RECENT PROGRESS IN LAGRANGIAN FIELD THEORY AND APPLICATIONS

Editors : C.P. KORTHALS-ALTES
E. de RAFAEL
R. STORA

Marseilles, June 24-28, 1974

Universités d'Aix-Marseille I et II

CENTRE DE PHYSIQUE THEORIQUE - C.N.R.S.
31 Chemin J. Aiguier - 13274 MARSEILLE CEDEX 2

Proceedings of the Colloquium

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F - 13274 MARSEILLE CEDEX 2 (France)

This Colloquium was devoted to recent developments in the study of Lagrangian models of quantum field theory :

- Renormalized perturbation theories,
- Supergauge fields,
- Asymptotic freedom and infrared slavery in gauge field models involving quarks,
- Gauge fields on lattices,
- Theory of critical exponents.

We wish to thank the Université de Provence for its interest in this enterprise and financial support , Prof. A. Visconti, Director of the Centre de Physique Théorique for encouraging us to organize this meeting, and Mrs D. Roche for her devotion and competence in taking care of all details as well as editing these proceedings.

C.P. KORTHALS-ALTES E. de RAFAEL R. STORA

Centre de Physique Théorique
C.N.R.S. Marseille

le 9 avril 1975

- PARTICIPANTS -

D. BAILIN	University of Sussex - Brighton
G. BANDELLONI	University of Genova
C. BECCHI	University of Genova - Centre de Physique Théorique, Marseille
J. BELLISSARD	Université de Provence - Centre de Physique Théorique, Marseille
A. BLASI	University of Genova
G. BORDES	Collège de France - Paris
P. BREITENLOHNER	Max Planck Institut - Münich
E. BREZIN	D.Ph.T. - CEN Saclay
Ch. CARIMALO	Collège de France - Paris
A. CHAKRABARTI	Ecole Polytechnique - Paris
S. CICCARIELLO	University of Padova
R. COLLINA	University of Genova
Ph. COMBE	U.E.R. de Luminy - Centre de Physique Théorique, Marseille
R.J. CREWTHER	CERN - Genève
Cl. DARZENS	Ecole Polytechnique - Paris
A. De RUJULA	Harvard University - Cambridge
J.A. DIXON	University of Oxford
C. de DOMINICIS	D.Ph.T. - CEN Saclay
J.M. DROUFFE	D.Ph.T. - CEN Saclay
H. EPSTEIN	I.H.E.S. - Bures sur Yvette
P. FAYET	University de Paris XI - Orsay
B. FERNANDEZ	C.N.R.S. - Strasbourg
S. FERRARA	CERN - Genève
R. FLUME	DESY - Hambourg
D. FORSTER	Universität Clausthal - Zellerfeld
D. GROSS	Institut for Advanced Studies - Princeton
G. 't HOOFT	CERN - Genève
J. ILIOPOULOS	Université de Paris XI - Orsay
B. IOCHUM	Centre de Physique Théorique - Marseille
G. IMMIRZI	University of Lecce
D.T. JONES	University of Oxford
R. JOSCHKE	Institut für Theoretische Physics - Aachen

H. KLUBERG-STERN	D.Ph.T. - CEN Saclay
C.P. KORTHALS-ALTES	Centre de Physique Théorique - Marseille
A. LAMBERT	Centre de Physique Théorique - Marseille
S.M. LEE	Michigan Technology University - Houghton
J.C. Le GUILLOU	D.Ph.T. - CEN Saclay
Cl. LEROY	Université de Louvain
D. MAISON	Max Planck Institut - Münich
R. MEULDERMANS	Universitet Leuven
Ph. MEYER	Université de Paris XI - Orsay
P.K. MITTER	Université de Paris VI - Paris
G. PARISI	Columbia University - New York
M. PERROTTET	Centre de Physique Théorique - Marseille
W. PIETSCH	Universität Inssbruck
O. PIGUET	CERN - Genève
J. PRENTKI	CERN - Genève
L. QUARANTA	Université de Provence - Centre de Physique Théorique, Marseille
E. de RAFAEL	Centre de Physique Théorique - Marseille
J.J. RAZAFITSIAROVANA	Centre de Physique Théorique - Marseille
K.D. ROTHE	CERN - Genève
A. ROUET	Centre de Physique Théorique - Marseille
S. SARKAR	CERN - Genève
P.N. SCHARBACH	Université de Paris XI - Orsay
Q. SHAFI	Institut für Theoretische Physics - Aachen
M. SIRUGUE-COLLIN	Université de Provence - Centre de Physique Théorique, Marseille
R. STORA	Centre de Physique Théorique - Marseille
R.L. STULLER	Max-Planck Institut - Münich
K. SYMANZIK	DESY - Hambourg
C. VAN DER KOLK	Amsterdam
G. VILASI	University of Salerno
A. VISCONTI	Université de Provence - Centre de Physique Théorique, Marseille
K. WILSON	Cornell University - Ithaca
K. YOSHIDA	University of Napoli
J. ZINN-JUSTIN	D.Ph.T. - CEN Saclay
J.B. ZÜBER	D.Ph.T. - CEN Saclay

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RENORMALIZATION OF GAUGE FIELD MODELS(*)

C. BECCHI* A. ROUET**

R STORA

Centre de Physique Théorique
CNRS - Marseille

* Chercheur Associé C N.R.S.

** Boursier de thèse C.E A.

(*) Talk delivered at the Marseille Colloquium on "Recent Progress in Lagrangian Field Theory and Applications", June 1974.

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POSTAL ADDRESS : Centre de Physique Théorique - C.N.R.S.
31, Chemin Joseph Aiguier
13274 MARSEILLE CEDEX 2 (France)

This communication is devoted to some recent technical progress in the study of renormalizable gauge field models[1],[2].

Gauge theories are among the most aesthetic renormalizable models so far investigated. The oldest version is quantum electrodynamics whose massive version has been extensively discussed in the litterature[3].

More sophisticated models involving Yang-Mills fields associated with non abelian gauge transformations have been discussed more recently[4].

We shall describe here a new approach to gauge field models which is based on the Bogoliubov-Parasiuk-Hepp-Zimmermann[5] (B P H Z) renormalization scheme making extensive use of the quantum action principle which has been recently proved in this framework[6],[7].

Our study is also based on a newly discovered symmetry property of the gauge field lagrangians[8],[1] (Slavnov invariance). In section I we summarize the quantum action principle in the framework of the B P H Z scheme and we discuss a simple application to a global symmetry problem[2]. In section II we exhibit the symmetry property of the gauge field lagrangians in the tree approximation and we discuss how this property is preserved at the quantum level.
Finally we make a rapid review of the main results relative to the abelian[1] and SU(2) Higgs-Kibble[2] models.

I - THE QUANTUM ACTION PRINCIPLE

In the B P H Z [5] scheme the renormalized perturbation series is expressed in terms of an effective lagrangian which we shall choose in the form :

$$\mathcal{L}^{eff}(\underline{\varphi}) = N_4\left[(\underline{\varphi}, L\underline{\varphi}) + \sum_i a_i \mathcal{M}_{\omega_i}(\underline{\varphi})\right] \equiv \mathcal{L}_0(\underline{\varphi}) + \mathcal{L}_{int}(\underline{\varphi}) \quad (1)$$

where N_4 defines the substraction procedure, $(\underline{\varphi}, L\underline{\varphi})$ determines the propagators and $\mathcal{L}_{int}(\underline{\varphi})$ the vertices of the Feynamn graphs, the $\mathcal{M}_{\omega_i}$'s are monomials of naive dimension $\omega_i \leq 4$

The Green functional of the theory is :

$$Z[\underline{J}] = \langle e^{\frac{i}{\hbar}\int dx\,[\mathcal{L}_{int}(\underline{\varphi}) + \underline{J}\,\underline{\varphi}](x)} \rangle_+ \equiv \langle X[\underline{J}] \rangle_+ \quad (2)$$

and its connected part is .

$$Z^c[\underline{J}] = \frac{\hbar}{i} \ln Z[\underline{J}] \quad (3)$$

The symbol $\langle \quad \rangle_+$ defines the renormalized time ordered product previously alluded to.

The most powerful result of the theory is doubtlessly the renormalized quantum action principle which describes the infinitesimal variation of the Green functional under an infinitesimal variation either of a parameter[6] (let it be space-time dependent) or of the quantized fields[7].

If $\lambda(x)$ is an external field upon which $\mathcal{L}^{eff}(\underline{\varphi})$ depends

$$\frac{\hbar}{i} \frac{\delta Z[\underline{J}]}{\delta \lambda(x)} = \langle N_4\left[\frac{\delta}{\delta\lambda(x)} \int dy\, \mathcal{L}^{eff}(\underline{\varphi})(x)\right] X[\underline{J}] \rangle_+ \quad (4)$$

Let on the other hand

$$\varphi_i \to \varphi_i + N_d[P_i] \equiv (1+T)\varphi_i \quad (5)$$

be a local variation of the quantized field φ_i, where $N_d[P_i]$ is a normal product local polynomial of power counting degree less than or equal to d , then the infinitesimal statement of the invariance of $Z[\underline{J}]$ under a local changement of variables reads :

$$\langle \int dx\,[T\mathcal{L}^{eff}(\underline{\varphi}) + J_i T\varphi_i](x)\, X[\underline{J}] \rangle_+ = 0 \quad (6)$$

where $T\mathcal{L}^{eff}(\underline{\varphi})$ is the quantum variation of $\mathcal{L}^{eff}(\underline{\varphi})$ and has the following form .

$$T\mathcal{L}^{eff}(\underline{\varphi}) = N_{4+\mathrm{Max}(d - \dim.\varphi_i)}\left[t\,\mathcal{L}^{eff}(\underline{\varphi}) + \hbar\, Q(\underline{\varphi})\right] \quad (7)$$

where $t\mathcal{L}^{eff}(\varphi)$ is the naive variation of $\mathcal{L}^{eff}(\varphi)$ under the transformation .

$$\varphi_i \to \varphi_i + P_i \tag{8}$$

and $\hbar Q(\varphi)$ is a polynomial of dimension not exceeding $4 + \max(d - \dim \varphi_i)$ summing up the quantum corrections. Although more detailed statements of the quantized action principle exist[7] the general structure indicated here will be sufficient for the applications we have in mind

The identity (6) can be written as a functional differential equation upon introducing a system of classical fields (sources) $\{\eta_i\}$ coupled to $\{T\varphi_i\}$.

$$\mathcal{T} Z[J] \equiv \int dx \left[J_i \frac{\delta}{\delta \eta_i}\right](x) Z[J] = \frac{i}{\hbar} B Z[J] \tag{9}$$

where :

$$B Z[J] = \left\langle \int dx \left[T\mathcal{L}^{eff}(\varphi) + O(\eta_i)\right](x) X[J] \right\rangle_+ \tag{10}$$

A typical problem in the renormalization program is the following :
Suppose that a theory which is invariant under the transformation (8) exists at the classical level, is it possible to extend such a theory to the quantum level ? That is, if we start with a tree approximation lagrangian $\mathcal{L}$ such that :

$$t\mathcal{L} = 0 \tag{11}$$

Is it possible to find a quantum extension $\mathcal{L}^{eff}$ of $\mathcal{L}$ such that

$$T\mathcal{L}^{eff} = 0 \qquad ? \tag{12}$$

The answer to this question turns out to be positive if we can prove that

$$B \equiv T\mathcal{L}^{eff} = t\mathcal{R} + O(\hbar B) \simeq t\mathcal{R} \tag{13}$$

for some $\mathcal{R}$. Here $O(\hbar B)$ means quantum connections wh ch vanish if $B = 0$.

A simple example of such a problem is the following :

$$\varphi_i \to \varphi_i + \delta\lambda_\alpha \, \Theta^\alpha_{ij} (\varphi_j + v_j) = (1 + \delta\lambda_\alpha T^\alpha)\varphi_i \tag{14}$$

where $\delta\lambda_\alpha \in \mathcal{G}$ and $\mathcal{G}$ is a semisimple Lie algebra.

The quantum action principle reads in terms of the connected Green functional

$$\mathcal{T}^\alpha Z^c[J] \equiv \int dx \left[J_i \Theta^\alpha_{ij}\left(\frac{\delta Z^c[J]}{\delta J_j} + v_j\right)\right](x) = B^\alpha Z^c[J] \tag{15}$$

with

$$B^\alpha = t^\alpha \mathcal{L}^{eff} + \hbar Q^\alpha \tag{16}$$

Let us introduce a system of classical fields $\{b_\alpha\}$ coupled to $\{B^\alpha\}$

$$\mathcal{L}^{eff}(\varphi, \{b_\alpha\}) = \mathcal{L}^{eff}(\varphi) + \sum_\alpha b_\alpha B^\alpha \tag{17}$$

After the transformation (14) we get :

$$\mathcal{T}^\beta Z^c[J, \{b_\alpha\}] = \left[B^\beta + \sum_\gamma b_\gamma t^\beta B^\gamma + O(\hbar B)\right] Z^c[J, \{b_\alpha\}] \tag{18}$$

Applying $\mathcal{T}^\alpha$ to both sides of Eq.(18) we obtain for $\{b_\alpha\} \equiv 0$

$$\begin{aligned} \mathcal{T}^\alpha \mathcal{T}^\beta Z^c[J, 0] &= \mathcal{T}^\alpha \int dx \frac{\delta}{\delta b_\beta(x)} Z^c[J, \{b_\alpha\}]\Big|_{\{b\}=0} = \\ &= \int dx\, dy \frac{\delta}{\delta b_\alpha(x)} \frac{\delta}{\delta b_\beta(y)} Z^c[J, \{b_\alpha\}]\Big|_{\{b_\alpha\}=0} + \left[t^\alpha B^\beta + O(\hbar B)\right] Z^c[J, 0] \end{aligned} \tag{19}$$

hence

$$[\mathcal{T}^\alpha, \mathcal{T}^\beta] Z^c[J] \cong [t^\alpha B^\beta - t^\beta B^\alpha] Z^c[J] \tag{20}$$

Now the τ^{α}'s satisfy the same commutation relations as the corresponding elements of the Lie algebra $\mathcal{G}$:

$$[\tau^\alpha, \tau^\beta] = f^{\alpha\beta}_{\ \ \gamma}\,\tau^\gamma \qquad (21)$$

Eq (20) together with Eq (21) leads to the consistency condition for $\{B^\alpha\}$:

$$t^\alpha B^\beta - t^\beta B^\alpha \cong f^{\alpha\beta}_{\ \ \gamma}\, B^\gamma \qquad (22)$$

which can be solved in the case of a semisimple Lie algebra(2)

$$B^\alpha \cong t^\alpha R = t^\alpha \frac{t_\beta B^\beta}{t^2} \qquad (23)$$

II - GAUGE FIELD MODELS

In this section we shall review the relevant properties of the classical lagrangians on which gauge theories are based in the tree approximation We shall in particular show that these lagrangians are invariant under a family of non linear transformations involving the Faddeev-Popov(9) ($\Phi\Pi$) ghost fields

We shall then show how this invariance property allows to extend the procedure exhibited in the preceding section to prove to all orders of perturbation theory the so-called Slavnov identities(8) which are in the present case the relevant substitutes of the Ward identities

The theory is based at the tree level on the classical lagrangian:

$$\mathcal{L} \equiv \mathcal{L}(\{\varphi_i\},\{a^\mu_\alpha\},\{c^\beta\},\{\bar c^\gamma\}) = \mathcal{L}_{inv}(\{\varphi_i\},\{a^\mu_\alpha\}) - \frac{1}{\alpha}\left[g_\alpha^2 - c^\alpha(x)\,\mathcal{M}^{\alpha\beta}(x,y)\,\bar c^\beta(y)\right] \qquad (24)$$

where $\{\varphi_i\}$ denotes a matter field multiplet, $\{a^\alpha_\mu\}$ is a Yang-Mills field, $\{\bar c^\beta\}$, $\{c^\gamma\}$ are the $\Phi\Pi$ ghost scalar Fermi fields labelled by the adjoint representation of the gauge group $\mathcal{G}$. $\mathcal{L}_{inv}(\{\varphi_i\}\{a^\alpha_\mu\})$ is the most general dimension four polynomial invariant under the local gauge transformation :

$$\delta\varphi_i(x) = \frac{\delta\varphi_i(x)}{\delta\Lambda_\alpha(y)}\,\delta\Lambda_\alpha(y)$$
$$\delta a^\alpha_\mu(x) = \frac{\delta a^\alpha_\mu(x)}{\delta\Lambda_\beta(y)}\,\delta\Lambda_\beta(y) \qquad (25)$$

The gauge function g^α , labelled by the adjoint representation is a polynomial of dimension two at most involving $\{\varphi_i\}\{a^\alpha_\mu\}$, one term of which is $\partial_\mu a^\alpha_\mu$, which is familiar in quantum electrodynamics.

$$\mathcal{M}^{\alpha\beta}(x,y) = \frac{\delta g^\alpha(x)}{\delta\Lambda^\beta(y)} \qquad (26)$$

is thus the kernel of an operator of the hyperbolic type which in general depends on $\{\varphi_i\}$ and $\{a^\alpha_\mu\}$.

In all the preceding formulae repeated indices are to be summed over and repeated space-time points are to be integrated over.

The essential property of the lagrangian $\mathcal{L}$ is its invariance under the following infinitesimal transformation which we shall call the Slavnov transformation :

$$\delta\varphi_i(x) = \delta\lambda\,\frac{\delta\varphi_i(x)}{\delta\Lambda^\alpha(y)}\,\bar c^\alpha(y) = \delta\lambda\, P_i$$
$$\delta a^\alpha_\mu(x) = \delta\lambda\,\frac{\delta a^\alpha_\mu(x)}{\delta\Lambda^\beta(y)}\,\bar c^\beta(y) = \delta\lambda\, P^\alpha_\mu$$
$$\delta c^\alpha(x) = \delta\lambda\, g^\alpha(x) \qquad (27)$$
$$\delta\bar c^\alpha(x) = \frac{\delta\lambda}{2}\, f^{\alpha\beta\gamma}\,\bar c_\beta(x)\,\bar c_\gamma(x) = \delta\lambda\, P^\alpha$$

(For simplicity we have chosen the non singular part of the Killing form of $\mathcal{G}$ to be the identity $\delta\lambda$ is a space-time independent infinitesimal parameter which commutes with $\{\varphi_i\}$, $\{a_\mu\}$, but anticommutes with $\{c^\alpha\}$, $\{\bar{c}^\beta\}$ and for two transformations labelled by $\delta\lambda_1$, and $\delta\lambda_2$, $\delta\lambda_1$ and $\delta\lambda_2$ anticommute

This invariance can be checked immediately, using the composition law for the gauge transformations :

$$\left[\frac{\delta}{\delta\Lambda^\alpha(x)}, \frac{\delta}{\delta\Lambda^\beta(y)}\right] = f^{\alpha\beta\gamma}\delta(x-y)\frac{\delta}{\delta\Lambda^\gamma(x)} \tag{28}$$

Conversely it can be shown[2] that in the case of a semisimple gauge group $\mathcal{L}$ is the most general lagrangian leading to an action invariant under Slavnov transformation and carrying no $\Phi\Pi^-$ charge

In the abelian case the invariance condition allows the addition of a term of the form :

$$\int dx\left[B(x) + \int dy\,\bar{c}(y)\,\mathcal{M}(x,y)\,c(x)\right] \tag{29}$$

for any B such that :

$$\frac{\delta B(x)}{\delta\Lambda(y)} = \mathcal{M}(x,y)\,g(x) \tag{30}$$

when $\mathcal{M}$ is the kernel of a possibly field dependent differential operator which does not upset the hyperbolic character of $\mathcal{M}$ For example in quantum electrodynamics we can add to $\mathcal{L}$

$$\frac{a_\mu a_\mu}{2} + e\,\bar{c} \tag{31}$$

which is nothing but the mass term for the photon

Another property of the Slavnov transformation which will be used in the following is that :

$$\delta P_i = \delta P^\alpha_\mu = \delta P^\alpha = 0 \tag{32}$$

and of course :

$$\delta g^\alpha(x) = \delta\lambda\,\mathcal{M}^{\alpha\beta}(x,y)\,\bar{c}^\beta(y) = \delta\lambda\frac{\delta}{\delta\bar{c}^\alpha(x)}\int\mathcal{L}(y)\,dy \tag{33}$$

We now discuss the extension of the Slavnov invariance to the quantum level.

