,\ell;00)\; C(\ell'_1\,\ell'_2\,\ell;00)$$

$$\times\, C(\ell_1\,\ell_2\,\ell;0,-m_2,m-\tau)\; C(\ell'_1\,\ell'_2\,\ell;0,-m_2,m-\tau)\left[C(\ell\,\tfrac{1}{2}j;m-\tau,\tau)\right]^2$$

$$\times\, d(\ell_1\ell_2\ell)\, d(\ell'_1\,\ell'_2\,\ell)\;\left(Y_{\ell'_2 m_2}(\hat{k}_2),\; Y_{\ell_2 m_2}(\hat{k}_2)\right) \qquad (2.39)$$

Orthonormality of the spherical harmonics gives $\delta_{\ell_2\,\ell'_2}$ so there is no interference between partial waves of the ejected electrons. From $C(\ell_1\,\ell_2\,\ell;0,-m_2,m-\tau)$ we have $\tau - m = m_2$; consequently we may rewrite (2.39) as

$$\sum_{m}\int\left|H_{fi}\right|^2 d\Omega_2 = (4\pi)^3\, (-i)^{\ell_1}\, (i)^{\ell'_1}\, \frac{(2\ell_1+1)(2\ell'_1+1)}{(2\ell_2+1)(2\ell+1)}$$

$$\times\, C(\ell'_1\,\ell_2\,\ell;00)\; C(\ell_1\,\ell_2\,\ell;00)\, d(\ell_1\,\ell_2\,\ell)\, d(\ell'_1\,\ell_2\,\ell)\sum_{m}\left[C(\ell\,\tfrac{1}{2}j;m-\tau,\tau)\right]^2$$

$$\times\, C(\ell_1\,\ell_2\,\ell;0,m-\tau)\; C(\ell'_1\,\ell_2\,\ell;0,m-\tau) \equiv \tfrac{1}{2}K \qquad (2.40)$$

For any element other than hydrogen it is necessary to multiply (2.40) by two to account for the two K shell electrons. As we are not principally interested in hydrogen this factor is inserted henceforth. We now carry out the sum over m. Define

$$S_0 = \sum_{m}\left[C(\ell\,\tfrac{1}{2}j;m-\tau,\tau)\right]^2 C(\ell_1\,\ell_2\,\ell;0,m-\tau)\; C(\ell'_1\,\ell_2\,\ell;0,m-\tau) \qquad (2.41a)$$

We use a symmetry relation to rewrite the last two Clebsch-Gordan coefficients.

$$C(\ell_1 \ell_2 \ell; 0, m-\tau) = (-)^{\ell_1+\ell_2+\ell} \; C(\ell_2 \ell_1 \ell; m-\tau, 0)$$

$$C(\ell_1' \ell_2 \ell; 0, m-\tau) = (-)^{\ell_1'+\ell_2+\ell} \; C(\ell_2 \ell_1' \ell; m-\tau, 0)$$

In each of these the phase may be dropped since in (2.40) the parity coefficients vanish unless the sum of their arguments is even. Now we may write:

$$S_o = \sum_m \left[C(\ell_2 \ell_1' \ell; m-\tau, 0)\, C(\ell \tfrac{1}{2} j; m-\tau, \tau) \right]$$

$$\times \left[C(\ell_2 \ell_1 \ell; m-\tau, 0)\, C(\ell \tfrac{1}{2} j; m-\tau, \tau) \right] \qquad (2.41b)$$

It is now necessary to rewrite each of the square brackets in (2.41b) using the Racah recoupling theorem.

$$C(\ell_2 \ell_1 \ell; m-\tau, 0)\, C(\ell \tfrac{1}{2} j; m-\tau, \tau) = \sum_v \left[(2\ell+1)(2v+1) \right]^{\frac{1}{2}}$$

$$\times\, W(\ell_2 \ell_1 j \tfrac{1}{2}; \ell v)\, C(\ell_1 \tfrac{1}{2} v; 0, \tau)\, C(\ell_2 v j; m-\tau, \tau)$$

Where $W(\ell_2 \ell_1 j \tfrac{1}{2}; \ell v)$ is a Racah coefficient. In exactly the same manner:

$$C(\ell_2 \ell_1' \ell; m-\tau, 0)\, C(\ell \tfrac{1}{2} j; m-\tau, \tau) = \sum_{v'} \left[(2\ell+1)(2v'+1) \right]^{\frac{1}{2}}$$

$$\times\, W(\ell_2 \ell_1' j \tfrac{1}{2}; \ell v')\, C(\ell_1' \tfrac{1}{2} v'; 0, \tau)\, C(\ell_2 v' j; m-\tau, \tau)$$

So we now have:

$$S_o = \sum_{v,v'} (2\ell+1)\left[(2v+1)(2v'+1)\right]^{\frac{1}{2}} W(\ell_2\ell_1 j\tfrac{1}{2};\ell v)\, W(\ell_2\ell_1' j\tfrac{1}{2};\ell v')$$

$$\times C(\ell_1'\tfrac{1}{2}v';0,\tau)\, C(\ell_1\tfrac{1}{2}v;0\,\tau) \sum_m C(\ell_2 v' j;m-\tau,\tau)\, C(\ell_2 v j;m-\tau,\tau) \qquad (2.41c)$$

The sum over m is accomplished by using a symmetry relation to rewrite both Clebsch-Gordan coefficients as follows:

$$C(\ell_2 vj;m-\tau,\tau) = (-)^{\ell_2-(m-\tau)}\left(\frac{2j+1}{2v+1}\right)^{\frac{1}{2}} C(j\,\ell_2 v;m,\tau-m)$$

$$C(\ell_2 v'j;m-\tau,\tau) = (-)^{\ell_2-(m-\tau)}\left(\frac{2j+1}{2v'+1}\right)^{\frac{1}{2}} C(j\,\ell_2 v';m,\tau-m)$$

Since $(m-\tau)$ is an integer the phases give unity and the sum over m is:

$$\frac{2j+1}{\left[(2v+1)(2v'+1)\right]^{\frac{1}{2}}} \sum_m C(j\,\ell_2 v;m,\tau-m)\, C(j\,\ell_2 v';m,\tau-m)$$

which by the orthogonality of the Clebsch-Gordan coefficients reduces to

$$\frac{2j+1}{2v+1}\,\delta_{vv'}$$

Therefore:

$$S_o = \sum_v (2\ell+1)(2j+1)\, W(\ell_2\ell_1 j\tfrac{1}{2};\ell v)\, W(\ell_2\ell_1' j\tfrac{1}{2};\ell v)$$

$$\times\ C(\ell_1'\tfrac{1}{2}v;0\,\tau)\, C(\ell_1\tfrac{1}{2}v;0\,\tau) \qquad (2.41d)$$

Now we again consider the two parity coefficients in (2.40). Since $\ell_1 + \ell_2 + \ell =$ even integer and similarly for $\ell_1' + \ell_2 + \ell$ it follows that $|\ell_1 - \ell_1'| = 0, 2, \ldots.$ In this case it is impossible to satisfy $\Delta(\ell_1'\tfrac{1}{2}v)$ and $\Delta(\ell_1\tfrac{1}{2}v)$ unless $\ell_1 = \ell_1'$.[6] Therefore, there is no interference between different partial waves in the cross section and we rewrite (2.41d) as

$$S_o = \sum_v (2\ell+1)(2j+1)\left[W(\ell_2\ell_1 j\tfrac{1}{2};\ell v)\right]^2\left[C(\ell_1\tfrac{1}{2}v;0,\tau)\right]^2 \qquad (2.41e)$$

At this point we assign to τ its value $\frac{1}{2}$ since we wish to specify the initial spin direction. We now use the explicit values of the Clebsch-Gordan coefficient as follows:

$$\left[C(\ell_1\tfrac{1}{2}v;0,\tfrac{1}{2})\right]^2 = \begin{cases} \dfrac{\ell_1+1}{2\ \ _1+1} & \text{for } v = \ell_1 + \tfrac{1}{2} \\[2ex] \dfrac{\ell_1}{2\ \ _1+1} & \text{for } v = \ell_1 - \tfrac{1}{2} \end{cases} \qquad (2.42a)$$

In either case

$$\left[C(\ell_1\tfrac{1}{2}v,0,\tfrac{1}{2})\right]^2 = \frac{1}{2}\left(\frac{2v+1}{2\ell_1+1}\right) \qquad (2.42b)$$

We now make use of a symmetry relation for the Racah coefficients to write

$$w(\ell_2\ell_1 j\tfrac{1}{2};\ell v) = W(\ell_2 j\,\ell_1\tfrac{1}{2};v\ell)$$

and rewrite S_o as:

$$S_o = \frac{2j+1}{2(2\ell_1+1)}\sum_v (2v+1)(2\ell+1)\left[W(\ell_2 j\,\ell_1\tfrac{1}{2};v\ell)\right]^2 \qquad (2.43)$$

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6. The symbol $\Delta(j_1j_2j_3)$ is used to indicate that the three angular momenta must for a triangle, meaning that $|j_1-j_2| \leq j_3 \leq |j_1+j_2|$.

By the orthonormality property of the Racah coefficients

$$\sum_{v} (2v+1)(2\ell+1)\left[W(\ell_2 j \ell_1 \tfrac{1}{2}; v\ell)\right]^2 = 1$$

and therefore

$$S_o = \frac{2j+1}{2(2\ell_1+1)} \tag{2.44}$$

Thus

$$K = (4\pi)^3 \frac{(2\ell_1+1)(2j+1)}{(2\ell+1)(2\ell_2+1)} \left[C(\ell_1 \ell_2 \ell; 0,0)\right]^2 \mathcal{J}(\ell_1 \ell_2 \ell)^2 \tag{2.45}$$

and

$$\sigma = \frac{v_2}{v_1} \frac{1}{(2\pi a_e)^2} K \tag{2.46}$$

At this stage several points may be mentioned. Except for the evaluation of the radial integrals, $\mathcal{J}(\ell_1 \ell_2 \ell)$, the partial cross sections for the Auger capture are given by (2.46). One notes that the two levels, $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$, belonging to a given ℓ are populated in accordance to their statistical weight. If the electron is ejected with zero orbital angular momentum then K becomes

$$K = (4\pi)^3 (2j+1) \mathcal{J}(\ell_1 0 \ell) \; \delta_{\ell_1 \ell} \tag{2.47}$$

and the cross section has a very simple form. Otherwise the Clebsch-Gordan coefficients are evaluated by the relation[7]

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7. Rose, <u>op</u>. <u>cit</u>. p. 47.

$$C(L_1L_2L_3;00) = (-)^{\frac{1}{2}(L_1+L_2-L_3)} \left(\frac{2L_3+1}{L_1+L_2+L_3+1}\right)^{\frac{1}{2}}$$

$$\frac{\tau(L_1+L_2+L_3)}{\tau(L_1+L_2-L_3)\,\tau(L_1-L_2+L_3)\,\tau(-L_1+L_2+L_3)} \tag{2.48}$$

where $\tau(x) = \frac{(\frac{1}{2}x)!}{\sqrt{x!}}$ and $L_1+L_2+L_3$ must be an even integer. We now discuss the evaluation of the radial integrals.

To evaluate the radial integral we must first specify the radial bound state functions. These are taken as those appropriate to a hydrogen-like atom and these are well known. Thus in (2.34) we set:

$$R(r_2) = 2\left(\frac{Z}{a_e}\right)^{3/2} e^{-Zr_2/a_e} \tag{2.49}$$

and

$$R_{n,\ell}(r_1) = -\left\{\left(\frac{2Z}{na_\mu}\right)^3 \frac{(n-\ell-1)!}{2n\left[(n+\ell)!\right]^3}\right\}^{\frac{1}{2}} e^{-\frac{1}{2}x}\, x^{\ell}\, L_{n+\ell}^{2\ell+1}(x) \tag{2.50}$$

where $L_{n+\ell}^{2\ell+1}(x)$ is the associated Laguerre polynomial given as:

$$L_{n+\ell}^{2\ell+1}(x) = \sum_{k=0}^{n-\ell-1} (-)^{k+1} \frac{\left[(n+\ell)!\right]^2 x^k}{(n-\ell-1-k)!\,(2\ell+1+k)!\,k!} \tag{2.51}$$

and

$$x = \frac{2Z}{na_\mu} r_1 \tag{2.52}$$

The normalization is

$$\int_0^\infty \left[R_{n,\ell}(r_1)\right]^2 r_1^2\, dr_1 = 1$$

We now define:

$$\mathcal{d}(\ell_1 \ell_2 \ell) = J - ZG \, \delta_{\ell_2 0} \tag{2.53}$$

Corresponding to the definition (2.32). We now consider the term J, and use the following definitions:

$$R_{n,\ell}(r_1) = \left(\frac{Z}{na_\mu}\right)^{3/2} X(r_1) \tag{2.54a}$$

$$R(r_2) = \left(\frac{Z}{a_e}\right)^{3/2} Y(r_2) \tag{2.54b}$$

The J is:

$$J = \left(\frac{Z}{na_\mu}\right)^{3/2} \left(\frac{Z}{a_e}\right)^{3/2} \int_0^\infty X(r_1)\, r_1^2\, j_{\ell_1}(k_1 r_1)\, V(r_1)\, dr_1 \tag{2.55}$$

where:

$$V(r_1) = \int_0^{r_1} \frac{r_2^{2+\ell_2}\, dr_2}{r_1^{\ell_2+1}}\, Y(r_2)\, j_{\ell_2}(k_2 r_2)$$

$$+ \int_{r_1}^{\infty} \frac{r_1^{\ell_2}\, dr_2}{r_2^{\ell_2-1}}\, Y(r_2)\, j_{\ell_2}(k_2 r_2) \tag{2.56}$$

We use the definitions (2.52) and the following

$$y = \frac{Zr_2}{a_e} \tag{2.57a}$$

$$b_2 = \frac{k_2 a_e}{Z} \tag{2.57b}$$

$$b_1 = \frac{nk_1 a_\mu}{Z} \tag{2.57c}$$

$$b_o = \frac{na_\mu}{2a_e} \tag{2.57d}$$

to change variables in (2.55); we then have:

$$J = \left(\frac{Z}{na_\mu}\right)^{3/2} \left(\frac{Z}{a_e}\right)^{3/2} \left(\frac{na_\mu}{2Z}\right)^3 \int_o^\infty X(x)\, x^2\, j_{\ell_1}(b_1 x)\, V(x)\, dx \tag{2.58}$$

where:

$$V(x) = \left(\frac{a_e}{Z}\right)^{\ell_2+3} \left(\frac{2Z}{na_\mu}\right)^{\ell_2+1} \int_o^{b_o x} Y(y)\, j_{\ell_2}(b_2 y)\, \frac{y^{\ell_2+2}\, dy}{x^{\ell_2+1}}$$

$$+ \left(\frac{Z}{a_e}\right)^{\ell_2-1} \left(\frac{na_\mu}{2Z}\right)^{\ell_2} \int_{b_o x}^\infty Y(y)\, j_{\ell_2}(b_2 y)\, \frac{x^{\ell_2}\, dy}{y^{\ell_2-1}} \tag{2.59}$$

Therefore J is rewritten as

$$J = 2\left(\frac{Z}{na_\mu}\right)^{3/2} \left(\frac{Z}{a_e}\right)^{3/2} \left(\frac{na_\mu}{2Z}\right)^3 \left(\frac{a_e}{Z}\right)^2 \left\{\left(\frac{1}{b_o}\right)^{\ell_2+1}\right.$$

$$\times \int_o^\infty x^{1-\ell_2}\, j_{\ell_1}(b_1 x)\, X(x)\, K_1(x)\, dx$$

$$\left. + b_o^{\ell_2} \int_o^\infty x^{2+\ell_2}\, j_{\ell_1}(b_1 x)\, X(x)\, K_2(x)\, dx \right\} \tag{2.60}$$

where

$$K_1(x) = \int_o^{b_o x} y^{\ell_2+2}\, e^{-y}\, j_{\ell_2}(b_2 y)\, dy \tag{2.61a}$$

and

$$K_2(x) = \int_{b_o x}^{\infty} y^{1-\ell_2} e^{-y} j_{\ell_2}(b_2 y) \, dy \tag{2.61b}$$

This defines the term J. It is possible to carry out all of the integrations analytically. Except for a special case, $\ell_2 = 0$ and a circular capturing orbit, this is of little value since the results are obtained as multiple sums in a form not suited for numerical evaluation. Therefore in the term J the integrals were done numerically. We defer discussion of the results until after the evaluation of G. For a purpose that will be obvious presently we define

$$J = 2 \left(\frac{Z}{na_\mu}\right)^{3/2} \left(\frac{Z}{a_e}\right)^{3/2} \left(\frac{na_\mu}{2Z}\right)^3 \left(\frac{a_e}{Z}\right)^2 I \tag{2.62}$$

Using the definitions (2.54) the term G is

$$G = \left(\frac{Z}{na_\mu}\right)^{3/2} \left(\frac{Z}{a_e}\right)^{3/2} \int_0^\infty \int_0^\infty X(r_1) \, r_1^2 \, j_{\ell_1}(k_1 r_1) \, Y(r_2) \, j_{\ell_2}(k_2 r_2) \, r_2 \times dr_2 \, dr_1 \tag{2.63}$$

and on changing variables as before and substituting for $Y(r_2)$ we get

$$G = 2 \left(\frac{Z}{na_\mu}\right)^{3/2} \left(\frac{Z}{a_e}\right)^{3/2} \left(\frac{na_\mu}{2Z}\right)^3 \left(\frac{a_e}{Z}\right)^2 \int_0^\infty X(x) \, x^2 \, j_{\ell_1}(b_1 x) \, dx \times \int_0^\infty e^{-y} \, y \, j_o(b_2 y) \, dy \tag{2.64}$$

and we now define

$$G = 2\left(\frac{Z}{na_\mu}\right)^{3/2}\left(\frac{Z}{a_e}\right)^{3/2}\left(\frac{na_\mu}{2Z}\right)^{3}\left(\frac{a_e}{Z}\right)^{2} F \qquad (2.65)$$

At this point we rewrite the cross section,(2.45),as

$$\sigma = \frac{v_2}{v_1}\,\frac{n^3}{Z^4}\,\frac{\pi a_e^2}{(207)^3}\,\frac{(2\ell_1+1)(2j+1)}{(2\ell+1)(2\ell_2+1)}\left[C(\ell_1\ell_2\ell\,;0,0)\right]^2\left[I\text{-}ZF\,\delta_{\ell_2 0}\right]^2 \qquad (2.66)$$

The quantities b_1 and b_2 are related by energy conservation. The conservation of energy requies that the change in energy of the mu meson equal the change in energy of the electron. Thus:

$$\frac{\hbar^2 k_1^2}{2m_\mu} + \frac{1}{2}\left(\frac{\alpha Z}{n}\right)^2 m_\mu c^2 = \frac{\hbar^2 k_2^2}{2m_e} + \frac{1}{2}(\alpha Z)^2 m_e c^2 \qquad (2.67)$$

and this may be rewritten as

$$b_2^2 = \frac{828\, b_1^2}{n^2} + \frac{207}{n^2} - 1 \qquad (2.68)$$

Using the definitions, (2.57), one may rewrite (2.66) as

$$\frac{\sigma}{\pi a_e^2} = \frac{b_2}{2b_1}\left(\frac{n}{Z}\right)^4\frac{1}{(207)^3}\,\frac{(2\ell_1+1)(2j+1)}{(2\ell+1)(2\ell_2+1)}\left[C(\ell_1\,\ell_2\,\ell;0,0)\right]^2\left[I\text{-}ZF\,\delta_{\ell_2,0}\right]^2 \qquad (2.69)$$

We must still consider the functions I and F although these are both defined explicitly above. The parameters b_1 and b_2 must satisfy (2.68). From (2.68) we have the interesting result that sufficiently slow mu mesons cannot be captured into states with principle quantum numbers greater than 14, since b_2^2 must remain positive. However, we will find in Chapter III that such slow mesons need not concern us.

We now carry out the integration indicated in (2.64). For this purpose we use the definition

$$j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac{1}{2}}(x)$$

where $J_{\ell+\frac{1}{2}}(x)$ is the standard Bessel function. We then have

$$F = \int_0^\infty X(x)\, x^2 \sqrt{\frac{\pi}{2b_1 x}}\; J_{\ell_1+\frac{1}{2}}(b_1 x)\, dx \int_0^\infty e^{-y}\, y \sqrt{\frac{\pi}{2b_2 y}}\, J_{\frac{1}{2}}(b_2 y)\, dy \qquad (2.70)$$

Integrals of this form may be evaluated using a result given by Watson[8]. Namely,

$$\int_0^\infty e^{-at}\, J_\nu(bt)\, t^{\mu-1}\, dt = \frac{\left(\frac{b}{2a}\right)^\nu \Gamma(\mu+\nu)}{a^\mu\, \Gamma(\nu+1)} \left(1 + \frac{b^2}{a^2}\right)^{\frac{1}{2}-\mu}$$

$$\times\; {}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu+2}{2}; \nu+1; -\frac{b^2}{a^2}\right)$$

${}_2F_1$ is the hypergeometric function. The condition that this result be valid is that $\nu + \mu > 0$. Application of this gives

$$F = \int_0^\infty X(x)\, x^2 \sqrt{\frac{\pi}{2b_1 x}}\, J_{\ell_1+\frac{1}{2}}(b_1 x)\, dx \left(\frac{1}{1+b_2^2}\right) \qquad (2.71)$$

Inserting $X(x)$ and taking $\ell_1 = \ell$, which follows from (2.53) gives

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8. G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, London, 1944) 2nd. ed., Chap. 13, p. 385.

$$F = -\frac{2}{1+b_2^2}\left\{\frac{(n-\ell-1)!\;(n+\ell)!\;\pi}{2\;nb_1}\right\}^{\frac{1}{2}}\sum_{k=0}^{n-\;-1}\frac{(-1)^{k+1}}{(n-\ell-1-k)!\;(2\ell+1+k!)\;k!}$$

$$\times\int_0^\infty e^{-x/2}\,x^{k+\ell+3/2}\,J_{\ell+\frac{1}{2}}(b_1x)\,dx \tag{2.72}$$

The integration is done according to the prescription above and we obtain

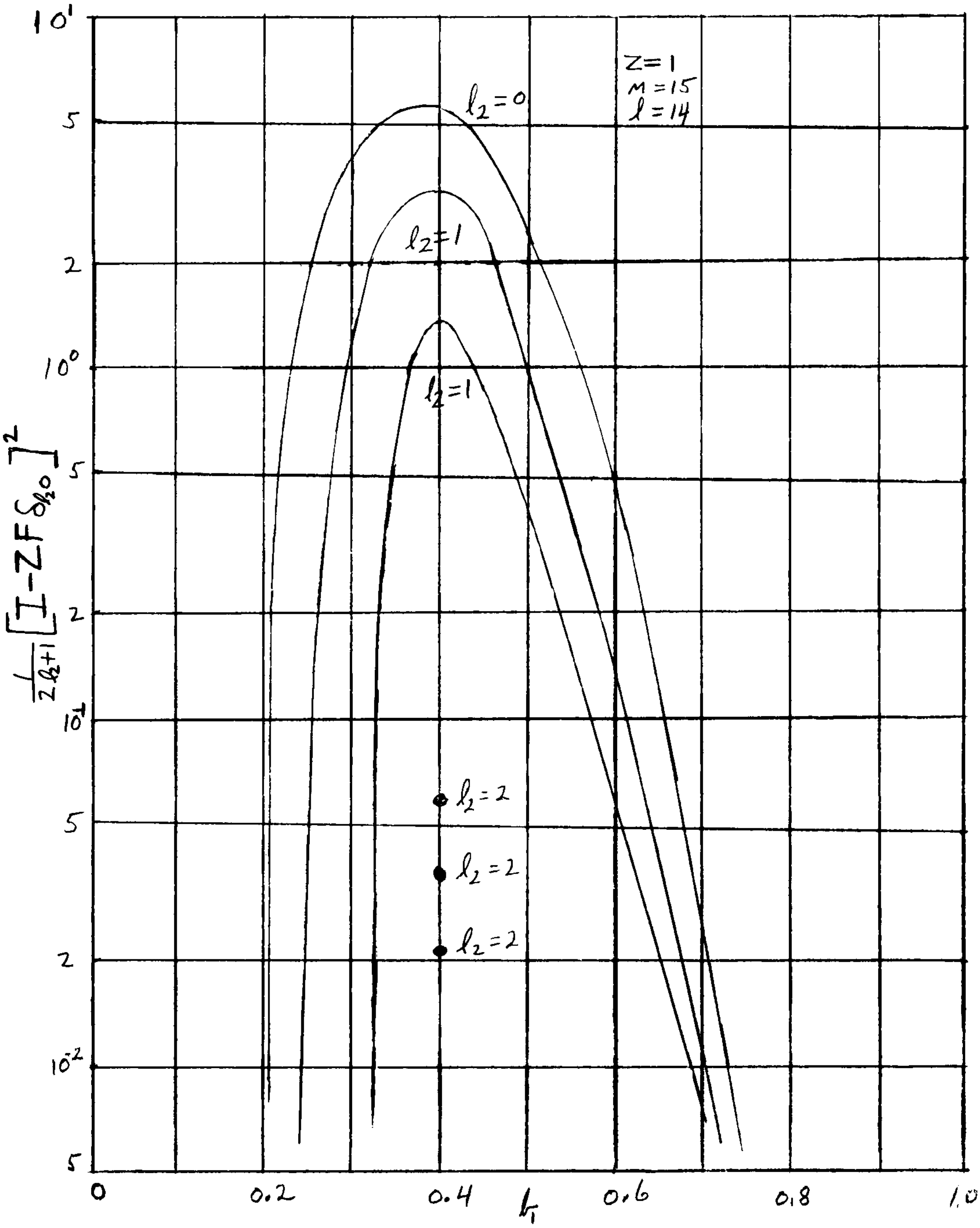
$$F = \frac{1}{2}\left\{\frac{(n-\ell-1)!\;(n+\ell)!\;\pi}{n}\right\}^{\frac{1}{2}}\frac{1}{1+b_2^2}\left(\frac{b_1}{2}\right)^{\ell}\left(\frac{1}{(\frac{1}{2})^2+b_1^2}\right)^{\ell+2}\frac{1}{\Gamma(\ell+3/2)}$$

$$\times\sum_{k=0}^{n-\ell-1}(-\tfrac{1}{2})^k\,\frac{(2\ell+k+2)}{(n-\ell-1-k)!\;k!}\left(\frac{1}{(\frac{1}{2})^2+b_1^2}\right)^{k}\;{}_2F_1\left(-\frac{k+1}{2},\;-\frac{k}{2};\;\ell+3/2;\;-4b_1^2\right)$$

(2.73)

Since k is never negative the ${}_2F_1$ terminates in all cases. One notes that the evaluation of (2.73) involves a double summation. The structure of the expression is such that very precise numerical work is required to get meaningful results.