Upon introducing a system of external fields $\{\gamma_i, \gamma^\alpha_\mu, \gamma^\alpha, \chi^\alpha, \zeta^\alpha\}$ coupled to $\{P_i, P^\mu_\alpha, P_\alpha, g^\alpha, \mathcal{M}^{\alpha\beta}\bar{c}_\beta\}$ and the sources $\{J_i, J^\alpha_\mu, \xi^\alpha, \bar{\xi}^\beta\}$ for the fields $\{\varphi_i, a^\alpha_\mu, \bar{c}^\alpha, c^\beta\}$ we can write the quantum action principle in the functional form

$$\begin{aligned}\delta Z^c &\equiv \int dx\left[J_i\frac{\delta}{\delta P_i} + J^\alpha_\mu\frac{\delta}{\delta P^\alpha_\mu} + \xi_\alpha\frac{\delta}{\delta P^\alpha} + \bar{\xi}_\alpha\frac{\delta}{\delta\chi_\alpha} + \chi^\alpha\frac{\delta}{\delta\zeta_\alpha}\right](x)\,Z^c \\ &= \int dx\left[t\mathcal{L}^{eff} + \hbar Q\right](x)\,Z^c = B Z^c\end{aligned} \tag{34}$$

Much in the same way as in the previous example the algebraic structure of the Slavnov transformation implies the consistency condition for B :

$$\delta^2 Z^c \equiv \int dx\left[\xi^\alpha\frac{\delta}{\delta\zeta_\alpha}\right](x)\,Z^c = tBZ^c + O(\hbar B) \simeq tBZ^c \tag{35}$$

After a Legendre transformation[10] Eq.(35) reads :

$$\int dx\,\frac{\delta}{\delta c^\alpha(x)}\Gamma\,\frac{\delta}{\delta\zeta_\alpha(x)}\Gamma = tB + O(\hbar B) \simeq tB \tag{36}$$

tB is now the local functional corresponding to the insertion tB. Since we have also

$$\Gamma = A + \Gamma' \tag{37}$$

when A is a local functional of dimension lower than five and $\Gamma' = O(\hbar A)$ we have

$$(1 + O(\hbar))\int dx\,\frac{\delta}{\delta c^\alpha(x)}A\,(\mathcal{M}\bar{c})_\alpha(x) \simeq tB \tag{38}$$

which is solved by

$$t\mathcal{B} \cong t^2 A \tag{39}$$

thus giving :

$$t(\mathcal{B} - tA) \cong 0 \tag{40}$$

This problem of showing that $\mathcal{B} \cong t\mathcal{R}$ is now reduced to prove that for any local functional $\mathcal{B}^4$ of dimension lower than six and carrying the $\Phi\Pi$ charge of $\bar{c}$ the equation

$$t\,\mathcal{B}^4 = 0 \tag{41}$$

implies that :

$$\mathcal{B}^4 = t\,\mathcal{R} \tag{42}$$

where $\mathcal{R}$ is a local functional of dimension lower than five

This result has been proved to be valid in the semisimple models in which the Adler-Bardeen anomaly(11) can be excluded a priori

Here we show as an example how the consistency condition (Eq (40)) allows to get rid of terms of the form :

$$\mathcal{B} = \int dx\; K^{\alpha}(x)\; \bar{c}_{\alpha}(x) \tag{43}$$

where $K^{\alpha}(x)$ depends only on $\{\varphi_i\}$ and $\{a_{\mu}^{\alpha}\}$
(Other possible terms are dealt with in a similar way).

The consistency condition for $K^{\alpha}(x)$ is :

$$\frac{\delta K^{\beta}(y)}{\delta \Lambda^{\alpha}(x)} - \frac{\delta K^{\alpha}(x)}{\delta \Lambda^{\beta}(y)} \cong f^{\alpha\beta\gamma}\delta(x-y)\,K^{\gamma}(x) \tag{44}$$

which turns out to be the Wess-Zumino(12) consistency condition for the current algebra anomalies.

One can show(2) that in the absence of the Adler-Bardeen anomaly

$$K^{\alpha}(x) \cong \frac{\delta}{\delta \Lambda^{\alpha}(x)}\, Y \tag{45}$$

where Y is a dimension four local functional.
Thus in this case

$$\mathcal{B} = t\, Y \tag{46}$$

which is the desired result.

III - RESULTS

The proof of the Slavnov identity is the essential tool for the physical interpretability of the theory. This has been investigated in all details in the following two cases : abelian(1) and SU(2)(2) Higgs-Kibble models.

The results are the following :

i) The parameters defining the theory can be specified in such a way that the theory be interpreted as an operator theory in the Fock space of the free fields (including the ghosts).

ii) A physical subspace of this Fock space is then defined. The restriction to this subspace of the S operator is then shown to be unitary and independent from the parameters which specify the gauge function to all orders of perturbation theory.

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WARD IDENTITIES AND SOME CLUES TO THE RENORMALIZATION OF GAUGE INVARIANT OPERATORS

by

H. KLUBERG-STERN and J.B. ZUBER

Département de Physique Théorique
C.E.N. - Saclay

Abstract

The problem of the renormalization of gauge-invariant operators in non-abelian Yang-Mills theory is tackled through the study of a specific example, $F^2_{\mu\nu}$, for which the explicit solution can be derived from renormalization group considerations. It is shown that the operator $F^2_{\mu\nu}$ mixes with non-gauge-invariant operators and that this mixing must be taken into account for the computation of the anomalous dimension of the renormalized gauge invariant operator. The explicit solution is examined with the help of Ward identities derived from a new type of gauge transformations which appear very convenient from a technical point of view. The multiplicatively renormalizable gauge-invariant operator is shown to satisfy Ward identities and to possess an α-independent anomalous dimension. As a by-product, we analyze the gauge dependence of the Callan-Symanzik function β.

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THE THIRRING MODEL WITH U(n) SYMMETRY *

P.K. MITTER

Laboratoire de Physique Théorique et
Hautes Energies - Université Paris VI

* Abstract of talk delivered at Marseille Colloquium on "Recent progress in Lagrangian field theo y and applications", June 1974.

A generalisation of the Thirring model with internal U(n) symmetry has been studied to all orders in renormalised perturbation theory. The model is characterised by three coupling constants characterising the most general set of renormalisable SU(n) invariant 4 - fermion interactions which are also Lorentz and C.P invariant. The model is studied with the aid of Callan Symanzik equations and Ward identities for Green functions of composite fields in the BPHZ renormalisation programme. It thus generalises the earlier work of Gomez and Lowenstein.

In contrast to the usual U(1) Thirring model one finds the presence of hard axial as well as scaling anomalies. However these anomalies are not independent. There exist identities relating them valid to all orders of perturbation theory. With the aid of these one finds locally a critical curve passing through the origin in the 3-dimensional coupling constant space on which all anomalies vanish. A region of ultraviolet attraction can be systematically found using qualitative methods from the theory of dynamical systems. In addition to the previously mentioned curve the identities (referred to above) are consistent with the existence of another critical curve not passing through the origin. However to prove its existence one has to go beyond perturbation theory.

The details can be found in :

P.K. MITTER and P.H. WEISZ : Phys. Rev. D 4410-4429 (1973)

SOME CONSIDERATIONS ON NON RENORMALIZABLE INTERACTIONS*

by

G. PARISI

Columbia University - New York **

ABSTRACT : A particular class of non renormalizable interactions is studied in the infinite cutoff limit.

We present arguments which suggest that the theory is finite after the introduction of a finite number of counter terms. The Green functions are not C^{∞} in the coupling costant at the origine. The same results are obtained using three different techniques :

infinite resummation of Feynmann graphs in perturbation theory,

analogies with second order phase transitions,

the use of the renormalization group equation to define the theory.

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In the past the properties of non renormalizable interactions have been the object of intensive studies(1),(2);(3), mainly in connection with weak interactions. After the discovery of renormalizable models of weak interactions(4), this subject is no more interesting as some years ago ; however we think that it is still worthwhile to try to understand if non renormalizable interactions do exist and which are the analiticity properties of the Green functions as functions of the coupling constant.

The aim of this talk is to present "euristic" arguments which suggest the existence of some non renormalizable quantum field theories. These arguments are similar to those used for renormalizable interactions : a formal series is constructed, each term is proved to have a finite limit when the cutoff goes to infinity, however the convergence of the expansion remains as usually unproved.

It is well known that standard perturbation theory is unable to cope with the problems presented by non renormalizable interactions : for example the i-th order of perturbation theory for the Fermi interaction diverges like $(G\Lambda^2)^i$. Finite results can be obtained only introducing an infinite number of counter terms ; no prediction can be obtained without assumptions on the magnitude of these arbitrary counterterms.

Many people have conjectured that non renormalizable interactions are finite without the introduction of extra counter-terms : the divergences found in perturbation theory are supposed to be spurious and to be connected to the lack of analyticity in the coupling constant(1-3),(4-6). We will argue that this conjecture is true, at least for a particular class of interactions. Various techniques can be used : we will describe them in increasing order of complexity and generality.

We assume that ultraviolet divergences arise in perturbation theory because the vertex has a "wrong" behaviour in the large momenta region : we strongly believe that no divergences are present in an "improved" perturbation theory in which asymptotic scale invariance is immemented at each stage. Non analytical term in the coupling constant arises naturally from the non uniform convergence of perturbation theory in the large momenta region.

As a first example we study the $\lambda N^{-1}(\phi_i \phi^i)^2$ interaction in the space-time dimensions D, $4 < D < 6$. The ϕ^i are a multiplet of N field and the Lagrangian is invariant under the transformation of the O(N) group. We use standard dimensional analysis : the length has dimension -1, the field ϕ, $\frac{D-2}{2}$; the coupling constant λ, 4-D.

Following Symanzik[8] we introduce an auxiliary field σ, whose zeroth order propagator is equal to 1 in momentum space. The interaction term in the new Lagrangian is $g N^{-\frac{1}{2}} \sigma(\phi_i \phi^i)$ where

1) $$g^2 = -\lambda \ .$$

The Green functions can be formally written as an infinite sum over Feynmann diagrams : the high order terms are undefined because of ultraviolet divergences. We try to give a meaning to this expansion performing partial resumations of diagrams. The technique we use, is the 1/N expansion[9,10] : at the first order the σ-self energy is computed and the resummed σ-propagator is introduced in high order diagrams. We stress that only a finite number of skeleton survive at a fixed order in 1/N .

The first order contribution to the σ-self energy is :

2) $$\Sigma(k) = g^2 \int \left[\frac{1}{(k+p)^2+m^2} \cdot \frac{1}{p^2+m^2} - \frac{1}{(p^2+m^2)^2} \right] d^D p \longrightarrow \underset{k \gg m}{\longrightarrow} -A g^2 k^{D-4} \quad ; \quad A \propto -\Gamma\left(\frac{4-D}{2}\right) \cong \frac{4}{(D-4)(6-D)} \ .$$

The corresponding propagator is for large values of k :

3) $$G(k) \cong \begin{cases} 1 & \left(k \ll g^{\frac{-2}{D-4}}\right) \\ A^{-1} g^{-2} k^{4-D} & \left(k \gg g^{\frac{-2}{D-4}}\right) \end{cases}$$

A simple analysis shows that when (3) is inserted in higher order diagrams, the large momenta behaviour of the integrals is just the one dictated by asymptotic scale invariance : the effective coupling constant turns out to be dimensionless, g-independent and proportional to A^{-1}.

The only divergent diagrams are related to the vertex and wave function renormalization and to the ϕ-self mass. The use of the improved propagator (3) avoids the appearance of additional divergences. The output of the 1/N expansion is not C^∞ in the coupling costant g . For example a contribution to the six point function at zero external momentum is roughly equal to :

4) $$g^6 \int d^D p \ \left[p^2+m^2\right]^{-3} \left[1+g^2 A k^{D-4}\right]^{-3} \ .$$

This integral is finite for any positive value of g^2, but it is not analytic around $g=0$: a term proportional to $g^{6+2\frac{D-4}{6-D}}$ is present (if D =5 the behaviour at small g is $g^8 \ln(g)$). These non analytic contributions come from the integration regions where $g^2 k^{D-4} \sim 1$; for such value of the momenta the effective dimensionless coupling constant is of order one and high order corrections are relevant also for small g . The coefficient of the singular term can be expanded in powers of 1/N : only in very few cases it can be written in a closed form.

The new perturbation expansion reproduces the results of standard perturbation theory when they are finite ; "divergent" counterterms can be computed in powers of 1/N : they are not C^∞ functions of the coupling constant.

The 1/N expansion is very simple and straightforward, however it is possible that a different resummation technique yields different results (remember Riemann's theorem !). It would be interesting to derive the same conclusions from more general arguments.

To this end the following procedure can be used : as a first step we cut off the theory introducing an extra term in the Lagrangian ; all divergences disappear and the interaction becomes superrenormalizable. We look for a value of the renormalized parameter for which renormalized Green functions are finite but all bare parameters are infinite ; it is clear that this particular situation corresponds to the infinite cutoff limit. A similar technique is used in the study of the infinite coupling constant limit of superrenormalizable theories[11].

To be more specific let us see how this idea can be applied to the $\sigma(\phi_i \phi^i)$ interaction. At the zeroth order in the coupling constant the σ-propagator is taken equal to $(k^2+\mu^2)^{-1}$. A renormalized dimensionless coupling constant (g_R) is introduced and the function β is defined as[12] :

$$5) \qquad m^2\left(\frac{\partial}{\partial m^2} + \mu^2\frac{\partial}{\partial \mu^2}\right)\Big|_{g_B} g_R = \beta\left(g_R, \frac{m}{\mu}\right) .$$

If the function β has a zero at $g = g_c$, all bare quantities are divergent for this value of the coupling constant, in particular the cutoof becomes infinite. The non renormalizable theory is finite iff the Green function of the superrenormalizable theory have a finite limit when g goes to g_c. The consistency of this last assumption can be checked looking to the large momenta behaviour of the Green function at $g = g_c$[13] (asymptotic scale invariance is valid in the large momenta region).

This technique is particular suited for the case of $6-\varepsilon$ dimensions. The analytic structure in the coupling constant is the same as the one found using the 1/N expansion.

The general "equivalence theorem" holds : "non renormalizable theories are superrenormalizable theories computed at the infrared stable fixed point". The existence of non renormalizable theories is related to the the existence of infrared stable fixed point in superrenormalizable theories.

Zero mass non renormalizable theories can be constructed using the renormalization group[14] (in the massive case we can use the improved renormalization group[15]). The idea is to use the output of the Dyson equation.as input for the r.g. equation for the vertex and to use the output of the r.g. equation as input for the Dyson equation. In this way, if the β function has a zero, the Green functions are automatically scale invariant and no divergences appear. Non analytic term in the coupling constant are obtained also in this case. An application of this technique to second order phase transition can be found in a recent paper of Sugar and White[16].

We mention 'en passant' that, if we change the dimensions of the space in such a way to transform a renormalizable interaction in a non renormalizable one, no pathology is shown by the conformal invariant self consistency conditions for the propagator and the vertex[17]. : the values of the coupling constant and of the anomalous dimensions seem to be regular function of the dimensions of the space[18,19].

The conclusions of this talk are the following : there is a particular class of non renormalizable theories which exist and are finite (stability problems are neglected) ; the Green functions are not C^∞ in the coupling constant, however in the large momenta region they are coupling constant independent and scale invariant. In particular situations (1/N or ε expansions) the effective coupling constant in the large momenta region becomes small and we can compute the coefficient of the term singular at $g = 0$.

It would be tempting to apply these considerations to a realistic model of weak interactions ; however we have no control on the existence of the zero of the relevant β function, the presence of a vector interaction may destroy asymptotic convformal invariance and introduce additionnal divergences.

Also if we skip these difficulties, we are still faced with the problem of fixing the low energy structure of the theory : self consistency techniques should be used like in ref. 3 and ref. 20 . Although the situation may seem quite promising, additionnal studies are needed to understand the properties of a finite non renormalizable model for weak interactions. It may be interesting to note that these problems disappear in less than 4 dimensions : the presence of Fermions does not introduce new difficulties.

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II - SUPERGAUGE FIELDS

SUPERGAUGE SYMMETRY IN LOCAL QUANTUM FIELD THEORY

by

S. FERRARA

CERN - Geneva

Extension of supergauge symmetry to four-dimensional space-time (1) has allowed to investigate the possible role of this symmetry in conventional local quantum field theory.
Supergauge transformations form a sort of extended Lie Algebra with both commuting and anticommuting parameters. There are both bosonic and fermionic generators among the charges of this symmetry. As a consequence the representations of this algebra combine fermions with bosons.

The supergauge algebra is obtained adding to the conformal group of space-time two Majorana spinor generators and the chiral charge (2). The two spinor charges Q_α , S_α , to be called restricted and special supergauge transformation respectively, admit the following commutation relations with the conformal sud chiral charges.

1) $[Q_\alpha, D] = \frac{i}{2} Q_\alpha \qquad [S_\alpha, D] = -\frac{i}{2} S_\alpha$

$[Q_\alpha, M_{\mu\nu}] = i(\sigma_{\mu\nu} Q)_\alpha \qquad [S_\alpha, M_{\mu\nu}] = i(\sigma_{\mu\nu} S)_\alpha$

$[Q_\alpha, P_\mu] = [S_\alpha, K_\mu] = 0$

$$[Q_\alpha, K_\mu] = -i(\gamma_\mu S)_\alpha \qquad [S_\alpha, \mathcal{P}_\mu] = i(\gamma_\mu Q)_\alpha$$

$$[Q_\alpha, \pi] = -i\tfrac{3}{4}(\gamma_5 Q)_\alpha \qquad [S_\alpha, \pi] = i\tfrac{3}{4}(\gamma_5 S)_\alpha$$

$$\{Q_\alpha, \bar{Q}_\beta\} = -2\not{P} \qquad \{S_\alpha, \bar{S}_\beta\} = 2\not{K}$$

$$\{Q_\alpha, \bar{S}_\beta\} = 2(\sigma^{\mu\nu}M_{\mu\nu} - D + 2\gamma_5\pi)$$

Of course this symmetry, due to the presence of conformal charges, is useful only for massless theories. However there is a non trivial subgroup, generated by Poincaré transformations and restricted supergauge transformations which commute with the mass operator $\mathcal{P}_\mu \mathcal{P}^\mu$ and which is therefore consistent with more realistic massive Lagrangian theories.

In order to construct field representations of supergauge symmetry one considers the action of an element of the group on the group manifold(3)

2) $$e^{-ix.\mathcal{P} - \bar{\theta}Q}$$

where x, θ are the coordinates of the superfield

3) $$\phi(x,\theta) = e^{-ix.\mathcal{P} - \bar{\theta}Q}\,\phi(0,0)$$

Using the commutation relation of the algebra one easily gets

$$\delta\phi(x,\theta) = i\,\partial_\mu\phi(x,\theta) \qquad \text{translation}$$

4) $$\delta\phi(x,\theta) = \left(-\frac{\partial}{\partial\bar{\theta}} + i\gamma^\mu\theta\partial_\mu\right)\phi(x,\theta) \qquad \text{supergauge}$$

$$\delta\phi(x,\theta) = -i\left(x_\mu\partial_\nu - x_\nu\partial_\mu + \bar{\theta}\sigma_{\mu\nu}\frac{\partial}{\partial\bar{\theta}}\right)\phi(x,\theta) \qquad \text{Lorentz}$$

It is convenient to use Weyl spinors $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ related to the Majorana spinor θ by the relation

5) $$\theta_\alpha = \tfrac{1}{2}(1 - i\gamma_5)\theta \qquad \bar{\theta}^{\dot{\alpha}}_{\alpha} = \tfrac{1}{2}(1 + i\gamma_5)\theta$$

It follows that the superfield can be written as

6) $$\phi(x,\theta,\bar{\theta}) = e^{-ix\mathcal{P} + i\theta Q + i\bar{\theta}\bar{Q}}\,\phi(0,0)$$

Due to the anticommuting properties of the spinor **parameters** the most general form of the superfield is

7) $$\phi(x,\theta,\bar{\theta}) = A + \theta\psi + \bar{\theta}\bar{\psi} + \theta\theta\mu + \bar{\theta}\bar{\theta}\mu^* + \theta\sigma_\mu\bar{\theta}v^\mu + \theta\theta\bar{\theta}\bar{\lambda} + \bar{\theta}\bar{\theta}\theta\lambda + \theta\theta\bar{\theta}\bar{\theta}D$$

where we imposed the invariant constraint

8) $$\phi(x,\theta,\bar{\theta}) = \bar{\Phi}(x,\theta,\bar{\theta}).$$

This field contains two Majorana spinors ψ, λ, two scalar fields A,D a chiral doublet M, M^* and a real vector field. This is called the vector superfield.
From eqs. 7) and 4), composing the various powers of $\theta, \bar{\theta}$ one gets the transformation properties of the various component of the multiplet under a supergauge transformation. We remark that the last component of a supermultiplet always transforms as a total derivative, i.e.