We now discuss the relative magnitude of the terms in (2.71). Clearly, if I and F are of comparable magnitude the term in F will be dominant for large values of Z, if we consider electrons emitted into partial waves with zero orbital angular momentum. If we consider other values of the electron orbital angular momentum only the quantity I exists. In Figure 2 we present some typical values for the expression $[I - F]^2$ and $\frac{1}{2\ell_2+1}I^2$. The ordinate is arbitrary. The curve labeled, $\ell_2 = 0$ is for $[I - F]^2$; the two curves labeled $\ell_2 = 1$ are the two

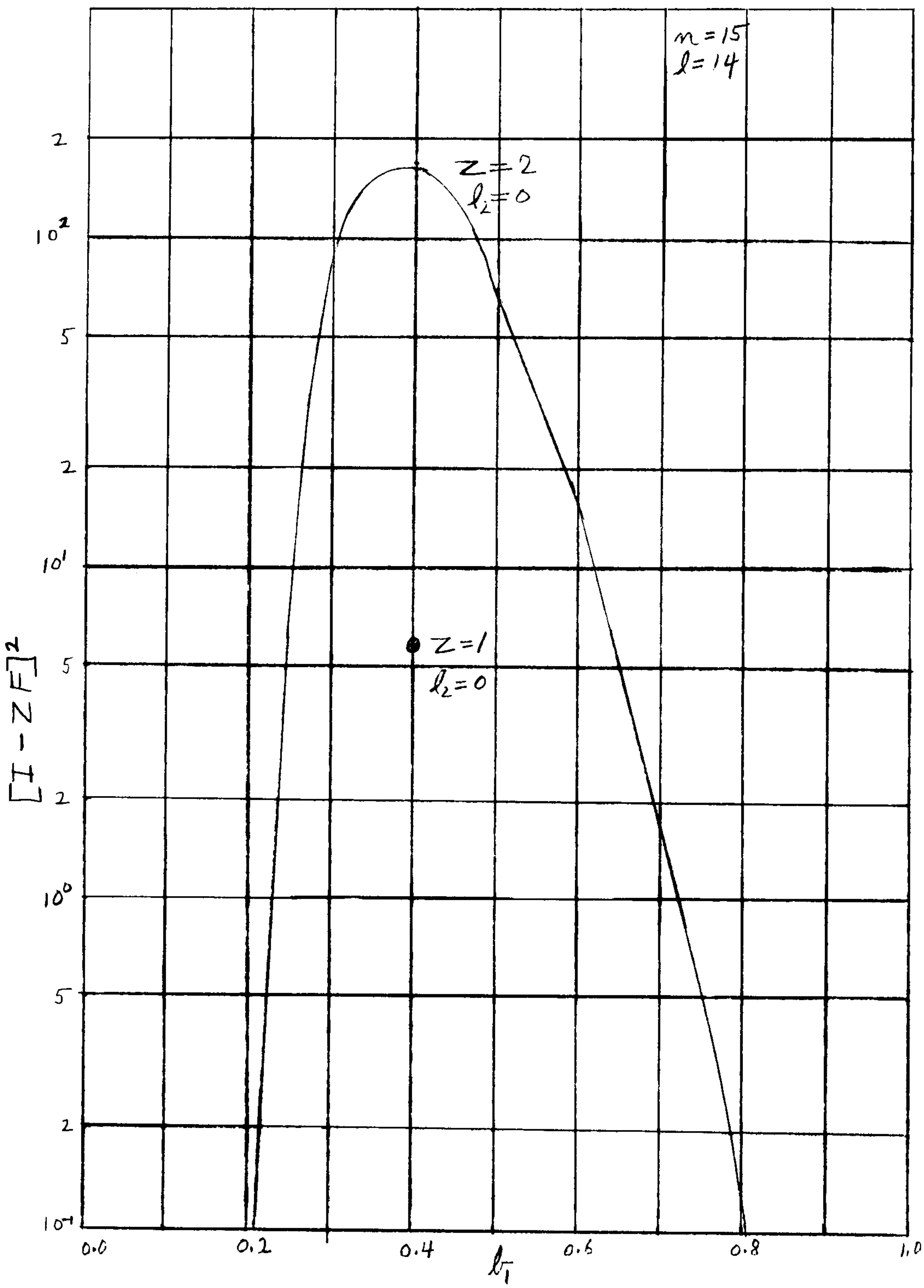


Auger Capture Radial Integrals for Different Electron
Partial Waves

Figure 2

possible cases for the electron being ejected with one unit of orbital angular momentum. The three points labeled $\ell_2 = 2$ are corresponding peaks for the ejection of electrons with two units of orbital angular momentum. Consequently, we conclude from Figure 1 that the important contributions to the total Auger capture cross section will be from the partial cross section involving ejection of electrons with zero and one unit of orbital angular momentum. However, all of Figure 2 is for Z = 1; and we have little interest in hydrogen. In Figure 3 we show the quantity $\left[I - 2F\right]^2$ corresponding to the case Z = 2. We also show the point corresponding to the peak of the $\ell_2 = 0$ curve in Figure 2. From this information we conclude that the ejection of electrons into partial waves other than those with zero orbital angular momentum is unimportant if $Z > 2$. We also conclude that the term F is the dominant term in the cross section for medium values of Z. A detailed numerical study was made concerning this point and it was found that for Z = 6 the replacement of Z by (Z - 1) in the term ZF and neglect of I led to less than 10 per cent inaccuracy in the cross sections obtained.

At this point we may remark that in the calculation of the partial cross sections 10 per cent errors in the numerical work need cause no concern. This is true for several reasons. We are calculating to first order using plane waves. This is therefore an approximate calculation, regardless of how precisely the various terms are computed. The relevant point is that we do not need very good absolute values of the partial cross sections. This will be discussed in the following chapter but we state here that the relevant quantities are the ratios

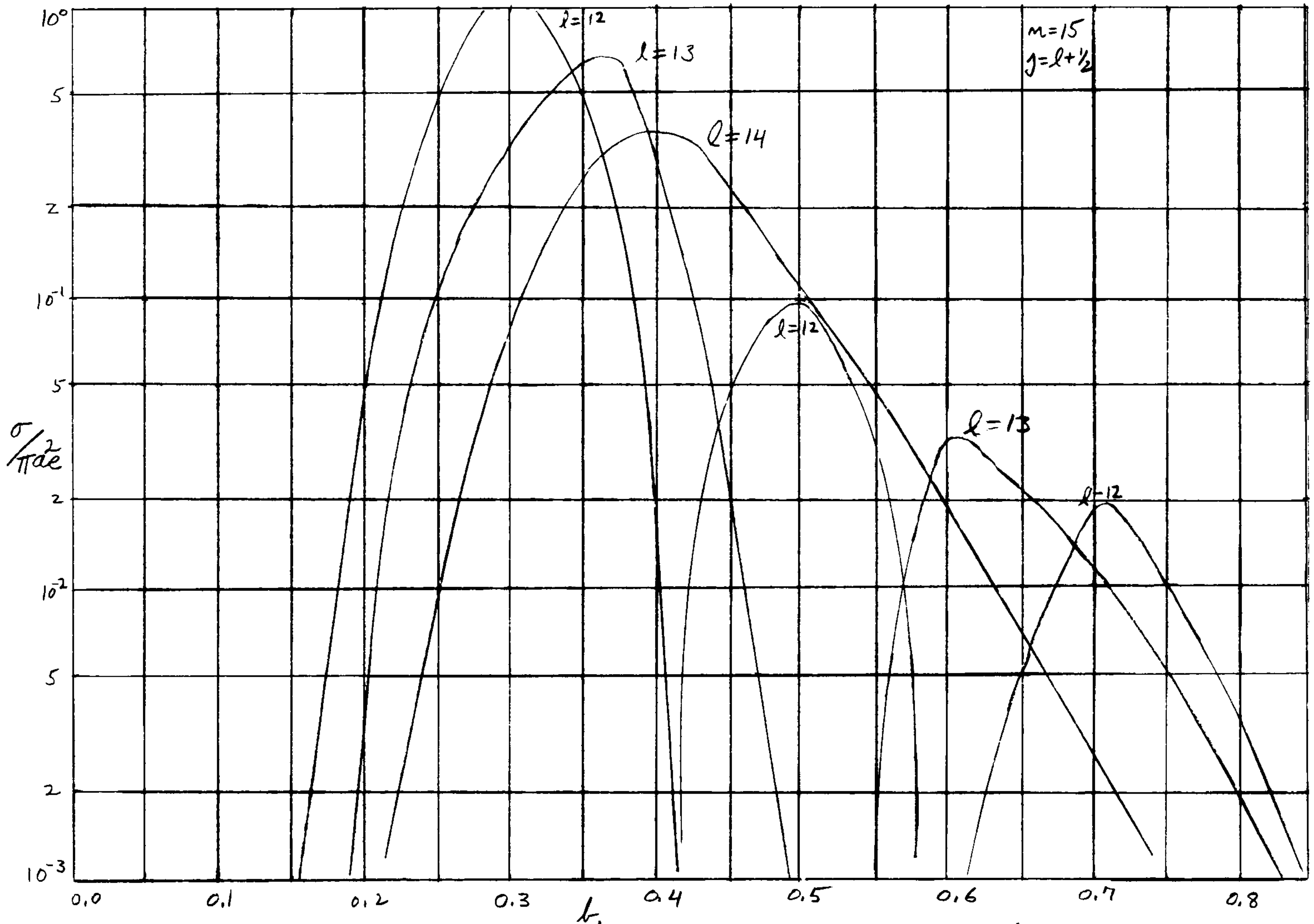


Variation of Auger Capture Radial Integrals with Z

Figure 3

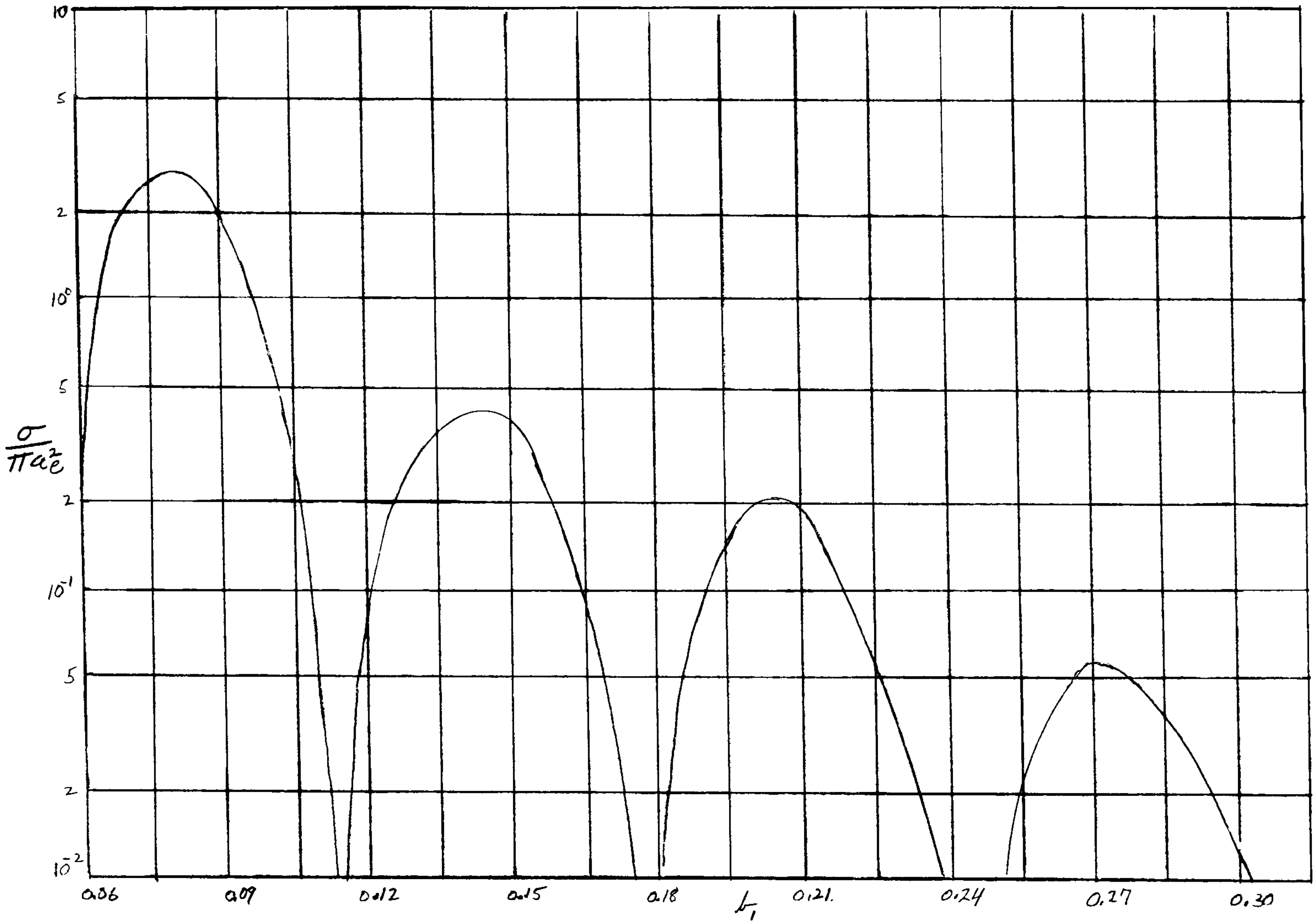
of the various partial cross sections to their sum. Further, we find that the quantity we are principally concerned with, namely, the meson polarization, is quite insensitive to the accuracy of the calculated cross sections. For these reasons we make use of the fact that the quantity F, defined above, is adequate for computing the cross sections as indicated.

It is of interest to consider some of the individual partial cross sections. These are presented in Figures 4, 5, and 6. These are all for capture of the mu mesons by carbon. In Figure 4 the partial cross sections are shown for capture of the meson into three states belonging to the level n = 15. The state, $\ell = 14$, is the circular orbit, namely, the state with radial quantum number $n_r = n - \ell - 1 = 0$. One notes that there is only one peak in the partial cross section for capture into this state. Further, this peak is higher than any peak belonging to the same n and any other ℓ at incident meson energies higher than the one at which the peak occurs. For the state with $\ell = 12$ there are three peaks. In general there are $n_r + 1 = n - \ell$ peaks in a specified partial cross section plot. Figures 5 and 6 show the partial cross section for capture into the state n = 14, $\ell = 0$. The figure does not cover the entire relevant energy range and so only shows nine of the fourteen peaks. The variation with n is not very rapid. For example, if Figure 4 had been constructed for n = 16 then the peak in the $\ell = 15$ partial cross section would have been at $\sigma/\pi a_e^2 = 0.48$ and at the same b_1 value as the peak in the $\ell = 14$ curve in Figure 4. A similar decrease in n would cause about the same decrease in σ. The



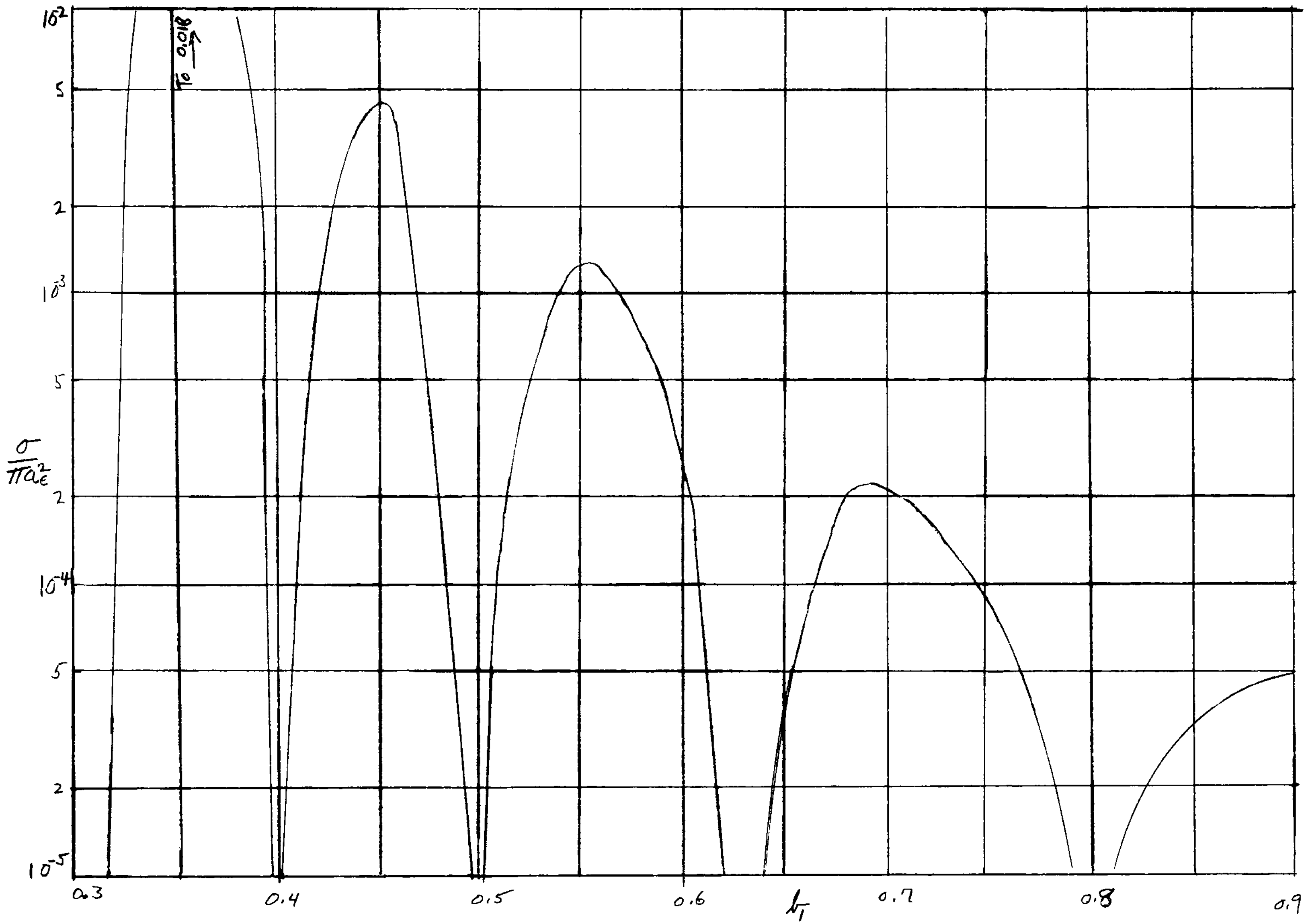
Partial Cross Sections for Auger Capture of Mu Mesons in States $n = 15$, $\ell = 12, 13, 14$ of Carbon

Figure 4



Partial Cross Section for Auger Capture of Mu Mesons in State n = 14, ℓ = 0, of Carbon

Figure 5



Partial Cross Section for Auger Capture of Mu Mesons in State n = 14, ℓ = 0, of Carbon

Figure 6

difference is essentially due to the different statistical weights of the states. The relevant point is that b_1 is a function of the principle quantum number, n, and therefore increasing n causes the peaks to be shifted to lower incident meson energy. The importance of this will be demonstrated in the following chapter.

CHAPTER III

THE DISTRIBUTION OF MU MESONS IN ATOMIC STATES FOLLOWING AUGER CAPTURE

1. The Capture Rate as a Function of the Meson Energy

It was demonstrated in the preceding chapter that the partial cross sections for Auger capture are strongly varying funtions of the incident meson energy. Consequently, the number of mesons captured into a specific state will depend on the rate at which the mesons lose energy in the slowing down process and the rate at which the mesons are captured at a specific energy. We must now formulate this process in terms of the Auger partial capture cross sections. This is done as follows:
Let N(E) be the number of free mu mesons at energy E; then

$$\frac{dN(E)}{dE} = \frac{dN(E)}{dX}\,\frac{dX}{dE} \qquad (3.1)$$

where dx is the path increment for the meson beam. Now,

$$\frac{dE}{dx} = -\langle \Delta E \rangle \, n\sigma_T(E) \qquad (3.2)$$

where n is the number of stopping atoms per unit volume, $\sigma_T(E)$ is the total elastic scattering cross section for mu mesons incident on the atoms of the stopping material, and $\langle \Delta E \rangle$ is the average energy loss in a collision between meson and atom at energy E. The minus sign occurs because a collision leads to a reduction of the meson energy. The energy loss in such an elastic collision is given by:

$$\Delta E = \kappa E(1-\cos\theta) \tag{3.3}$$

where $\kappa = \frac{2 m_{\mu} M}{(m_{\mu}+M)^2}$ and M is the mass of the atom. Thus $\langle E \rangle$ is given by the average of (3.3) over the scattering angle, θ. This is:

$$\langle \Delta E \rangle = \kappa E \frac{2\pi \int_0^{\pi} \sigma(\theta)\,(1-\cos\theta)\,\sin\theta\, d\theta}{2\pi \int_0^{\pi} \sigma(\theta)\,\sin\theta\, d\theta} \tag{3.4}$$

where $\sigma(\theta)$ is the differential elastic scattering cross section. However the denominator of (3.4) is just σ_T and the numerator (apart from κE) is commonly called the transport cross section. Thus, we write

$$\langle \Delta E \rangle = \kappa E \frac{\sigma_{Tr}(E)}{\sigma_T(E)} \tag{3.5}$$

The reduction in the number of free mesons may now be expressed as

$$\frac{dN}{dx} = - n\, N(E)\, \sigma_A(E) \tag{3.6}$$

where n and N(E) have been previously defined. $\sigma_A(E)$ is the total Auger capture cross section at energy E; namely, the partial cross sections of Chapter II summed over all j, ℓ, and n.[1] We now have:

$$\frac{dN(E)}{dE} = \frac{N(E)\,\sigma_A(E)}{\kappa E\, \sigma_{Tr}(E)} \tag{3.6a}$$

which may be put into the form of an elementary differential equation:

$$\frac{dN(E)}{N(E)} = \frac{\sigma_A(E)\, dE}{\kappa E\, \sigma_{Tr}(E)} \tag{3.6b}$$

The appropiate boundary condition is that

$$N(E) = N(E_o) \qquad \text{where } E_o < E$$

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1. There should be no confusion due to the use of n for the principle quantum number and also for the number of scattering centers per unit volume, since the context is always adequate for a clear distinction.

Thus we have

$$N(E) = N(E_o) \exp\left[-\frac{1}{\kappa} \int_E^{E_o} \frac{\sigma_A(E)}{\sigma_{Tr}(E)} \frac{dE}{E} \right] \tag{3.7}$$

with the requirement that $E \leq E_o$.

In (3.7) we will consider E_o to be an energy sufficiently high so that no mesons have been captured; then we may use (3.7) to determine the number remaining at energy E down to energies sufficiently low so that essentially all of the mesons have been captured. Thus we know the number of mesons captured at energy E, namely $N(E + \Delta E) - N(E)$. Define the number of mesons captured into a specified state n, ℓ, j, in the energy range ΔE to be $\Delta P_E(n, \ell, j)$. Then

$$\Delta P_E(n, \ell, j, E) = \frac{\sigma(n, \ell, j, E)}{\sigma_A(E)} \quad N(E + \Delta E) - N(E) \tag{3.8}$$

Since

$$\sum_{n, \ell, j} \sigma(n, \ell, j, E) = \sigma_A.$$

Summation of ΔP_E over all relevant ΔE then gives the distribution of the mu mesons in the initial states.

Thus we can find all we need from (3.7); however, we must do the integration in (3.7) numerically and then sum (3.8) over the energy. We now determine $\sigma_{Tr}(E)$ in a form suitable for machine computation.

2. The Elastic Scattering of Mu Mesons by Carbon Atoms

Since the best experimental data concerning the asymmetry coefficients is for mu mesons stopping in Carbon we calculate σ_{Tr} for Carbon. The method we use is clearly applicable to any atom. The problem of elastic scattering from atoms has been treated by many writers. The differential cross sections is given by[2]

$$\sigma(\theta) = \left(\frac{m_\mu e^2}{2\, p^2 \sin^2 \theta/2} \right)^2 \left[Z - F(\theta) \right]^2 \qquad (3.9)$$

where m_μ is the mass of a mu meson having incident momentum p and Z is the atomic number of the scattering atom. The form factor F (θ) is usually given as

$$F(\theta) = 4\pi \int_0^\infty p(r) \frac{\sin Kr}{Kr} r^2\, dr \qquad (3.10)$$

where p(r) is the electron density and $\hbar$ K is the momentum transfer; namely $K = 2\, k \sin \theta/2$ where k is the incident meson wave number.

The normalization is such that

$$\frac{1}{4\pi} \int_0^\infty r^2 p(r) dr = \text{number of electrons}$$

We must now evaluate this form factor for the Carbon atom. Using hydrogen like wave functions gives

$$p(r) = \psi^* \psi = (\psi^* \psi)_{1s} \quad (\psi^* \psi)_{2s} \quad (\psi^* \psi)_{2P_{\frac{1}{2}}} \qquad (3.11)$$

where the contribution to the electron density from each of the filled electron states in Carbon has been explicity indicated. It is clear that (3.10) is valid for scattering from $\ell = 0$ states; however, one may be inclined to question its validity when there is a $\ell = 1$ state involved. Since this point is not discussed in the literature we give the following proof.

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2. N. F. Mott and H. S. W. Massey, The Theory of Atomic Collisions (Clarendon Press, Oxford, 1949) 2nd ed., Chap. 7, p. 117. The derivation of this formula may be found in many other standard treatments of quantum mechanics.

First we construct the proper antisymmetric wave function for the $2P_{\frac{1}{2}}$ state, namely:

$$\psi = \sum_{m_1, m_2} C(\tfrac{1}{2}\,\tfrac{1}{2}\,0;\, m_1\, m_2)\, \chi^{m_1}(1)\, \chi^{m_2}(2) \tag{3.12}$$

so that the total angular momentum in the shell is zero.

$$\chi^{m_1}(1) = \sum_{\tau} C(1\tfrac{1}{2}\,\tfrac{1}{2};\, m_1 - \tau, \tau)$$

$$\times \quad Y_{1,\, m_1 - \tau}(\hat{n}_1)\, R_{n,\ell}(r_1)\psi^{\frac{1}{2}} \tag{3.13}$$

where

$$\psi^{\frac{1}{2}}_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \psi^{-\frac{1}{2}}_{\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{3.14}$$

Note that (3.12) is antisymmetric since interchange of electrons 1 and 2 gives:

$$C(\tfrac{1}{2}\,\tfrac{1}{2}\,0;\, m_2\, m_1) = -\, C(\tfrac{1}{2}\,\tfrac{1}{2}\,0;\, m_1\, m_2)$$

Also we have that $m_1 = -\, m_2$.

In complete generality we may write the form factor as:[3]

$$F(\theta) = \int p(r) e^{i\, \vec{K}\cdot\vec{r}}\, d^3 r \tag{3.15}$$

In (3.15) we take $p(r) = (\psi^* \psi)_{2P_{\frac{1}{2}}}$ and use

$$C(\tfrac{1}{2}\,\tfrac{1}{2}\,0;\, m_1, m_2) = (-)^{\frac{1}{2}-m_1} \sqrt{\tfrac{1}{2}}\; C(\tfrac{1}{2}\,0\,\tfrac{1}{2};\, m_1\, 0) = \frac{(-)^{\frac{1}{2}-m_1}}{\sqrt{2}}$$

to get

$$(\psi^* \psi)_{2P_{\frac{1}{2}}} = \sum_{1,\,2} \sum_{m_1,\, m_1'} \frac{(-)^{1-m_1-m_1'}}{2}$$

$$\chi^{m_1'*}(1)\, \chi^{m_1}(1)\, \chi^{m_2'*}(2)\, \chi^{m_2}(2)$$

---

3. Ibid.

and

$$F(\theta) = \sum_{m_1, m_1'} \tfrac{1}{2} \int r_2^2 r_1^2 \, dr_1 \, dr_2 \int \Big[\chi^{m_1'*}(1) \chi^{m_1}(1) \chi^{m_2'*}(2) \, e^{i \vec{K} \cdot \vec{r}_2} \chi^{m_2}(2)$$

$$+ \chi^{m_1'*}(1) \, e^{i \vec{K} \cdot \vec{r}_1} \chi^{m_1}(1) \chi^{m_2'*}(2) \chi^{m_2}(2) \Big] \, d\Omega_1 \, d\Omega_2 \, (-)^{1-m_1-m_1'} \qquad (3.16)$$

where the sum over 1 and 2 has been written out. The angular integration over Ω_1 in the first term of (3.16) gives $\delta_{m_1 \, m_1'}$ and over Ω_2 in the second term gives $\delta_{m_2' \, m_2}$; thus the two terms are equivalent and we have

$$F(\theta) = \sum_{m_1} \int r^2 \, dr \quad \chi^{m_1*} \, e^{i \vec{K} \cdot \vec{r}} \chi^{m_1} \, d\Omega \qquad (3.17)$$

since the other radial integration in each term yielded unity because we are dealing with normalized wave functions. We now use

$$e^{i \vec{K} \cdot \vec{r}} = 4\pi \sum_{L,M} i^L \, j_L(Kr) \quad Y_{LM}(\hat{r}) \quad Y^*_{LM}(\hat{K}) \qquad (3.18)$$

and obtain

$$F(\theta) = 4\pi \sum_{\substack{L,M \\ m_1, \tau}} Y^*_{LM}(\hat{K}) \, i^L \Big[C(1 \tfrac{1}{2} \tfrac{1}{2}; m_1 - \tau, \tau) \Big]^2 \int r^2 \, R_{n,1}(r) \, j_L(Kr) dr$$

$$\times \Big(Y_{1, m_1 - \tau}(\hat{r}), \quad Y_{L,M}(\hat{r}) \quad Y_{1, m_1 - \tau}(\hat{r}) \Big) \qquad (3.19)$$

This becomes

$$F(\theta) = 4\pi \sum_{\substack{L,M \\ m_1, \tau}} Y^*_{LM}(\hat{K}) \, i^L \Big(C(1 \tfrac{1}{2} \tfrac{1}{2}; m_1 - \tau, \tau) \Big)^2 C(1 \, L \, 1; 0 \, 0) \left(\frac{2L+1}{4\pi} \right)^{\frac{1}{2}}$$

$$\times \quad C(1 \, L \, 1; m_1 - \tau, M, m_1 - \tau) \int_0^\infty r^2 \, R^2_{n,1}(r) \, j_L(Kr) dr \qquad (3.20)$$

From the last Clebsch-Gordan coefficient in (3.20) we have $M = 0$, and we see that $L = 0,2$. We now do the sums over τ, and m_1; for this purpose we make the substitution $m_1 - \tau = \mu$ and we then have

$$\sum_{\tau,\mu}\left[C(1\ \tfrac{1}{2}\ \tfrac{1}{2};\mu,\tau)\right]^2 \quad C(1\ L\ 1;\mu\ 0)$$ which we define as S.

Using the symmetry relation

$$C(1\ \tfrac{1}{2}\ \tfrac{1}{2};\mu,\tau) = (-)^{\frac{1}{2}+\tau}\sqrt{2/3}\quad C(\tfrac{1}{2}\ \tfrac{1}{2}\ 1;\ -(\mu+\tau),\tau)$$

yields

$$S = 2/3\sum_{\mu} C(1\ L\ 1;\mu 0)\ \sum_{\tau}\left[C(\tfrac{1}{2}\ \tfrac{1}{2}\ 1;\ -\mu+\tau,\tau)\right]^2 \qquad (3.21)$$

and by the orthonormality of the Clebsch-Gordan coefficients this is

$$S = 2/3\sum_{\mu} C(1\ L\ 1;\mu 0) \qquad (3.22)$$

The most satisfying way to do this sum is as follows:

$$C(1\ 0\ 1;\mu\ 0) = 1$$

Thus

$$S = 2/3\sum_{\mu} C(1\ 0\ 1;\mu\ 0)\quad C(1\ L\ 1;\mu 0) \qquad (3.23)$$

Applying a symmetry relation to each of the Clebsch-Gordan coefficients yields

$$S = \frac{2}{\sqrt{2L+1}}\sum_{\mu}(-)^{1-\mu}(-)^{1-\mu}\ C(1\ 1\ L;\mu,\ -\mu)\quad C(1\ 1\ 0;\mu,\ -\mu) \qquad (3.24)$$

and again by the orthonormality relation we have

$$S = \frac{2}{\sqrt{2L+1}}\ \delta_{L,0} \qquad (3.25)$$

thus

$$F_{2P\frac{1}{2}}(\theta) = 2\int_0^{\infty} r^2\ R^2_{n,1}\ (r)\ jo\ (Kr)\ dr \qquad (3.26)$$

Consequently the form factor given by (3.10) is correct, even for scattering from the $P_{\frac{1}{2}}$ shell. The factor 2 in (3.26) is of course due to the presence of 2 electrons. We now evaluate the form factor for Carbon.

The following radial wave functions are used:[4]

$$R_{10}(r) = 2\left(\frac{Z}{a_e}\right)^{3/2} e^{-\frac{Zr}{a_e}} \tag{3.27a}$$

$$R_{20}(r) = \left(\frac{Z}{2a_e}\right)^{3/2}\left(2 - \frac{Zr}{a_e}\right) e^{-\frac{Zr}{2a_e}} \tag{3.27b}$$

$$R_{21}(r) = \left(\frac{Z}{2a_e}\right)^{3/2} \frac{Zr}{a_e\sqrt{3}} e^{-\frac{Zr}{2a_e}} \tag{3.27c}$$

with the understanding that Z will be set equal to 6 at a convenient point. Using these the form factor for Carbon becomes:

$$F(\theta) = F_{1s} + F_{2s} + F_{2P\frac{1}{2}} \tag{3.28}$$

where

$$F_{1s} = \frac{32B}{K}\int_0^\infty re^{-2\gamma r}\frac{(e^{iKr} - e^{-iKr})}{2i}\,dr \tag{3.29a}$$

$$F_{2s} = B/K\int_0^\infty r\left[4 - 4\gamma r + \gamma^2 r^2\right]\frac{(e^{iKr} - e^{-iKr})}{2i}e^{-\gamma r}\,dr \tag{3.29b}$$

$$F_{2P\frac{1}{2}} = \frac{B\gamma^2}{3K}\int_0^\infty r^3 e^{-\gamma r}\frac{(e^{iKr} - e^{-iKr})}{2i}\,dr \tag{3.29c}$$

where $B = 2\left(\frac{Z}{2ae}\right)^3$ and $\gamma = \frac{Z}{a_e}$.

The integrals are elementary and may be evaluated as:

$$F_{1s} = 16\,\frac{8B\gamma}{(4\gamma^2 + K^2)^2} \tag{3.30a}$$

$$F_{2s} = \frac{8B\gamma}{(\gamma^2 + K^2)^2}\left[1 - \frac{3\gamma^2 - K^2}{\gamma^2 + K^2} \quad \frac{3\gamma^2(\gamma^2 - K^2)}{(\gamma^2 + K^2)^2}\right] \tag{3.30b}$$

$$F_{2P\frac{1}{2}} = \frac{8B\gamma}{(\gamma^2 + K^2)^2}\;\frac{\gamma^2(\gamma^2 - K^2)}{(\gamma^2 + K^2)^2} \tag{3.30c}$$

---

4. These are the same as the hydrogen-like wave functions used in Chapter II; except here they contain the electron mass.

So now we have

$$F(\theta) = 8B\gamma \left[\frac{16}{(4\gamma^2 + K^2)^2} + \frac{2(\gamma^4 - 2\gamma^2 K^2 + K^4)}{(\gamma^2 + K^2)^4} \right] \tag{3.31}$$

Now the differential cross section becomes:

$$\sigma(\theta) = \frac{1}{4a\mu^2} \left(\frac{1}{k^2 \sin^2 \theta/2} \right)^2 \left[Z - F(\theta) \right]^2 \tag{3.32}$$

Using $K = 2\, k \sin \theta/2$ one may show that:

$$\sin^4 \theta/2 = \frac{K^4}{16k^4} \tag{3.33a}$$

$$1 - \cos\theta = \frac{K^2}{2k^2} \tag{3.33b}$$

$$\sin\theta \; d\theta = \frac{K\,d\,K}{k^2} \tag{3.33c}$$

So we obtain:

$$\sigma_{Tr} = \frac{2\pi}{4a\mu^2} \int_0^{2k} \frac{1}{k^4} \frac{16\,k^4}{K^4} \frac{K^2}{2k^2} \frac{K\,d\,K}{k^2} \left[Z - F(\theta) \right]^2 \tag{3.34a}$$

which may be rewritten as

$$\sigma_{Tr} = \frac{4\pi}{a\mu^2 k^4} \int_0^{2k} \frac{dk}{K} \left[Z - F(\theta) \right]^2 \tag{3.34b}$$

Using $8B\gamma = 2\gamma^4$ in (3.31) and setting $Z = 6$ gives:

$$\left[6 - F(\theta) \right]^2 = \left[\frac{8\gamma^2 K^2 + 2K^4}{(4\gamma^2 + K^2)^2} + \frac{24\gamma^6 K^2 + 20\gamma^4 K^4 + 16\gamma^2 K^6 + 4K^8}{(\gamma^2 + K^2)^4} \right]^2 \tag{3.35}$$

We note that this has the proper limiting values; namely:

$$\lim_{K \to 0} (6 - F(\theta)) = 0 \tag{3.36a}$$

$$\lim_{K \to \infty} \left(6 - F(\theta)\right) = 6 \tag{3.36b}$$

It is obvious that it would be possible to substitute (3.35) into (3.34b) and carry out the indicated integration; this leads to a complicated function not ideally suited for machine computation. Therefore, we proceed in the following fashion. In Figure 7 we have plotted the integrand of (3.34b). The dotted line is for $6-F(\theta) = 6$. Thus with reasonable accuracy we may use this value when $K \geq 4\gamma$. So we have:

$$\sigma_{Tr} = \frac{4\pi}{a_\mu^2 k^4} \left[I_1 + I_2 \right] \tag{3.37}$$

where $I_1 = \int_0^{4\gamma} \frac{dK}{K} \left[6-F(\theta) \right]^2$

and $I_2 = \int_{4\gamma}^{2k} 36 \frac{dK}{K} = 36 \ln \left(\frac{k}{2\gamma}\right)$

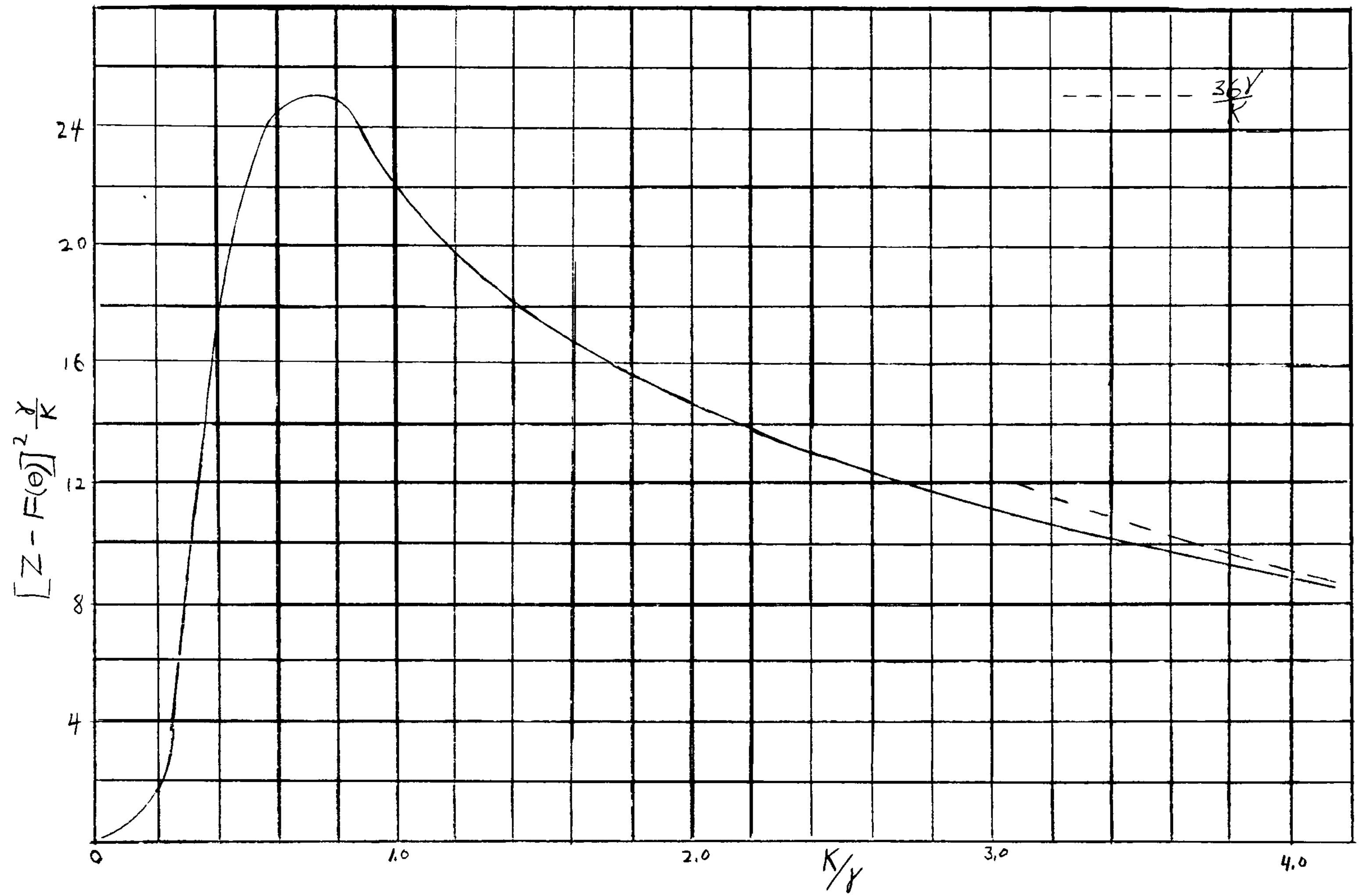
I_1 is obtained by applying Simpsons rule and using the points given in Figure 7. The formula obtained is given by:

$$\sigma_{Tr} = 4\pi a_e^2 \left(\frac{207}{36}\right)^2 \frac{\left[54.5 + 36 \ln \left(\frac{k}{2\gamma}\right) \right]}{(k/\gamma)^4} \tag{3.38}$$

where we have multiplied and divided by γ^4 and used

$$\gamma^4 a_\mu^2 = \left(\frac{36}{207}\right)^2 \frac{1}{a_e^2} \tag{3.39}$$

We must note that (3.38) is not correct when $2k < 4\gamma$; however, it will be seen later that we have no need to concern ourselves with such low energy mesons. The expression given by (3.38) is a simple function of its arguments and may therefore be used efficiently in a machine calculation.



Integrand for Transport Cross Section

Figure 7

At this point we may make several remarks concerning the accuracy of (3.38). The error introduced by the numerical treatment is very small and therefore of no concern. It is, of course, known that the treatment used; namely, using hydrogen-like wave functions to calculate the form factor, is not the best. Nevertheless, we feel that the transport cross section we obtain is suitable for the purpose at hand. We are not striving for great numberical accuracy in the distribution we obtain since it will be evident in Chapter IV that small errors in the distribution will not affect the final results. In fact, we will see presently that the energy range of importance is such that the scattering is principally nuclear ($k >> 4\gamma$) so the form factor corrections one might make would alter the distribution obtained in a negligible manner.

3. Calculation of the Distribution

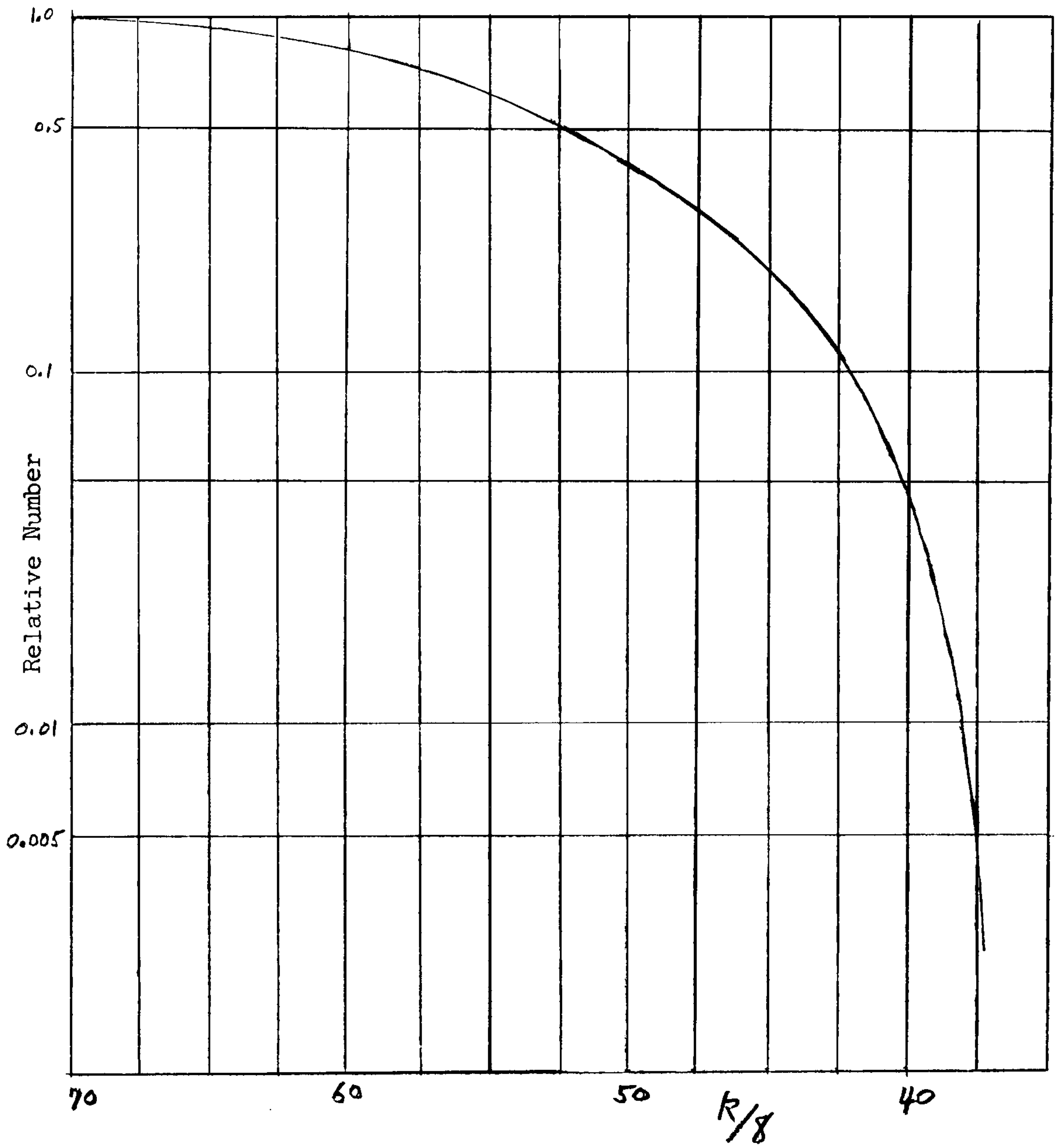
In (3.7) the initial distribution, $N(E_o)$ was taken as unity corresponding to the assumption that all of the mesons have a given energy E_o. The expression (3.7) was then programed for the electronic computer using the expression for partial capture cross sections derived in Chapter II and (3.38) above.[5] The relevant range of energies over which the mesons are captured strongly was found by trial. The mu mesons are not captured to any appreciable extent when their energy is such that $k > 70\gamma$; this is about 11.5 kev. The mesons are captured strongly at energies around 8 kev. and lower; essentially none of the mesons remain free until their energy

5. The computer used for all of these calculations was the IBM-704 located at the Oak Ridge Gaseous Diffusion Plant.

is reduced to 2.5 kev. The number of free mesons as a function of k/γ is shown in Figure 8.

It was found desirable to modify the notation of Chapter II and express all momentum dependent quantities in terms of γ. A suitable increment for the integration of (3.7) was the energy increment corresponding to $\Delta k = 0.4\gamma$. This was determined by trial. The actual integration was done by using Simpson's rule and recomputing $N(E_0)$ at each inteval. It was not necessary to compute the distribution as a function of j since if the number of mu mesons in a state n, ℓ is given the two corresponding states $j = \ell \pm \frac{1}{2}$ are populated according to their satistical weight.

We have indicated at several points that certain approximations were made. If one should attempt to improve on the work presented here, presumably by using Coulomb waves for the free particles in the cross section calculation, then the following problem would arise. In the course of evaluating N(E) as a function of E, it was necessary to compute 21,600 values for the partial cross sections. The expression we used for the partial cross section is not very complicated as may be judged by the fact that the final run took slightly less than one hour of computer time. It is our feeling that the expressions involving Coulomb waves would be much more complicated than these and that the evaluation of N(E) with these expressions would take an unreasonable amount of computer time in view of the relatively small sensitivity of the final polarization results on N(E).



Relative Number of Free Mu Mesons in Carbon with Momentum $\hbar k$

Figure 8

4. The Distribution

Table 4 gives the calculated distribution of mu mesons in atomic states. We found that there was no need to consider states for which $n > 16$. Figures 9 through 11 show how this distribution varies with the quantum numbers involved. From Figure 9 it is seen that the mesons are captured into states around $\ell = 5$. Figure 10 shows that the states around $n = 7$ get the maximum number of mesons. In Figure 11 we show the distribution as a function of the radial quantum number, $n_r = n-\ell-1$. These are the results we calculate; however, as there has been some qualitative discussion in the literature concerning what one might expect we present the following remarks.

Since there has been no previous analytical treatment of the distribution, several writers have attempted to guess its form.[6] Their arguments go something like this. The meson orbit has nearly the same Bohr radius as the electron orbit if the meson goes into a state for which $n = 14$; this means that the meson wave function and the electron wave function have maximum overlap for this case. Since maximum overlap sometimes leads to large transition rates, it is asserted that the mesons will be captured into states around $n = 14$. As the statistical weight of a

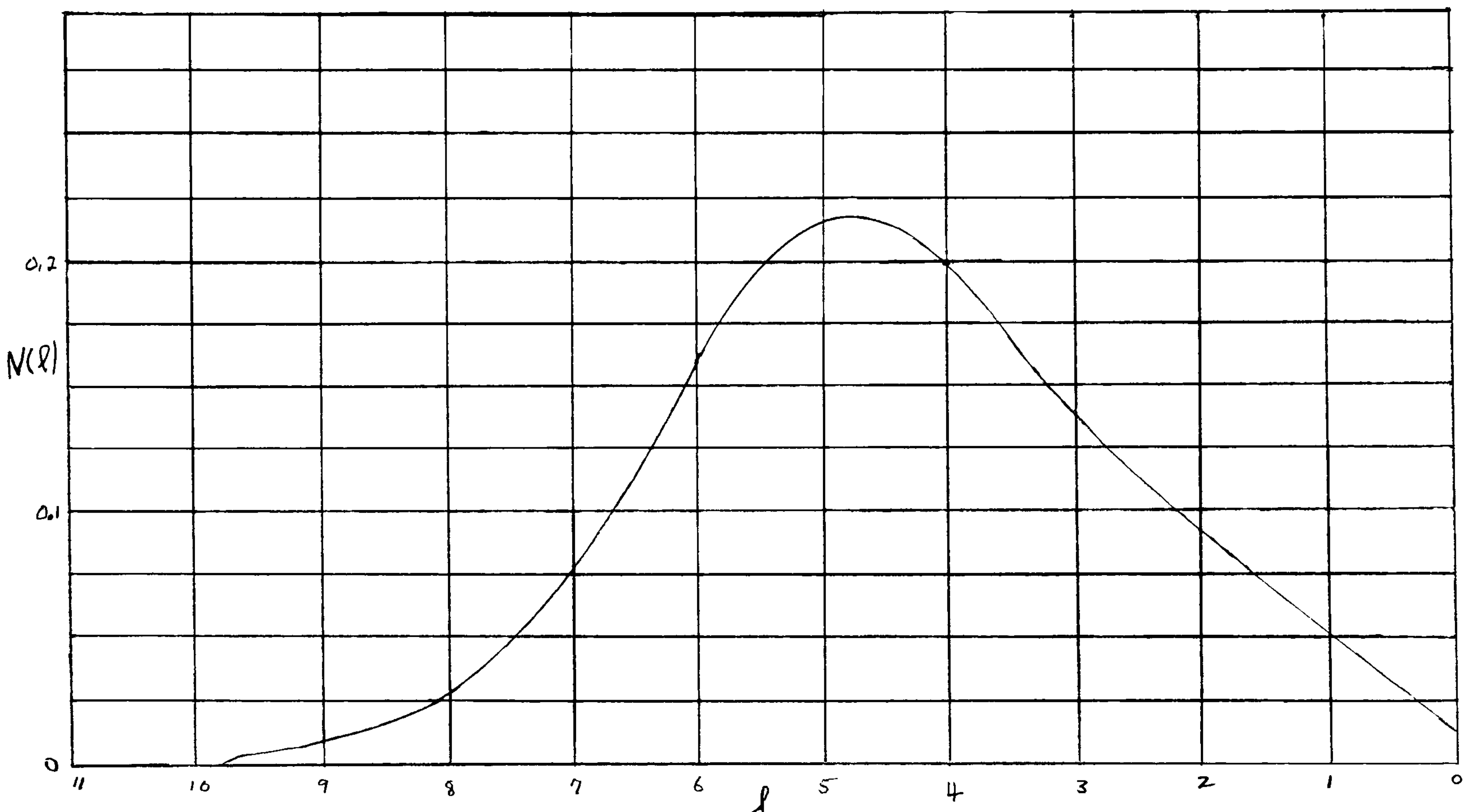
6. G. R. Burbidge and A. H. de Borde, Phys. Rev. 89, 189 (1953). See M. Demeur, Nuclear Phys. 1, 516 (1956). Assumptions concerning the initial distribution appear in many other places in the literature. It is usually assumed that the mesons are captured into states around $n = 14$ and then the distribution in ℓ is varied to suit the problem at hand. As will be seen in the subsequent chapters, the observable facts are rather insensitive to these assumptions, and for this reason the inaccuracy of such a procedure has not been brought to light previously.