9) $$\delta D = \partial_\mu \chi^\mu \qquad \text{where } \chi^\mu \text{ is some Fermi field.}$$

This property will be crucial in order to build up invariant Lagrangian with supergauge multiplets. In order to develop further the tensor calculus with superfields we observe that the group-manifold can be parametrized in three different but equivalent ways(4), i.e.

10)
$$e^{-ix\mathcal{P} + i\theta Q + i\bar{\theta}\bar{Q}}$$
$$e^{-ix\mathcal{P} + i\theta Q}\,e^{i\bar{\theta}\bar{Q}}$$
$$e^{-ix\mathcal{P} + i\bar{\theta}\bar{Q}}\,e^{i\theta Q}$$

They correspond to three different realization of supergauge representations i.e. to different definition of superfields related by the following identity

$$11)\quad \phi(x,\theta,\bar\theta) = \phi_1(x+i\theta\sigma_\mu\bar\theta,\theta,\bar\theta) = \phi_2(x-i\theta\sigma_\mu\bar\theta,\theta,\bar\theta).$$

Note, that once the superfields are defined as type I or type II fields, (corresponding to the second and third definition of the group manifold), they become intrinsically complex so the reality condition cannot still be imposed.

We introduce now the following spinor derivatives[4]

$$12)\quad D_\alpha = \frac{\partial}{\partial\theta} + i\sigma_\mu\bar\theta\partial^\mu \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta} - i\theta\sigma_\mu\partial^\mu$$

which have the property of commuting with supergauge transformations. They are therefore covariant operations. It is important to observe that they have the following anticommutation relations

$$13)\quad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0 \qquad \{D_\alpha, \bar D_{\dot\alpha}\} = -2i\,\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu .$$

We call $\phi(x,\theta,\bar\theta)$ a left-handed superfield if it satisfies the equation

$$14)\quad \bar D_{\dot\alpha}\,\phi_L = 0 .$$

In analogous way a field such that $D_\alpha\phi_R = 0$ will be called right-handed superfield. Note that $\bar\phi_L$ is right-handed in fact

$$15)\quad \overline{\bar D_{\dot\alpha}\,\phi_L} = D_\alpha\,\bar\phi_L = 0 .$$

As a consequence of 11), a left-handed type I superfield is only a function of θ because $\bar D_{\dot\alpha}\phi_1(x,\theta,\bar\theta) = \frac{\partial}{\partial\bar\theta}\phi_1(x,\theta,\bar\theta)$ and analogously a right-handed type II superfield is only a function of $\bar\theta$.

It follows that a left-handed superfield can be always written as

$$16)\quad \phi_1(x,\theta) = \phi_L(x-i\theta\sigma_\mu\bar\theta,\theta,\bar\theta) = A(x) + \theta\,\psi(x) + \theta.\theta.F(x).$$

The representation $\phi_L \oplus \bar\phi_L$ is now a real representation which contains two scalars and to pseudoscalar superfields A , F , B , G and a Majorana spinor ψ . This will be called the scalar multiplet according to the original definition[1].

We end the basic ingredients of tensor calculus with the observation that also multiplication of superfields is well defined.

The following multiplication rules can be explicitely verified by the reader as an exercise. Call ϕ,ψ two superfields, then

$$\phi_L\psi_L = (\phi\psi)_L \qquad \phi_R\psi_R = (\phi\psi)_R$$

$$17)\quad \phi_L\psi_R \pm \phi_R\psi_L = (\phi\psi)_\pm \quad \text{are vector superfields}$$

$$\phi_L D_\alpha\psi_L = V_\alpha \quad \text{verifies the property} \quad \bar D\bar D V_\alpha = 0 .$$

The covariant derivatives previously defined, together with the scalar and the vector supermultiplets are the basic ingredients in order to build up supergauge symmetric Lagrangians. We will confine ourselves with with the construction of such Lagrangians making use of the powerful techniques previously introduced. Professor Iliopolous will discuss in detail, in his talk, the consistency of this invariance with the renormalization procedure and moreover the problem of spontaneous symmetry breaking[5],[6].

The previously derived techniques make us sure that the only possible fields which can be considered as supergauge invariant Lagrangians are the F and D components of the scalar and vector multiplets respectively.

For the self-interacting supermultiplet[7] we have the paradox that the less intuitive term which appears in the Lagrangian is the kinetic energy term. To build this term[4] we remark that the superfield given by $\bar{\mathcal{D}}_{\dot\alpha}\ \bar{\mathcal{D}}^{\dot\alpha}\ \bar{\phi}_L$ is a left-handed multiplet with components F^*, $\sigma_\mu \partial^\mu \bar{\psi}$, $\mathcal{D} A^*$ respectively. From this follows that the Γ component of the left-handed superfield $\phi_L \bar{\mathcal{D}}_{\dot\alpha} \bar{\mathcal{D}}^{\dot\alpha} \bar{\phi}_L$ is the kinetic term of the supergauge invariant Lagrangian. The full Lagrangian is given by the formula

$$18)\qquad \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\left(\phi_L \bar{\mathcal{D}}\bar{\mathcal{D}}\bar{\phi}_L + m\phi_L^2 + g\phi_L^3 + \lambda\phi_L\right) + h.c$$

where the various terms are easily identified making use of the explicit expression of the superfields.

Interactions of vector multiplets are more involved due to the simultaneous presence of supergauge and local gauge symmetry.
However there is no problem for what concerns the supergauge symmetric free Maxwell equations[1]. In fact, starting with the vector multiplet, one can construct the left-handed spinor superfield

$$19)\qquad W_{L\alpha} = \bar{\mathcal{D}}_{\dot\alpha}\bar{\mathcal{D}}^{\dot\alpha}\mathcal{D}_\alpha \phi$$

then the F component of the left-handed superfield

$$20)\qquad \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta} W_{L\alpha} W_L^{\alpha} + h.c.$$

gives the kinetic energy terms for the vector supermultiplet

$$21)\qquad \mathcal{L} = -\frac{1}{4}v_{\mu\nu}v^{\mu\nu} - \frac{i}{2}\bar{\lambda}\gamma\cdot\partial\lambda + \frac{1}{2}\mathcal{D}^2 .$$

To construct an interaction with the vector multiplet we first consider the supergauge invariance extension of Q.E.D., i.e. the interaction of a vector multiplet with a complex left-handed superfield[8].
The role of gauge transformations in a supergauge invariant theory is taken by an entire scalar supermultiplet. In fact usual gauge transformations are not preserved by supergauge transformations[8].

Under a local infinitesimal gauge transformation one has

$$(22)\qquad \delta S = -i\Lambda S \qquad \delta S^* = i\Lambda^* S^* \qquad \delta V = i(\Lambda - \Lambda^*)$$

or in the integrated form

$$(23)\qquad e^V \rightarrow e^{-i\Lambda^+} e^V e^{i\Lambda}$$

$$(24)\qquad S \rightarrow e^{-i\Lambda}S \qquad\qquad S^* \rightarrow S^* e^{i\Lambda^+}$$

It follows that the interaction term $S^* e^V S$ is actually invariant. However, as parity changes $V \rightarrow -V$ then it follows that the Lagrangian must contain another piece $T^* e^{-V} T$ when $T^* \rightarrow T^* e^{i\Lambda}$ $T \rightarrow e^{-i\Lambda^*} T$ under local gauge transformations.
Then it follows that $\mathcal{S}$ and T^* are left-handed superfields.
One can put

$$\mathcal{S} = S_1 + i S_2 \qquad\qquad T^* = S_1 - i S_2$$

then it follows that, provided one chooses a special gauge, where $C = \psi = M = N = 0$ (Wess, Zumino gauge) the Lagrangian becomes manifestly renormalizable and takes the form

$$\mathcal{L} = \frac{\partial}{\partial\theta}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\bar\theta}\frac{\partial}{\partial\bar\theta}\Big(2(S_1\bar{S}_1 + S_2\bar{S}_2) + 2i(S_1\bar{S}_2 - \bar{S}_1 S_2)V + V^2(S_1\bar{S}_2 + \bar{S}_1 S_2)\Big).$$

The renormalization of such a model, consistent with supergauge symmetry was shown by Wess and Zumino up to one loop[8].

Finally we mention the self-interaction of the vector multiplet, consistent with supergauge invariance. This was discovered independently by Salam, Strathdee and Zumino and the present author[9].

Let us consider for instance a Yang-Mills supermultiplet, i.e. a vector multiplet whose component transform according to the regular representation of some group, say SU(N).

In a supergauge invariant theory the Yang-Mills gauge transformation has to be replaced by a Yang-Mills left-handed multiplet Λ .

$$(25) \qquad e^{V} \to e^{-i\Lambda^{+}} e^{V} e^{iV} \quad , \quad e^{-V} \to e^{-i\Lambda} e^{-V} e^{i\Lambda^{+}} .$$

In infinitesimal form we get

$$(26) \qquad V \to V + i(\Lambda - \Lambda^{+}) - \tfrac{1}{2}\left([\Lambda,\Lambda^{+}] + i[\Lambda+\Lambda^{+}, V]\right) + \cdots$$

Note that V always stands for the NxN matrix $\lambda_i V_i$.

From the previous formulae one easily derives that

$$(27) \qquad e^{-V} D_\alpha e^{V} \to e^{-i\Lambda}(e^{-V} D_\alpha e^{V}) e^{i\Lambda} + e^{-i\Lambda} D_\alpha e^{i\Lambda}$$

under a Yang-Mills transformation. From this it follows that the left-handed multiplet given by

$$(28) \qquad W_\alpha = \bar{D}\bar{D}(e^{-V} D_\alpha e^{V})$$

transforms as $W_\alpha \to e^{-i\Lambda} W_\alpha e^{i\Lambda}$ under a (generalized) Yang-Mills transformation. It follows immediatly that the F component of the left-handed field $Tr(W_\alpha W^\alpha)$ generates a Lagrangian which is invariant under supergauge transformations and generalized Yange-Mills transformations. In the special gauge, where $C_i = M_i = N_i = \Psi_i = 0$ one gets

$$\mathcal{L}_{YM} = Tr\left(-\tfrac{1}{4} v_{\mu\nu} v^{\mu\nu} - \tfrac{i}{2} \lambda \gamma^\mu \mathcal{D}_\mu \lambda + \tfrac{1}{2} D^2\right)$$

where
$$v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu + ig[v_\mu, v_\nu]$$
$$\mathcal{D}_\mu \lambda = \partial_\mu \lambda + ig[v_\mu, \lambda] .$$

As a consequence one gets the result that the interaction of a Yang-Mills field with a Majorana spinor belonging to the regular representation of an internal symmetry group is supergauge invariant As a final remark let us mention the possible geometrical meaning of supergauge transformations. We have already seen that the representations of the restricted algebra can be realized on fields $\phi(Z)$ defined on the superspace $Z = (x, \theta, \bar{\theta})$. Now on this space there is a line element which is invariant under translations and restricted supergauge transformations(10), i.e.

$$(29) \qquad \omega_\mu = dx_\mu + i(\theta \sigma_\mu d\bar{\theta} - d\theta \sigma_\mu \bar{\theta}) .$$

So its square $\omega^2 = \omega_\mu \omega^\mu$ is actually invariant under the whole restricted supergauge algebra. It can be shown that one can enlarge the representation of the restricted supergauge algebra $\phi(x, \theta, \bar{\theta})$ to a representation of the enlarged algebra. This is because additional generators S_α , K_λ, D and Π belong (together with $M_{\mu\nu}$) to the stability algebra of the point Z = (0,0).

Due to the structure of such an algebra one can always consider representations of this algebra such that

$$(30) \qquad S_\alpha = K_\lambda = 0$$

this is in fact the condition that ensures that the lowest component of the superfield $\phi(x, 0)$ transforms irreducibly under Lorentz transformation. In this way, for any transformation of the group one gets the corresponding transformation of the superfield

$$\delta\phi(x,\theta,\bar{\theta}) = \left(\left(\delta x^\mu \partial_\mu + \delta\theta \frac{\partial}{\partial\theta} + \delta\bar{\theta}\frac{\partial}{\partial\bar{\theta}} + f(x,\theta,\bar{\theta})\right)\phi(x,\theta,\bar{\theta})\right)$$

where $f(x, \theta, \bar{\theta})$ is some weight associated to the field and δx^μ , $\delta\theta$, $\delta\bar{\theta}$ are themselves functions of x , θ $\bar{\theta}$. For example, under a special supergauge transformation one finds(10)

$$\delta\theta = i\sigma\cdot x\bar{\beta} + 2\beta\theta\theta - \sigma^{\mu}\bar{\beta}\,\theta\sigma_{\mu}\bar{\theta}$$

$$\delta\bar{\theta} = i\bar{\sigma}x\beta + 2\bar{\beta}\bar{\theta}\bar{\theta} + \bar{\sigma}^{\mu}\beta\theta\sigma_{\mu}\bar{\theta} \tag{31}$$

$$\delta x_{\mu} = -\beta\sigma\cdot x\bar{\sigma}_{\mu}\theta - \bar{\beta}\bar{\sigma}\cdot x\sigma_{\mu}\bar{\theta} + i(\theta\theta\beta\sigma_{\mu}\bar{\theta} - \bar{\theta}\bar{\theta}\theta\sigma_{\mu}\bar{\beta})$$

then it follows that the line element (29) transforms as

$$\delta\omega_{\mu} = -2(\beta\sigma_{\lambda}\bar{\sigma}_{\mu}\theta + \bar{\beta}\bar{\sigma}_{\lambda}\sigma_{\mu}\bar{\theta})\,\omega^{\lambda} \tag{32}$$

under supergauge transformation of special type. Therefore

$$\delta\omega^{2} = 4(\beta\theta + \bar{\beta}\bar{\theta})\,\omega^{2}\,. \tag{33}$$

Analogously under a conformal transformation one finds

$$\delta\omega^{2} = 4c\cdot x\,\omega^{2}\,. \tag{34}$$

Of course $\delta\omega^{2} \rightarrow 2\varepsilon\,\omega^{2}$ under dilatation, and $\delta\omega^{2} = 0$ under chiral transformations.

Then we have that supergauge transformations can be regarded as the subset of general coordinate transformations $Z \rightarrow f(Z)$ which dilate the metric ω^{2} by a function of the point $Z(x,\theta,\bar{\theta})$

$$\delta\omega^{2}(Z) = f(Z)\,\omega^{2}(Z)\,. \tag{35}$$

So we see that supergauge transformations are those transformations of the enlarged space $(x,\theta,\bar{\theta})$ which eaves invariant the "light-like" distance $\omega^{2}(Z) = 0$.
The previous analysis suggests the possibility of generalizing supergauge transformations in curved superspaces.

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A) SUPERGAUGE INVARIANCE AND RENORMALIZATION

Presented by

J. ILIOPOULOS

"Broken Supergauge Symmetry and Renormalization"
J. Iliopoulos and B. Zumino
Nuclear Physics B76 (1974) 310-332.

Abstract : A field theory model invariant under supergauge transformations is shown to be renormalizable to all orders in perturbation theory. Renormalization is shown to be consistent with supergauge invariance. It is further shown that only one renormalization constant is needed, a common wave function renormalization for all fields. A symmetry breaking term is introduced which breaks the symmetry explicitly but so smoothly that the renormalization procedure of the symmetric case can still be applied. Relations among masses and coupling constants emerge. Among other topics discussed, the possibility that the supergauge symmetry is spontaneously broken and that a Goldstone spinor appears is examined.

"Supergauge Invariance and the Gell-Mann-Low Eigenvalue"
S. Ferrara, J. Iliopoulos and B. Zumino
Nuclear Physics B77 (1974) 413-419.

Abstract : The connections among supergauge, scale and conformal invariance are elucidated on the example of a renormalizable field theory model. For the massless model, at the Gell-Mann-Low eigenvalue, a non-perturbative argument leads to the contradictory result that it can only describe a free field theory. A study of the Callan-Symanzik equations for the massive model clarifies the situation from the point of view of perturbation theory. It is argued that the eigenvalue equation has no non-trivial solution and that the effective coupling constant increases without bound with energy.

B) SPONTANEOUSLY BROKEN SUPERGAUGE SYMMETRY AND GOLDSTONE SPINORS

Presented by

P. FAYET

"Spontaneously Broken Supergauge Symmetries and Goldstone Spinors"
P. Fayet and J. Iliopoulos
Physics Letters 51B (1974) 461-464.

Abstract : We present a model with a spontaneously broken supergauge symmetry which results in the appearance of a massless Goldstone spinor. The model combines supergauge invariance with ordinary gauge invariance. After the breaking the gauge boson acquires a mass as a result of the Higgs mechanism.

III - GAUGE FIELDS, QUARKS
ASYMPTOTIC FREEDOM,
INFRARED SLAVERY

√

Asymptotic Near Freedom

D. Bailin, University of Sussex

The work which this report is based upon was done in collaboration with Alex Love. It came about as a response to the shock we felt when we studied the experimental data on scaling. It seemed to us that the data do not rule out a $-q^2$ variation of the structure functions (at fixed q^2/ν) of perhaps as much as $(-q^2)^{-0.3}$, if we parametrize the variation as a power of $-q^2$. Of course the asymptotically free gauge theories predict logarithmic deviations from scaling[1] and the problems of distinguishing between these two types of deviation have been discussed by Gross elsewhere in these Proceedings. The N.A.L. experiments using muon beams should clarify the situation. At any rate, we asked ourselves the question: what would be the theoretical consequences if the N.A.L. experiments actually show a power deviation from exact scaling with an exponent less than about -0.3?

Such a variation is still considerably slower than any known form factor so probably we should not simply drop the apparatus which has been built up to understand "exact" scaling. Presumably we should say that the underlying strong interaction field theory was asymptotically "nearly" free rather than asymptotically free. More precisely we should say that $\beta(\lambda)$ has a zero at $\lambda = \lambda_F$, near the origin, which controls the ultra-violet behaviour of the theory. Then the effective coupling constant $\bar{\lambda}(t, \lambda) \to \lambda_F \ll 1$ as $t \to \infty$. But for λ_F to be an ultra-violet stable fixed point we want

$$\beta(\lambda) > 0 \qquad \lambda < \lambda_F$$

$$\text{and} \qquad \beta(\lambda) < 0 \qquad \lambda > \lambda_F ,$$

and since $\beta(0) = 0$ the simplest possibility is that shown in Fig. 1a. Of course $\beta(\lambda)$ <u>might</u> have two zeros near the origin with the second one attractive as in Fig. 1b. We thought this latter behaviour unlikely (and certainly unpredictable) so we considered only the first possibility. Thus we are considering theories in which $\lambda = 0$ is an infra-red stable point.