TABLE 4

NUMBER OF MESONS, IN 10,000, CAPTURED INTO ATOMIC STATE n, ℓ IN CARBON

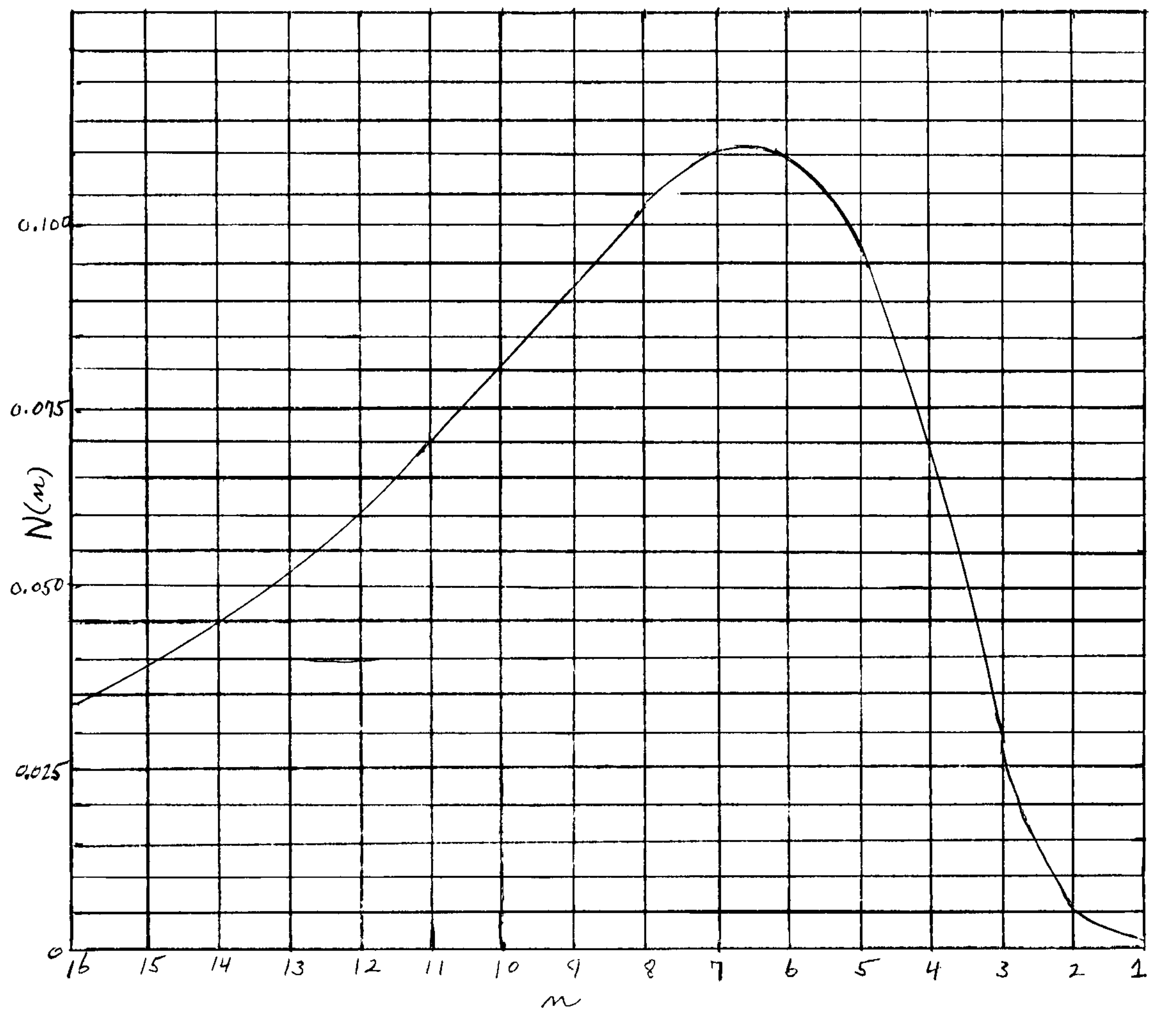
$n-\ell$	Circular 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
n																
1	1															
2	35	13														
3	232	38	21													
4	499	54	101	12												
5	525	256	82	87	15											
6	338	449	134	102	64	16										
7	151	401	302	98	93	48	15									
8	50	227	346	204	82	77	38	13								
9	13	91	236	273	145	70	62	31	11							
10	3	27	110	215	212	108	59	49	26	9						
11	-	6	37	113	186	165	84	50	40	21	7					
12	-	1	10	42	107	157	130	67	41	32	17	6				
13	-	-	2	12	44	97	132	104	54	35	27	14	5			
14	-	-	-	3	13	42	87	110	85	44	29	22	12	4		
15	-	-	-	1	3	14	40	77	93	69	37	25	19	10	3	
16	-	-	-	-	1	4	14	37	67	79	58	31	21	16	9	3
Total	1847	1563	1381	1152	0965	0798	0661	0538	0417	0289	0175	0098	0057	0030	0012	0003

TOTAL NUMBER SHOWN: 9986



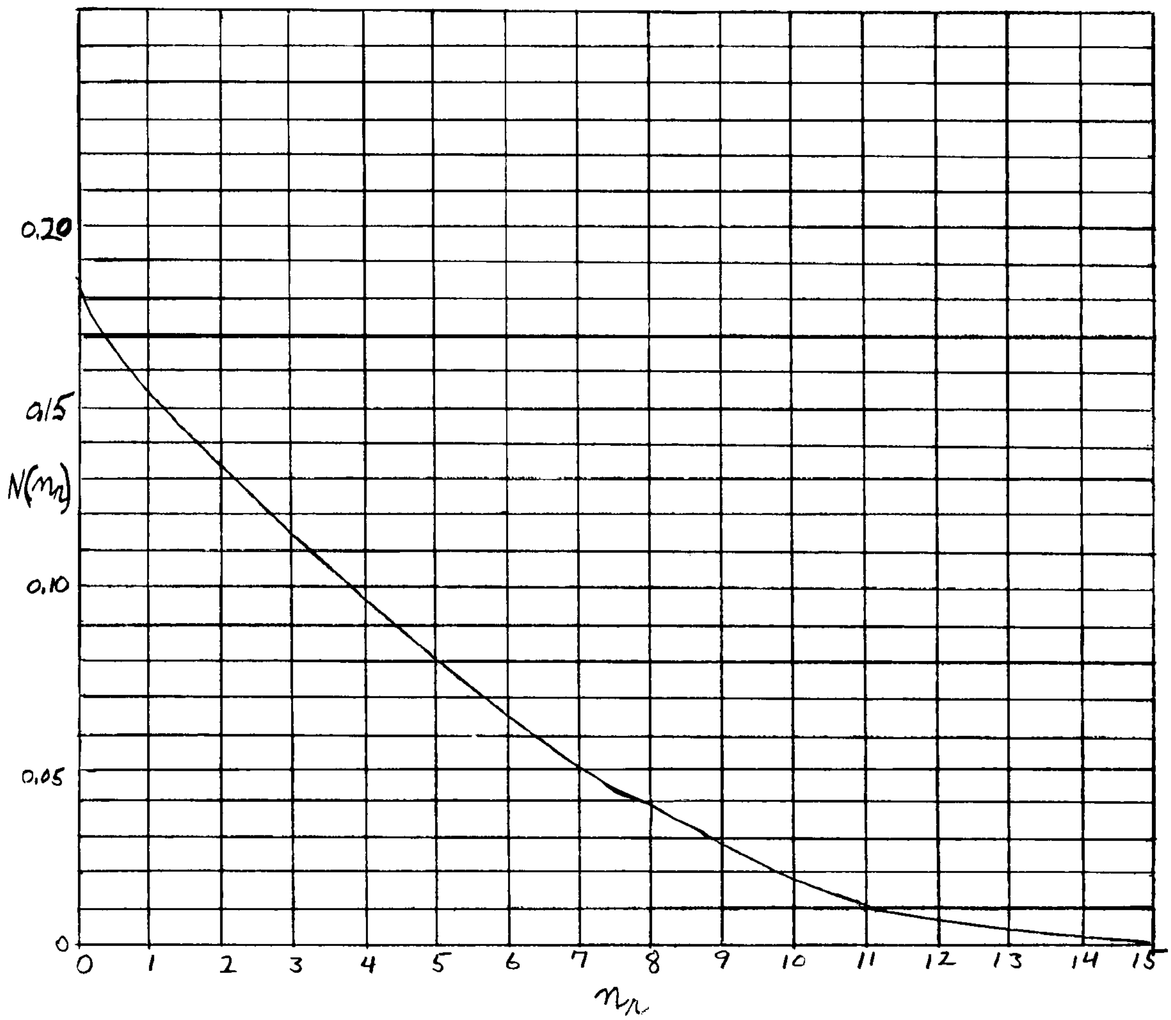
Initial Distribution of Mu Mesons in States ℓ of Carbon

Figure 9



Initial Distribution in n of Mu Mesons Captured in Carbon

Figure 10



Initial Distribution in n_r of Mu Mesons Captured in Carbon

Figure 11

state having given n is largest when $\ell =$ n-1 it is further assumed that these circular orbits will get the most mesons. We do not disagree with these considerations; we merely state that they are not adequate. From Figure 11 it is evident that the circular orbits do get the most mesons. The reason that the states of high n, $n > 12$, get so few mesons is that the mesons are captured before they are slowed down enough to be captured by these states. It is in this respect that the qualitative arguments are inadequate; they do not take into account the competition of the various capturing states as a function of the meson energy.

In view of the foregoing we make the following statements concerning the distribution given in Table 4. There is no theoretical argument that contradicts the results we have presented; in fact, the distribution we have is in several ways in agreement with the expectations. There is no experimental information that can be used to test this distribution directly; any such test using the results of the reported observations is indirect. In Chapter V we obtain results which are to some extent a test of this distribution and we find that, in so far as the results we get constitute a test of the distribution, the agreement is very good. Therefore we assert that the mu mesons are captured into the states of mu-mesic Carbon as given in Table 4.

CHAPTER IV

THE POLARIZATION OF MU MESONS IN THE INITIAL STATES OF MU-MESIC ATOMS

In the preceding chapter we have found how the mu mesons are distributed in atomic states when they are initially captured. Although the problem of the distribution had not been previously solved, we are principally concerned with the polarization of the mu mesons. Thus the results to be derived in the present and following chapter constitute an adequate solution of one of the several problems that require knowledge of such a distribution. In this chapter we show how one may determine the polarization of the mu mesons in their initial states following capture if we know their polarization previous to capture. As was explained in Chapter I, we have excellent reasons to believe that the mu mesons are completely polarized along their original beam direction. We will see here that the important consideration is the amount of scattering the mu mesons suffer before capture. To show this in detail we show how to obtain the polarization after capture for the case of no scattering and for the case of much scattering. We then prove that one of these cases is an extremely good approximation to the actual situation. We now turn to an analytical treatment of the problem.

1. Definition of Polarization

We define the polarization of the mu mesons as[1]:

$$\vec{P} = \frac{(\psi, \vec{\sigma}\psi)}{(\psi, \psi)} \tag{4.1}$$

where ψ is the meson wave function and $\vec{\sigma}$ is the vector composed of the Pauli spin matrices. We have indicated previously that the muon beam is along the z axis and that it is initially completely polarized along the beam direction. Thus, initially we have that

$$\vec{P}\cdot\hat{e}_z = +1 \tag{4.2}$$

where e_z is a unit vector along the z axis. The process of depolarization will reduce $\vec{P}\cdot\hat{e}_z$; of course a strong spin flip mechanism could reverse the sign in (4.2), however, we do not treat a case where this occurs. We use the fact that the meson wave functions are normalized and write

$$P = \langle \sigma_z \rangle \hat{e}_z \tag{4.3}$$

where we have used the standard notation for the expectation value. Thus we are going to calculate $\langle \sigma_z \rangle$ in the following and at any stage this will correspond to the surviving average spin orientation along the initial beam direction. If one should wish to consider an incident beam not fully polarized then the only change would be to multiply (4.3) by the appropriate factor.

---

1. M. E. Rose, Relativistic Electron Theory (John Wiley and Sons, Inc., New York, to be published) Chap. I. This reference contains a complete discussion of the description of the polarization in the Pauli representation. Since the mu mesons have small velocity when they are captured we need not consider relativistic effects.

Although it was just stated that we intend to calculate $\langle\sigma_z\rangle$, we do not do so directly; since it will be easier to work in terms of $\langle j_z\rangle$. The relationship between $\langle\sigma_z\rangle$ and $\langle j_z\rangle$ may be obtained by using the projection theorem for first rank tensors[2]. Thus,

$$\langle\sigma_z\rangle = \frac{\langle j_z\rangle \; (j//\vec{\sigma}\cdot\vec{j}//j)}{j(j+1)} \tag{4.4}$$

and one may take the reduced matrix element as the expectation value of $\vec{\sigma}\cdot\vec{j}$, namely

$$\langle\vec{\sigma}\cdot\vec{j}\rangle = j(j+1) - \ell(\ell+1) + 3/4 \tag{4.5}$$

Then we obtain

$$\langle\sigma_z\rangle = \frac{\langle j_z\rangle}{j} \quad \text{when } j = \ell + \tfrac{1}{2} \tag{4.6a}$$

$$\langle\sigma_z\rangle = -\frac{\langle j_z\rangle}{j+1} \quad \text{when } j = \ell - \tfrac{1}{2} \tag{4.6b}$$

When we discuss bound states, the values of j and ℓ are known, therefore having $\langle j_z\rangle$ is equivalent to having $\langle\sigma_z\rangle$. Since we will find that $\langle\sigma_z\rangle$ is required only for the final atomic state of the mu mesons, namely, the $1s_{\frac{1}{2}}$ state we will only need to use (4.6a). Therefore the quantity which we shall deal with will be $\frac{\langle j_z\rangle}{j}$; and we shall speak of this as the polarization. Thus, with the restriction that we only apply this to the $1s_{\frac{1}{2}}$ state, the asymmetry coefficient of Chapter I is:

---

2. This and the other important theorems that we use to treat angular momenta are reproduced in Appendix 1, suitable references are given there.

$$a = \frac{|P|}{3} = \frac{1}{3} \frac{\langle j_z \rangle}{j} \tag{4.7}$$

In Chapter V we shall see how to obtain $\frac{\langle j_z \rangle}{j}$ for the mu mesons in the $1s_{\frac{1}{2}}$ state. Here we must find this quantity for the states that the mu mesons are initially captured into.

2. Depolarization of Mu Mesons Captured from a Beam

In this section we consider the depolarization of the mu mesons if they undergo Auger capture from a beam. By capture from a beam we mean that the mu mesons are incident on an absorber as a well collimated beam and that we assume them to be captured before their initial direction is altered by scattering. Discussion of the extent to which this represents the physical facts is postponed to section 4 below.

We now note that

$$\frac{\langle j_z \rangle}{j} = \frac{\sum_m m \text{ pop}(m)}{j \sum_m \text{pop}(m)} \tag{4.8}$$

where pop(m) means the number of mu mesons that go into a state of j and ℓ having the magnetic quantum number m. In Chapter II we saw that the partial cross section for capture into a state j and ℓ was given by

$$\sigma = \left(\frac{v_2}{v_1}\right) \frac{1}{(2\pi a_e)^2} \int \left| H'_{fi} \right|^2 d\Omega_{k_2}$$

which is just (2.29) with the integration over electron directions indicated. Since there is no dependence on the magnetic quantum numbers in the front factors we may take

$$\mathrm{pop}(m) \propto \int |H'_{fi}|^2 \, d\Omega_{k_2} \tag{4.9}$$

Thus we use the result

$$\int |H_{fi}|^2 \, d\Omega_2 = (4\pi)^3 (-i)^{\ell_1} (i)^{\ell'_1} \frac{(2\ell_1+1)(2\ell'_1+1)}{(2\ell_2+1)(2\ell+1)} C(\ell'_1 \ell_2 \ell;00)$$

$$C(\ell_1 \ell_2 \ell;00) \, I(\ell_1 \ell_2 \ell) \, I(\ell'_1 \ell_2 \ell) \left[C(\ell \tfrac{1}{2} j;m-\tau,\tau)\right]^2$$

$$C(\ell_1 \ell_2 \ell;0,m-\tau) \, C(\ell'_1 \ell_2 \ell;0,m-\tau) \tag{4.10}$$

which is just (2.40) with the $\sum_m$ removed. In Chapter II, K was defined as

$$K = 2 \sum_m \int |H_{fi}|^2 \, d\Omega_2 \tag{4.11a}$$

and we now define

$$K_1 = 2 \sum_m m \int |H_{fi}|^2 \, d\Omega_2 \tag{4.11b}$$

so that

$$\frac{\langle j_z \rangle}{j} = \frac{K_1}{j\,K} \tag{4.12}$$

We now define S_1 such that

$$K_1 = 2(4\pi)^3 (-i)^{\ell_1} (i)^{\ell'_1} \frac{(2\ell_1+1)(2\ell'_1+1)}{(2\ell_2+1)(2\ell+1)} C(\ell'_1 \ell_2 \ell;00) \, C(\ell_1 \ell_2 \ell;00)$$

$$I(\ell_1 \ell_2 \ell) \, I(\ell'_1 \ell_2 \ell) \, S_1 \tag{4.13}$$

Thus

$$S_1 = \sum_m m \left[C(\ell \tfrac{1}{2} j; m-\tau, \tau) \right]^2 C(\ell_1 \ell_2 \ell; 0, m-\tau)$$

$$C(\ell_1' \ell_2 \ell; 0, m-\tau) \qquad (4.14)$$

Now, τ must be taken as $+\frac{1}{2}$ and we do so in the following. We define a new quantity, an integer,

$$M = m - \tfrac{1}{2} \qquad (4.15)$$

and substitute this into (4.14) to obtain

$$S_1 = \sum_M (M+\tfrac{1}{2}) \left[C(\ell \tfrac{1}{2} j; M, \tfrac{1}{2}) \right]^2 C(\ell_1 \ell_2 \ell; OM)\, C(\ell_1 \ell_2 \ell; OM) \qquad (4.16)$$

and we now use the explicit values of the Clebsch-Gordan coefficients to obtain

$$\left[C(\ell \tfrac{1}{2} j; M, \tfrac{1}{2}) \right]^2 = \frac{\ell + M + 1}{2\ell + 1} \qquad \text{for } j = \ell + \tfrac{1}{2} \qquad (4.17a)$$

$$\left[C(\ell \tfrac{1}{2} j; M, \tfrac{1}{2}) \right]^2 = \frac{\ell - M}{2\ell + 1} \qquad \text{for } j = \ell - \tfrac{1}{2} \qquad (4.17b)$$

We substitute these into (4.16) to obtain

$$S_1^+ = \sum_M \left[\frac{1}{2} \frac{\ell + 1}{2\ell + 1} + M \frac{\ell + 1 + \frac{1}{2}}{2\ell + 1} + \frac{M^2}{2\ell + 1} \right] C(\ell_1 \ell_2 \ell; 0, M)\, C(\ell_1' \ell_2 \ell; 0, M)$$

$$(4.18a)$$

$$S_1^- = \sum_M \left[\frac{1}{2} \frac{\ell - \frac{1}{2}}{2\ell + 1} + M \frac{\ell - \frac{1}{2}}{2\ell + 1} - \frac{M^2}{2\ell + 1} \right] C(\ell_1 \ell_2 \ell; 0, M)\, C(\ell_1' \ell_2 \ell; 0, M)$$

$$(4.18b)$$

Where the superscript + and - refer to the $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$ cases respectively.

Use of the symmetry relations and the fact that $\ell_1 + \ell_2 + \ell$ and $\ell_1' + \ell_2 + \ell$ must both be even integers as was shown in Chapter II will verify that

$$C(\ell_1\, \ell_2\, \ell;0,M)\; C(\ell_1'\, \ell_2\, \ell;0,M) = C(\ell_1\, \ell_2\, \ell;0,-M)\; C(\ell_1'\, \ell_2\, \ell;0,-M) \tag{4.19}$$

Consequently, the terms linear in M in (4.18a) and (4.18b) vanish in the summation. Now we use the symmetry relations to obtain

$$\sum_M C(\ell_1\, \ell_2\, \ell;0,M)\; C(\ell_1'\, \ell_2\, \ell;0,M) = \sum_M (-)^{2\ell_2-2m} \left[\frac{(2\ell+1)^2}{(2\ell_1+1)(2\ell_1'+1)}\right]^{\frac{1}{2}}$$

$$\times\; C(\ell_2\, \ell\, \ell_1;M,-M)\; C(\ell_2\, \ell\, \ell_1';M,-M) = \frac{2\ell+1}{2\ell_1+1}\, \delta_{\ell_1' \ell_1} \tag{4.20}$$

as the last sum is immediately evaluated by the orthonormality condition. This allows the evaluation of the contributions due to the first terms in S_1^+ and S_1^- and also those due to the additional constant term that arises when the explicit value of $C(\ell\, 2\, \ell;M,0)$ is used to find how we must write M^2. Thus

$$M^2 = \frac{1}{3}\left[(\ell+1)(2\ell-1)(2\ell+3)\right]^{\frac{1}{2}} C(\ell\, 2\, \ell;M0) + \frac{1}{3}\,\ell(\ell+1) \tag{4.21}$$

Therefore, the final summation required to evaluate S_1 is

$$\sum_M C(\ell\, 2\, \ell;M,0)\; C(\ell_1\, \ell_2\, \ell;0,M)\; C(\ell_1'\, \ell_2\, \ell;0,M)$$

and the symmetry relations may be used to show that this is equivalent to

$$\sum_M C(\ell_1'\, \ell_2\, \ell;0,M)\; C(\ell_2\, \ell_1\, \ell;M0)\; C(\ell\, 2\, \ell;M0) \tag{4.22}$$

We use the Racah recoupling formula to rewrite the last two Clebsch-Gordan coefficients in (4.22) as

$$C(\ell_2 \ell_1 \ell; M0)\, C(\ell\, 2\, \ell; M0) = \sum_s \left[(2s+1)(2\ell+1)\right]^{\frac{1}{2}} W(\ell_2 \ell_1 \ell 2; \ell s)$$

$$\times\ C(\ell_2 s \ell; M,0)\, C(\ell_1 2 s; 0,0) \qquad (4.23)$$

and the sum over M we are now requir.ed to evaluate is

$$\sum_M C(\ell_1' \ell_2 \ell; 0,M)\, C(\ell_2 s \ell; M,0)$$

and by the symmetry relations and the orthonormality condition this is

$$\sum_M (-)^{2\ell_2 - 2m} \left[\frac{(2\ell+1)^2}{(2\ell_1'+1)(2s+1)}\right]^{\frac{1}{2}} C(\ell_2 \ell \ell_1'; M,-M)\, C(\ell_2 \ell s; M,-M0)$$

$$= \frac{2\ell+1}{2\ell_1'+1}\, \delta_{\ell_1' s} \qquad (4.24)$$

Now (4.22) may be written as

$$C(\ell_1 2 \ell_1'; 0,0)\, \frac{(2\ell+1)^{3/2}}{(2\ell_1'+1)^{\frac{1}{2}}}\, W(\ell_2 \ell_1 \ell 2; \ell \ell_1') \qquad (4.25)$$

We use a symmetry relation for permutation of the arguments of the Racah coeffient as follows

$$W(\ell_2 \ell_1 \ell 2; \ell \ell_1') = (-)^{\ell_2 + 2 - \ell - \ell_1'}\, W(\ell \ell \ell_1' \ell_1; 2\ell_1) \qquad (4.26)$$

Clearly the phase is unity and we may write out S_1^+ and S_1^-.

$$S_1^+ = \left[\frac{1}{2}\, \frac{(+1}{2\ell_1+1} + \frac{\ell(\ell+1)}{3(2\ell_1+1)}\right] \delta_{\ell_1 \ell_1'} + \frac{\left[\ell(\ell+1)(2\ell-1)(2\ell+3)\right]^{\frac{1}{2}}}{3}$$

$$\times \left[\frac{2\ell+1}{2\ell_1'+1}\right]^{\frac{1}{2}} C(\ell_1 2 \ell_1'; 0,0)\, W(\ell \ell \ell_1' \ell_1; 2\ell_2) \qquad (4.27a)$$

$$S_1^- = \left[\frac{1}{2} \frac{\ell}{2\ell_1+1} - \frac{(\ell+1)}{3(2\ell_1+1)} \right] \delta_{\ell_1 \ell_1'} - \frac{\left[\ell(\ell+1)(2\ell-1)(2\ell+3)\right]^{\frac{1}{2}}}{3}$$

$$\times \left[\frac{2\ell+1}{2\ell_1'+1}\right]^{\frac{1}{2}} C(\ell_1 2 \ell_1'; 0,0)\, W(\ell \ell \ell_1' \ell_1; 2\ell_2) \tag{4.27b}$$

We use

$$\frac{2j+1}{2} = \ell + 1 \qquad j = \ell + \tfrac{1}{2}$$

$$\frac{2j+1}{2} = \ell \qquad j = \ell - \tfrac{1}{2}$$

and

$$(-1)^{\ell - j + \frac{1}{2}} = 1 \qquad j = \ell + \tfrac{1}{2}$$

$$(-1)^{\ell - j + \frac{1}{2}} = -1 \qquad j = \ell - \tfrac{1}{2}$$

to write

$$S_1 = \left[\frac{1}{4} \frac{2j+1}{2\ell_1+1} + (-)^{\ell - j + \frac{1}{2}} \frac{(\ell+1)}{3(2\ell_1+1)} \right] \delta_{\ell_1 \ell_1'} + (-)^{\ell - j + \frac{1}{2}}$$

$$\times \frac{\left[\ell(\ell+1)(2\ell-1)(2\ell+3)\right]^{\frac{1}{2}}}{3} \left(\frac{2\ell+1}{2\ell_1'+1}\right)^{\frac{1}{2}} C(\ell_1 2 \ell_1'; 0,0)\, W(\ell \ell \ell_1' \ell_1; 2\ell_2)$$

(4.28)

So from this result, we see that although there was no interference between the different partial waves in the cross section, there is interference in the polarization. The only values possible for ℓ_1', given ℓ_1, are $\ell_1' = \ell_1$, $\ell_1 \pm 2$.

We may now write

$$\frac{\langle j_z \rangle}{j} = \frac{\sum_{\ell_1', \ell_1 \ell_2} i^{\ell_1' - \ell_1} \frac{(2\ell_1+1)(2\ell_1'+1)}{(2\ell_2+1)(2\ell+1)}}{j \sum_{\ell_1, \ell_2} \frac{(2\ell_1+1)^2}{(2\ell_2+1)(2\ell+1)}}$$

$$\times \frac{C(\ell_1' \ell_2 \ell; 0,0)\, C(\ell_1 \ell_2 \ell; 0,0)\, I(\ell_1 \ell_2 \ell)\, I(\ell_1' \ell_2 \ell)\, S_1}{C(\ell_1 \ell_2 \ell; 00)^2\, I(\ell_1 \ell_2 \ell)^2\, S_0} \tag{4.29}$$

from which one can obtain the polarization of the mu mesons captured into a state j, ℓ.[3] The Racah coefficients occurring in S_1 may be evaluated by using formulas given in the literature[4].

In Chapter II we found that the majority of the mu mesons were captured by emission of an electron with zero orbital angular momentum. If we set $\ell_2 = 0$ in (4.29) we see that: $\ell_1 = \ell_1' = \ell$, the sums contain one term each and the interference disappears. So, for the case $\ell_2 = 0$, we may write

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3. In (4.29) there occur products of $I(\ell_1 \ell_2 \ell)$. Since our radial functions are real this is correct. However, in obtaining (4.29) we make no assumptions concerning radial functions and the calculation would be the same for Coulomb waves if the $I(\ell_1 \ell_2 \ell)$ were redefined. This would involve redefining F_λ in Chapter II. In such a calculation the Coulomb phase would be kept with the radial functions and the relevant products in (4.29) would be replaced by $|I(\ell_1 \ell_2 \ell)|^2$ and $I(\ell_1' \ell_2 \ell)^* I(\ell_1 \ell_2 \ell)$.

4. L. C. Biedenharn, J. M. Blatt, and M. E. Rose, Revs. Modern Phys. 24, 249 (1952).

$$\frac{\langle j_z \rangle}{j} = \frac{S_1}{j S_o} \tag{4.30}$$

S_o was evaluated in Chapter II and is

$$S_o = \frac{2j+1}{2(2\ell_1+1)} \tag{2.44}$$

For $\ell_2 = 0$, S_1 becomes

$$S_1 = \frac{1}{4}\ \frac{2j+1}{2\ell+1} + \frac{(-)^{\ell-j+\frac{1}{2}}}{3} \left[\frac{\ell(\ell+1)}{(2\ell+1)} + \left[\ell(\ell+1)(2\ell-1)(2\ell+3)\right] \right]^{\frac{1}{2}}$$

$$\times\ C(\ell 2 \ell;00)\ W(\ell\ell\ell\ell\ ;\ 20) \tag{4.31}$$

and using a symmetry relation for the Racah coefficient we have

$$W(\ell\ell\ell\ell;20) = W(\ell\ell\ell\ell;02) \tag{4.32}$$

which may be evaluated as follows[5]

$$W(\ell\ell\ell\ell;02) = \frac{1}{2\ell+1} \tag{4.33}$$

The Clebsch-Gordan coefficient may be evaluated by using (2.48) to obtain

$$C(\ell 2 \ell;00) = -\left[\frac{\ell(\ell+1)}{(2\ell+3)(2\ell-1)}\right]^{\frac{1}{2}} \tag{4.34}$$

Thus we see that, if $\ell_2 = 0$, S_1 reduces to

$$S_1 = \frac{1}{4}\ \frac{2j+1}{2\ell+1} \tag{4.35}$$

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5. When one of the arguments of a Racah coefficient is zero it may be evaluated by using the symmetry relations and:

$$W(abcd;of) = \frac{(-)^{f-b-d}\ \delta_{ab}\ \delta_{cd}}{\left[(2b+1)(2d+1)\right]^{\frac{1}{2}}}\ .$$

and (4.30) becomes, on setting $\ell_1 = \ell$ in S_o,

$$\frac{\langle j_z \rangle}{j} = \frac{1}{2j} \tag{4.36}$$

Consequently, for the problem we are concerned with, (4.36) gives the remaining polarization after capture if the mu mesons are completely polarized and if they are captured from a beam. We make a rough estimate of the remaining polarization under these circumstances. In Figure 9 as discussed in Chapter III we show that the peak in the distribution of the mesons in states of ℓ is around $\ell = 4, 5$. So a fair value for j is $\frac{9}{2}$ and (4.36) tells us that the remaining polarization is about $\frac{1}{9}$. Therefore, since we know from experiment that the depolarization is not this severe, we know that we have not described the physical situation correctly[6].

So that we may understand why the result, (4.36) gives such a small polarization it is worthwhile to look for this reason in the mechanics of the calculation. The reason may be found by looking at the last two Clebsch-Gordan coefficients in (4.10). Since we took the original beam along the z axis $\ell_{1z} = 0$ and the maximum value of j_z in the bound state is determined by the requirement that $\ell_{2z} = m - \tau$. Since ℓ_2 is taken as zero there is clearly a severe restriction on the values that m assumes in (4.14). Indeed, the only case where there is

6. By this statement we mean that we have not described the situation that prevails when the asymmetry coefficients are measured. It is easy to think of a case where one would expect (4.36) to apply; namely, for the mu mesons captured, which would be very few, when a fast beam penetrates a thin foil.

no restriction on m, namely $j = \frac{1}{2}$, gives the maximum value $\frac{\langle j_z \rangle}{j} = 1$. Thus we may ask how the physics of the problem must be so that when we calculate, this restriction on j_z will be lifted. The answer is clear from the above statements; the mesons must not move along the axis of quantization. If this is the case then ℓ_{1z} can have values of the same magnitude as ℓ. Since the axis of quantization is fixed by the initial beam direction we are led to consider the scattering of the mu mesons before they are captured. In section 4 we will treat this scattering quantitatively; we now determine how it affects the polarization.

3. Depolarization of Mu Mesons Captured after Scattering

In Chapter II and therefore in the above treatment we represented the mu mesons by plane waves along the z axis. We could now take them as plane waves moving in the direction $\hat{k}_1$. However, as will be shown, we need not limit ourselves to the use of plane waves. Therefore we will take the free meson wave function to be given by

$$\psi_{\text{free}}(\mu) = 4\pi\, \chi_{\frac{1}{2}}^{\tau_1} \sum_{\ell_1, m_1} i^{\ell_1} f_{\ell_1}(k_1 r_1)\, Y^*_{\ell_1, m_1}(\hat{k}_1)\, Y_{\ell_1, m_1}(\hat{r}_1) \tag{4.37}$$

where we neglect to write normalization volumes, since these always cancel. The function $f_1(k_1 r_1)$ is defined to be the proper radial wave function; if one wished, it could be a Coulomb function with phase.

There is another improvement which we use here. In Chapter II we took the wave function of the bound electron to be a hydrogen-like function and we considered only electrons in the K shell. We now take the wave function of the bound electron to be

$$\Psi_{\text{bound}}(e) = \sum_{\tau_3} C(\ell_3 \tfrac{1}{2} j_3; m_3 - \tau_3, \tau_3)\, Y_{\ell_3, m_3 - \tau_3}(\hat{r}_2)\, \chi_{\frac{1}{2}}^{\tau_3}\, R(r_2) \tag{4.38}$$

with the stipulation that $R(r_2)$ is the correct radial wave function for the bound electron. Thus, for example, $R(r_2)$ might include the effects of screening.

The wave function for the ejected electron is taken as

$$\Psi_{\text{free}}(e) = 4\pi\, \chi_{\frac{1}{2}}^{\tau_2} \sum_{\ell_2, m_2} i^{\ell_2}\, f_{\ell_2}(k_2 r_2)\, Y^*_{\ell_2, m_2}(\hat{k}_2)\, Y_{\ell_2, m_2}(\hat{r}_2) \tag{4.39}$$

where the same remarks apply to $f_{\ell_2}(k_2 r_2)$ as to $f_{\ell_1}(k_1 r_1)$. The wave function for the final meson state is

$$\Psi_{\text{bound}}(\mu) = \sum_{\tau} C(\ell \tfrac{1}{2} j; m - \tau, \tau)\, Y_{\ell, m-\tau}(\hat{r}_1)\, \chi_{\frac{1}{2}}^{\tau}\, R_{n,\ell}(r_1) \tag{4.40}$$

where $R_{n,\ell}(r_1)$ is not necessarily a hydrogen-like function. For the interaction we keep the definition of Chapter II, since the only conceivable modification would be to change the definition of F_λ. Thus we write

$$V_{\text{int}} = \sum_{\lambda, M_\lambda} \frac{4\pi}{2\lambda+1}\, Y^*_{\lambda, M_\lambda}(\hat{r}_2)\, Y_{\lambda, M_\lambda}(\hat{r}_1)\, F_\lambda(r_1, r_2) \tag{4.41}$$

with the understanding that the choice of F_λ is determined by the proper formulation of the perturbation; which depends on the choice of f_{ℓ_2} and f_{ℓ_1}. We are now ready to begin the calculation.

The quantity which contains all the information concerning polarization is $\int |H'_{fi}|^2 \, d\Omega_{k_2}$. Thus we must first find H_{fi}.

$$H_{fi} = (4\pi)^3 \sum (-i)^{\ell_2} (i)^{\ell_1} C(\ell_3 \tfrac{1}{2} j_3; m_3-\tau_3, \tau_3) \, C(\ell \tfrac{1}{2} j; m-\tau, \tau)$$

$$\times \left(\chi_{\frac{1}{2}}^{\tau_2}, \chi_{\frac{1}{2}}^{\tau_3}\right)\left(\chi_{\frac{1}{2}}^{\tau}, \chi_{\frac{1}{2}}^{\tau_1}\right) \frac{1}{2\lambda+1} \left(Y^*_{\ell, m-\tau}(\hat{r}_1), Y_{\lambda, M_\lambda}(\hat{r}_1)\, Y_{\ell_1, m_1}(\hat{r}_1)\right)$$

$$\times \left(Y^*_{\ell_2, m_2}(\hat{r}_2), Y^*_{\lambda, M_\lambda}(\hat{r}_2)\, Y_{\ell_3, m_3-\tau_3}(\hat{r}_2)\right) I(\ell \lambda \ell_2 \ell_1 \ell_3)$$

$$\times \quad Y^*_{\ell_1, m_1}(\hat{k}_1)\, Y_{\ell_2, m_2}(\hat{k}_2) \qquad (4.42)$$

where the sum is over τ_2, τ_3, λ, M_λ, m_1, m_2, ℓ_1, and ℓ_2; we do not write the sum on m and m_3 yet since we will form $\langle j_z \rangle$ before we do these sums. Also we have used the definition

$$I(\ell \lambda \ell_2 \ell_1 \ell_3) = \int_0^\infty \int_0^\infty r_1^2 \, r_2^2 \, dr_1 \, dr_2 \, F_\lambda(r_1, r_2) \, R_{n,\ell}(r_1) \, R(r_2)$$

$$\times \quad f^*_{\ell_2}(k_2 r_2) \, f_{\ell_1}(k_1 r_1) \qquad (4.43)$$

Henceforth we denote (4.43) by I, with the understanding that the arguments are implied. In (4.42) we use

$$\left(\chi_{\frac{1}{2}}^{\tau_2}, \chi_{\frac{1}{2}}^{\tau_3}\right) = \delta_{\tau_2 \tau_3} \qquad (4.44a)$$

$$\left(\chi_{\frac{1}{2}}, \chi_{\frac{1}{2}}^{\tau_1}\right) = \delta_{\tau \tau_1} \qquad (4.44b)$$

$$(Y^*_{\ell_2,m_2}(\hat{r}_2),\ Y^*_{\lambda,M_\lambda}(\hat{r}_2)\ Y_{\ell_3,m_3-\tau_3}(\hat{r}_2)) = (-)^{M_\lambda}\left[\frac{(2\ell_3+1)(2\lambda+1)}{(2\ell_2+1)\ 4\pi}\right]^{\frac{1}{2}}$$

$$\times\ C(\ell_3\lambda\ \ell_2;0,0)\ C(\ell_3\lambda\ \ell_2;m_3-\tau_3,-M_\lambda,m_2) \qquad (4.44c)$$

$$(Y^*_{\ell,m-\tau}(\hat{r}_1),\ Y_{\lambda,M_\lambda}(\hat{r}_1)\ Y_{\ell_1,m_1}(\hat{r}_1)) = \left[\frac{(2\ell_1+1)(2\lambda+1)}{(2\ell+1)\ 4\pi}\right]^{\frac{1}{2}} C(\ell_1\lambda\ell;0,0)$$

$$\times\ C(\ell_1\lambda\ell;m_1,M_\lambda,m-\tau) \qquad (4.44d)$$

We see that the spin sum is now over τ_3. Eventually we will take $\tau = \frac{1}{2}$ so there will be no sum on τ. Now we may write out $|H'_{fi}|^2$, taking account of the following consideration. The problem is formulated so that ℓ_3, j_3, ℓ and j are specified in advance. Thus we have

$$|H_{fi}|^2 = (4\pi)^4 \sum (-i)^{\ell_2}\, i^{\ell'_2}\, i^{\ell_1}\, (-i)^{\ell'_1}\left[C(\ell_3\tfrac{1}{2}j_3;m_3-\tau_3,\ \tau_3)\right]^2$$

$$\times\left[C(\ell\ \tfrac{1}{2}j;m-\tau,\ \tau)\right]^2 \frac{1}{(2\lambda+1)(2\lambda'+1)}\left[\frac{(2\ell_1+1)(2\lambda+1)}{(2\ell+1)}\ \frac{(2\ell'_1+1)(2\lambda'+1)}{(2\ell+1)}\right]^{1/2}$$

$$\times C(\ell'_1\lambda'\ell;0,0)\ C(\ell_1\lambda\ell;0,0)\ C(\ell'_1\lambda'\ell;m'_1,M'_\lambda,m-\tau)\ C(\ell_1\lambda\ \ell;m_1,M_\lambda,m-\tau)$$

$$\times\left[\frac{(2\ell_3+1)(2\lambda'+1)}{(2\ell'_2+1)}\ \frac{(2\ell_3+1)(2\lambda+1)}{(2\ell_2+1)}\right]^{\frac{1}{2}} (-)^{M_\lambda+M'_\lambda}\ C(\ell_3\lambda'\ \ell'_2;m_3-\tau_3,-M'_\lambda,m'_2)$$

$$\times C(\ell_3\lambda\ \ell_2;m_3-\tau_3,-M_\lambda,m_2)\ C(\ell_3\lambda'\ \ell'_2;00)\ \ C(\ell_3\lambda\ \ell_2;0,0)$$

$$Y^*_{\ell_1,m_1}(\hat{k}_1)\ Y^*_{\ell'_2,m'_2}(\hat{k}_2)\ Y_{\ell_2,m_2}(\hat{k}_2)\ I^{*'}\ I\ Y_{\ell'_1,m'_1}(\hat{k}_1) \qquad (4.45)$$

The meaning of the prime on $I^{*\prime}$ is that the appropriate arguments of I^* are primed. The sum is over τ_3, λ, λ', M_λ, M'_λ, m_1, m'_1, m_2, m'_2, ℓ_1, ℓ'_1, ℓ_2 and ℓ'_2. We now integrate over the unobserved electron direction; this involves only the last two spherical harmonics.

$$\int Y^*_{\ell'_2\, m'_2}(\hat{k}_2)\, Y_{\ell_2, m_2}(\hat{k}_2)\, d\Omega_{k_2} = \delta_{m_2\, m'_2}\, \delta_{\ell_2\, \ell'_2} \qquad (4.46)$$

This results in considerable simplification of (4.45) since M_λ must now be M'_λ, and therefore $m'_1 = m_1$.

At this point we must discuss the fashion in which we will proceed. We could expand the product $Y_{\ell'_1, m'_1}(\hat{k}_1)\, Y^*_{\ell_1, m_1}(\hat{k}_1)$ using a standard technique and proceed eventually to the evaluation of $\frac{\langle j_z \rangle}{j}$. However, we will prove, in the following section, that this is unnecessary. For the present we assume that the mu mesons have random directions. This assumption is equivalent to stating that all directions of $\hat{k}_1$ are equally probable. Therefore we shall average over $\hat{k}_1$. We define

$$Q_o = \sum_{m, m_3} \int \left| H'_{fi} \right|^2 d\Omega_{k_2}\, d\Omega_{k_1}\, \frac{1}{4\pi} \qquad (4.47)$$

The factor, 4π, is just $\int d\Omega_{k_1}$. The integral over $d\Omega_{k_1}$ involves only the spherical harmonics just mentioned.

$$\int Y^*_{\ell_1, m_1}(\hat{k}_1)\, Y_{\ell'_1, m'_1}(\hat{k}_1)\, d\Omega_{k_1} = \delta_{m_1\, m_1}\, \delta_{\ell_1\, \ell'_1} \qquad (4.48)$$

So, we now obtain

$$Q_o = (4\pi)^3 \sum_{m,m_3} \sum_{\substack{\ell_2,\ell_1 \\ \lambda,\lambda'}} \sum_{m_2,m_1,\tau_3} \left[C(\ell_3 \tfrac{1}{2} j_3; m_3-\tau_3, \tau_3)\right]^2$$

$$\times\left[C(\ell \tfrac{1}{2} j; m-\tau, \tau)\right]^2 C(\ell_1 \lambda \ell; 00)\ C(\ell_1 \lambda' \ell; 00)\ C(\ell_1 \lambda \ell; m_1, M_\lambda, m-\tau)$$

$$\times C(\ell_1 \lambda' \ell; m_1, M_\lambda, m-\tau)\ C(\ell_3 \lambda' \ell_2; m_3-\tau_3, -M_\lambda, m_2)\ C(\ell_3 \lambda \ell_2; m_3-\tau_3, -M_\lambda, m_2)$$

$$\times C(\ell_3 \lambda' \ell_2; 0,0)\ C(\ell_3 \lambda \ell_2; 0,0)\ \frac{2\ell_3+1}{2\ell_2+1}\ \frac{2\ell_1+1}{2\ell+1}\ I^{*'}\ I \qquad (4.49)$$

Now the sums over the magnetic quantum numbers must be carried out. We consider the sum involving these four Clebsch-Gordan coefficients

$$\sum_{m_1,m_2} C(\ell_1 \lambda \ell; m_1, M_\lambda, m-\tau)\ C(\ell_1 \lambda' \ell; m_1, M_\lambda, m-\tau)\ C(\ell_3 \lambda' \ell_2; m_3-\tau_3, -M_\lambda, m_2)$$

$$\times\ C(\ell_3 \lambda \tau_2; m_3-\tau_3, -M_\lambda, m_2) \qquad (4.50)$$

We consider two of these as follows:

$$M_\lambda = m-\tau-m_1 \qquad (4.51a)$$

Since we also have

$$m_2 = m_3-\tau_3-M_\lambda \qquad (4.51b)$$

m_2 is determined by m_1 and we need sum only over m_1. We use the symmetry relations to obtain

$$C(\ell_1 \lambda \ell; m_1, m-\tau-m_1) = C(\lambda \ell_1 \ell; m-\tau-m_1, m_1) \qquad (4.52a)$$

since $\ell_1 + \lambda + \ell$ is even.

$$C(\ell_3 \lambda \ell_2; m_3 - \tau_3, -(m - \tau - m_1)) = (-)^{\ell_3 - m_3 + \tau_3} \left(\frac{2\ell_2+1}{2\lambda+1} \right)^{\frac{1}{2}}$$

$$\times \; C(\ell_3 \ell_2 \lambda; m_3 - \tau_3, m - \tau - m_1 - m_3 + \tau_3) \qquad (4.52b)$$

Applying the Racah recoupling formula:

$$(-)^{\ell_3 - m_3 + \tau_3} \left(\frac{2\ell_2+1}{2\lambda+1} \right)^{\frac{1}{2}} C(\ell_3 \ell_2 \lambda; m_3 - \tau_3, m - \tau - m_1 - m_3 + \tau_3) \; C(\lambda \ell_1 \ell; m - \tau - m_1, m_1)$$

$$= (-)^{\ell_3 - m_3 + \tau_3} \left(\frac{2\ell_2+1}{2\lambda+1} \right)^{\frac{1}{2}} \sum_{v} \left[(2v+1)(2\lambda+1) \right]^{\frac{1}{2}} C(\ell_3 v \ell; m_3 - \tau_3, m - \tau - m_3 + \tau_3)$$

$$\times \quad C(\ell_2 \ell_1 v; m - \tau - m_1 - m_3 + \tau_3, m_1) \; W(\ell_3 \ell_2 \ell \ell_1; \lambda v) \qquad (4.53)$$

We do exactly the same with the other pair of Clebsch-Gordan coefficients in (4.50); we then obtain the equivalent expression

$$\sum_{v,v'} (2\ell_2+1) \left[(2v+1)(2v'+1) \right]^{\frac{1}{2}} W(\ell_3 \ell_2 \ell \ell_1; \lambda v) \; W(\ell_3 \ell_2 \ell \ell_1; \lambda' v')$$

$$\times C(\ell_3 v \ell; m_3 - \tau_3, m - \tau - m_3 + \tau_3) \; C(\ell_3 v' \ell; m_3 - \tau_3, m - \tau - m_3 + \tau_3)$$

$$\times \sum_{m_1} C(\ell_2 \ell_1 v; m - \tau - m_1 - m_3 + \tau_3, m_1) \; C(\ell_2 \ell_1 v'; m - \tau - m_1 - m_3 + \tau_3, m_1) \quad (4.54)$$

The sum over m_1 gives $\delta_{vv'}$ by the orthonormality relation, so Q_o is reduced to:

$$Q_o = (4\pi)^3 \sum_{\substack{\ell_2, \ell_1 \\ \lambda, \lambda'}} \sum_{m, m_3} \sum_{\tau_3} \sum_{v} \frac{(2\ell_1+1)(2v+1)(2\ell_3+1)}{2\ell+1} W(\ell_3 \ell_2 \ell \ell_1; \lambda v)$$

$$\times \; W(\ell_3 \ell_2 \ell \ell_1; \lambda' v) \left[C(\ell \tfrac{1}{2} j; m - \tau, \tau) \right]^2 C(\ell_1 \lambda \ell; 00) \; C(\ell_1 \lambda' \ell; 00)$$

$$\times \; C(\ell_3 \lambda' \ell_2; 00) \; I^* \; I \left[C(\ell_3 v \ell; m_3 - \tau_3, m - \tau - m_3 + \tau_3) \; C(\ell_3 \tfrac{1}{2} j_3; m_3 - \tau_3, \tau_3) \right]^2$$

$$\times \; C(\ell_3 \lambda \ell_2; 00) \qquad (4.55)$$

where the last four Clebsch-Gordan coefficients have been written with the brackets to indicate how we will do the sum on τ_3 and m_3. Using a symmetry relation we may write

$$C(\ell_3 v \ell; m_3-\tau_3, m-\tau-m_3+\tau_3) = (-)^{v-(m_3-\tau_3)+m-\tau}\left(\frac{2\ell+1}{2\ell_3+1}\right)^{\frac{1}{2}}(-)^{\ell+v-\ell_3}$$

$$\times C(\ell v \ell_3; +(m-\tau), -(m-\tau)+(m_3-\tau_3)) \qquad (4.56)$$

and we now recouple the last Clebsch-Gordan coefficient in (4.56) with the last in (4.55) to obtain

$$(-)^{v-(m_3-\tau_3)+m-\tau+\ell+v-\ell_3}\left(\frac{2\ell+1}{2\ell_3+1}\right)^{\frac{1}{2}}\sum_s\left[(2s+1)(2\ell_3+1)\right]^{\frac{1}{2}} W(\ell v j_3 \tfrac{1}{2}; \ell_3 s)$$

$$C(\ell s j_3; m-\tau, m_3-(m-\tau))\, C(v\tfrac{1}{2}s; m_3-\tau_3-(m-\tau), \tau_3) \qquad (4.57)$$

and we do exactly the same with the other pair in the last bracket of (4.55), clearly the phase must be unity

$$\sum_{m_3,\tau_3}\left[\quad\right]^2 = \sum_{s,s'}\left[(2s+1)(2s'+1)\right]^{\frac{1}{2}}(2\ell+1)\, W(\ell v j_3\tfrac{1}{2}; \ell_3 s)\, W(\ell v j_3 \tfrac{1}{2}; \ell_3 s')$$

$$\times\sum_{m_3} C(\quad 2 j_3; m-\tau, m_3-(m-\tau))\, C(\ell s' j_3; m-\tau, m_3-(m-\tau))$$

$$\times\sum_{\tau_3} C(v\tfrac{1}{2}s; m_3-\tau_3-(m-\tau), \tau_3)\, C(v\tfrac{1}{2}s'; m_3-\tau_3-(m-\tau), \tau_3) \qquad (4.58)$$

Applying a symmetry relation to each of these last two Clebsch-Gordan coefficients and then using the orthonormality relation yields $\delta_{ss'}$. For the Clebsch-Gordan coefficients in the sum over m_3 we use two symmetry relations to write:

$$\sum_{m_3}\left[C(\ell s j_3;m-\tau,m_3-(m-\tau))\right]^2 = \frac{2j_3+1}{2\ell+1}\sum_{m_3}\left[C(j_3 s\ell;m_3,(m-\tau)-m_3)\right]^2 \quad (4.59)$$