But Zee[2] has shown that there are lots of theories which cannot be asymptotically free so we considered those theories which seemed to us to be a priori the most reasonable. There are advantages in basing the strong interaction Lagrangian on a group which commutes with the ordinary classification SU(3), since we gauge parts of classification SU(3) to generate weak and electromagnetic interactions. An appealing choice of strong interaction group[3] is the group which transforms the colour indices of the three triplets of quarks required to give the correct rate for $\pi^0 \to \gamma\gamma$. This rate and the total e^+e^- annihilation cross-section do <u>not</u> depend upon the choice of colour group, but only on the fact that there are *three* classification SU(3) triplets of quarks with conventional charges and weak charges. Thus we may take the 3 colour indices $\alpha = 1, 2, 3$ of the quark q_α as transforming as the three dimensional representation of a group **G**. This group is usually taken to be SU(3) and the three dimensional representation is the fundamental representation. However, we can also take **G** = SU(2) which is the same as the rotation group. This too has a three dimensional representation namely the regular representation in which the quarks have "colour spin" unity. The next option open to us is whether to use vector gluons or scalar gluons transforming according to the regular representation of G. If G is SU(3) there will be eight gluons while if G is SU(2) there are three. The Yukawa coupling of scalar gluons is known to be a theory in which the fixed point at the origin is <u>not</u> ultra-violet stable[2] so that the ultra-violet behaviour must be controlled by a non-trivial fixed point, such as we are looking for, which we

assume to be <u>near</u> the origin. In the vector gluon case we have a non-abelian gauge theory which, if G = SU(3), gives an ultra-violet <u>stable</u> origin[4] provided we do not generate the gauge field's masses by means of Higgs scalars. Thus $\beta(\lambda)$ does not have the form of Fig. 1a. and we discard this possibility. On the other hand if G = SU(2) we have[4]

$$\beta(g) = A\frac{g^3}{16\pi^2} + B\frac{g^5}{(16\pi^2)^2} + \ldots\ldots$$

where

$$A = -\left[\frac{11}{3}C_2(G) - \frac{4}{3}mT(R)\right]$$

and

$$C_2(G)\,\delta_{ij} = f_{ike}\,f_{jke},$$

$$T(R)\,\delta^{ab} = \mathrm{Tr}\,[R^a R^b];$$

R^a are the matrices representing G in the (3-dimensional) representation to which the fermions belong. m is the number of fermion multiplets, 3 in our case. Now for G = SU(2)

$$C_2(G) = 2 \quad , \quad T(R) = 2$$

Thus

$$A = -\left[\frac{22}{3} - \frac{24}{3}\right] = +\frac{2}{3}$$

and we see that the origin $g = 0$ is no longer ultra-violet stable, and again the asymptotic behaviour must be controlled by a non-trivial fixed point. Notice that the small value of A above results from extensive cancellation between the two terms so that if B is <u>negative</u> and of "normal" size we may expect a zero of β near the origin. In fact the results of Jones, presented elsewhere in these Proceedings, and of Caswell[5] show that

$$B \approx +80$$

for G = SU(2) with three triplets. Thus it is unlikely that in this theory β has a zero near the origin, and in this report we shall henceforth confine ourselves to the scalar gluon theories. Notice however that the form of A does provide a "natural" mechanism for cancellations and hence for nearby zeros, and it may be that some other reasonable choice of m, R and G could yield the behaviour we are seeking; as Wilson has observed earlier in this Meeting, the e^+e^- annihilation cross-section may force us to entertain theories with large numbers of fermion multiplets.

We now calculate the anomalous dimensions of the tensor operators which occur in the light cone expansion for e- or ν-production. We assume that the scalar fields φ^a have a Yukawa coupling to the quark fields q_α:

$$\mathcal{L}_Y = g_o \sum_{\alpha,\beta,a} \bar{q}_\alpha R^a_{\alpha\beta} q_\beta \varphi^a$$

where R^a are matrices in the 3 dimenional representation of G, and a runs from 1.... 8 if G = SU(3) and a = 1...3 if G = SU(2). In addition there is a four scalar coupling, denoted $\lambda_o \varphi^4$, which however makes no contribution in the single loop order in which we shall be calculating. We are concerned with the light cone expansion for the product of two currents, vector or axial. For a spin averaged target and energies below the threshold for the production of colour non-singlet states the only terms in the light cone expansion which we need to worry about are

$$O_m^{\mu_1 \dots \mu_n} \equiv i^{n-1} \sum_{s,\alpha} \bar{q}_\alpha \, \partial^{\mu_1} \dots \gamma^{\mu_s} \dots \partial^{\mu_n} \tfrac{1}{2} \lambda_m q_\alpha \qquad (m = 0, \dots 8)$$

Continued Overleaf/....

$$O_9^{\mu_1 \dots \mu_n} \equiv i^n \sum_a \varphi^a \, \partial^{\mu_1} \dots \partial^{\mu_n} \varphi^a ,$$

where λ_m acts in the space of ordinary classification SU(3); the summation over α and a ensures that the above operators are colour singlets. We see that for classification SU(3) non-singlets there is only one operator (for a given n) to worry about, while for SU(3) singlets there are two (O_o and O_9). We denote the inserted Green's functions by $\Gamma^{\psi_i \psi_i O_j^n}$ where $\psi_i = q_\alpha$ or φ_a and j = 0...9. The SU(3) non-singlet insertions are multiplicatively renormalized but the singlet insertions are not because of mixing between O_o and O_9. In general the renormalized Green's functions $\bar{\Gamma}$ are related to Γ by

$$\Gamma^{\psi_i \psi_i O_j^n} = \sum_k Z^2(\psi_i) \, Z^n_{jk} \, \bar{\Gamma}^{\psi_i \psi_i O_k^n}$$

They satisfy a Callan-Symanzik equation[6] which in the deep euclidean region (relevant for deep inelastic e- and ν- production) takes the form

$$[D + 2\gamma(\psi_i)] \, \bar{\Gamma}^{\psi_i \psi_i O_j^n} + \sum_k \gamma^n_{jk} \, \bar{\Gamma}^{\psi_i \psi_i O_k^n} = 0$$

where $\gamma(\psi_i)$ is the anomalous dimension of the field ψ_i and

$$\gamma^n_{jk} = \sum_\ell (Z^n)^{-1}_{j\ell} \, D Z^n_{\ell k}$$

is the anomalous dimension matrix;

$$D = \mu \frac{\partial}{\partial \mu} + m \frac{\partial}{\partial m} + \beta_\lambda \frac{\partial}{\partial \lambda} + \beta_g \frac{\partial}{\partial g}$$

μ and m are the masses of the scalar and fermion while λ and g are the renormalized coupling constants. To calculatue the γ^n_{jk} in lowest order we simply have to calculate the divergent parts of the diagrams shown in Fig. 2 which are proportional to

$$\sum_s p_{\mu_1} \cdots \gamma_{\mu_s} \cdots p_{\mu_n} \qquad \text{for external fermions,}$$

or

$$p_{\mu_1} \cdots p_{\mu_n} \qquad \text{for external scalars.}$$

Fig. 2d. has a convergent contribution which is why λ does not enter in lowest order. The upshot is that for SU(3) non-singlets

$$\gamma^n_m (g) = \frac{g^2}{16\pi^2} \quad C_2 (R) \left[1 - \frac{2}{n(n+1)} \right] \qquad (m = 1 \ldots 8) \tag{1}$$

For SU(3) singlets we get a 2 x 2 anomalous dimension matrix

$$\underset{\approx}{\gamma}^n (g) = \frac{g^2}{16\pi^2} \begin{array}{cc} (o) & (9) \\ \left(\begin{array}{cc} C_2 (R) \left[1 - \frac{2}{n(n+1)} \right] & \frac{24}{\sqrt{6}} \, T(R) \left[1 + (-1)^n \right] \\ \frac{\sqrt{6}\, C_2(R)}{2n(n+1)} \left[1 + (-1)^n \right] & 12\, T(R) \end{array} \right) \end{array} \tag{2}$$

where

$$C_2 (R)\, I = \sum_a R^a R^a = \begin{cases} 4/3 & : SU(3) \\ 2 & : SU(2) \end{cases}$$

$$T(R)\, \delta^{ab} = \mathrm{Tr}\, [R^a R^b] = \begin{cases} 1/2 & : SU(3) \\ 2 & : SU(2) \end{cases}$$

Using the Callan-Symanzik equation and the Wilson light cone expansion it then follows that the Fourier transformed coefficients of O^n_k, $\tilde{C}^n_k (q^2)$, satisfy

$$D\, \tilde{C}^n_j (q^2) = \sum_k \gamma^n_{kj} \, \tilde{C}^n_k (q^2)$$

These equations can be solved by diagonalization of $\underset{\approx}{\gamma}^n$ with the result (writing $\tilde{C}^n_j$ as the elements of a vector $\underline{\tilde{C}}^n$):

$$\underline{\tilde{C}}^n = \sum_i c^n_i \, (-q^2)^{-g^n_i/2} \, \underline{e}^n_i$$

where g^n_i are the eigenvalues and $\underline{e}^n_i$ eigenvectors of $\underset{\approx}{\gamma}^{n\,T}$. In the ultra-violet

region we take g to have it's fixed point value g_F and we see that the SU(3) non-singlets are controlled by one power of $-q^2$ and the SU(3) singlets by two powers. Notice that in the limit $n \to \infty$ the anomalous dimensions approach fixed non-zero values:

$$\frac{g_F^2}{16\pi^2} \quad C_2 (R) \qquad \text{for SU(3) non-singlets} \tag{3}$$

$$\frac{g_F^2}{16\pi^2} \quad (C_2 (R),\ 12\, T(R)) \qquad \text{for SU(3) singlets} \tag{4}$$

It is this property of Yukawa theories which gives rise to the characteristic threshold behaviour discussed by Gross elsewhere in these Proceedings.

These statements may be translated directly into statements about the moments of the structure functions. The SU(3) non-singlet combinations of the currents are always controlled by one operator as we have seen and

$$\int d\xi \, \frac{1}{2\xi} \quad F_2^{ij} \, \xi^n \ \propto \ (-q^2)^{-\frac{1}{2}\gamma^{n+1}_m} \qquad (m = \ldots 8)$$

the same is true of the F_3^{ij} moments for all SU(3) quantum numbers including singlet combinations - this follows from G-invariance. Thus

$$\int d\xi \ F_3^{ij} \ \xi^n \ \propto \ (-q^2)^{-\frac{1}{2}\gamma^{n+1}_m} \qquad (m = 0 \ldots 8)$$

In both of these expressions γ^n_m is given in equation (1). The only complicated behaviour arises when we consider the moments of F_2 arising from SU(3) singlet combinations. In general they are controlled by two powers of $(-q^2)$ corresponding to the two eigenvalues of equation (2). The only exception is the n = 2 case where one of the eigenvalues vanishes. This occurs because one linear combination of the operators $O_0^{\mu_1 \mu_n}$, $O_9^{\mu_1 \mu_n}$ is the energy-momentum tensor whose conservation ensures that it has no anomalous dimensions. The known matrix element of this tensor enables us to derive a modified version of the Fritzsch Gell-Mann[7] pure quark sum rule:

$$6\int_0^1 d\xi\,(F_2^{ep}+F_2^{en}) - \int_0^1 d\xi\,(F_2^{\nu p}+F_2^{\nu n}) - \frac{4}{3}R + O(g_F^2)$$

$$\propto (-q^2)^{-\gamma^2_{(2)}/2}$$

where $\gamma^2_{(2)} = \dfrac{g_F^2}{16\pi^2}\left[12\,T(R) + \dfrac{2}{3}C_2(R)\right] \cong \dfrac{g_F^2}{16\pi^2}\begin{cases}6.9 : SU(3)\\ 25.3 : SU(2)\end{cases}$

is the non-zero eigenvalue of $\underset{\approx}{\chi}^2$.

$$R = 18\,T(R)\left[18\,T_2(R) + C_2(R)\right]^{-1} \qquad \begin{cases}27/31 : SU(3)\\ 9/10 : SU(2)\end{cases}$$

Conclusions

The characteristic power deviations from scaling of theories which are not asymptotically free should be detectable in the N.A.L. muon experiments. The Yukawa theories which we have considered have SU(3) non-singlet structure function moments varying as a power of $-q^2$, namely $(q^2)^{-p}$. The maximum value of p is determined from (3) to be

$$p = \frac{1}{2}\frac{g_F^2}{16\pi^2}C_2(R) = \frac{g_F^2}{16\pi^2}\begin{cases}\frac{2}{3} : SU(3)\\ 1 : SU(2)\end{cases}$$

(the moments of F_3, even the SU(3) singlet combination, are bounded by the same power variation). Thus these tests do not provide a very sensitive way of distinguishing between the two theories.

If we assume that the scaling deviations are something less than $(-q^2)^{-0.3}$, we should conclude from these tests that

$$\frac{\alpha_F}{\pi} = \frac{g_F^2}{4\pi^2} \lesssim \begin{cases}1.8 : SU(3)\\ 1.2 : SU(2)\end{cases}$$

On the other hand tests of these theories using SU(3) singlet combinations of the currents provide more stringent restrictions on α_F and distinguish more easily between the theories we are considering. It is easy to see that the largest of the two eigenvalues of $\underset{\approx}{\chi}^n$, given in (2), is bounded and that the maximum variation of SU(3) singlet moments is given by

$$p = \frac{1}{2}\frac{g_F^2}{16\pi^2}\left[12\,T(R) + C_2(R)\right] = \frac{g_F^2}{16\pi^2}\begin{cases}\frac{11}{3} : SU(3)\\ 13 : SU(2)\end{cases}$$

If $p < 0.3$ this gives

$$\frac{\alpha_F}{\pi} \lesssim \begin{cases}0.33 : SU(3)\\ 0.092 : SU(2)\end{cases}$$

Thus the outstanding question is whether the Yukawa theories we have been considering do in fact have fixed points satisfying these inequalities. The only way we know of to tackle this problem is to calculate the β function in two loop order. At this level we have a two coupling constant problem because we have to include a $\lambda\,\varphi^4$ interaction for renormalizability. (In fact for the SU(3) case we have to introduce a $h\varphi^3$ coupling which however is presumably negligble asymptotically[8]). Thus we are concerned to find simultaneous (non-trivial) zeros of β_g and β_λ. The calculation is straight forward but not yet completed: there are of course no problems connected with gauge invariance which have been referred to by other contributors.

We have restricted our attention in this article to three triplet models of the hadrons. However, the well known problem concerning the e^+e^- annihilation cross-section has made three quartet models much more popular since these yield a larger value for $R = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$. It is easy to generalize our results to the case when SU(4) is the classification symmetry. This makes the SU(3) singlet behaviour even more complicated since in this case there are three operators instead of two controlling the asymptotic behaviour. However, it is easy to see that the SU(3) non-singlet behaviour is unmodified so long as we continue to use SU(3) or SU(2) for the colour group. For this reason a safer but less rigorous test of our ideas is obtained from a study of the scaling deviations in SU(3) non-singlet combinations.

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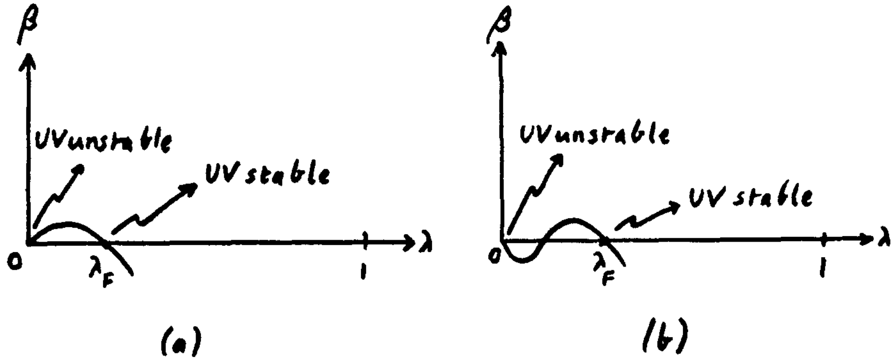

Fig. 1. Graphs of β having non-trivial ultra-violet stable fixed point "near" 0.

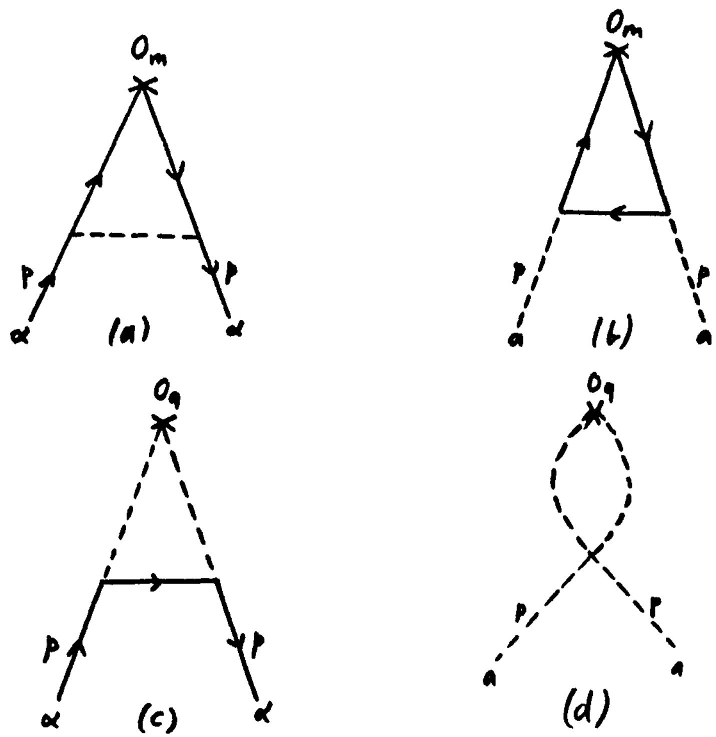

Fig. 2. Inserted diagrams for calculating anomalous dimensions.

"LEPTON PHYSICS AND GAUGE THEORIES"

Presented by

A. De RUJULA

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D. GROSS

(PRINCETON UNIVERSITY)

Professor D. Gross' communication has not reached us.

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58-67

QUARKS AND GAUGE FIELDS

G. 't Hooft *)
CERN -- Geneva

A B S T R A C T

Quark confinement in four space-time dimensions can be obtained in a simple model described by a local but not renormalizable Lagrangian. The interactions can be made such that the electric components of a gauge field are squeezed into vortex lines so that there are stringlike forces between quarks. The Lagrangian can perhaps be looked upon as an "effective" Lagrangian of the real world, and the model may give some insight in the dynamics of quark confinement.

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*) Address after August 1974 : Institute for Theoretical Physics, Sorbonnelaan 4, De Uithof, Utrecht, Netherlands.

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1) INTRODUCTION

To obtain a detailed understanding of the mechanism that keeps the quarks permanently bound is one of today's major problems in particle physics. It would be most satisfactory if the feature could be explained in terms of infra-red effects in a renormalizable field theory. And then we are naturally led to a non-Abelian gauge theory without Higgs-Kibble mechanism, for three reasons :

a) The theory contains massless fields interacting with each other and is therefore infra-red divergent. The absence of mass follows from the symmetry and needs not be put in by hand.

b) The theory is infra-red unstable and therefore the long-distance behaviour is not described by the classical limit of small coupling, and there need not be physical massless particles.

c) The theory contains vector fields. These could form vortex lines which behave like strings.

It is this latter point to which we want to focus attention. Vortex line solutions are known to exist in an Abelian theory with Higgs mechanism [1]. However, in that case, it is the magnetic field lines which are trapped in a vortex. If quarks are to sit at the end points of such a vortex, so that they will be permanently bound, then they have to be magnetic monopoles [2]. The quantum rules necessary to exclude exotic states are then not very elegant.

We think that it should be the electric (i.e., time-space) components of $F_{\mu\nu}$ that are trapped in a vortex. In that case the triality zero selection rule comes out more naturally, as we shall see. We asked ourselves whether electric vortices can occur in a classical Lagrangian field theory. The answer is yes, but the Lagrangian is not renormalizable. It deviates from a renormalizable one only for small values of certain fields. Since these fields have the dimension of a mass, this is a modification in the infra-red region. It is not inconceivable that higher order quantum corrections from zero mass particles give rise effectively to such modifications. This is why we call our Lagrangian an effective Lagrangian.

2) CONSTRUCTION OF THE LAGRANGIAN (ABELIAN CASE)

We can follow the guide of the renormalization group. We want to describe an asymptotically free, infra-red unstable theory. Let us assume that for momenta going to zero, the effective coupling goes to infinity :

$$\text{if} \quad k \to 0, \qquad \text{then} \quad g_{eff} \to \infty . \tag{2.1}$$

This we now take as an input. So from now on we can forget that the theory was non-Abelian, and first construct an effective Lagrangian with this property for Abelian gauge fields.

All charged particles will be represented by an external source function $J(x)$. Consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} Z F_{\mu\nu} F_{\mu\nu} - J_\mu A_\mu , \tag{2.2}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

and Z is just a constant. Of course one could scale Z out of the kinetic term :

$$A_\mu \to Z^{-1/2} A_\mu , \qquad \mathcal{L} \to -\tfrac{1}{4} F_{\mu\nu} F_{\mu\nu} - Z^{-1/2} J_\mu A_\mu . \tag{2.3}$$

So we see that the interactions are proportional to Z^{-1}. Therefore we want :

$$\text{if} \quad k \to 0 , \qquad \text{then} \quad Z \to 0 . \tag{2.4}$$

Now if we took Z to be momentum dependent then we would have a non-local Lagrangian, and (in the non-Abelian case) gauge invariance would be destroyed. The trick is to let Z depend on an auxiliary scalar field φ with the dimension of a mass :

$$\text{if} \quad \varphi \to 0, \qquad \text{then} \quad Z(\varphi) \to 0. \tag{2.5}$$

Thus we are led to consider the following Lagrangian,

$$\mathcal{L}(A,\varphi) = -\tfrac{1}{4}Z(\varphi)F_{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi) - J_\mu A_\mu . \tag{2.6}$$

By definition we shall require

$$\begin{aligned} V(\varphi) &> 0 \quad \text{if} \quad \varphi \neq 0 , \\ V(0) &= 0 . \end{aligned} \tag{2.7}$$

In practice one can take

$$V(\varphi) = \tfrac{1}{2}m^2\varphi^2 + \frac{\lambda}{24}\varphi^4 . \tag{2.8}$$

(there is no reason to require symmetry breaking here). And further

$$\begin{aligned} Z(\varphi) &\to 1 \quad \text{for large } |\varphi| , \\ Z(\varphi) &\to 0 \quad \text{for } \varphi \to 0 . \end{aligned} \tag{2.9}$$

The exact form of Z for small values of φ shall be left open for the time being. For simplicity we take Z and V both to be increasing functions of φ^2.

3) WHAT HAPPENS TO COULOMB'S LAW

From the Lagrangian (2.6) we derive the Lagrange equation,

$$\partial_\mu E_{\mu\nu} = J_\nu , \tag{3.1}$$

where $E_{\mu\nu}$ is the induction field,

$$E_{\mu\nu} = Z(\varphi)F_{\mu\nu} . \tag{3.2}$$

It is convenient to take $E_{\mu\nu}$ as a separate variable and write

$$\begin{aligned} \mathcal{L}(E,A,\varphi) &= \\ &= \frac{1}{4Z(\varphi)}E_{\mu\nu}E_{\mu\nu} - \tfrac{1}{2}E_{\mu\nu}F_{\mu\nu} - \tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi) - J_\mu A_\mu \\ &= \frac{1}{4Z(\varphi)}E_{\mu\nu}E_{\mu\nu} - \tfrac{1}{2}(\partial_\mu\varphi)^2 - V(\varphi) + A_\nu(\partial_\mu E_{\mu\nu} - J_\nu) \\ &\quad + \text{total derivative.} \end{aligned} \tag{3.3}$$

Variation with respect to $E_{\mu\nu}$ gives (3.2) and the original Lagrangian (2.6). Now the vector potential acts as a Lagrange multiplier.

Suppose now that the charged particles are at rest and we are interested in the stationary solution :

$$\begin{aligned} J_i &= 0 , \qquad i,j = 1,2,3 , \\ J_4(x) &= i\rho(\vec{x}) , \\ E_{ij} &= 0 , \\ E_{i4} &= iE_i . \end{aligned} \tag{3.4}$$

Then the Lagrangian equals minus the energy density, which is

$$\begin{aligned} \mathcal{H} &= \frac{E_i^2}{2Z(\varphi)} + \tfrac{1}{2}(\partial_i\varphi)^2 + V(\varphi) ; \\ \partial_i E_i &= \rho(\vec{x}) . \end{aligned} \tag{3.5}$$

The field configuration is found by requiring the total energy to be a minimum under the additional condition $\partial_i E_i = \rho$.

Since we want to find the Coulomb force between two charges far apart, we are mainly interested in fields $\vec{E}$ that are nearly constant as a function of space also. In that case the derivative term for the φ field is unimportant. The field φ is then determined by the requirement that

$$\frac{\frac{1}{2}E^2}{Z(\varphi)} + V(\varphi) \tag{3.6}$$

be minimal, so finally all that is relevant is the relation between Z and V. Suppose that for $\varphi \to 0$,

$$Z = C \cdot V^{\alpha} \quad ; \qquad C > 0 \qquad \alpha > 0 . \tag{3.7}$$

(so if we take $V = \frac{1}{2}m^2\varphi^2 + \mathcal{O}(\varphi^4)$ then $Z(\varphi) \to C(\frac{1}{2}m^2\varphi^2)^{\alpha}$)

(3.6) is minimal if

$$[V(\varphi)]^{\alpha+1} = \frac{\alpha}{2C} E^2 . \tag{3.8}$$

So the energy density for small values of E is

$$\mathcal{H}(E) = \left(1 + \frac{1}{\alpha}\right)\left(\frac{\alpha E^2}{2C}\right)^{\frac{1}{1+\alpha}} = C'|E|^{\frac{2}{1+\alpha}} . \tag{3.9}$$

For large values of E we have

$$\mathcal{H}(E) = \frac{1}{2}E^2 , \tag{3.10}$$

so we have roughly,

$$\mathcal{H}(E) = C'|E|^{\frac{2}{1+\alpha}} + \frac{1}{2}|E|^2 \tag{3.11}$$

Now consider a flux line going from particle A to particle B carrying a small flux Φ [Fig. 1. Remember that due to (3.1) we have flux conservation]. Let it have a cross-section Ω. The electric induction is then

$$E = \Phi / \Omega , \tag{3.12}$$

and the energy U per unit of length,

$$U = \Omega \mathcal{H} = \Omega\left(C'\left|\frac{\Phi}{\Omega}\right|^{\frac{2}{1+\alpha}} + \frac{1}{2}\left|\frac{\Phi}{\Omega}\right|^2\right) . \tag{3.13}$$

In Fig. 2 the function $U(\Omega)$ is sketched in three cases : $\alpha > 1$, $\alpha = 1$ and $\alpha < 1$.

In the case $\alpha < 1$ the flux lines tend to take as large a volume as possible and Ω will be of the order r^2 where r is the distance between the particles. We get a power law for the potential $V(r)$:

$$V(r) \propto r\, U(r^2) \propto r^{\frac{3\alpha - 1}{1+\alpha}} \qquad (\alpha < 1) . \tag{3.14}$$

leading to quark confinement if $\alpha > 1/3$.

If $\alpha = 1$ the energy of a flux line will become independent of its width and just proportional to its length. We get

$$V(r) \propto r \qquad (\alpha = 1) . \tag{3.15}$$

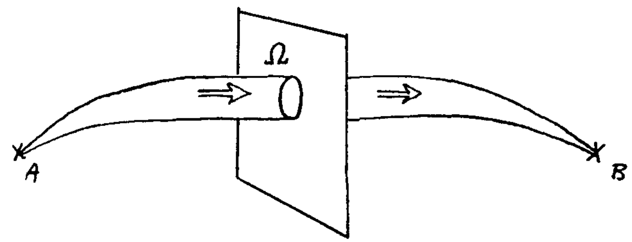

Fig. 1

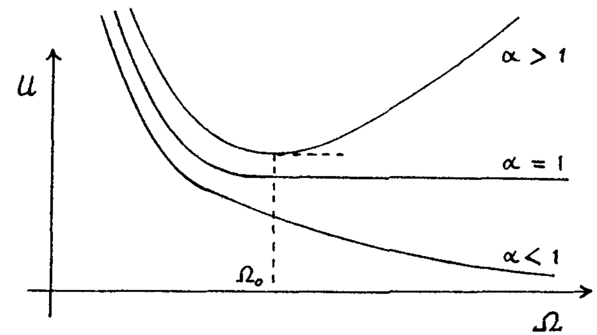

Fig. 2

If $\alpha > 1$ then the flux lines will not spread further than the cross-section Ω_o where the energy is minimal (Fig. 2). The whole field E will be confined in a tube with a definite width and arbitrary length. These are the electric vortices meant in the beginning. They will behave exactly as the dual string. So here also

$$V(r) \propto r \qquad (\alpha > 1) . \tag{3.16}$$

The success of the dual models in describing the phenomenology of the strong interactions indicates that most likely $\alpha > 1$.

Let us now turn back to the term $(\partial_i \varphi)^2$ in Eq. (3.5). It does not change our arguments. But in the case $\alpha > 1$ it is the only term that raises the energy if we try to split the tube. If it would be absent then the field would drop sharply to zero at the edge of a tube. So its task is to make the vortex soft at the edge, and continuous inside.

We observe from the solution that, in fact, a symmetry breaking mechanism takes place : the gauge field term in the Hamiltonian (3.5) forces φ to become non-zero if a vector field is present. In other regions $\varphi = 0$, and there the gauge field cannot penetrate. Thus our theory resembles the bag theory [3] : inside we have $\varphi \neq 0$, gauge fields present, and outside $\varphi \to 0$. But we have a soft bag : everything is continuous. The hard bag [3] would correspond to neglecting the kinetic term for φ and taking $Z(V)$ to be a step function $(\alpha = \infty)$. Charged particles are automatically confined to the bag because of the field they drag along.

4) THE NON-ABELIAN CASE

In the non-Abelian case the induction field $E^a_{\mu\nu}$ satisfies the equation

$$D^{ab}_\mu E^b_{\mu\nu} = J^a_\nu , \tag{4.1}$$

where D^{ab}_μ is the covariant derivative. So flux is no longer conserved, and we cannot extend our classical solution to this case : there are no classical, stable vortex lines. Nevertheless we believe that also this theory will have vortex lines if $\alpha > 1$. The argument goes as follows.

Let us take one isospin direction, say the 8 direction in SU(3) space, and consider the electric field $\partial_\mu A^8_\nu - \partial_\nu A^8_\mu$ separately. All other fields, i.e., the charged Fermi particles and the charged gauge vector particles, are described by the source J_μ. The difficulty we had in the beginning can now be formulated as follows : given a pair of charged particles with an electric vortex line in between. Then the charged gauge bosons can be created in pairs and thus neutralize the electric field, and the vortex will fall into pieces.

However, the quarks have charges 1/3, 1/3 and -2/3 with respect to this colour electric field. The gauge bosons, on the other hand, have charges 0, and ± 1 only. In Fig. 3a we depicted what we expect to happen if a pair of gauge particles tends to neutralize a vortex between a single quark-antiquark pair. In Fig. 3b we see that three vortex lines can be eliminated if they are parallel. Only colourless, triality zero states have no vortex lines emerging.

a)

b)

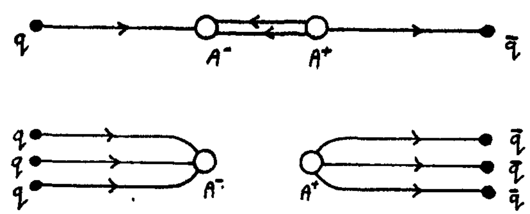

Fig. 3

After completion of these notes the author became aware of the work of Kogut and Susskind [4], who describe essentially the same model.

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68-77

Two Loop Diagrams in Yang-Mills Theory

D.R.T. Jones

Department of Theoretical Physics
University of Oxford

ABSTRACT

A calculation of the renormalisation constants of the Yang-Mills field to $O(g^4)$ is presented. The function $\beta(g)$ is hence evaluated to $O(g^5)$ and possible implications for gauge theories of the strong interactions discussed.

Introduction

Great interest has been aroused recently in non-abelian gauge theories of the strong interactions because of the observation[1] that such theories can exhibit free-field asymptotic behaviour at large Euclidean momenta, hence providing a natural field theoretic framework for the explanation of the scaling phenomena observed in electroproduction and neutrino-production experiments.[2)-4)] However, it seems that the introduction of scalar mesons with non-vanishing vacuum expectation values to give masses to the gauge fields destroys the free field behaviour, because of the quartic interactions inevitably present in a renormalisable theory.[2)] This observation led to the conjecture[2),5)] that the strong gauge symmetry is in fact exact, and that the absence of massless vector bosons is associated with the infra-red divergences involved in such a theory.

Asymptotic freedom for non-abelian theories is a consequence of the behaviour near the origin of the calculable function $\beta(g)$; in view of the above conjecture its behaviour away from the origin is also of interest; a zero of β for some finite value of g would correspond to an infra-red stable fixed point, hence rendering the conjecture suspect. In the absence of non-perturbative approaches a two-loop calculation of β is therefore of interest, and is performed here, using the dimensional regularisation technique of 't Hooft and Veltman[6)].

We find that for the favoured model (SU(3) with three fermion triplets) the second term in β has the same sign as the first, suggesting that the domain of attraction of the origin is large.

2. Calculation of the Renormalisation Constants

The Lagrangian we shall consider consists of a set of gauge fields $W_\mu^a(x)$ and a multiplet of spin one-half fields $\Psi^i(x)$, and possesses local gauge invariance with respect to a compact semi-simple Lie group G, of dimension r. The bare Lagrangian is therefore

$$\mathcal{L} = -\frac{1}{4}\tilde{G}^a_{\mu\nu}\tilde{G}^a_{\mu\nu} - \bar{\tilde{\Psi}}\left(\gamma_\mu D_\mu + \tilde{M}\right)\tilde{\Psi} \tag{1}$$

where

$$\tilde{G}^a_{\mu\nu} = \partial_\mu \tilde{W}^a_\nu - \partial_\nu \tilde{W}^a_\mu + \tilde{g} f^{abc} \tilde{W}^b_\mu \tilde{W}^c_\nu \tag{2}$$

and

$$(D_\mu \tilde{\Psi})^i = \partial_\mu \tilde{\Psi}^i + i\tilde{g}(R^a)^{ij}\tilde{\Psi}^j \tilde{W}^a_\mu \tag{3}$$

($\sim$ means unrenormalised)

Here f^{abc} (a: 1,2...r) are the real, totally antisymmetric structure constants of G; $(R^a)_{ij}$ is the matrix representation of the a'th generator of G on the fermion multiplet. (i = 1,2...d(R), where d(R) is the dimension of the representation). We assume that G is non-chiral, to avoid difficulty with Adler anomalies.

In terms of renormalised quantities the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4} Z_3 \left(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu\right)^2 - \frac{1}{2\alpha}\left(\partial_\mu W^a_\mu\right)^2 \\ & - Z_1 g f^{abc} \partial_\mu W^a_\nu W^b_\mu W^c_\nu - \frac{1}{4} Z_1^2/Z_3\, g^2 f^{abc} f^{ade} W^b_\mu W^c_\nu W^d_\mu W^e_\nu \\ & - Z_3^F \partial_\mu \phi^{a*} \partial_\mu \phi^a - Z_1^F g f^{abc} \partial_\mu \phi^{a*} W^b_\mu \phi^c \ldots\end{aligned}$$

$$-Z_2^M\,\overline{\Psi}(\gamma_\mu\partial_\mu + M)\Psi \;-\; iZ_1^M g\,\overline{\Psi}\gamma_\mu R^a\Psi W_\mu^a$$

$$-Z_2^M(Z_M - 1)\,\overline{\Psi}\Psi M \tag{4}$$

where we have added a gauge term and a ghost Lagrangian in the usual way. The renormalised quantities g, α and M are given by

$$\tilde{g} = Z_1/Z_3^{3/2}\, g \tag{5}$$

$$\tilde{M} = Z_M M \tag{6}$$

$$\tilde{\alpha} = Z_3\,\alpha \tag{7}$$

(Note that the longitudinal part of the W propagator is unrenormalised). We will work throughout in the renormalised Feynman gauge ($\alpha = 1$).

The function $\beta(g)$, as we shall see in the next section, is related to the renormalisation of the coupling constant g. We choose to calculate this to $O(g^4)$ by considering renormalisation of the W∅∅ vertex, and therefore calculate only Z_3, Z_3^F and Z_1^F to $O(g^4)$. Clearly the calculation of Z_1 would provide a non-trivial check on the algebra through the Slavnov-Taylor[7] identity:—

$$Z_1/Z_3 = Z_1^F/Z_3^F \tag{8}$$

This calculation has in fact been performed by W. Caswell[8], with results in agreement with those presented here.

We use the metric conventions and Feynman rules as given by

't Hooft[9]; and since the Feynman integrals are to be performed in n dimensions it is convenient to define a new coupling constant u by

$$g = m^{\epsilon/2}\, u \tag{9}$$

where $\epsilon = 4-n$ and m is an arbitrary mass.

The subtraction constants (Z_1-1), (Z_3-1) etc. are chosen (following 't Hooft[10]) to contain only inverse powers of ϵ in an expansion about $\epsilon = 0$. It follows[10),11)] that the subtraction constants are mass independent; accordingly we set $M = 0$ throughout.

For details of how the pole terms are extracted from the Feynman integrals see reference (12); here we present only the results.

From one-loop diagrams we obtain

$$\begin{aligned}
(Z_3-1)^{(1)} &= \left\{\frac{10}{3}C_2(G) - \frac{8}{3}T(R)\right\}\frac{u^2}{16\pi^2\epsilon}\\
(Z_3^F-1)^{(1)} &= C_2(G)\,\frac{u^2}{16\pi^2\epsilon}\\
(Z_2^M-1)^{(1)} &= -2C_2(R)\,u^2/16\pi^2\epsilon\\
(Z_1-1)^{(1)} &= \left\{\frac{4}{3}C_2(G) - \frac{8}{3}T(R)\right\}\frac{u^2}{16\pi^2\epsilon}\\
(Z_1^F-1)^{(1)} &= -C_2(G)\,u^2/16\pi^2\epsilon\\
(Z_1^M-1)^{(1)} &= -2\left\{C_2(R) + C_2(G)\right\}u^2/16\pi^2\epsilon
\end{aligned} \tag{10}$$

where

$$
\begin{aligned}
C_2(G)\,\delta^{ab} &= f^{acd} f^{bcd} \\
C_2(R)\, I &= R^a R^a \\
T(R)\,\delta^{ab} &= \mathrm{Tr}\left[R^a R^b\right]
\end{aligned}
\tag{11}
$$

Using these results we obtain from two-loop diagrams

$$
\left(Z_3 - 1\right)^{(2)} = \left(\frac{u^2}{16\pi^2}\right)^2 \left[\left(-\frac{25}{3\epsilon^2} + \frac{23}{4\epsilon}\right) C_2^2(G) - \frac{4}{\epsilon} C_2(R)\,T(R) + \left(\frac{60}{9\epsilon^2} - \frac{5}{\epsilon}\right) C_2(G)\,T(R)\right]
$$

$$
\left(Z_3^F - 1\right)^{(2)} = \left(\frac{u^2}{16\pi^2}\right)^2 \left[\left(\frac{49}{24\epsilon} - \frac{4}{\epsilon^2}\right) C_2^2(G) + \left(\frac{2}{\epsilon^2} - \frac{5}{6\epsilon}\right) C_2(G)\,T(R)\right]
$$

$$
\left(Z_1^F - 1\right)^{(2)} = \left(\frac{u^2}{16\pi^2}\right)^2 \left(\frac{5}{2\epsilon^2} - \frac{3}{4\epsilon}\right) C_2^2(G) \tag{12}
$$

3. <u>Calculation of β</u>

The renormalisation group equation for a single-particle irreducible Green's function $\Gamma(p_i; u, m, \alpha, M)$ is

$$
\left[m\frac{\partial}{\partial m} + \beta(u)\frac{\partial}{\partial u} - \gamma_M(u)\, M\frac{\partial}{\partial M} + \delta(u,\alpha)\frac{\partial}{\partial \alpha} - \gamma_\Gamma(u,\alpha)\right]\Gamma = 0 \tag{13}
$$

where

$$
\beta(u) = m\frac{\partial u}{\partial m}\bigg|_{\hat g, \hat\alpha, \hat M \text{ const.}} \tag{14}
$$

$$
\gamma_M(u) = m\frac{\partial}{\partial m} \ln Z_M \bigg|_{\hat g, \hat\alpha, \hat M \text{ const.}} \tag{15}
$$

Note that we have written β and γ_M as functions of u only (and not α). For discussion of this point see references (12) and (13).

The solution of equation (13) is well known :—

$$
\Gamma(\lambda p, u, M, m, \alpha) = \Gamma(p, \bar u, \bar M, m, \bar\alpha)\, \lambda^{\delta_\Gamma} \otimes \exp - \int_0^t \gamma_\Gamma\left(\bar u(x), \bar\alpha(x)\right) dx \tag{16}
$$

where $t = \ln\lambda$ and

$$
\frac{d\bar u}{dt} = \beta(\bar u) \quad ; \quad \bar u(0) = u \tag{17}
$$

$$
\frac{d\bar M}{dt} = -\left\{1 + \gamma_M(\bar u)\right\}\bar M \quad ; \quad \bar M(0) = M \tag{18}
$$

$$
\frac{d\bar\alpha}{dt} = \delta(\bar u, \bar\alpha) \quad ; \quad \bar\alpha(0) = \alpha \tag{19}
$$

(δ_Γ is the mass dimension of Γ).

(16) expresses the fact that rescaling the external momenta is equivalent to changing the coupling constant, mass, and the gauge. (17) shows that the zeroes of β control the behaviour of the effective coupling constant $\bar u$ in the high or low momentum limits.