So, by the orthonormality relation the sum over m_3 is $\frac{2j_3+1}{2\ell+1}$. Now Q_o becomes:

$$Q_o = (4\pi)^3 \sum_{m}\sum_{\substack{\ell_2\,\ell_1\\ \lambda\,\lambda'}}\sum_{v,s}\frac{(2\ell_1+1)(2v+1)(2\ell_3+1)(2j_3+1)(2s+1)}{(2\ell+1)}$$

$$\times\left[W(\ell v j_3\tfrac{1}{2};\ell_3 s)\right]^2 W(\ell_3\ell_2\ell\ell_1;\lambda v)\,W(\ell_3\ell_2\ell\ell_1;\lambda' v)\,C(\ell_1\lambda\ell;00)$$

$$C(\ell_1\lambda'\ell;00)\,C(\ell_3\lambda\ell_2;0,0)\,C(\ell_3\lambda'\ell_2;0,0)\quad I'^{*}\, I\,\left[C(\ell\tfrac{1}{2}j;m-\tau,\tau)\right]^2$$

(4.60)

We use a symmetry relation for the Racah coefficients to write

$$\left[W(\ell v j_3\tfrac{1}{2};\ell_3 s)\right]^2 = \left[W(\ell j_3 v\tfrac{1}{2};s\,\ell_3)\right]^2 \quad (4.61)$$

and the sum over s is, by the orthonormality of the Racah coefficients,

$$\sum_{s}(2s+1)(2\ell_3+1)\left[W(\ell j_3 v\tfrac{1}{2};s\,\ell_3)\right]^2 = 1 \quad (4.62$$

The sum over v is done in a similar fashion. From the symmetry relations

$$W(\ell_3\ell_2\ell\ell_1;\lambda v)\,W(\ell_3\ell_2\ell\ell_1;\lambda' v) = W(\ell_3\ell\ell_2\ell_1;v\lambda)\,W(\ell_3\ell\ell_2\ell_1;v\lambda')$$

(4.63)

and the sum over v is

$$\sum_{v}(2v+1)(2\lambda+1)\,W(\ell_3\ell\ell_2\ell_1;v\lambda)\,W(\ell_3\ell\ell_2\ell_1;v\lambda') = \delta_{\lambda\lambda'} \quad (4.64)$$

Consequently, Q_o becomes

$$Q_o = (4\pi)^3 \sum_{\substack{\ell_2 \ell_1 \lambda \\ m}} \frac{(2\ell_1+1)(2j_3+1)}{(2\ell+1)(2\lambda+1)} \left[C(\ell_1\lambda\ell;00)\right]^2 \left[C(\ell_3\lambda\ell_2;00)\right]^2$$

$$\times \, |I|^2 \left[C(\ell\tfrac{1}{2}j;m-\tau,\tau)\right]^2 \qquad (4.65)$$

Now, if we wished, we could use Q_o to calculate a cross section; however, we wish to find $\frac{\langle j_z \rangle}{j}$. We find this quantity just as we did before, by calculating $\frac{\sum_m m \, \mathrm{pop}(m)}{j \sum_m \mathrm{pop}(m)}$. One should note that m occurs in (4.65) only in the last Clebsch-Gordan coefficient and that the quantum numbers, ℓ and j, that are the arguments of the coefficient are not summed over in evaluating Q_o. Thus when we form the expression for $\frac{\langle j_z \rangle}{j}$ we have

$$\frac{\langle j_z \rangle}{j} = \frac{\sum_m m \left[C(\ell\tfrac{1}{2}j;m-\tau,\tau)\right]^2}{\sum_m \left[C(\ell\tfrac{1}{2}j;m-\tau,\tau)\right]^2} \qquad (4.66)$$

and all the other factors have canceled. We now evaluate this expression. First we use the symmetry relations to rewrite the Clebsch-Gordan coefficients in the denominator; thus

$$\sum_m \left[C(\ell\tfrac{1}{2}j;m-\tau,\tau)\right]^2 = \frac{2j+1}{2} \sum_m \left[C(j\tfrac{1}{2}\ell;m,\tau-m)\right]^2 = \frac{2j+1}{2} \qquad (4.67)$$

To accomplish the sum in the numerator of (4.66) we use

$$C(j1j;m,0) = \frac{m}{\sqrt{j(j+1)}} \qquad (4.68)$$

and the sum we now have is

$$\sum_{m} \sqrt{j(j+1)}\, C(\ell \tfrac{1}{2} j; m-\tau, \tau)\, C(j1j; m0)\, C(\ell \tfrac{1}{2} j, m-\tau, \tau) \tag{4.69}$$

We recouple the first two Clebsch-Gordan coefficients in (4.69), thus

$$C(\ell \tfrac{1}{2} j; m-\tau, \tau)\, C(j1j; m,0) = \sum_{v} \left[(2v+1)(2j+1)\right]^{\frac{1}{2}} W(\ell \tfrac{1}{2} j1; jv)$$

$$C(\ell vj; m-\tau, \tau)\, C(\tfrac{1}{2} 1v; \tau, 0) \tag{4.70}$$

When we substitute (4.70) in (4.69) the remaining sum is done as follows:

$$\sum_{m} C(\ell vj; m-\tau, \tau)\, C(\ell \tfrac{1}{2} j; m-\tau, \tau) = \left(\frac{2j+1}{2v+1}\right)^{\frac{1}{2}} \left(\frac{2j+1}{2}\right)^{\frac{1}{2}}$$

$$\times \sum_{m} C(j\ell v; m, \tau-m)\, C(j\ell \tfrac{1}{2}, m, \tau-m) = \frac{2j+1}{2}\, \delta_{v\frac{1}{2}} \tag{4.71}$$

So we now have

$$\frac{\langle j_z \rangle}{j} = \frac{1}{j} \left[(2j+1)\, j(j+1)2\right]^{\frac{1}{2}} C(\tfrac{1}{2} 1 \tfrac{1}{2}; \tau, 0)\, W(\ell \tfrac{1}{2} j1; j\tfrac{1}{2}) \tag{4.72}$$

and we now use

$$C(\tfrac{1}{2} 1 \tfrac{1}{2}; \tau, 0) = \frac{\tau}{\sqrt{\frac{1}{2}\ \frac{3}{2}}} \tag{4.73}$$

and the explicit value of the Racah coefficient

$$W(\ell \tfrac{1}{2} j1; j\tfrac{1}{2}) = (-)^{\ell+1-j-\frac{1}{2}}\, W(jj\tfrac{1}{2}\tfrac{1}{2}; 1\ell) = \frac{j(j+1) + \frac{3}{4} - \ell(\ell+1)}{\left[2j(j+1)(2j+1)3\right]^{\frac{1}{2}}} \tag{4.74}$$

Consequently we obtain our result:

$$\frac{\langle j_z \rangle}{j} = \frac{2\tau}{3j} \left[j(j+1) + \tfrac{3}{4} - \ell(\ell+1)\right] \tag{4.75}$$

We note that we still must choose the value of τ. First we remark that (4.75) clearly is an odd function of τ, thus $\frac{\langle j_z \rangle}{j}$ changes sign if the polarization of the initial beam is taken opposite to the beam instead of along the beam. So, even though we have averaged over the meson direction the initial direction of the meson's spin is still with us. One also notes, that had the initial beam been unpolarized then we would now have to sum over τ and we would find that $\langle j_z \rangle = 0$ as we should. τ is now taken as $\frac{1}{2}$ and (4.75) evaluated for the two possible cases:

$$\frac{\langle j_z \rangle}{j} = \frac{1}{3}\left(1 + \frac{1}{j}\right) \quad ; \quad j = \ell + \tfrac{1}{2} \tag{4.76a}$$

$$\frac{\langle j_z \rangle}{j} = -\frac{1}{3} \quad ; \quad j = \ell - \tfrac{1}{2} \tag{4.76b}$$

We wish to emphasize that these results, (4.76), depend only on the assumption that the mu mesons have random direction when they are captured. The results do not depend on which electron is ejected, which partial wave in the incoming beam is captured, or which partial wave the electron is emitted into. Further, in setting up the wave functions used in obtaining (4.76) we were careful to point out that the wave functions were defined in such a way that one could introduce any refinements desired; therefore the results do not depend on the atomic model. Of course, when we deal with an ensemble of mesons we must know the relative number of mesons captured into each state j; for this purpose we will use the distribution obtained in Chapter III. We must now determine if the assumption of random directions is valid.

4. Randomization of a Mu Meson Beam by Scattering

We wish to determine what the effect of scattering is on the mu meson beam. For this purpose the quantity $\langle \hat{k}_E \cdot \hat{k}_{E_o} \rangle$ is defined. This quantity is just the expectation value of cos θ where θ is the angle between the original beam direction of the mu mesons with energy E_o and the direction of the mu meson when it has energy E. If $\langle k_E \cdot k_{E_o} \rangle = 1$ the mu mesons form a beam; if $\langle k_E \cdot k_{E_o} \rangle = 0$ the mesons have random directions. To calculate this quantity we proceed as follows. Define:

$$s = \cos\theta = \frac{\hat{k}_{z\ \text{current}}}{\hat{k}_{z\ \text{initial}}} \tag{4.77a}$$

$$s' = \cos\theta' = \frac{\hat{k}_{z\ \text{current}+1}}{\hat{k}_{z\ \text{initial}}} \tag{4.77b}$$

where θ is the angle between the initial direction and the direction after some scattering has occurred. Then θ' is the angle between the direction of the meson after one more scattering event and the initial direction. Then we may write

$$\Delta s = s'-s = \cos\Theta \cos\theta + \sin\Theta \sin\theta \cos\phi - \cos\theta \tag{4.78}$$

where Θ is the angle that the meson was scattered through by the scattering event current + 1. Now we may average Δs over an increment of path Δx, and obtain

$$\frac{\Delta\langle s\rangle}{\Delta x} = n\int\sigma(\theta)\, d\Omega\, \Delta s \tag{4.79}$$

where $\sigma(\theta)$ is the differential scattering cross section, n, the number of scattering centers per unit volume and $d\Omega$ the surface element.

Using (4.78), and observing that the second term of (4.78) does not contribute, we have

$$\frac{\Delta \langle s \rangle}{\Delta x} = -n \langle s \rangle \int \sigma(\theta)(1 - \cos\theta)\, d\Omega \tag{4.80}$$

where we have used the definition, (4.77a). We note that the integral in (4.80) is just the definition of the transport cross section, σ_{Tr}; so we have

$$\frac{d\langle s \rangle}{dx} = -n \langle s \rangle \sigma_{Tr} \tag{4.81}$$

Now we use the identity

$$dx = dE/(dE/dx) \tag{4.82}$$

and dE/dx is given by

$$\frac{dE}{dx} = - \langle \Delta E \rangle\, n\, \sigma_T(E) \tag{3.2}$$

as in Chapter III; use of (3.5),

$$\langle \Delta E \rangle = \kappa E \frac{\sigma_{Tr}(E)}{\sigma_T(E)} \tag{3.5}$$

then yields

$$dx = - \frac{dE}{n\, \kappa E\, \sigma_{Tr}(E)} \tag{4.83}$$

where

$$\kappa = \frac{2m_\mu M}{(m_\mu + M)^2} \tag{4.84}$$

M is the mass of the scattering atom. We now integrate (4.81) to obtain

$$\langle s \rangle = \exp\left(-n \int_0^x \sigma_{Tr}\, dx\right) \tag{4.85}$$

we interchange the limits in (4.85) to remove the minus sign and use (4.83) to find

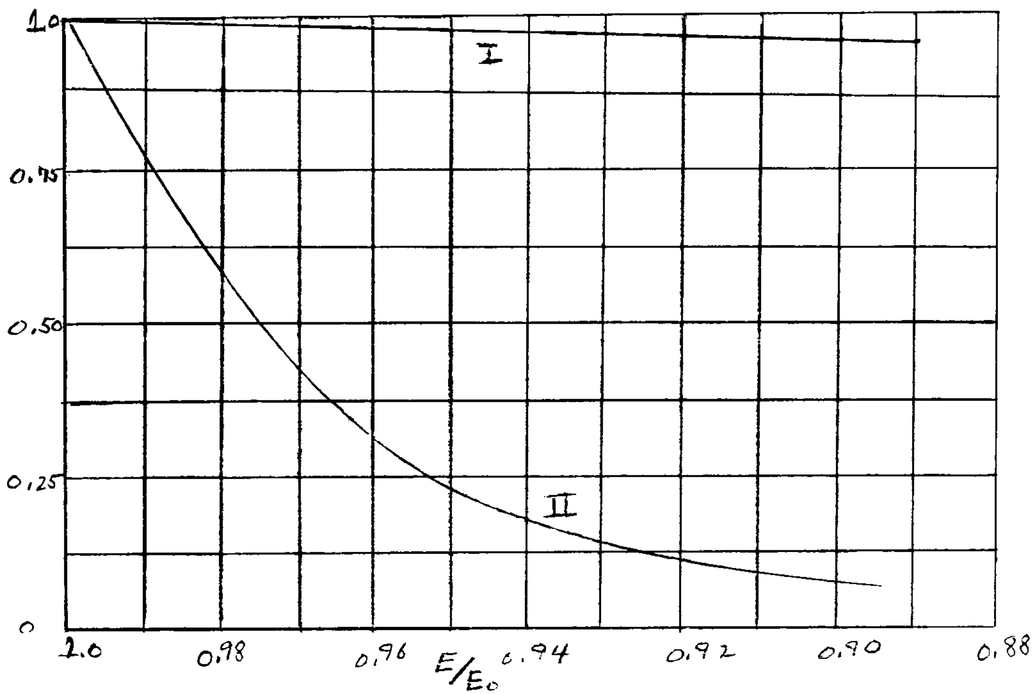
$$\langle s \rangle = \exp\left(-\frac{1}{\kappa}\int_{E}^{E_0}\frac{dE}{E}\right) \qquad (4.86)$$

and clearly the right side of (4.86) may be evaluated as $(E/E_0)^{1/\kappa}$. Thus we obtain the result,

$$\langle \hat{k}_E \cdot \hat{k}_{E_0} \rangle = (E/E_0)^{1/\kappa} \qquad (4.87)$$

This expression tells us, in effect, how well the mesons remember their initial direction. To see what one may say about the direction of the mu mesons when they are captured it is necessary to compare the energy dependence of their direction memory with their capture rate. In Figure 12 we have taken E_0 to correspond to $p = 70\ \gamma\hbar$ and taken carbon as the scattering atom, as was done in Chapter III, when we derived the initial distribution. The nearly flat curve in Figure 12 shows the number of mu mesons remaining free at energy E; the other curve represents (4.87). Thus we see from Figure 12 that the vast majority of the mu mesons will have forgotten their initial direction before they are captured. Therefore we have shown that we have a very good description of the physical situation if we assume that the mu mesons have random directions when they are captured. Further, we understand why we obtained a result that could not be compatible with the experimental facts when we assumed that the mesons were captured from a beam.

5. Discussion



I. Relative number of free mu mesons at energy E

II. $\langle \hat{k}_E \cdot \hat{k}_{E_o} \rangle$; memory of initial beam direction at energy E

Effect of Scattering on Meson Beam

Figure 12

The results of sections 3 and 4, above, lead us to a very definite conclusion concerning the amount of polarization the mu mesons retain after they are captured[7]. We reproduce these results here so that we may point out some of their consequences.

$$\frac{\langle j_z \rangle}{j} = \frac{1}{3}\left(1 + \frac{1}{j}\right) \quad ; \qquad j = \ell + \tfrac{1}{2} \qquad (4.76a)$$

$$\frac{\langle j_z \rangle}{j} = -\frac{1}{3} \quad ; \qquad j = \ell - \tfrac{1}{2} \qquad (4.76b)$$

First we note that if a mu meson is captured into a state $\ell = 0$, then it remains completely polarized. Next we point out that there appears to be an asymmetry in the expressions (4.76a) and (4.76b), this is because we calculate $\frac{\langle j_z \rangle}{j}$; in particular the minus sign in (4.76b) does not mean spin reversal for the mesons that go into states with $j = \ell - \frac{1}{2}$. For the purpose of this discussion we write (4.76) in terms of $\langle \sigma_z \rangle$

$$\langle \sigma_z \rangle = \frac{1}{3}\left(1 + \frac{1}{j}\right) \quad ; \qquad j = \ell + \tfrac{1}{2} \qquad (4.88a)$$

$$\langle \sigma_z \rangle = \frac{1}{3}\left(1 - \frac{1}{j+1}\right) \quad ; \qquad j = \ell - \tfrac{1}{2} \qquad (4.88b)$$

From these we see that the mesons never suffer a spin flip in capture; however, we note that those mesons captured into states $j = \ell - \frac{1}{2}$ are always depolarized to some extent. This is understandable

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7. There have been previous attempts to determine the depolarization due to capture. None of the articles are very comprehensive. See: J. Von Behr and H. Marshall, Nuclear Phys. 14, 342 (1959); also I. M. Shmuskevitch, Nuclear Phys. 11, 419 (1959).

when one recalls that there are values of m, the projection of j_z, for which the bound state wave function is an eigenfunction of σ_z when $j = \ell + \frac{1}{2}$; when $j = \ell - \frac{1}{2}$ eigenfunctions of σ_z do not occur.

We now discuss a statement that we have made several times previously; namely that the depolarization on capture is insensitive to the distribution of the mesons among the atomic states. As we mentioned in section 2 above, a fair value for j according to our distribution is $\frac{9}{2}$. Therefore the term $\frac{1}{j}$ in (4.76a) represents a correction of about 20 per cent. Consequently, a 10 per cent change in our distribution would amount to only 2 per cent change in the depolarization due to capture.

The results of Chapter III and sections 3 and 4 above constitute an adequate solution of the depolarization suffered when the mu mesons are captured; it is now necessary to study the events that occur after the mesons are captured.

CHAPTER V

DEPOLARIZATION OF MU MESONS IN THE ATOMIC CASCADE

1. The Nature of the Atomic Cascade

As shown in Chapter III the mu mesons are captured into highly excited states of the mu mesic atom. The mesons in such states make transitions to the lower states and in general suffer some depolarization in these transitions. There are only two possible types of transitions that the mu meson may make; namely, radiative transitions and Auger transitions. The mesic Auger transition was defined in Chapter I, where typical rates for these two processes were also presented. The matters to be discussed in this section are the details of the cascade as determined by the selection rules and the available energy.

First, it is necessary to recall that the states of the mesic atom are described by the three quantum numbers n, ℓ , j. The principle quantum number n gives the energy of the state. A given state n is n-fold degenerate in states of ℓ ; that is, for a given state n, ℓ takes on the value $0 \leq \ell \leq n-1$. The degeneracy in energy is lifted by the finite nuclear size but we have no need to consider this small effect. For each state ℓ there are two states of j, $j = \ell + \frac{1}{2}$ and $j = \ell - \frac{1}{2}$; except when $\ell = 0$ where there is one state $j = \frac{1}{2}$. We now temporarily confine ourselves to a discussion of the radiative transitions. It is well known that the electric dipole transitions are the only transitions that occur with significant probability in hydrogen-like atoms, excluding meta-stable states of course.[1] The selection rules for electric dipole transitions are

$$\Delta \ell = \pm 1 \tag{5.1a}$$

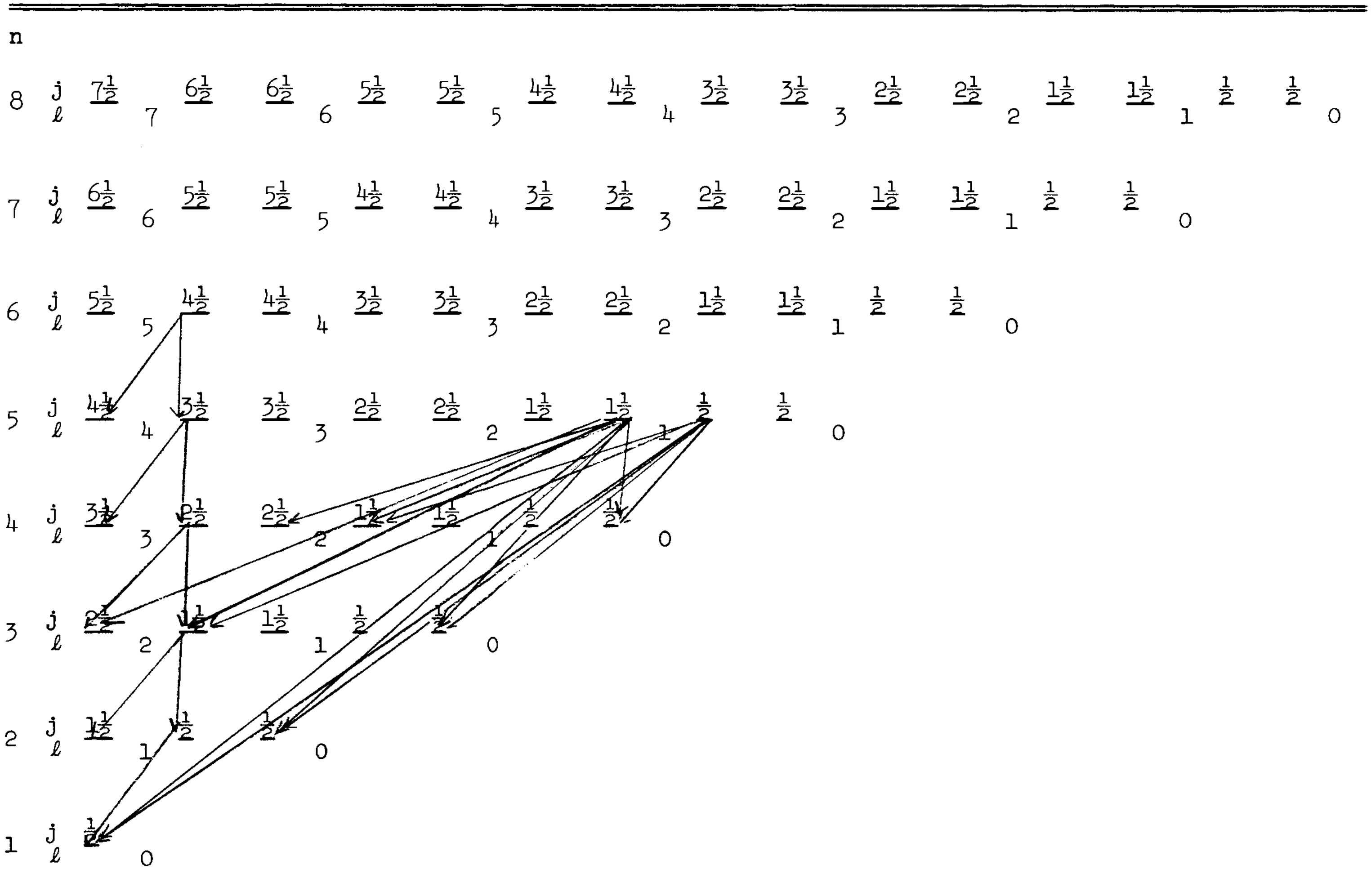
$$\Delta j = 0, \pm 1 \tag{5.1b}$$

Since we will see, presently, that the transition rate depends strongly on the available energy we exclude transitions with $\Delta n = 0$ because the splitting between states of different ℓ, belonging to a given n, is never sufficient to make such transitions competitive. Comsequently, we may add the selection rule,

$$\Delta n \leq -1 \tag{5.1c}$$

Now that we have the classification of states and the selection rules we can discuss the possible transitions. In Figure 13, we show the state of a mu-mesic atom in a schematic manner. The possible paths that a mu meson may take in reaching the ground state are shown for two assumed initial states. There is a significant amount of understanding to be gained by a study of Figure 13. First one will note that the possible transitions are either straight down or further to the left in such a representation. This means that the eccentricity of the orbit of the meson is never increased by the transition. We use the term eccentricity as a measure of $n - \ell$ because in the Bohr theory the states in ℓ at the

1. Hans A. Bethe and Edwin E. Salpeter, Quantum Mechanics of One and Two-Electron Atoms (Academic Press, Inc., New York, 1957) Chap. 4. This includes a comprehensive treatment of radiation by hydrogen like atoms. The radiative decay of the meta-stable state is discussed beginning with p. 285 of the chapter cited.



SOME STATES INVOLVED IN THE ATOMIC CASCADE SHOWING POSSIBLE BRANCHING
FROM TYPICAL STARTING POINTS

Figure 13

extreme left of Figure 13 are the circular orbits ($n-\ell = 1$); as one goes across Figure 13 toward the right the orbit becomes less circular until one reaches the most penetrating orbit ($\ell = 0$). Further, it is to be noticed that once a meson is in a state $j = \ell + \frac{1}{2}$ then only a $\Delta \ell = +1$ transition can cause it to go into a state $j' = \ell' - \frac{1}{2}$. In particular, for states for which $n-\ell \leq 2$ there is no possibility of a meson leaving a state of type $j = \ell + \frac{1}{2}$.

There is an important fact that may be deduced from Figure 13 that we wish to point out before we go further. Consider a meson captured into a state $n-\ell = 1$. Then it must be in one of the two possible states of j. For mesons in these circular orbits, the selection rules require $\Delta n = -1$, $\Delta j = 0,-1$. Since the two states in j are degenerate in energy the branching ratio for a meson in a state $j = \ell - \frac{1}{2}$ to go to a state $j' = \ell - 3/2$ or to a state $j' = \ell' + \frac{1}{2} = j$ is determined only by angular momentum considerations. This fact is utilized in some previous work concerning the problem at hand. We will discuss this in section 6 below.

There is one question that arises from the preceeding; how does a mu meson leave the $2s_{\frac{1}{2}}$ state? The answer is given in section 4, below.