From equations (5), (8), (9) and (14) we find

$$\beta(u) = \frac{-\frac{1}{2}\epsilon u}{1 + u^2 \frac{d}{du^2} \ln\left\{\frac{1}{Z_3}\left(Z_1^F / Z_3^F\right)^2\right\}} \tag{20}$$

Substituting from (10) and (12) we find, in the limit $\epsilon \to 0$

$$\beta(u) = \frac{Au^3}{16\pi^2} + \frac{Bu^5}{(16\pi^2)^2} + O(u^7) \tag{21}$$

where

$$A = 4/_3\, T(R) - 11/_3\, C_2(G)$$

$$B = 20/_3\, C_2(G)T(R) + 4C_2(R)T(R) - \frac{34}{3} C_2^2(G)$$

4. Discussion

Let us consider the case SU(N), with f multiplets of fermions in (a) the adjoint and (b) the vector representation.

For case (a) we have

$$C_2(G) = C_2(R) = N\ ; \quad T(R) = fN \tag{22}$$

Thus

$$A = \frac{4f-11}{3} N$$

$$B = \frac{2}{3}(16f-17)N^2 \tag{23}$$

For f $\geqslant$ 3, A $>$ 0 so that asymptotic freedom is lost. This model (the particular case f = 3, N = 2) was proposed by Bailin and Love[14] as a possible model of strong interactions, with the

observation that if β had a zero sufficiently near the origin the power violations of scaling predicted might still be compatible with the data. The expression (23) for β clearly does not support this conjecture.

For case (b) we have

$$C_2(G) = N\ ; \quad C_2(R) = \frac{N^2-1}{2N}\ ; \quad T(R) = f/_2\ . \tag{24}$$

$$A = 2f/_3 - 11/_3\, N$$

$$B = \frac{13}{3} Nf - f/_N - 34/_3\, N^2 \tag{25}$$

$$\therefore \text{ for } \frac{34N^3}{13N^2-3} \leqslant f \leqslant \frac{11}{2} N$$

we have A $<$ 0 and B $>$ 0, and the possibility of a theory with calculable infra-red and ultra-violet behaviour. The number of fermions required is too large for strong interaction models, however - for example in SU(3) we would require 15 or 16 triplets to give a zero at a believably small value of the expansion parameter.

For the case SU(3), with three (or four) fermion triplets we have A = -9 ($\frac{-25}{3}$) and B = -64 ($\frac{-154}{3}$). For these cases, therefore, the domain of attraction of the origin is clearly large; we may even consider the result as supporting to some extent the conjecture that the strong gauge group is unbroken.

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MIXING OF OPERATORS IN WILSON EXPANSIONS

S. Sarkar
CERN -- Geneva

ABSTRACT

The tree-loop theorem of 't Hooft and Veltman is used in Fermi type gauges to show that gauge invariant operators in Wilson expansions mix in general with non-gauge invariant operators. Background field gauges are proposed in which this does not occur. Calculations of anomalous dimensions in these gauges are discussed.

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1. INTRODUCTION

Recently field theories have been discovered which at large momenta behave as if they were free[1]. More precisely, in the language of the renormalization group, the effective coupling constant moves towards zero as the momentum increases. In applying these theories to deep inelastic phenomena it is necessary to consider currents J and in particular the short distance behaviour of the product of two currents. When the Wilson[2] or light-cone expansion

$$J(x)\,J(-x) \underset{x^2 \approx 0}{\sim} \sum_j C^{(j)}(x^2, g)\, x^{\mu_1} \ldots x^{\mu_j} R^{(j)}_{\mu_1 \ldots \mu_j}(0) \tag{1.1}$$

is written, where for convenience we have suppressed all Lorentz and group indices of the J's, we have to consider all operators $R^{(j)}_{\mu_1 \ldots \mu_j}(0)$ which mix with each other under renormalization. The problem of mixing is present in all field theories. In ϕ^4 theory, for example, if we are interested in the Green functions

$$\langle 0|\, T(\phi^4(x)\, \phi(x_1)\, \phi(x_2) \cdots \phi(x_r)\,|0\rangle \tag{1.2}$$

which are generated by adding to the Lagrangian a source term $K(x)\phi^4(x)$, we must consider counter-terms linear in K but proportional to operators such as $(\partial_\mu \phi)^2$ which are different from ϕ^4. This is basically all that the phenomenon of mixing is. Non-Abelian gauge theories are the only theories which can be asymptotically free. They are much more complicated than ϕ^4 theory because of gauge invariance.

In order to be specific we shall first consider the action[3] S

$$S[A] = -\tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar\psi\, i\gamma^\mu (\partial_\mu + g\,\sigma^a A^a_\mu)\,\psi - \chi^\mu \chi^\lambda F^a_{\nu\mu} F^a_{\nu\lambda} \tag{1.3}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu$; A^a_μ is the gauge field; g is the bare gauge coulping constant; χ^μ, χ^λ are c-number sources such that $\chi^\mu \chi_\mu$ vanishes; ψ are quark fields transforming under a representation matrix σ of the (colour gauge) group; and f_{abc} are the structure constants of the gauge group. [Moreover, the supercondensed notation[4] is used for which indices are supposed implicitly to carry coordinate labels. In particular, summation over indices implies space-time integrations as well (in an obvious way).]

The reason for considering initially the operator $F^a_{\nu\mu} F^a_{\nu\lambda}$ is that it is the simplest operator of twist two which is invariant under the bare gauge transformation

$$A^{\mu}_a(x) = A^{\mu\prime}_a(x) + g\, f_{acb} A^c_\mu(x) \Lambda^b(x) - \partial_\mu \Lambda_a(x) \tag{1.4}$$

with Λ_a a c-number infinitesimal group transformation parameter. At short distances it is the lowest twist operators that dominate. There are of course other operators of twist two, e.g.

$$R^{(s)} = i^{s-2} F^a_{\mu_1\nu}\, D_{\mu_2} \cdots\cdots D_{\mu_{s-1}} F^a_{\nu\mu_s}\, \chi^{\mu_1} \chi^{\mu_2} \cdots \chi^{\mu_s} \tag{1.5}$$

where D_μ is a covariant derivative. In general we know that operators of the same twist and Lorentz structure will mix. We will find some differences in the mixing of operators with s larger than two and the operators with s equal to two. The action given in (1.3) as it stands cannot be used without specifying a gauge fixing term $C_\alpha(A)$ and the associated Faddeev-Popov Lagrangian.

The most straightforward way of seeing which operators mix with $F^a_{\nu\mu} F^a_{\nu\lambda} \chi^\mu \chi^\lambda$ is just to insert it into various Green functions and obtain counter-terms[5] of the form

$$\begin{aligned}
& U\, \partial_\alpha A^\alpha_a A^\beta_b A^\gamma_c f^{abc} \chi_\beta \chi_\gamma \\
&+ V\, \chi^\beta \chi^\gamma A^\alpha_a A^\beta_b A^\gamma_c A^\alpha_d\, \delta_{ab} \delta_{cd} \\
&+ W\, \chi^\beta \chi^\gamma A^\alpha_a A^\beta_b A^\alpha_a A^\gamma_c f^{abe} f^{cde} \\
&+ X\, \partial_\rho c^\dagger_c f^{aes} f^{skd} f^{cde} \chi^\alpha \chi^\rho A^\alpha_a c_k \\
&+ Z\, \chi^\alpha \partial_\alpha A^\mu_a \chi^\beta \partial_\beta A^\mu_a \\
&+ Y\, c^\dagger_c f^{ekd} f^{cde} \chi^\beta \chi^\rho \partial_\beta \partial_\rho c_k
\end{aligned} \tag{1.6}$$

+ possible permutations of indices (where c_a denotes the Faddeev-Popov field, and U, V, W, X, Y, and Z are divergent constants).

In this note we will show how invariance arguments[6,7] can tell us the connection between the above constants, at least at the one-loop level. Taylor[8] has recently adopted a similar philosophy but we are unable to follow his methods. The formalism and techniques that we adopt are those of Ref. 6. We find in Fermi-type gauges for s=2 that the Lagrangian containing one-loop counter-terms is invariant under a χ-dependent gauge transformation, and in particular the Faddeev-Popov mixing is nothing other than the consequent χ-dependence of the Faddeev-Popov term arising from this gauge transformation. For higher s we find that the Lagrangian in general is invariant under a transformation of A which is non-linear. The gauge invariance of the Lagrangian then becomes more difficult to disentangle. Since the one-loop Lagrangian is invariant under some transformation and the bare Lagrangian is invariant under a different transformation (namely the bare gauge transformation), the difference (i.e. the set of one-loop counter-terms) is not invariant under either of these transformations. Consequently there is much mixing between gauge-invariant and non-gauge-invariant operators. We will discuss, however, a class of gauges which do not suffer from such problems; these are the background field gauges[9]. Gauge-invariant operators mix only with gauge-invariant operators in such gauges.

In Section 2 we shall give the invariance arguments in Fermi-type gauges. In Section 3 we will introduce the background field gauge and discuss its consequences.

2. FERMI-TYPE GAUGES

We will consider a bare Lagrangian $\mathscr{L}_0$ invariant under the gauge transformation

$$A^{\mu}_{a} = A'^{\mu}_{a} + g\,\hat{s}^{\mu}_{ab}(A')\Lambda_b + \hat{t}^{\mu}_{ab}\Lambda_b \qquad (2.1)$$

where Λ is an infinitesimal parameter, $\hat{s}$ is field-dependent but $\hat{t}$ is not. We must also choose a gauge-fixing function $C_a(A)$ for $\mathscr{L}_0$. Under (2.1) we have

$$C_a(A) = C_a(A') + g\,\hat{l}_{ab}(A')\Lambda_b + \hat{m}_{ab}\Lambda_b \qquad (2.2)$$

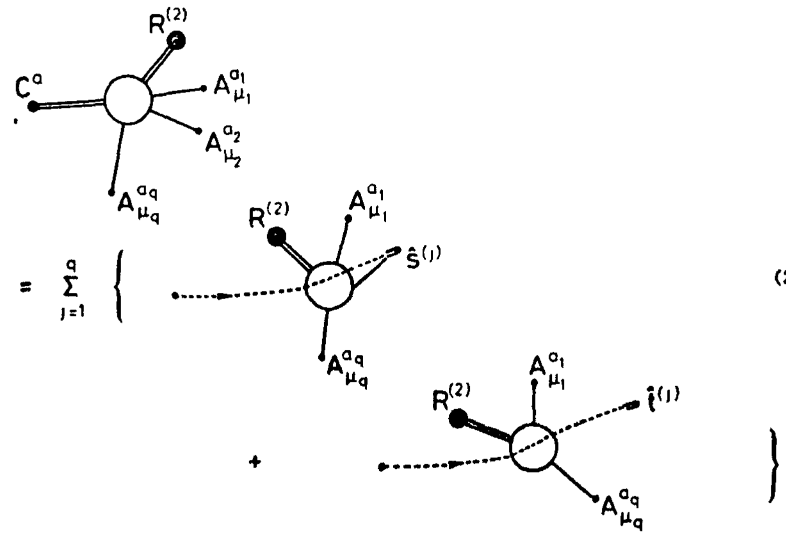
When $C_a(A)$ is $(1/\sqrt{\xi})\,\partial_\mu A^a_\mu$, ξ being the gauge-fixing parameter,

$$\hat{s}^{\mu}_{ab} = f_{acb}\,\delta_{\mu\nu}\,A^c_\nu$$

and

$$\hat{t}^{\mu}_{ab} = -\delta_{ab}\,\partial_\mu$$

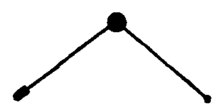
Taking (1.3) for $\mathscr{L}_0$, regularized Ward identities can be written down in n-dimensions. Diagrammatically they are represented as



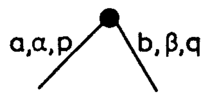
(2.3)

Bold lines denote vector mesons and dotted lines ghosts. These diagrams often seem to cause confusion and so we shall explain their meaning using simple examples. We would first like to note that (2.3) is valid for the total amplitudes on the left- and the right-hand side. In particular it is true at the tree diagram level and at the one-loop level. Since we shall only be interested in the one-loop counter-terms of the theory, this is all that we shall need. Below we shall omit fermion lines because they are not essential to the argument.

We will first consider q=1 and the tree approximation; (2.3) then becomes just

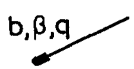
 $= 0$ (2.4)

where the central vertex is



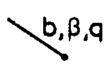
$$-2\,\delta_{ab}\,\big(p\cdot q\,\chi^\alpha\chi^\beta-\chi\cdot q\,\chi^\alpha p^\beta-\chi\cdot p\,q^\alpha\chi^\beta+\chi p\,\chi\cdot q\,\delta^{\alpha\beta}\big) \qquad (2.5)$$

the left vertex is



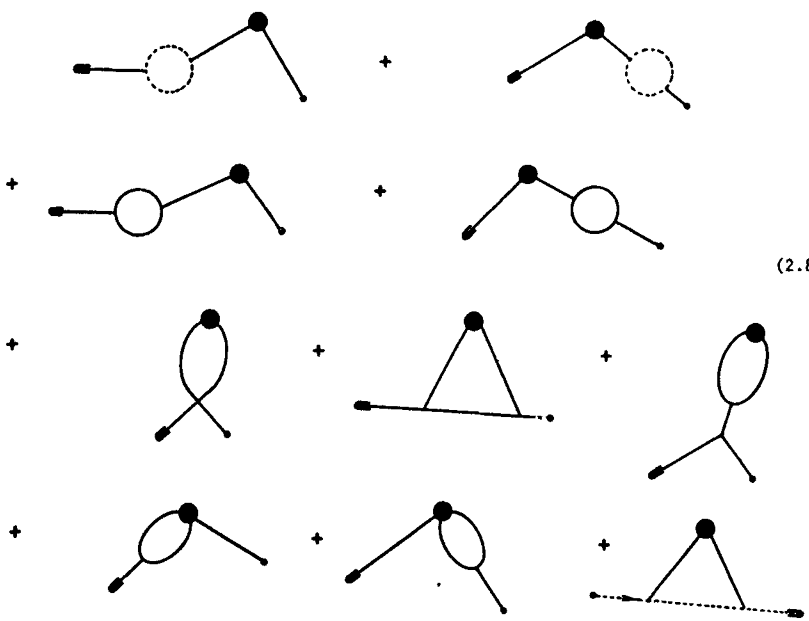
$$-\,i\,q_\mu\,\delta_{\beta\mu} \qquad (2.6)$$

and the right vertex is

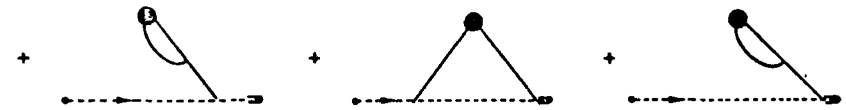


$$\delta_{a,b}\,\delta_{\beta\mu,} \qquad (2.7)$$

(Our conventions are such that all momenta are taken pointing into the vertex; p and q denote momenta; α,β Lorentz indices; and a,b group indices.) In order to understand the conventions for the right-hand side of (2.3) we will consider the one-loop version of (2.3)

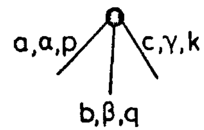


(2.8)



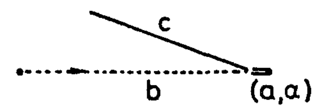
$= 0$

In (2.8) the new vertices that appear (apart from the standard vertices of $\mathscr{L}_0$) are

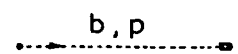


$$2igf^{abc}\big(\chi^\alpha\chi^\gamma(-p^\beta+k^\beta)+\chi^\alpha\chi^\beta(p^\gamma-q^\gamma)+\chi^\beta\chi^\gamma(q^\alpha-k^\alpha)+\chi^\gamma\,\chi\cdot(p-q)\,\delta^{\alpha\beta}+\chi^\beta\,\delta^{\alpha\gamma}\,\chi\cdot(k-p)+\chi^\alpha\,\delta^{\beta\gamma}\,\chi\cdot(q-k)\big) \qquad (2.9)$$

and the non-Lagrangian vertices



$$g\,f^{abc} \qquad (2.10)$$

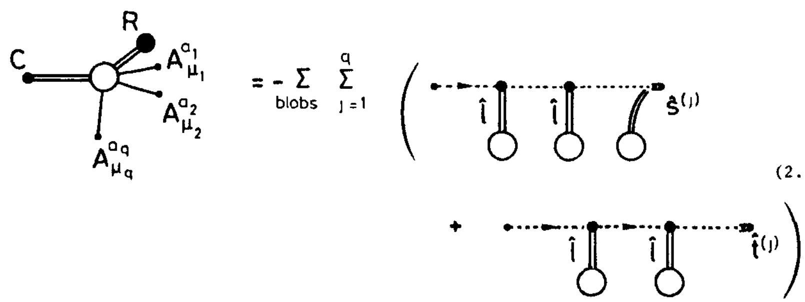


$$i\,\delta_{ab}\,p_\alpha \qquad (2.11)$$

One method of neatly deriving the Ward identities is by using the path integral representation of the generating functional and regarding a gauge transformation simply as a change of variables of the path integral[7,10]. The identities obtained in this way are a general expression of the gauge invariance of (1.3); however, our problem is to learn something not about the Lagrangian in (1.3) but about the Lagrangian obtained from it by adding the counter-terms (1.6), $\mathscr{L}_1$ say. The counter-term Lagrangian is defined by saying that it is the sum of terms required to cancel the poles at n=4 of one-loop diagrams of (1.3). Here n is the continued dimension of space-time required in dimensional regularization.

So far we have -- knowing the invariance properties of a Lagrangian -- derived identities between Feynman diagrams. We would like to invert this procedure and ask whether we can deduce the invariance properties of a Lagrangian if we know something about identities between its diagrams. Such an inversion is provided by the tree-loop theorem of 't Hooft and Veltman[6]. This is not to be confused with another theorem of the same name due to Feynman.

Equation (2.3) when written in terms of tree diagrams is given by



(2.12)

This is a simple consequence of the fact that the ghost interaction vertex is given by $c^{\dagger}_a \ell_{ab}(a) c_b$. On the right-hand side of (2.12) we have intentionally not explicitly marked the R and A legs. These are distributed in all possible ways in the blobs.