It is clear that to determine how the mesons reach the ground state from the states shown in Figure 13 one must be able to compute the various branching ratios. The branching ratios are determined from the transition rates for the various specific transitions; and the theory of these transition rates is well known. The reciprocal lifetime for a transition from a state j', ℓ' to a state j, ℓ with emission of an electric multipole, E-L, is given by:[2]

$$\frac{1}{\tau} = 2\,\frac{L+1}{L}\,\frac{(2\ell+1)\,(2j+1)\,(2\ell'+1)}{\left[(2L+1)\,!\,!\right]^2}\Big[C(\ell\ell'L;\,0\,0)$$

$$\times W(j\ell j'\ell';\,\tfrac{1}{2}L)\Big]^2$$

$$\times\,\omega\,\frac{e^2}{\hbar c}\,(kR)^{2L} M_L^2 \qquad (5.2)$$

where

$$M_L = \int_0^\infty R_{n,\ell}(r)\left(\frac{r}{R}\right)^L R_{n',\ell'}(r)\,r^2\,dr \qquad (5.3)$$

and where $R_{n\ell}(r)$ is a radial hydrogen-like wave function. The meaning of R need not concern us; since we are only interested in electric dipole transitions ($L = 1$) and R cancels in all cases. Therefore, we have

$$M = \int_0^\infty R_{n,\ell}\,R_{n',\ell'}\,r^3\,dr \qquad (5.4)$$

This integral may be evaluated in general and is given by:[3]

$$\int_0^\infty R_{n,\ell}\,R_{n',\ell-1}\,r^3\,dr = \frac{(-)^{n'-\ell}}{4\,(2\ell-1)!}\left[\frac{(n+\ell)\,!\,(n'+\ell-1)\,!}{(n-\ell-1)\,!\,(n'-\ell)\,!}\right]^{1/2}$$

$$\times\,\frac{(4\,n\,n')^{\ell+1}\,(n-n')^{n+n'-2\ell-2}}{(n+n')^{n+n'}}\,G \qquad (5.5a)$$

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2. M. E. Rose, Multipole Fields (John Wiley and Sons, 1955) p. 78. This formula was derived for application to nuclear radiation, thus the parameter R. We note that the condition $kR \ll 1$ is well satisfied for the radiation we discuss when R is taken as the meson Bohr radius.

3. Bethe and Salpeter, *op. cit.* p. 262.

where

$$G = \left[{}_2F_1\left(-n_r, -n_r'; 2\ell ; \frac{-4\, n\, n'}{(n - n')^2} \right) - \left(\frac{n - n'}{n + n'} \right)^2 {}_2F_1\left(-n_r - 2, -n_r'; 2\ell ; -\frac{4\, n\, n'}{(n - n')^2} \right) \right] \quad (5.5b)$$

and

$$n_r = n - \ell - 1 \quad (5.5c)$$

$$n_r' = n - \ell \quad (5.5d)$$

We can discard the constants in (5.2) since we need only the branching ratios. Then for the electric dipole radiation the relative rates are obtained from:

$$\frac{1}{\tau} \propto (2\ell + 1)(2j + 1)(2\ell' + 1) \times \left[C(\ell \ell' 1; 0\, 0) \quad W(j \ell j' \ell'; \tfrac{1}{2} 1) \right]^2 E^3 M^2 \quad (5.6)$$

where for our purpose E is defined as:

$$E = \left(\frac{1}{n^2} - \frac{1}{n'^2} \right) \quad (5.7)$$

Thus the quantity E is proportional to the energy available in a given transition and we note that the transition rate is proportional to E^3. For this reason, the branching ratios that may be derived from (5.6) will tend to favor the maximum decrease in n. One also notes that the branching between states in j for a specific transition in ℓ and n is determined by the angular momenta involved and is independent of E and M.

The rate of the Auger transitions in the cascade is discussed in section 4, below. It suffices to state here that, the Auger transitions do not cause any transitions that violate the selection rules for radiative transitions, within the limits of a good approximation.

2. Depolarization in Radiative Transitions

We now give a derivation of the depolarization due to radiative transitions. After we have found how the different transitions affect the polarization, we may continue with the study of the cascade.

By definition:

p_m is the meson population of a final state m belonging to j and ℓ;

$p_{m'}$ is the population of an initial state m' belonging to j' and ℓ'.

$\lambda_{m\,m'}$ is proportional to the transition probability from state m' to m.

Thus we have[4]

$$\lambda_{m\,m'} = \sum_{P} \left| \left(\psi^{m}_{j,\ell} \,\middle|\, \vec{V}\cdot\vec{A}_{L\,M} \,\middle|\, \psi^{m'}_{j',\ell'} \right) \right|^{2} \quad p(E_{final}) \tag{5.8}$$

Where $\vec{V}$ is the current operator and $\vec{A}_{L\,M}$ is the vector potential for the radiation field. The vector potential is taken as

$$\vec{A} = \sqrt{2\pi} \sum_{L,M} i^{L} (2L+1)^{\frac{1}{2}} D^{L}_{M\,P} (\phi, \theta, 0)$$
$$\times \left[\vec{A}_{L\,M} (\text{mag}) + i\,P\,\vec{A}_{L\,M} (\text{el}) \right] \tag{5.9}$$

where $D^{L}_{M\,P}$ is a rotation matrix, $P = \pm 1$. Since we are only concerned

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4. For information concerning the rotation matrices and the multipole potentials see: M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, New York 1957) Chapters 4 and 7 respectively. For proof of the Wigner-Eckart theorem see also Chapter 5.

with electric dipole radiation we may rewrite (5.8) as

$$\lambda_{m\,m'} \propto \left| D^{L}_{M\,P}(\phi, \theta, 0)\right|^{2} \left[C(j'Lj;\, m', m - m')\right]^{2}$$
$$\times \left| (j \,||\, \vec{V}\cdot\vec{A}_{L} \,||\, j')\right|^{2} d\Omega_{\text{final}} \tag{5.10}$$

where we have used the Wigner-Eckart theorem.

There will be no need to consider the reduced matrix element in (5.10) further since it is independent of m. We proceed as follows. The product of $D^{L}_{M\,P}$ may be expanded;

$$D^{L*}_{M\,P}\, D^{L}_{M\,P} = (-)^{M-P}\, D^{L}_{-M,\,-P}\, D^{L}_{M\,P} = \sum_{\nu} (-)^{M-P}\, C(L\,L\,\nu;\, -M, M)$$
$$\times C(L\,L\,\nu;\, -P, P)\, D^{\nu}_{0,\,0} \tag{5.11}$$

But

$$D^{\nu}_{0,\,0} = P_{\nu}(\cos\theta) \tag{5.12}$$

Now the population of the state m' is expanded;

$$p_{m'} = \sum_{n} a_{n}\, C(j'\, n\, j';\, m', 0) \tag{5.13}$$

which is just a power series in m.

The population of state m is given in terms of $p_{m'}$ as

$$p_{m} = \sum_{m'} p_{m'}\, \lambda_{m\,m'} \tag{5.14}$$

Substitution of (5.11) into (5.10) and then (5.10) into (5.13) yields

$$p_{m} = \sum_{m} \sum_{n} a_{n}\, C(j'\, n\, j';\, m, 0) \sum_{\nu} (-)^{M-P}\, C(L\,L\,\nu;\, P, -P)$$
$$\times C(L\,L\,V;\, M, -M) \left[C(j'\,L\,j;\, m', m-m')\right]^{2} \int P_{\nu}(\cos\theta)\, d\Omega \tag{5.15}$$

Since,

$$\int P_\nu (\cos \theta)\, d\Omega = \delta_{\nu, 0} \quad \text{we obtain}$$

$$p_m = \sum_{m'} \sum_n a_n \; C(j' \, n \, j'; m', 0) \left[C(j' \, L \, j; m', m - m') \right]^2 \frac{1}{2L+1}$$

(5.16)

We may drop the factor $\frac{1}{2L+1}$. Now, the notation is altered as follows.

Define S_n such that

$$p_m = \sum_n a_n \; S_n \tag{5.17}$$

Thus,

$$S_n = \sum_{m'} C(j \, n \, j'; m', 0) \left[C(j' \, L \, j; m', m - m') \right]^2 \tag{5.18}$$

Use of a symmetry relation yields

$$C(j' \, L \, j; m, m - m') = (-)^{L + m - m'} \left(\frac{2j + 1}{2j' + 1} \right)^{\frac{1}{2}} C(L \, j \, j'; m' - m, m)$$

(5.19)

Next a Racah recoupling is used.

$$(-)^{L+m-m'} \left(\frac{2j+1}{2j'+1} \right)^{\frac{1}{2}} C(L \, j \, j'; m' - m, m) \; C(j' n \, j'; m', 0) =$$

$$(2j + 1)^{\frac{1}{2}} (-)^{L + m - m'} \sum_\nu (2\nu + 1)^{\frac{1}{2}} C(j \, n \, \nu; m, 0) \; C(L \quad j', m' - m, m)$$

$$\times \; W(L \, j \, j' \, n; j' \nu) \tag{5.20}$$

where ν is now different from the previous ν. S_n becomes

$$S_n = (2j + 1)^{\frac{1}{2}} \sum_\nu W(L \, j \, j' \, n; j' \nu) \, (2\nu + 1)^{\frac{1}{2}} \; C(j \, n \, \nu; m, 0)$$

$$\times \sum_{m'} (-)^{L + m - m'} C(j' \, L \, j; m', m - m') \; C(L \, \nu \, j'; m' - m, m) \tag{5.21}$$

Using the symmetry relations one obtains,

$$\sum_{m'} (-)^{L+m-m'} C(j'\ L\ j;\ m',\ m-m')\ C(L\nu j';\ m'-m,\ m) =$$

$$\sum_{m'} \left(\frac{2j'+1}{2\nu+1}\right)^{\frac{1}{2}} C(j'\ L\ j;\ m',\ m-m')\ C(j'\ L\ \nu;\ m',\ m-m')$$

$$= \left(\frac{2j'+1}{2\nu+1}\right)^{\frac{1}{2}} \delta_{j\nu} \tag{5.22}$$

Using a symmetry relation to permute the arguments of the Racah coefficient in (5.21) now allows us to write

$$S_n = (-)^{L+n-j'-j} \left[(2j+1)\ (2j'+1)\right]^{\frac{1}{2}} W(j'\ j'\ j\ j;\ n\ L)$$

$$\times C(j\ n\ j;\ m,\ 0) \tag{5.23}$$

We define

$$S_n = T_n\ C(j\ n\ j;\ m,\ 0) \tag{5.24}$$

for the sake of compactness. The condition on n is that $n \leq 2\ j_{min.}$. Where $j_{min.}$ is the minimum possible value of j. Here $j_{min.} = \frac{1}{2}$ so $n = 0,\ 1$. We now use P to designate $\frac{\langle j_z \rangle}{j}$. Thus for the final state

$$P = \frac{\sum_m m\ p_m}{j \sum_m p_m} \tag{5.25}$$

Since

$$p_m = a_o\ S_o + a_1\ S_1 \tag{5.26}$$

and since S_o is an even function of m and S_1 is an odd function of m we have

$$P = \frac{\sum_m a_1\ S_1\ m}{j \sum_m a_o\ S_o} \tag{5.27}$$

taking m = $\sqrt{j\,(j+1)}$ C(j 1 j; m, 0) and using the definition of T_n yields

$$P = \frac{\sum_m a_1 T_1 \sqrt{j\,(j+1)}\,\left[C(j\;1\;j;\;m,\;0)\right]^2}{j \sum_m a_o T_o} \tag{5.28}$$

and we have immediately

$$\sum_m a_o T_o = (2\,j+1)\,a_o T_o \tag{5.29}$$

Further; use of the symmetry relations gives

$$\sum_m \left[C(j\;1\;j;\;m,\;0)\right]^2 = \frac{2j+1}{3} \sum_m \left[C(j\;j\;1;\;m,\;-\,m)\right]^2 = \frac{2j+1}{3} \tag{5.30}$$

Therefore

$$P = \frac{\sqrt{j\,(j+1)}}{3j}\,\frac{a_1 T_1}{a_o T_o} \tag{5.31}$$

For the initial state

$$P' = \frac{\sum_{m'} (a_o + a_1\,C(j'\;1\;j';\;m',\;0) \quad m'}{j' \sum_{m'} (a_o + a_1\,C(j'\;1\;j';\;m',\;0))} \tag{5.32}$$

The numerator and denominator of (5.32) are evaluated in the same fashion as was done for (5.28); thus

$$P' = \frac{\sqrt{j'\,(j'+1)}}{3j'}\,\frac{a_1}{a_o} \tag{5.33}$$

The quantity that concerns us is the change in the polarization in a given transition, therefore we form

$$\frac{P}{P'} = \left[\frac{(j+1)\;j'}{(j'+1)\;j}\right]^{\frac{1}{2}} \frac{T_1}{T_o} \tag{5.34}$$

To evaluate this we need only take the part of (5.23) that is defined as T_n and evaluate the Racah coefficients for n = 0 and n = 1. We find,

$$T_o = 1 \tag{5.35a}$$

$$T_1 = \frac{j'(j'+1) + j(j+1) - L(L+1)}{\left[4j'(j'+1)(j+1)j\right]^{\frac{1}{2}}} \tag{5.35b}$$

and consequently

$$\frac{P}{P'} = \frac{j'(j'+1) + j(j+1) - L(L+1)}{2j(j'+1)} \tag{5.36}$$

We set L = 1 since we wish to consider electric dipole ratiation. Now we may evaluate P/P' for the three possible cases $\Delta j = 0, \pm 1$. We find:

$$\frac{P}{P'} = 1 \quad ; \quad \Delta j = -1 \tag{5.37a}$$

$$\frac{P}{P'} = 1 - \frac{1}{j(j+1)} \quad ; \quad \Delta j = 0 \tag{5.37b}$$

$$\frac{P}{P'} = 1 - \frac{1}{(j'+1)^2} = 1 - \frac{1}{j^2} \quad ; \quad \Delta j = +1 \tag{5.37c}$$

Thus we can find the change in $\frac{\langle j_z \rangle}{j}$ for any possible radiative transition. We note that for $\Delta j = -1$ transitions there is no depolarization. For $\Delta j = +1$ transitions the maximum depolarization corresponds to $P/P' = \frac{5}{9}$. The most severe depolarization is due to the $\Delta j = 0$, $j = \frac{1}{2}$ transitions; here $P/P' = -1/3$. This type transition is of considerable importance since it is the final transition for many of the mesons $(2p_{\frac{1}{2}} \rightarrow 1s_{\frac{1}{2}})$.

3. Pure Radiative Cascades

Now that we know how much the various transitions depolarize the mu mesons we can determine what the remaining polarization in the $1s_{\frac{1}{2}}$

state should be. In order to treat the cascade, we need the following formulation. We denote a quantity proportional to the partial width of a state n', ℓ', j' for a transition to a state n, ℓ, j by $\Gamma^{n\ \ell\ j}_{n'\ \ell'\ j'}$. Then we may use (5.6);

$$\Gamma^{n,\ \ell,\ j}_{n',\ \ell',\ j'} = \left[C(\ell\, 1\, \ell';\, 0\, 0)\right]^2 (2\ell + 1)\,(2j + 1) \times \left[W(j\, \ell\, j'\, \ell';\, \tfrac{1}{2}\, 1)\right]^2 E^3\, M^2 \tag{5.38}$$

Then the total width of the state $n', \ell'\ j'$ is

$$\Gamma_{n',\ \ell',\ j'} = \sum_{n,\ \ell,\ j} \Gamma^{n,\ \ell,\ j}_{n',\ \ell',\ j'} \tag{5.39}$$

It is possible to do the sum on j as follows:

We use a symmetry relation to rewrite the Racah coefficient in (5.38);

$$W(j\ \ell\, j'\, \ell';\, \tfrac{1}{2}\, 1) = (-)^{j + \ell' - \frac{1}{2} - 1}\, W(\tfrac{1}{2}\ \ell\, j'\, 1;\, j\, \ell') \tag{5.40}$$

Then for the sum over j we have

$$\frac{1}{2\ell' + 1} \sum_{j} (2j + 1)\,(2\,\ell' + 1) \left[W(\tfrac{1}{2}\,\ell\, j'\, 1;\, j\, \ell')\right]^2 = \frac{1}{2\ell' + 1} \tag{5.41}$$

by the orthonormality of the Racah coefficients. Therefore

$$\Gamma_{n'\ \ell'\ j'} = \sum_{n,\ell} \frac{2\ell + 1}{2\ell' + 1} \left[C(\ell\, 1\, \ell';\, 0\, 0)\right]^2 E^3\, M^2 \tag{5.42}$$

The Clebsch-Gordan coefficient in (5.42) has values such that

$$\frac{2\ell + 1}{2\ell' + 1} \left[C(\ell\, 1\, \ell';\, 0\, 0)\right]^2 = \begin{cases} \dfrac{\ell'}{2\ell' + 1} & \ell' > \ell \\[2ex] \dfrac{\ell' + 1}{2\ell' + 1} & \ell > \ell' \end{cases} \tag{5.43}$$

So (5.42) may be written as

$$\Gamma_{n' \ell' j'} = \frac{1}{2\ell' + 1}\left[\sum_{n, \ell<\ell'} \ell' E^3 M^2 + \sum_{n, \ell>\ell'} (\ell' + 1) E^3 M^2\right] \tag{5.44}$$

Evaluation of this requires evaluation of the quantities M for all the possible transitions. Consider now an initial state containing N mesons Then the fraction of these mesons that make the transition $n', \ell' j' \rightarrow n, \ell, j$ is given by

$$F_{n, \ell, j} = \frac{\Gamma^{n, \ell, j}_{n', \ell', j'}}{\Gamma_{n', \ell', j'}} \tag{5.45}$$

We mention that there are in general $n - \ell$ states n that are accessible from a state n'.

The cascade problem was programed for the electronic computer. Quantities equivalent to (5.45) were computed along with the change in polarization for the various transitions. This was done by setting up arrays in the computer memory similar to Figure 13. The initial distribution as a function of j and the corresponding initial polarization as determined by (4.76) were computed for each state. The resulting ensemble was then treated in a static fashion by considering first the states $n = 16$. The program computed the change in population and polarization in all the other states due to transitions out of the states $n = 16$; states with $n = 15$ were then considered and so on until all of the mesons were in either the $2s_{\frac{1}{2}}$ or $1s_{\frac{1}{2}}$ state. For reasons given in section 4, we assert that the Auger transition from the $2s_{\frac{1}{2}}$ to the $1s_{\frac{1}{2}}$ state causes no additional depolarization. This transition will always proceed by electron ejection since the $2s_{\frac{1}{2}}$ state is metastable against radiative transitions.

Some results derived for pure radiative transitions in Carbon are given in Figures 14 and 15. In Figure 14 the final polarization of mu mesons, assumed to be captured in specific states n and n_r is given. This includes the depolarization due to capture. If we take the mu mesons to be captured as given by our initial distribution the polarization in the $1s_{\frac{1}{2}}$ state is

$$P = 0.24 \tag{5.46a}$$

This leads to an asymmetry coefficient of

$$a = 0.08 \tag{5.46b}$$

which is not in agreement with experiment, since the experimental asymmetry coefficients do not exceed 0.06.

In Figure 15 we show the relative number of mu mesons that pass through the 2s state when they are captured into states of different belonging to $n = 8$. As we are considering only radiative transitions here, these are the relative numbers of mesons that would not produce a mesonic K - X-ray. We mention this point because of its bearing on the yield of X-rays per captured mu mesons, which we discuss briefly in section 6 below. It is clear from Figure 13 that the complexity of the cascade is such that any attempt to use the observed X-ray intensities to extrapolate back to the initial distribution will fail.

There have been two papers, dealing with the problem just discussed which did not contain obvious errors.[5] In both of these the problem was

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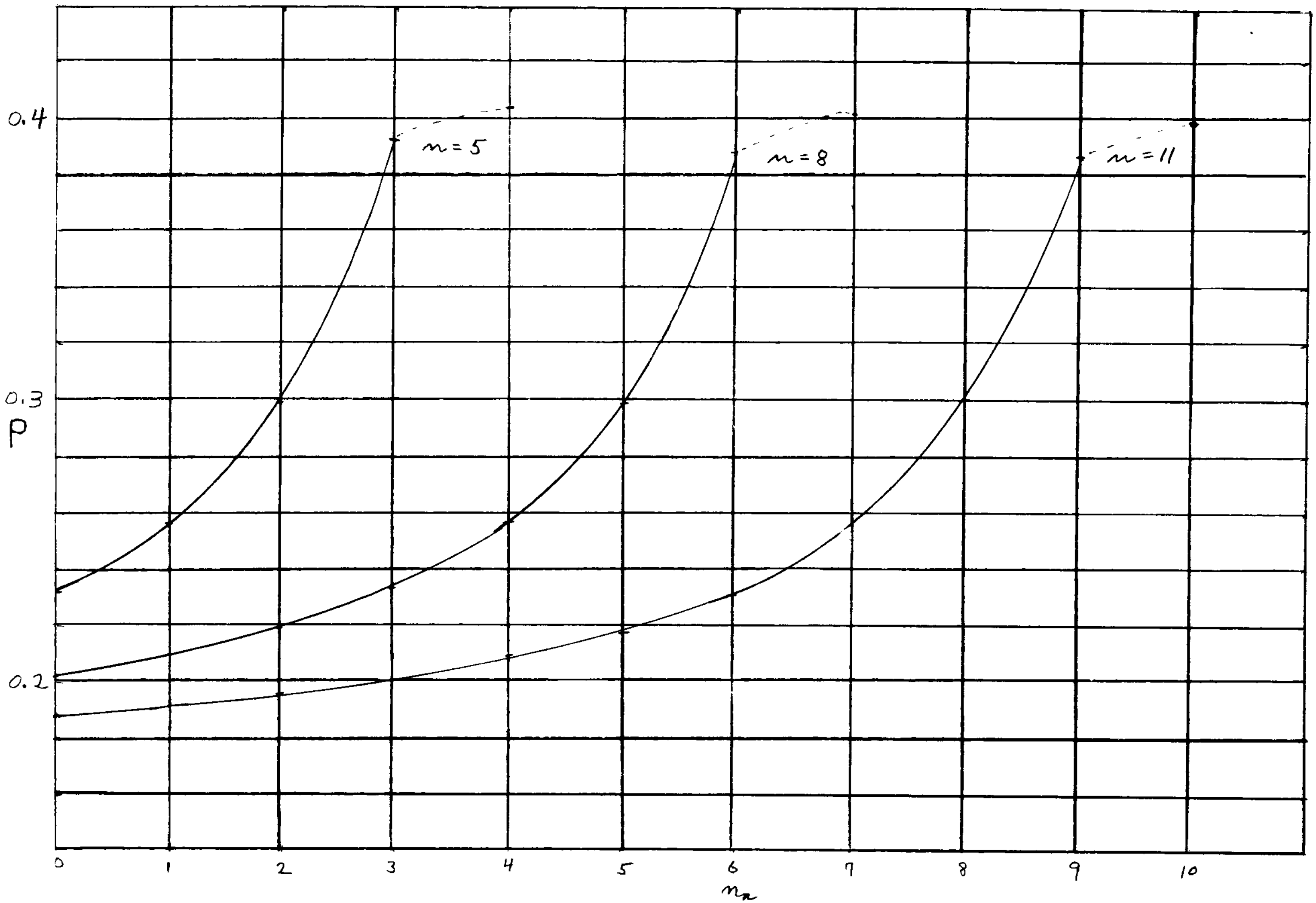
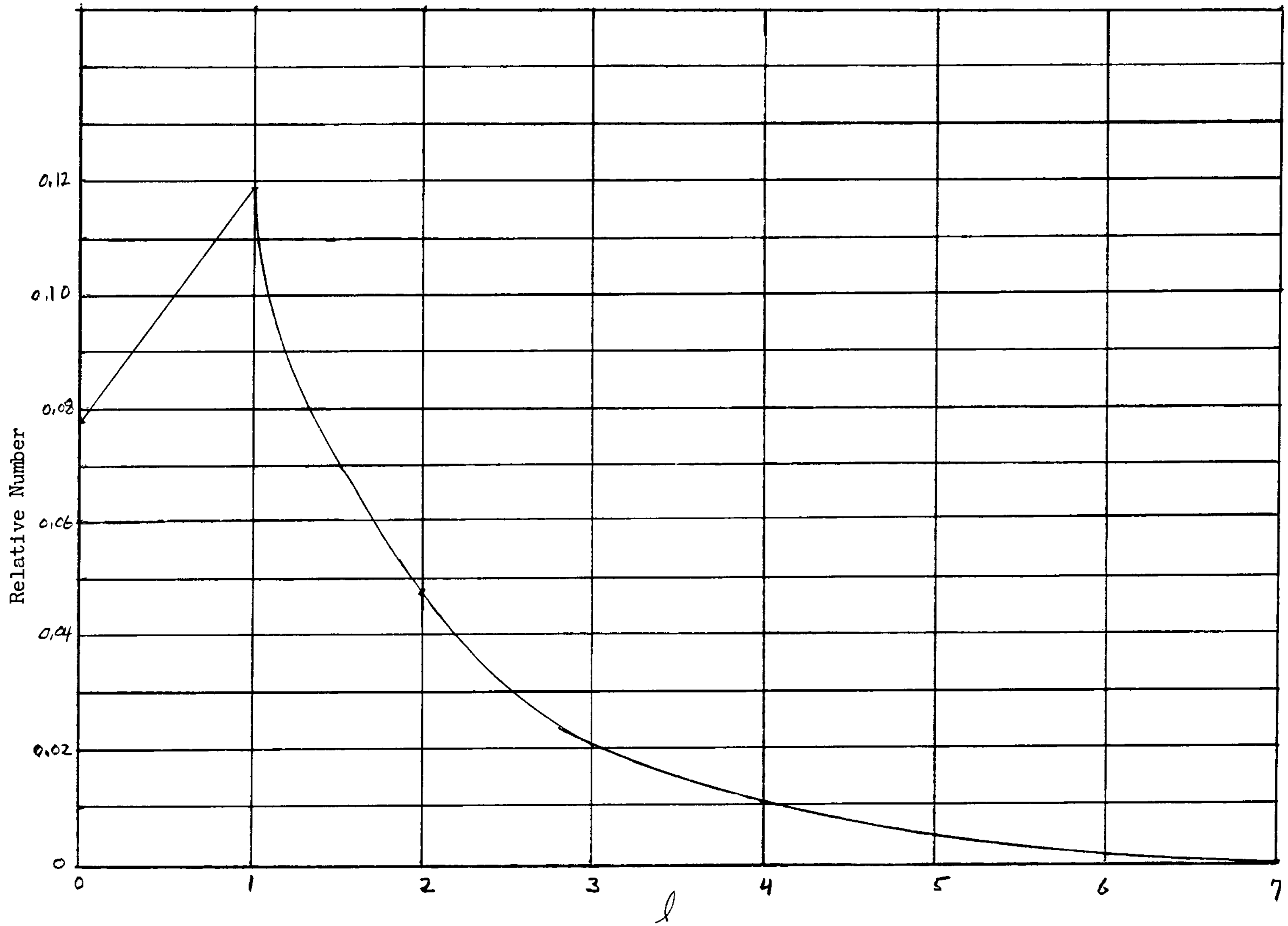
5. M. E. Rose, Bull. Am. Phys. Soc. 4, 80 (1959); see also I. M. Shmushkevitch, Nuclear Phys. 11, 419 (1959). An additional paper, V. A. Bzhrlashyan, Soviet Physics-JETP 36(8), 188 (1959), contains the same approximations but obtains an incorrect result.



Final Polarization in Ground State for Mu Mesons Captured in Specific States n, n_r. Radiative Transitions Only

Figure 14



Relative Number of Mu Mesons Entering the $2s_{\frac{1}{2}}$ State when Starting in State ℓ Belonging to $n = 8$. Radiative Transitions Only

Figure 15

considered in such a manner that the radial matrix elements were ignored. This can be done by, either restricting one's attention to the circular orbits or by just assuming all the matrix elements to be equal (no $\Delta \ell = +1$). Using either approximation one finds that the polarization is reduced by about $\frac{1}{2}$ due to radiative transitions. In the study discussed in this section we did not attempt to seperate the depolarization due to capture from that due to transitions. Our results, however, are rigourous in that nothing has been left out.

Since the asymmetry coefficient we obtain is not in agreement with experiment, it is clear that we must consider the Auger transitions as well as the radiative transitions.

4. The Auger Transitions

We must include the Auger transitions in competition with the radiative transitions in order to determine the final polarization. It has been shown by Burbidge and de Borde that the principle contribution to the mesonic Auger transition rate occurs when the Auger transition satisfies dipole selection rules. In other words, the same selection rules as those for emission of electric dipole radiation, apply to the ejection of electrons.[6] Now, this yields the following significant simplification when we consider the Auger transitions. When we derived the expressions for the polarization change in a given transition in section 2 above

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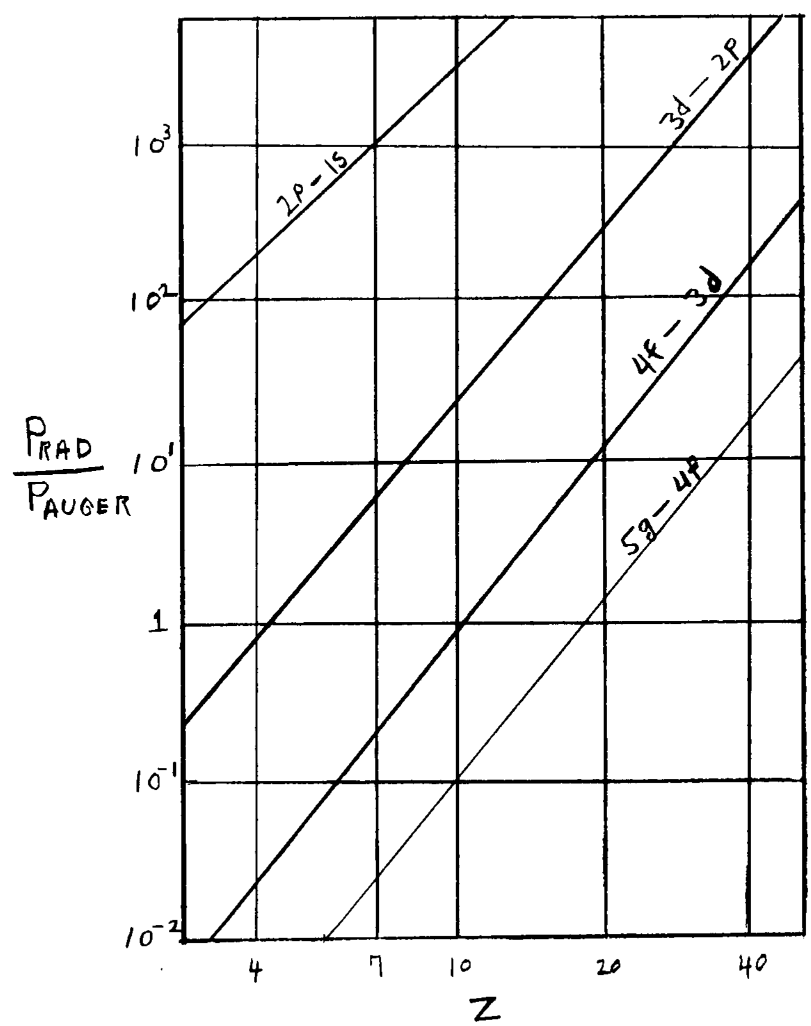
6. G. R. Burbidge and A. H. de Borde, Phys. Rev. 89, 189 (1953) and also A. H. de Borde, Proc. Phys. Soc. (London) A67, 57 (1954).

we did not need to consider the reduced matrix elements but obtained all of the results from considerations of the angular momentum. Thus the results obtained depend only on the multipolarity of the transitions. Therefore the change in polarization in a dipolar Auger transition is given by (5.37). Further, as we mentioned in the preceeding section, if we consider a monopolar Auger transition, then L in (5.36) is zero and $j = j'$. This leads to

$$\frac{P}{P'} = 1; \qquad \Delta\vartheta = 0 \tag{5.47}$$

So there is no depolarization in the $2s_{\frac{1}{2}} \rightarrow 1s_{\frac{1}{2}}$ transition. Thus we know how to calculate the change in the polarization for the Auger transitions.

Next we must determine how the competition of the Auger transitions with the radiative transitions is to be handled. Burbidge and de Borde also show that the Auger transitions favor $\Delta n = -1$, whereas the radiative transitions favor $\Delta n = -$ maximum. This is an important difference for it is clear that, in general, the more transitions a mu meson makes, the greater the depolarization. There is, however, no simple formula for the Auger transition rates, such as (5.6) for the radiative rates. Therefore we use a schematic method to handle the competion. Figure 16 shows the relative probability of radiation to Auger transitions for certain transitions and different elements. The Z dependence comes in because the radiative rates vary as Z^4 whereas the Auger rates are essentially independent of Z. The graph is based on the work of Burbidge



Relative Probability of Auger Transition to Radiative
Transition in Mesic Atoms

Figure 16

and de Borde and is taken from a paper by Rainwater.[7] To deal with the Auger transitions we use the data of this graph and one other fact. Demeur has shown that for $\Delta n = -1$ the dipolar Auger transitions have the same branching ratio between transitions that involve $\Delta \ell = \pm 1$ as the radiative transitions.[8] In view of the information just given, we were able to include the Auger transitions as follows. The computer program for the radiative transitions was rewritten so that for states $n > M$ only radiative transitions of the type $\Delta n = -1$ were allowed, where M is an integer. For $n \leq M$ the program computed as it did originally. This scheme includes the effect of the Auger transitions with reasonable accuracy. M is determined from Figure 16 as a function of Z.

5. Theoretical Asymmetry Coefficients for the Decay of Bound Mu Mesons

For Carbon, Figure 16 tells us that all of the transitions should be Auger transitions until the mu mesons reach the states $n = 3$. From then on the the normal radiative transitions occur. By utilizing the scheme outlined in section 4 above, we find that the final polarization in the $1s_{\frac{1}{2}}$ state of mu mesons stopping in Carbon is

$$P = 0.133 \tag{5.47a}$$

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7. Rainwater, Ann. Rev. Nuclear Sci. 7, 1 (1957). The graph appears in this source because of the bearing of these relative transition probabilities on the still unexplained experiments of Stearns and Stearns.

8. M. Demeur, Nuclear Phys. 1, 516 (1956). This work was also an attempt to understand the data of Stearns and Stearns.

This gives a theoretical asymmetry coefficient,

$$a = 0.044 \tag{5.47b}$$

this should be compared with the two experimental values given in Chap. 1:

$$a = 0.04 \pm 0.005 \tag{5.48a}$$

$$a = 0.054 \pm 0.006 \tag{5.48b}$$

and a more recent value[9]

$$a = 0.045 \tag{5.48c}$$

The agreement is satisfying. Only one of the reported values of the three asymmetry coefficients is inconsistent with out result, and this is only a very small difference.

Although our initial distribution was derived for Carbon, we can assume that it will not vary significantly when we go to heavier elements If we make this assumption, then we can predict the asymmetry coefficients for heavier elements by stopping the Auger transitions at higher n values. With the last Auger transition being into the state $n = 4$ we find

$$P = 0.153 \tag{5.49a}$$

$$a = 0.051 \tag{5.49b}$$

and with the last Auger transition being into the state $n = 5$ we find

$$P = 0.183 \tag{5.50a}$$

$$a = 0.061 \tag{5.50b}$$

The limiting value is, of course,

$$a = 0.08 \tag{5.46}$$

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9. Oral communication from the floor, Session O, Washington meeting of American Physical Society, April 1960. No limits of error were stated.

which we found for pure radiative transitions.

The values (5.49) and (5.50) would apply to elements with spin zero nuclei around $Z = 14$ and $Z = 30$, respectively. If the elements contain non spin zero contaminants then the observed asymmetry coefficient should be corrected for this before comparing with the values given. Since the depolarizing effect of nuclear spin is very strong an approximate correction can be made by dividing the observed asymmetry coefficients by the fractional abundance of the zero spin isotope when the element contains predominately spin zero nuclei. Applying this procedure to Cadium, we correct the observed asymmetry coefficient, 0.055 ± 0.012 by a factor 1.33 to obtain

$$a = 0.073 \pm 0.016 \tag{5.51}$$

since $Z = 48$ for Cd this is just about what we would predict. There are two asymmetry coefficients given in Table 1 for Mg. These are 0.036 ± 0.003 and 0.058 ± 0.008. Since the difference is so great, we only point out that we would expect the observed asymmetry coefficient of Mg. to be about

$$a = 0.045 \tag{5.52}$$

after correction for a 10 per cent spin 5/2 impurity.

6. Discussion and Conclusions Concerning the Theoretical Asymmetry Coefficients

In connection with our treatment of the atomic cascade, we wish to present the following remarks. The experiments of Stearns and Stearns indicate a discrepancy between theory and experiment concerning the Auger transition rate.[10] They were able to deduce that the Auger rates should be of the order 10^2 times the calculated rates in order to account for the observed yield of X-rays per stopped mu meson. They attempt to explain this by assuming that the mu mesons pass through the 2s state. Therefore, we give the following information; using our initial distribution and the appropriate schematic description of the Auger transitions in Carbon we find that 5.4 per cent of the mu mesons pass through the 2s state and 55.1 per cent pass through the 2p state; this is not consistant with what they report. As was mentioned in Chapter 1; their data has been questioned and it would be desirable to have their results verified. However, as we treated the Auger transitions in a schematic manner, we can make no stronger statements concerning their results. Part of the treatment we used for the Auger effect is based on the assumption that the atomic electrons are replenished as fast as they are ejected. This is a common assumption; see, for example, the paper of Demeur which we mentioned earlier. No substantial evidence concerning this replenishment of electrons has appeared in the literature.

We have just pointed out certain unsettled questions that might have some bearing on our results; nevertheless, we feel that we have presented for the first time an adequate and comprehensive treatment of the depolarization mechanisms. We conclude that there is no lack of understanding concerning the observed asymmetry coefficients; by this we mean that the negative mu mesons are created with complete polarization

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10. M. B. Stearns and M. Stearns, Phys. Rev. 105, 1573 (1957).

and that they retain this complete polarization until they are captured by an atom. With these remarks we conclude the consideration of the asymmetry coefficients.

In connection with the atomic cascade, there is a possibility of observing the polarization of the x-rays. We now determine the magnitude of this polarization.

7. Circular Polarization of the Mu-Mesic X-Rays

We consider a radiative transition from the state j', ℓ' to the state j, ℓ . The intensity of the emitted x-rays is

$$I = \sum_{mm'} \left| \left(\psi^{m}_{j,\ell} \middle| \vec{V}\cdot\vec{A}_{LM} \middle| \psi^{m}_{j',\ell'} \right) \right|^2 p_{m'} \, \rho(E_{final}) \qquad (5.53)$$

where $p_{m'}$ is the population of the state j', ℓ', m' and is defined as before

$$p_{m'} = \sum_{n} a_n \, C(j'nj';m',0) \qquad (5.13)$$

Then using the Wigner-Eckart theorem we have

$$I \propto \sum_{mm'} \left| D^L_{MP} \right|^2 \left(C(j'Lj;m',m-m') \right)^2 p_{m'} \qquad (5.54)$$

where we have dropped all quantities that do not depend on the magnetic quantum numbers. If P = + 1 the X-ray is right circular polarized; if P = - 1 the X-ray is left circular polarized. We also have that M, the projection of L is m-m'. Now expanding the product $\left| D^L_{MP} \right|^2$ as before we have

$$I \propto \sum_{mm'} P_{m'} \, (-)^{M-P} \sum_{v} C(LLv;P,-P)\, C(LLv;M,-M)$$

$$\times \left[C(j'Lj;m',m-m')\right]^2 P_v(\cos\theta) \qquad (5.55)$$

where θ is the angle between the axis of quantization (the initial beam direction) and the direction of the x ray.

We now use $(-)^P = (-)^1$ and write

$$S = \sum_{m} (-)^{M+1}\, C(LLv;M,-M) \left[C(j'Lj;m',m-m')\right]^2 \qquad (5.56)$$

Using the symmetry relations

$$C(LLv;M,-M) = (-)^v\, C(LLv;-M,M) \qquad (5.57a)$$

$$C(j'Lj;m',m-m') = (-)^{j'-m'} \left(\frac{2j+1}{2L+1}\right)^{\frac{1}{2}} C(j'jL;m',-m) \qquad (5.57b)$$

and a Racah recoupling then yields the following for the first pair of Clebsch-Gordan coefficients in (5.56)

$$(-)^{j'-m'+v} \left(\frac{2j+1}{2L+1}\right)^{\frac{1}{2}} C(j'jL;m',-m)\, C(LLv;-M,M) =$$

$$(-)^{j'-m+v} \left(\frac{2j+1}{2L+1}\right)^{\frac{1}{2}} \sum_{s} \left[(2s+1)(2L+1)\right]^{\frac{1}{2}} C(j'sv;m',-m')\, C(jLs;m,m-m')$$

$$\times\, W(j'jvL;Ls) \qquad (5.58)$$

where we used $M = m-m'$. Substitution of (5.58) in (5.56) gives for the sum on m

$$\sum_{m} C(j'Lj;m',m-m')\, C(jLs;-m',m-m')\, (-)^{m-m'} \qquad (5.59)$$

Use of the symmetry relations gives this as

$$\left(\frac{2j+1}{2j'+1}\right)^{\frac{1}{2}} \sum_{m} (-)^{L+m-m'} \; (-)^{m-m'+1} \; C(jLj',-m,m-m') \; C(jLs;-m,m-m')$$

$$= \left(\frac{2j+1}{2j'+1}\right)^{\frac{1}{2}} (-)^{L+1} \delta_{j's} \tag{5.60}$$

The Racah coefficient in (5.58) becomes

$$W(j'jvL;Lj') = (-)^{j+v-L-j'} \; W(j'j'LL;vj) \tag{5.61}$$

and S is

$$S = (-)^{j+1-m'} (2j+1) \; C(j'j'v;m',-m') \; W(j'j'LL;vj) \tag{5.62}$$

But $(-)^{1-m'} = (-)^{m'}$ and we rewrite I as

$$I = \sum_{v,n,m'} (2j+1) \; C(LLv;P,-P) \; W(j'j'LL;vj) \; P_v(\cos\Theta) \; (-)^{j+m'}$$

$$\times \; C(j'j'v;m',-m') \; a_n \; C(j'nj';m',0) \tag{5.63}$$

where we used (5.13). We do the sum on m':

$$\sum_{m'} (-)^{j+m'} \; C(j'j'v;m',-m') \; C(j'nj';m',0) =$$

$$\sum_{m'} (-)^{j+m'} \; (-)^{j'-m'} \; C(j'j'v;m',-m') \; C(j'j'n;m',-m') \left(\frac{2j'+1}{2n+1}\right)^{\frac{1}{2}} \tag{5.64}$$

After we have applied the usual procedure to (5.64) we find

$$\sum_{m'} = (-)^{j'+j} \left(\frac{2j'+1}{2n+1}\right)^{\frac{1}{2}} \delta_{v,n} \tag{5.65}$$

So

$$I \propto \sum_v (2j+1)\ W(j'j'LL;vj)\ P_v(\cos\theta)\ C(LLv;P,-P)\ a_v \left(\frac{2j'+1}{2v+1}\right)^{\frac{1}{2}} (-)^{j'+j} \tag{5.66}$$

We need only be concerned with the quantities that contain v; therefore we have

$$I \propto \sum_v a_v\ P_v(\cos\theta)\ W(j'j'LL;vj)\ C(LLv;P,-P)\ \frac{1}{\sqrt{2v+1}} \tag{5.67}$$

But v is n and n is limited to be $\leq 2j_{min}$. j_{min} is $\frac{1}{2}$. So we have

$$I \propto \sum_v I_v\ P_v(\cos\theta) = I_o + I_1 \cos\theta \tag{5.68}$$

We evaluate I_o and I_1 as follows:

$$I_o = a_o\ C(LL0;P,-P)\ W(j'j'LL;0j) \tag{5.69a}$$

$$I_1 = a_1\ C(LL1;P,-P)\ W(j'j'LL;1j)\ \frac{1}{\sqrt{3}} \tag{5.69b}$$

which may be determined to be:

$$I_o = a_o\ (-)^{j-j'+1}\ \frac{1}{(2L+1)(2j'+1)^{\frac{1}{2}}} \tag{5.70a}$$

$$I_1 = a_1\ P\ (-)^{j'-j}\ \frac{j'(j'+1) + L(L+1) - j(j+1)}{L(L+1)(2L+1)\left[4j'(j'+1)(2j'+1)\right]^{\frac{1}{2}}} \tag{5.70b}$$

and the ratio I_1/I_o is

$$\frac{I_1}{I_o} = - \frac{a_1}{a_o} P \frac{j'(j'+1) + L(L+1) - j(j+1)}{L(L+1) \left[4j'(j'+1)\right]^{\frac{1}{2}}} \tag{5.71}$$

Now we found before, that the polarization of either the initial or final state could be expressed in terms of a_1/a_o. Thus from (5.33) we have

$$\frac{a_1}{a_o} = \frac{3j'}{j'(j'+1)} P_\mu \tag{5.72}$$

where P_μ is just $\frac{\langle j_z \rangle}{j}$ for the mu mesons in the emitting state. Thus:

$$\frac{I_1}{I_o} = - \frac{3}{2} \frac{j'(j'+1) + L(L+1) - j(j+1)}{L(L+1)(j'+1)} P P_\mu \tag{5.73}$$

and L is taken as 1 since we deal with electric dipoles. The transitions of interest are $2p_{3/2} \to 1s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}} \to 1s_{\frac{1}{2}}$; since the K-x rays are those that one might observe. Therefore we evaluate (5.73) for $j' = j+1$ and $j' = j$, respectively.

$$\frac{I_1}{I_o} = - \frac{3}{2} P P_\mu \qquad ; \qquad j' = j+1 \tag{5.74a}$$

$$\frac{I_1}{I_o} = - \frac{3}{2} \frac{P P_\mu}{(j+1)} \qquad ; \qquad j' = j \tag{5.74b}$$

For the K-x rays $j = \frac{1}{2}$ in (5.74b).

The experimental determination of these ratios may be quite difficult because one cannot get enough mesons to have a high counting rate. To get good results, the $2p_{3/2} - 2p_{\frac{1}{2}}$ splitting should be well resolved. For carbon, we predict that 55.1 per cent of the mesons pass through the 2p state, with the statistical distribution. For these states, we predict

$$P_\mu = \frac{\langle j_z \rangle}{j} = 0.17 \qquad \text{in } 2p_{3/2} \qquad (5.75a)$$

$$P_\mu = \frac{\langle j_z \rangle}{j} = -0.25 \qquad \text{in } 2p_{\frac{1}{2}} \qquad (5.75b)$$

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APPENDIX A

The following relations are those that were used in the various calculations. Derivations, references and additional information may be found in the place cited.

The basic symmetry relations for the Clebsch-Gordan coefficients are:[1]

$$C(j_1\ j_2\ j_3;\ m_1\ m_2\ m_3) = (-)^{j_1 + j_2 - j_3}\ C(j_1\ j_2\ j_3;\ -m_1,\ -m_2,\ -m_3) \tag{A1.1a}$$

$$C(j_1\ j_2\ j_3;\ m_1\ m_2\ m_3) = (-)^{j_1 + j_2 - j_3}\ C(j_2\ j_1\ j_3;\ m_2,\ m_1\ m_3) \tag{A1.1b}$$

$$C(j_1\ j_2\ j_3;\ m_1\ m_2\ m_3) = (-)^{j_1 - m_1} \left(\frac{2j_3 + 1}{2j_2 + 1}\right)^{\frac{1}{2}}$$

$$\times\ C(j_1\ j_3\ j_2;\ m_1, -m_3,\ -m_2) \tag{A1.1c}$$

From these one may obtain:

$$C(j_1\ j_2\ j_3;\ m_1,\ m_2,\ m_3) = (-)^{j_2 + m_2} \left(\frac{2j_3 + 1}{2j_1 + 1}\right)^{\frac{1}{2}}\ C(j_3 j_2 j_1; -m_3, m_2, -m_1) \tag{A1.2a}$$

$$C(j_1\ j_2\ j_3;\ m_1,\ m_2,\ m_3) = (-)^{j_1 - m_1} \left(\frac{2j_3 + 1}{2j_2 + 1}\right)^{\frac{1}{2}}$$

$$\times\ C(j_3\ j_1\ j_2;\ m_3,\ -m_1,\ m_2) \tag{A1.2b}$$

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1. M. E. Rose, Elementary Theory of Angular Momentum (John Wiley and Sons, Inc., New York, 1957) Chap. 3, p. 38.

$$C(j_1\ j_2\ j_3;\ m_1,\ m_2,\ m_3) = (-)^{j_2 + m_2} \left(\frac{2j_3 + 1}{2j_1 + 1}\right)^{\frac{1}{2}}$$

$$\times\ C(j_2\ j_3\ j_1;\ -m_2,\ m_3,\ m_1) \qquad \text{(A1.2c)}$$

In all cases the third projection quantum number is the sum of the first two.

The orthonormality relation for the Clebsch-Gordan coefficients is:[2]

$$\sum_{m'} C(j_1\ j_2\ j_3;\ m_1,\ m - m_1)\ C(j_1\ j_2\ j_3';\ m_1\ m - m_1) = \delta_{j_3\ j_3'} \qquad \text{(A1.3)}$$

The coupling rule for the rotation matrices is the Clebsch-Gordan series;[3]

$$D^{j_1}_{\mu_1\ m_1}\ D^{j_2}_{\mu_2\ m_2} = \sum_j C(j_1\ j_2\ j;\ \mu_1,\ \mu_2)$$

$$\times\ C(j_1\ j_2\ j;\ m_1,\ m_2)\ D^{j}_{\mu_1 + \mu_2,\ m_1 + m_2} \qquad \text{(A1.4)}$$

Use of (A1.4) may be shown to lead to[4]

$$Y_{\ell_1\ m_1}(\hat{n})\ Y_{\ell_2\ m_2}(\hat{n}) = \sum_{\ell} \left[\frac{(2\ell_1 + 1)\ (2\ell_2 + 1)}{4\pi\ (2\ell + 1)}\right]^{\frac{1}{2}}$$

$$\times\ C(\ell_1\ \ell_2\ \ell;\ m_1\ m_2)\quad C(\ell_1\ \ell_2\ \ell;\ 0\ 0)\ Y_{\ell,\ m_1 + m_2}(\hat{n})$$

(A1.5)

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2. Ibid., p. 34.
3. Ibid., p. 58.
4. Ibid., p. 61.

Multiplication of (A1.5) by $Y^*_{\ell_3 m_3}(\hat{\Omega})$ and integration over the solid angle leads to

$$\int Y^*_{\ell_3 m_3} Y_{\ell_2 m_2} Y_{\ell_1 m_1} d\Omega = \left(\frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi(2\ell_3 + 1)} \right)^{\frac{1}{2}}$$

$$\times C(\ell_1 \ell_2 \ell_3; m_1, m_2, m_3) \qquad C(\ell_1 \ell_2 \ell_3; 0\ 0) \qquad \text{(A1.6)}$$

because of the orthornomality of the spherical harmonics.

The dependence on the magnetic quantum numbers of the matrix element of any irreducible tensor, T_L is given by the Wigner-Eckart theorem;[5]

$$(j'\ m' \mid T_{L\ M} \mid j\ m) = C(j\ L\ j', m, M, m') \quad (j' \| T_L \| j) \qquad \text{(A1.7)}$$

where $(j' \| T_L \| j)$ is a reduced matrix element independent of the magnetic quantum numbers.

The basic relation used in performing sums over the magnetic quantum numbers is the Racah recoupling theorem;[6]

$$C(a\ b\ e; \alpha, \beta)\ C(e\ d\ c; \alpha+\beta, \delta) = \sum_f \left[(2e+1)(2f+1)\right]^{\frac{1}{2}}$$

$$W(a\ b\ c\ d; e\ f) \qquad C(b\ d\ f; \beta, \delta)\ C(a\ f\ c; \alpha, \beta+\delta) \qquad \text{(A1.8)}$$

Some symmetry relations for the Racah coefficients are:[7]

$$W(a\ b\ c\ d; e\ f) = W(b\ a\ d\ c; e\ f) = W(c\ d\ a\ b; e\ f) = W(a\ c\ b\ d; f\ e) \qquad \text{(A1.9a)}$$

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5. Ibid., p. 85.
6. Ibid., p. 110.
7. Ibid., p. 111.

$$W(a\,b\,c\,d;\,e\,f) = (-)^{e+f-a-d}\,W(e\,b\,c\,f;\,a\,d) = (-)^{e+f-b-c}$$

$$\times\,W(a\,e\,f\,d;\,b\,c) \qquad \text{(A1.9b)}$$

The orthornormality relation for the Rocah coefficients is:[8]

$$\sum_{e}(2e+1)\,(2f+1)\,W(a\,b\,c\,d;\,e\,f)\,W(a\,b\,c\,d;\,e\,g) = \delta_{f\,g}$$

(A1.10)

The projection theorem for first rank tensors may be written as:[9]

$$(j'\,m'\,|\,T_{1\,M}\,|\,j\,m) \quad \frac{(j\,m'\,|\,J_M\,|\,j\,m)\,(j\,\|\,J\cdot T_1\,\|\,j)}{j\,(j+1)}$$

(A1.11)

where $T_{1\,M}$ is the Mth component of T_1.

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8. Ibid., p. 113.

9. Ibid., p. 94.

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