We shall now state the tree-loop theorem:

Given

i) the existence of functions C'_a as well as $\hat{s}'$, $\hat{t}'$, $\hat{\ell}'$, and $\hat{m}'$ such that

$$C'_a(A_b + g\hat{s}'_{bc}(A)\Lambda_c + \hat{t}'_{bc}\Lambda_c) = C'_a(A) + g\hat{\ell}'_{ab}(A)\Lambda_b + \hat{m}'_{ab}\Lambda_b + O(\Lambda^2)$$

and that

ii) the equation (2.12) is satisfied with C replaced by C', $\hat{\ell}$ replaced by $\hat{\ell}'$, and the ghost propagator by $-\hat{m}^{-1}$,

and, provided all the hierarchy of identities expressed by (2.12) are satisfied, then the Lagrangian $\mathscr{L}_1$ under consideration can be written as

$$\mathcal{L}_1 = \mathcal{L}_{\text{invariant}} - \tfrac{1}{2}(C'_a)^2$$

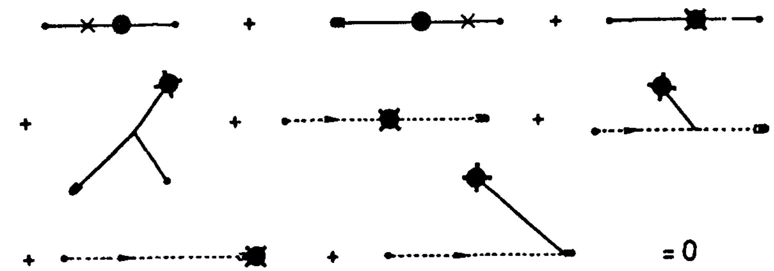
where $\mathscr{L}_{\text{invariant}}$ is invariant under the transformation

$$A_b = A'_b + \hat{s}'_{bc}(A')\Lambda_c + \hat{t}'_{bc}\Lambda_c$$

To $\mathscr{L}_1$ a Faddeev-Popov part is added which is given by

$$- c^{\dagger}_a \left(g\hat{\ell}'_{ab}(A) + \hat{m}'_{ab} \right) c_b$$

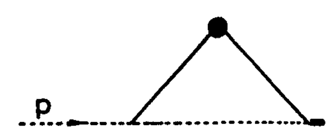
Now that we have stated the theorem we will see how we can obtain tree identities for $\mathscr{L}_1$. From (2.8) we can consider the pole parts at 'n=4' and immediately deduce



(2.13)

Crosses replace loops in (2.8). In particular a cross above a big black dot denotes replacing a loop involving a source vertex. The factor associated with a cross is just the pole part (at n=4) of the bubble being replaced. To illustrate this we will obtain the factor associated with

This is the pole part (P.P.) of



(where p is the momentum flowing in the ghost line). All calculations involved can be performed using

$$\int d^n l \, \frac{l_\mu l_\nu}{(l^2 - m^2 - i\varepsilon)^\alpha} = \frac{1}{2} \frac{i\pi^{n/2}}{(-m^2)^{\alpha - \frac{n}{2} - 1}} \frac{\delta_{\mu\nu}}{\Gamma(\alpha)} \Gamma(\alpha - 1 - \tfrac{n}{2}) \qquad (2.14)$$

and

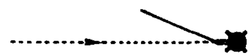
$$\int d^n l \, \frac{l_\mu l_\nu l_\lambda l_\delta}{(l^2 - m^2 - i\varepsilon)^\alpha} = \frac{1}{4} \frac{i\pi^{n/2}}{(-m^2)^{\alpha - \frac{n}{2} - 2}} \frac{\Gamma(\alpha - 2 - \frac{n}{2})}{\Gamma(\alpha)} \left\{ \delta_{\mu\nu}\delta_{\lambda\delta} + \delta_{\mu\lambda}\delta_{\nu\delta} + \delta_{\nu\lambda}\delta_{\mu\delta} \right\} \qquad (2.15)$$

We find in the Feynman gauge that

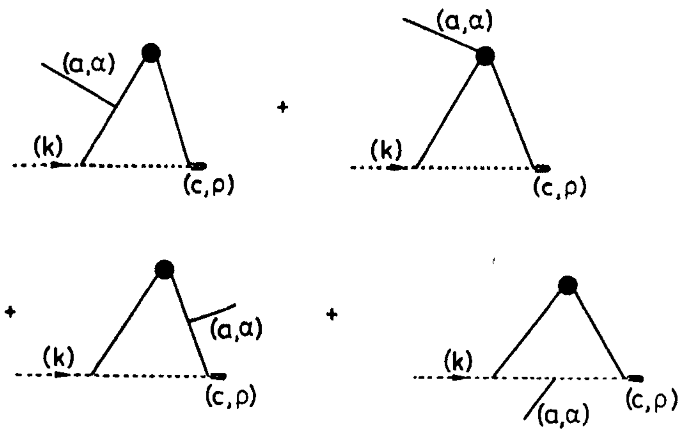
$$\frac{2g^2}{4-n} f^{ebd} f^{ade} n^2 (\chi \cdot p) \chi^0 \qquad (2.16)$$

Here (a,α) is the quantum number associated with ✖ and b with the ghost line.

If we add (2.4) to (2.13) we indeed have a tree identity for $\mathcal{L}_1$ of the type mentioned in the tree-loop theorem. However, we need to consider the hierarchy of tree Ward identities of $\mathcal{L}_1$. So far we have extracted information out of (2.3) for q=1. In fact for q=2 we can follow through exactly the same procedure to obtain Fig. 1. This is a rather long expression but we think it is instructive to see it in full. The interesting feature of these identities is that of the non-Lagrangian vertices. In addition to (2.16) we have now

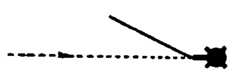


which arises as the pole part of

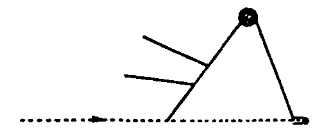


(2.17)

The quantities in brackets are associated with the appropriate line or vertex. A straightforward calculation reveals that

 $= \; 0$ (2.18)

The pole part of (2.17) is not in general zero for 'higher s' operators. We can write down the Ward identities for q=3, q=4, etc., but the purpose of showing q=2 identity in detail is to obtain some intuition about the nature of these tree identities. It is clear that apart from vertices (having crosses) with a ghost line entering but not leaving, all the other vertices can be readily identified with counter-terms calculated in the usual approach. The question is whether we get any new non-Lagrangian vertices as we go to q=3 and higher Ward identities. The only type of graphs that can give rise to such 'cross' vertices are, for example,



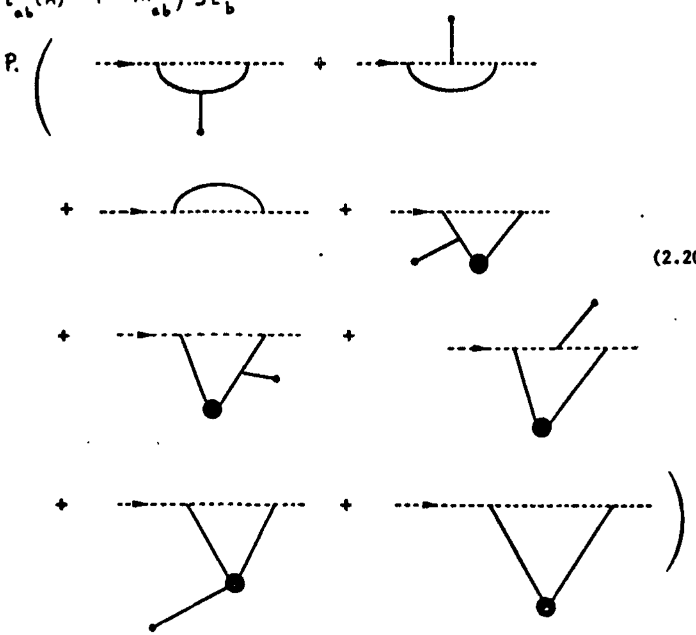
and other topological arrangements of the vector prongs. From power counting we know that this is finite and consequently there is no corresponding cross vertex. The higher 'q' Ward identities are clearly of the topological form required for the tree loop theorem. We have now to check that the condition (i) of the tree-loop theorem is satisfied.

Looking at the structure of (2.13) and Fig. 1 as well as the 'higher' tree identities we see that

$$C'_a(A) = C_a(A) = \frac{1}{\sqrt{\xi}} \; \partial\cdot A_a \qquad (2.19)$$

Moreover, from the identities we note that

$$(g\,\hat{t}'_{ab}(A) + \hat{m}'_{ab})\,\Lambda_b$$
$$= (g\,\hat{t}_{ab}(A) + \hat{m}_{ab})\,\Lambda_b$$
$$+ \text{P.P.}$$



(2.20)

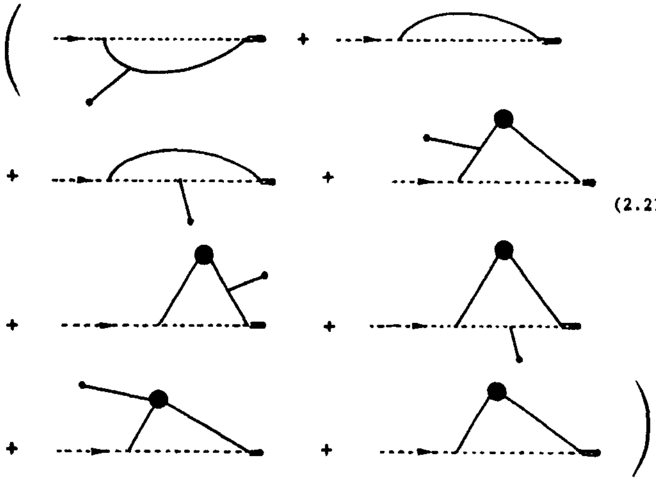
The infinities in the non-Lagrangian vertices that we have already discussed cannot be absorbed by terms in $\mathscr{L}_1$. They have to be removed by renormalization of $\hat{s}$ and $\hat{t}$, i.e. by a renormalization of the gauge transformation.

Hence

$$\left(g\,\hat{s}'_{ab}(A) + \hat{t}'_{ab}\right)\Lambda_b$$

$$= \left(g\,\hat{s}_{ab}(A) + \hat{t}_{ab}\right)\Lambda_b$$

+ P.P.

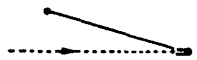


(2.21)

In order to have condition (i) holding, we need to show schematically that

$$\xi^{-\frac{1}{2}}\,\partial_\mu\,\left[\text{Expression of (2.21)}\right] = \left[\text{Expression of (2.20)}\right] \quad .$$

Fortunately it is necessary only to do combinatorics to verify this. We can see that the identity is valid from noting that

∂_μ  = 

Then all the integrals become identical and so condition (i) is satisfied. We now have shown that the tree-loop theorem can be applied to $\mathscr{L}_1$. Consequently the whole Lagrangian at the one-loop level is

$$\mathscr{L}_1 + (\text{gauge part}) \quad .$$

Here the gauge part is

$$-\tfrac{1}{2}\,\xi^{-1}\,(\partial . A)^2 \qquad \text{+ Faddeev-Popov part corresponding to this gauge fixing.}$$

Explicitly the Faddeev-Popov part is

$$c_a^{\dagger}\left(g\,\hat{l}'_{ab}(A) + \hat{m}'_{ab}\right)c_b$$

Now the χ-dependent parts of $\hat{\ell}'$ and $\hat{m}'$ are calculated from the χ-dependent parts of $\hat{s}'$ and $\hat{t}'$:

$$\hat{S}'^{\,g}_{ck} = 0 + \chi\text{ - independent terms} \tag{2.22}$$

and

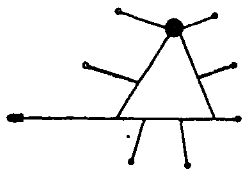
$$\hat{t}'_{ck} = \frac{2}{n-4}\,g^2\pi^2 f^{ekd} f^{cde}\chi^\beta\chi^\rho\partial_\beta + \chi\text{ - independent terms} \tag{2.23}$$

Thus the χ-dependent Faddeev-Popov contribution is

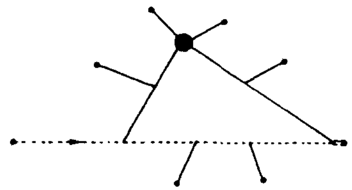
$$-\,c_e^{\dagger}\left(\frac{2g^2\pi^2}{(n-4)}\,f^{ekd}f^{cde}\right)\chi^\beta\chi^\rho\,\partial_\beta\partial_\rho\,c_k \tag{2.24}$$

The χ-independent terms in $\hat{s}'$ and $\hat{t}'$ reproduce the bare Faddeev-Popov interactions together with the one-loop renormalizations of the ghost propagator and vertex. There is a question whether the invariance of $\mathscr{L}_1$ proved using the tree-loop theorem is a group invariance. A proof can be given using the tree identities, but we shall not reproduce it here.

There are other twist-two operators such as those given in (1.5). The procedure followed here would go through exactly for them. However, in general we would expect in the Ward identities new varieties of infinities. Apart from the expected ones such as



for high enough s, we also have the new non-Lagrangian infinities, which we recall are responsible for the renormalized invariances of the theory. These are of the type given below:



Such infinities would give rise to χ-dependent invariance transformations which are non-linear in A, and so the Lagrangian would no longer be invariant under a simple gauge transformation. We thus see, just from invariance arguments, that Fermi-type gauges do not allow us to consider just only the mixings between gauge-invariant operators. Because of such difficulties we feel that the problem of mixing should be looked at in a different gauge. In the next section we shall discuss such gauges.

3. BACKGROUND FIELD QUANTIZATION

There exist renormalizable gauges in which the gauge renormalization found in Section 2 does not occur. These are the background field gauges[9] which are based on splitting the vector field into a classical part $A^{c\ell\, a}_{\mu}(x)$ and a quantum part $Q^a_\mu(x)$. It is the quantum parts of the fields which flow in loops, and the classical parts of the field flow in any tree structures attached to loops.

The invariance (1.4) can be regarded as being made up of

$$A^{cl\,a}_{\mu}(x) = A^{cl'\,a}_{\mu}(x) + g f^{acb} A^{cl'\,c}_{\mu}(x)\Lambda^b(x) - \partial_\mu \Lambda^a(x) \qquad (3.1)$$

$$Q^a_\mu(x) = Q'^{\,a}_{\mu}(x) + g f^{acb} Q'^{\,c}_{\mu}(x)\Lambda^b(x)$$

Alternatively the invariance (1.4) can be thought of as

$$A^{cl\,a}_{\mu}(x) = A^{cl'\,a}_{\mu}(x) \qquad (3.2)$$

$$Q^a_\mu(x) = Q'^{\,a}_{\mu}(x) + g f^{acb} Q'^{\,c}_{\mu}(x)\Lambda^b(x) - (\bar{D}_\mu \Lambda(x))^a = Q'^{\mu\,[A]}_{a}$$

where

$$\bar{D}^{ac}_{\mu} = \partial_\mu \delta^{ac} + g f^{acb} A^{cl\,b}_{\mu}$$

The gauge-fixing condition is taken to be

$$(\bar{D}_\mu Q^\mu(x))_a = C_a(x)$$

The attractive feature of such a term is that it is invariant under (3.1) but not under (3.2); so although we are breaking a gauge symmetry we are still maintaining a very helpful degeneracy. This forces the one-loop counter-terms (which are functions of just $A^{c\ell}_\mu$) to be invariant under (3.1) and consequently gauge invariant. At this stage we must take note of the fact that the classical field concept is defined with respect to sources; so, for example, to (1.3) we must add $J^a_\mu(x) A^a_\mu(x)$. We want J^a_μ to be field-independent. However, the equations of motion give

$$- D^{a'a}_{\mu'} F^a_{\mu'\nu'} - \chi^{\mu'}\chi^\lambda D^{a'a}_{\mu'} F^a_{\nu'\lambda} + \chi^{\nu'}\chi^\lambda D^{a'a}_{\mu'} F^a_{\mu'\lambda} - J^{a'}_{\nu'} = 0 \qquad (3.3)$$

This has the immediate consequence that

$$(D_{\nu'} J_{\nu'})^{a'} = 0$$

and so J is field-dependent, contrary to our starting assumption. Instead of considering the equation (3.3) we can take as our classical field a solution of

$$- D^{a'a}_{\mu'} F^a_{\mu'\nu'} - \chi^{\mu'}\chi^\lambda D^{a'a}_{\mu'} F^a_{\nu'\lambda} + \chi^{\nu'}\chi^\lambda D^{a'a}_{\mu'} F^a_{\mu'\lambda} - J^{a'}_{\nu'} + (D_{\nu'} \bar{G} D_\sigma J_\sigma)^{a'} = 0 \qquad (3.4)$$

where $\bar{G}$ is defined by

$$(D_\tau D^\tau)^{ab} \bar{G}_{bc}(x,x';A^{cl}) = \delta_{ac}\,\delta(x,x')$$

Following standard arguments[9] we are led finally to the generating functional Z

$$Z(J_\mu, \chi)$$
$$= \exp i \left(S(A^{cl}) + \int d^n x\, J^a_\mu(x) A^{cl\,a}_{\mu}(x) \right)$$
$$\cdot \int \prod_x \prod_\mu \prod_a dQ^a_\mu(x)\, \delta(\bar{D}_\mu Q^\mu - C)\, \bar{\Delta}(Q_\mu, A^{cl}_\mu) \qquad (3.5)$$
$$\exp -i (\bar{D}_\sigma J^\sigma)_a \bar{G}_{ab} (\bar{D}_\mu Q^\mu)_b \exp \tfrac{1}{2} i\, S^{ab}_{,\mu\nu} Q^\mu_a Q^\nu_b (1 + \cdots)$$

where S is the action of (1.3), $S^{ab}_{,\mu\nu}$ is $\delta^2 S/\delta A^a_\mu \delta A^b_\nu$, and

$$\Delta(Q^\mu, A^{cl}_\mu)\int d\Lambda\ \delta(\bar{D}_\mu Q^{\mu\,[\Lambda]}_a - C_a) = 1$$

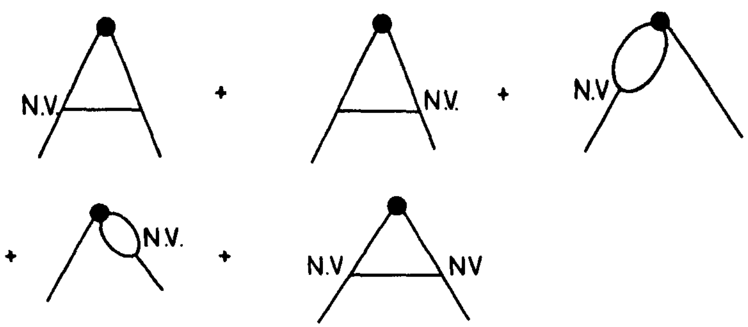
this being the Faddeev-Popov prescription. If we now introduce the weight factor exp $(-i/2\xi\ C_a^2)$ and integrate over C (owing to the C-independence of Z) we still retain a term involving JQ. In the background field method we want the external lines to be classical and so this could, in general, be unsatisfactory, at least for off-shell matrix elements. The simplest way of avoiding a JQ term is not to integrate over C but rather to take a particular value C=0. This has the effect of replacing $S^{ab}_{,\mu\nu}Q^\mu_a Q^\nu_b$ by $Q^\mu_a S^{ab}_{,\mu\nu}Q^\nu_b - (\bar{D}_\mu Q^\mu_a)^2$. Unlike in the Fermi-type gauges, the background field gauge condition helps not only to define the propagator, but also gives rise to interaction terms, namely

$$g\, f^{abc}\, A^b_\nu\, Q^c_\nu\, \partial_\mu Q^\mu_a$$

and

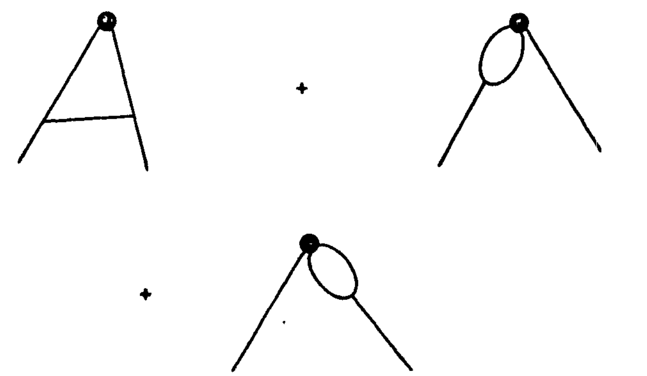
$$g\, f^{abc}\, A^b_\mu\, Q^c_\mu\, D_\nu Q^\nu_a \qquad (3.6)$$

The second of these terms does not contribute owing to the δ-function constraint in (3.5), but the first term certainly leads to additional diagrams for the mixing of the operators $R^{(s)}$. These are



N.V. denotes the new vertex arising from the first interaction given in (3.6). (We have for clarity not indicated in the diagrams the momentum or quantum numbers of the external legs.)

Owing to the specially attractive property of the background field gauge we know that the sum of these diagrams plus the diagrams of the Fermi gauges

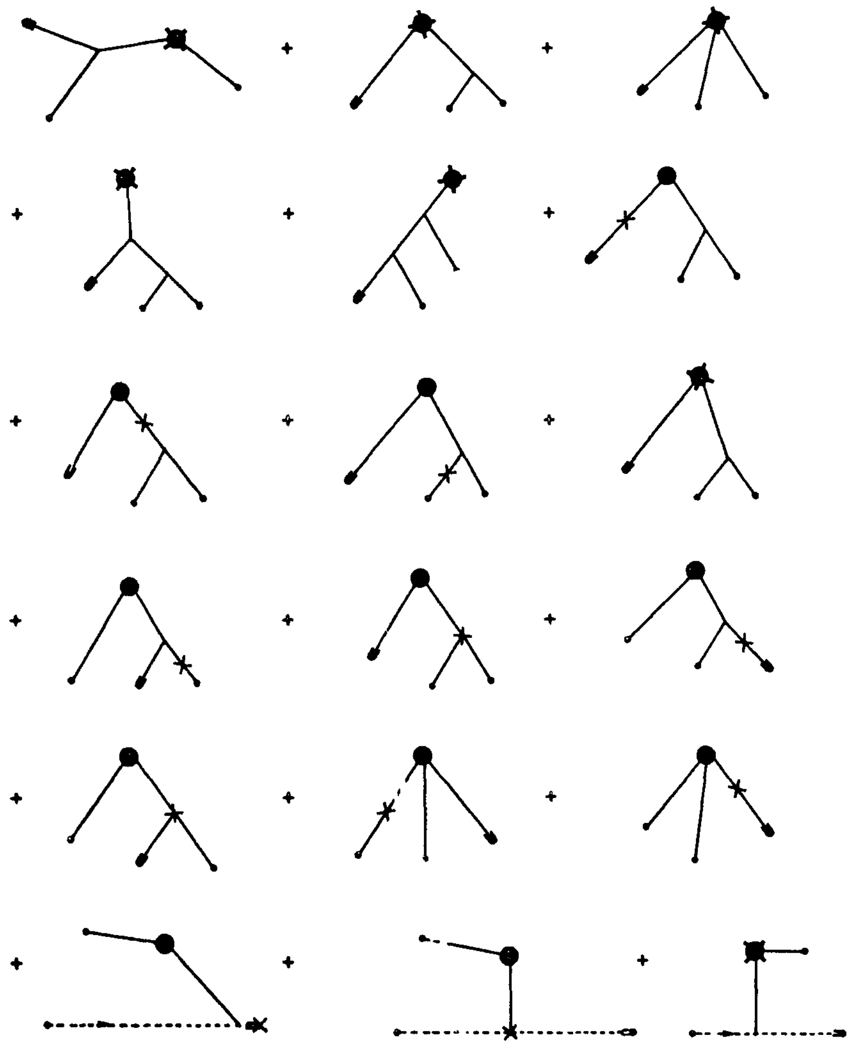


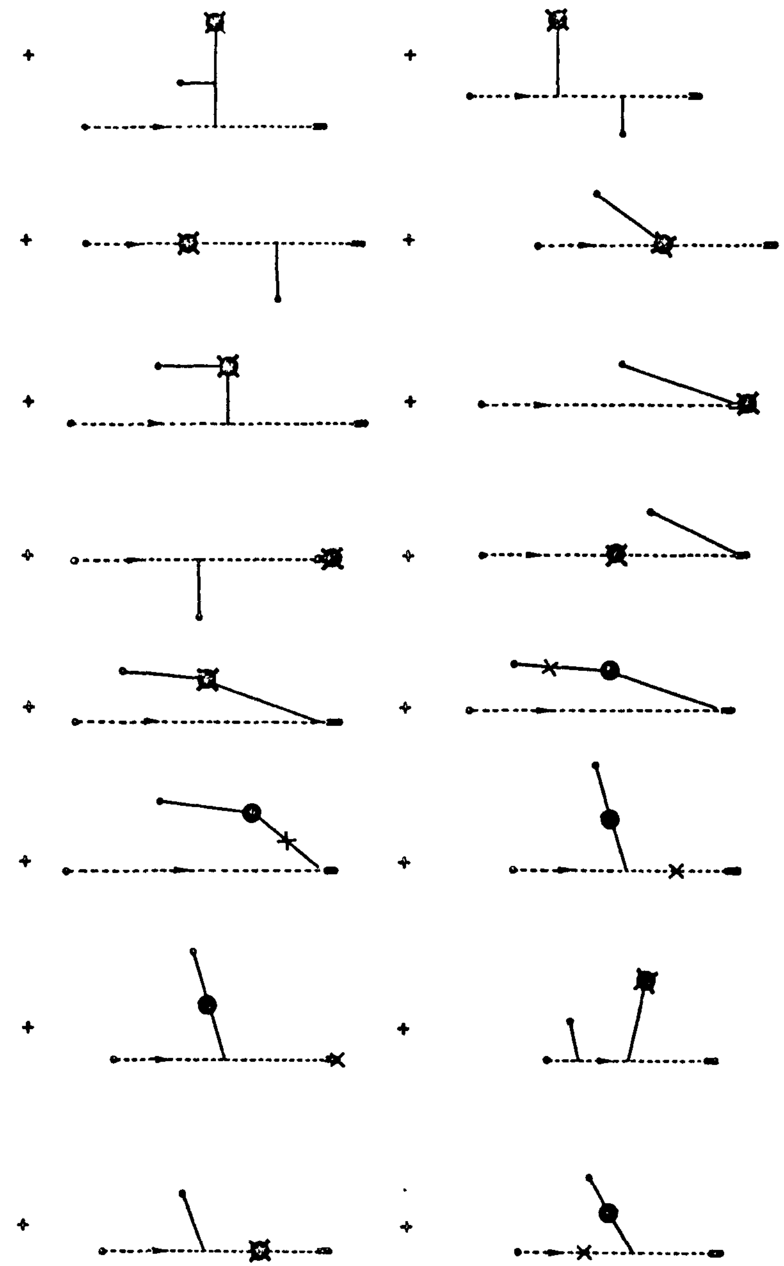
[where no vertex of (3.6) occurs] gives the true gauge-invariant mixing. (For simplicity we have left out the fermion contribution.) Owing to the intricacies of the calculation we shall report the numerical value of the anomalous dimensions of the operators $R^{(s)}$ elsewhere. However, any equality of our number with those obtained from calculations in Fermi-type gauges would seem to be fortuitous.

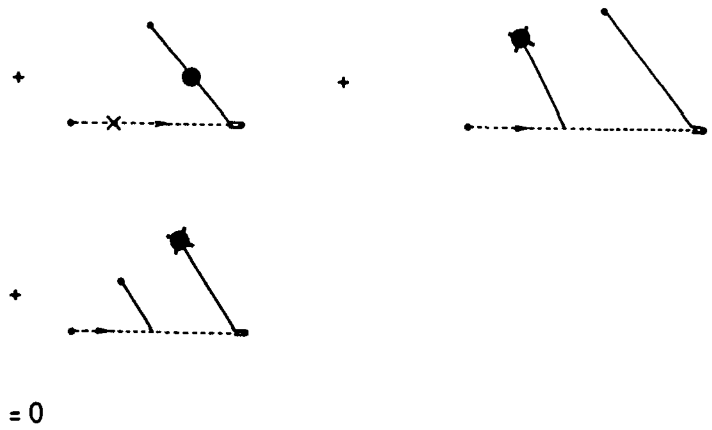
I would like especially to thank G. 't Hooft for many discussions. C.H. Llewellyn Smith and R.J. Crewther have also made helpful comments.

After this work was completed, H. Kluberg-Stern and J.B. Züber showed me their work on the mixing of the twist-four operator $F^a_{\mu\nu}F^a_{\mu\nu}$ at the Marseilles Conference, June 1974.

Figure 1







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IV - G A U G E F I E L D S O N L A T T I C E S

GAUGE FIELDS ON A LATTICE

Presented by

J.M. DROUFFE

"I - General Outlook"
R. Balian, J.M. Drouffe and C. Itzykson
Phys. Rev. D 10, 3376 (1974).

Abstract : We present Wilson's model of gauge-field theory on a lattice, including a coupling to a matter field. The algebraic structure is surveyed for both commutative and noncommutative groups. Various regimes are suggested by mean-field theory according to the relative values of coupling constants. In particular the gauge field undergoes a first-order transition while the matter-field transition is of second order.

"II - Gauge Invariant Ising Model"
R. Balian, J.M. Drouffe and C. Itzykson
Saclay Preprint DPHT 74/74. (To be published in Phys. Rev.)

Abstract : We study the case of a discrete local gauge Z_2 in order to discuss the existence of a transition in dimension $d \geq 3$. We compute the critical constant for $d = 3$ and 4 and show that in three dimensions the transition is a second order one.

"III - Strong Coupling Expansions and Transition Points"
R. Balian, J.M. Drouffe and C. Itzykson
Saclay Preprint DPHT 74/89. (To be published in Phys. Rev.)

Abstract : We discuss the principles of the high temperature expansion leading to a variation-perturbation method. For pure gauge fields, diagrams are two dimensional manifolds. As an application, we compute the critical coupling constants for discrete, abelian and SU(2) gauge groups and compare them to some earlier results.

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YANG-MILLS FIELDS ON A LATTICE AND DUAL AMPLITUDES

C P. KORTHALS-ALTES *

(Extended version of a talk given at the Marseilles Colloquium on Lagrangian Field theory , 24-29 Juin 1974)

Abstract :

Coloured $U(n)$ Yang-Mills fields are coupled through a coupling constant g to quarks The theory is quantized on a d-dimensional cubic lattice , Expanding in n^{-1}-keeping g^2n fixed - gives as leading contribution in the quark-antiquark sector a string type amplitude The theory obeys the area rule of Wilson , Kogut and Susskind for values of g with $2g^2n > x(d)$, where $x(d)$ is related to the number of ways a simply connected surface with a given border and fixed large area can be realized . From the requirement that the lattice length be much smaller then a typical hadronic distance follows $2g^2n \gtrsim 1$. For x(d) we have only a very generous bound : $x(d) \geq 2d-3$

POSTAL ADDRESS : Centre de Physique Théorique - CNRS
31, Chemin Joseph Aiguier
13274 - MARSEILLE - Cedex 2

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1. INTRODUCTION

Tradition wants it, that the spectrum of a perturbative quantum field theory is determined by the free part of the Lagrangian S-matrix elements in perturbation theory can be calculated by referring to this spectrum. A notable exception [1] is non Abelian Yang-Mills theory. there the infrared divergences do not cancel and obliterate the perturbative calculation of an S-matrix referring to the massless Yang-Mills quanta A qualitative picture of what might happen at long distance in such theories[1] is found in ref. 8; due to the peculiar properties of the vacuum polarisation dielectric constant can be zero everywhere except in a tube between quark and antiquark. This means that only in between the quarks there is flux and the field energy grows therefore linearly with increasing quark antiquark distance (like in 2 dimensional E D). This thin tube of flux between the quarks would be the basis for the dual string model.

The justification of this description is yet to be found One way of attacking the problem has been developed by Wilson [2,5] and consists in building a small distance cut-off into the theory, by putting the quark field on a lattice, with the gauge field defined on the bonds between the lattice points. One can make a perturbation expansion for large coupling constant g and find [5] certain qualitative features of the model described in ref. 8 On the other hand t' Hooft [3] has argued on basis of Feynman diagram analysis that an expansion in the dimension n of the gauge group, keeping g^2n fixed, precisely reproduces the hierarchy of dual surfaces with more and more complicated topologies.

We want to advocate in this paper the latter approach, but realized in Wilson's lattice version of gauge field theory discussed in section 2[2]. Again, we will find in section 3b a classification of the kind met in the dual model : the parameter n distinguishes different classes of diagrams, each class having a given fixed topology.

The leading term in n is again the sum of all planar contributions Crucial is then the question how does the energy of a quark antiquark pair vary, when pulled far apart (i.e. many lattice distances) This is discussed in section 3C)

In section 4 we look for possible critical points We find that for very large coupling constant the free energy is analytic in the mass of the fermions and therefore, that evertual critical points should be searched for g not too large For bosonic quarks we show that the situation might be different

2 The model and its regularization

In this section we will show how a field theory of Yang-Mills quanta coupled to quarks can be quantized with the quarks put on a lattice, and the Yang-Mills fields cut-off in a natural way. It then turns out that a strong coupling expansion is the natural thing to do, and we establish the graphical rules associated to this expansion. This section is subdivided into:

a) The classical action
b) The quantized theory
c) Graphical rules

a) The classical action of a gauge field A^α_μ carrying local $U(n)$ and coupling to quarks q in the fundamental representation of $U(n)$ ($\alpha = 1, \dots, n^2$) is in Euclidean four-space :

$$S = \int d^4x \left\{ -\tfrac{1}{4} G^\alpha_{\mu\nu} G^\alpha_{\mu\nu} + q^\dagger \left(\overset{\leftrightarrow}{\frac{\partial\!\!\!/}{2}} - m - g A\!\!\!/_\alpha t_\alpha \right) q \right\} \qquad 2\ 1$$

Here $G^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu - g f^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu$ and t_α are a set of Hermitian generators of the fundamental representation of $U(n)$ with.

$$[t^\alpha, t^\beta] = 2i f^{\alpha\beta\gamma} t^\gamma$$
$$\mathrm{Tr}\, t^\alpha t^\beta = 2\delta^{\alpha\beta} \qquad 2\ 2$$

$f^{\alpha\beta\gamma}$ being real and antisymmetric ×

The density in 2 1 is clearly invariant under local $U(n)$ gauge transformations:

$$t^\alpha A^\alpha_\mu(x) \to t^\alpha \hat{A}^\alpha_\mu(x) = e^{-i\Lambda(x)} t^\alpha A^\alpha_\mu(x) e^{i\Lambda(x)} + \frac{1}{g} e^{-i\Lambda(x)} \partial_\mu e^{i\Lambda(x)}$$

with $q^a(x) \to \hat{q}(x) = e^{-i\Lambda(x)} q(x)$. $\qquad 2.3$

× $\partial\!\!\!/ = \gamma_\mu \partial_\mu$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

We want the regularization such, that the gauge invariance 2 3 is kept

The first step is to put the quarks on a simple hypercubic lattice In each point i of this lattice we have an $SU(n)$ group $\{e^{i\Lambda(i)}$, $\Lambda(i)$ (traceless) Hermitean $n \times n$ matrix

We will consider a finite lattice with a finite number N of lattice points Calculations of physical quantities will always be done for finite N, where upon the limit $N \to \infty$ is taken

How to have a gauge invariant interaction between quarks and gauge fields is well known[7]. We take two adjacent points i and i' on the lattice and define :

$$q^{\dagger}(i)\,\gamma_{(ii')}\,T \exp\left[ig\int_i^{i'} A_{(ii')}(\xi)\,d\xi\right] q(i') \qquad 2\ 4$$

The integration is along the straight line segment between i and i'
The symbol T mean line-ordering along the segment ii' The direction corresponding to the directed bond (ii') is μ.
We will denote the line-ordered unitary matrix by $U(i,i')$
It is clear that :

$$U^{\dagger}(i,i') = U(i',i) \qquad (a)$$

$$U(i,i')\,U(i',i'') = U(i,i'') \qquad (b) \qquad 2\ 5$$

The part of the action referring to the quarks is now written :

$$S(q^{\dagger},q,U) = \sum_{\langle ii'\rangle} \frac{1}{2} \sum_{\mu=1}^{4} q^{\dagger}(i)\,\gamma_{\mu(i')}\,U(i,i')\,q(i') \qquad 2\ 6$$

and it is easy to check that in the limit $\delta_\mu \to 0$ (δ_μ being the lattice length vector in the μ-direction) we recover the quark part of the action 2 1 , except for the mass term

For later convenience 2 6 is written in a short hand notation :

$$S(q^{\dagger},q,U) = \sum_{\langle ii'\rangle} q^{\dagger}(i)\,M(i,i')\,q(i') \qquad 2\ 7$$

where $M(i,i')$ has the matrix value $\frac{1}{2}\gamma_{(ii')}U(i,i')$ and $\gamma_{(ii')} = -\gamma_{(i'i)} = \gamma_\mu$ if (ii') is directed along the μ axis .
If i'=i : $M(i,i) = -m\varepsilon$; that is , we included a mass term for the quarks

The field $A^{\alpha}_{\mu}(x)$ has been replaced by the unitary $n \times n$ matrix $U(i,i')$, which is a bilocal object furnishing the interactions between quarks at

neighbouring sites i and i'.
We are left with the Yang-Mills self interaction:

$$-\frac{1}{4} G^{\alpha}_{\mu\nu} G^{\alpha}_{\mu\nu} = -\frac{1}{8} \mathrm{Tr}\, G_{\mu\nu} G_{\mu\nu} \qquad 2.8$$

and we should find an expression in term of the $U(ii')$, which approximates 2.8(or its space-time integral) . The bilocals transform under a gauge transformation 2-3 as :

$$\tilde{U}(i,i') = e^{-i\Lambda(i)}\,U(i,i')\,e^{i\Lambda(i')} \qquad 2.9$$

So invariant are traces (determinants) of all products of $U(i,i')$ along closed circuits . The simplest circuit that gives a non-trivial result is the elementary square C_e .

The choice of what scalar function to take of the product of 4 $U(i,i')$'s around an elementary circuit C_e is easy in the case of an Abelian theory :

$$U(C_e) \equiv U(1,2)\,U(2,3)\,U(3,4)\,U(4,1) = \exp\left[ig\oint_{C_e} A_\mu(\xi)\,d\xi_\mu\right] \qquad 2\ 10$$

Use of Stokes theorem introduces the curl of the vector field A_μ and we find that :

$$\lim_{\delta\to 0} \sum_{C_e} \frac{1}{4}\left[U(C_e) + \text{c.c.}\right] = -\frac{1}{4} g^2 \int d^4x\, G_{\mu\nu} G_{\mu\nu} + \text{constant} \qquad 2\ 11$$

The sum on the left side of eqn. 2.11 is over all elementary circuits of one definite orientation ; adding the complex conjugate means adding all the circuits of the opposite orientation The appearance of the square of the coupling constant in the right side of 2.11 is important and is due to the form of $U(i,i')$ in eq. 2 4
The non-abelian case can be delt with in the same way ; we get the same relation as in eq. 2.11 .
To resume this section : The action 2.1 is replaced by the lattice action

$$S_L = -\frac{1}{2g^2} \sum_{C_e} \mathrm{Tr}\, U(C_e) + \sum_{\langle ii'\rangle} q^{\dagger}(i)\,M(ii')\,q(i') \qquad 2.\ 2$$

The sum $\sum_{C_e}$ is now over <u>all</u> elementary oriented circuits C_e and the matrix $M(ii')$ specifies the bonds and their corresponding strength between adjacent quarks . If the lattice length is let to zero , we recover from 2.12 the continuous classical action 2 1 The lattice action still obeys a gauge invariance , with in every lattice point i a local $U(n)$ group $\{e^{i\Lambda(i)}\}$ acting . Apart from the mass term, $M(ii')$ is anti-Hermitean

If the matter field consists of bosons b(i) , transforming under the gauge group like quarks :

$$b(i) \to e^{i\Lambda(i)} b(i)$$

we find easily the following lattice action :

$$S_L(b^+, b, U) = \frac{1}{2g^2}\sum_{C_e} \mathrm{Tr}\, U(C_e) + \sum_{(ii')} b^+(i) M(ii') b(i') \qquad 2.12'$$

where $M(ii') = U(ii')$ if i and i' are nearest-neighbour points . If i = i' then $M(ii') = -m^2\delta^2$. So in the boson case M(ii')is Hermitean . Although bosonic quarks are of no interest physically – the colorgauge group was invented just for the sake of having Fermion quarks – we mention them for later comparison in section 4

b) Quantization of the lattice action

Basic for the quantization is the Euclidean Feynman path integral

$$Z(J\ldots) = \int DA^\alpha_\mu(x)\, Dq^+(x)\, Dq(x) \exp -[S(q^+ q A) + J.A + \ldots] \qquad 2.13$$

where dots indicate eventual other source terms . Usually one gives 2.13 a meaning by expanding it in the coupling constant g^2 and by regulating the ensuing Feynman diagrams . The renormalization procedure then gives the physical answer . For Yang-Mills fields without spontaneous breaking the last step is only partially realizable ; we can only renormalize at unphysical points . Infrared divergencies do forbid the perturbative construction of an S-metrix with massless Yang-Mills particles in the spectrum .

The approach followed here is radically different[2,5] . Instead of a perturbation expansion in g^2 we will do an expansion in $\frac{1}{g^2}$, using the lattice action in the exponant of 2-13 ; and the integration variables DA^α_μ, Dq^+ and Dq are replaced by $DU(ii')$ and $dq^+(i)$ and $dq(i)$ where i and i' are points on the lattice and where $DU(ii')$ is the (left and right) invariant volume element of $U(n)$. It is not difficult to see that this replacement reduces in the $\delta \to 0$ limit

to the formal expression 2.13 :

$$Z_L(J\ldots) \to \int_{-\pi/g\delta}^{\pi/g\delta} DA^\alpha_\mu \int_{-\infty}^{\infty} Dq^+\, Dq \exp -(S + J.A + \ldots) \qquad 2.14$$

The boundary of the integrations over the gauge-fields comes from the integration $DU(ii')$ over the unitary matrices . Since the A^α_μ fields serve as parameters for these matrices in the $\delta \to 0$ limit , we must have a cut-off for the gauge-fields of the order of $\pi/g\delta$. Thus , when doing the $\delta \to 0$ limit , not only the lattice becomes dense , but also the integration range of the A^α_μ fields becomes infinite , as in 2.13.

c) The $\frac{1}{g^2}$ expansion

We will now go into a detailed description of the strong coupling (g^{-2}) expansion , when the sources J are coupled to gauge invariant (i.e colorless) combinations of quark-antiquark pairs . Simple possibilities for such pairs are: $\bar{q}(i)q(i)$, $\bar{q}(i)\gamma_{(ii')}q(i)$ and $q^+(i)\gamma_{(ii')}U(ii')q(i')$. We shall opt for the last possibility since it is mathematically the simplest . So we are faced with the problem of calculating

$$\frac{\delta}{\delta J(i_1 i_1')} \cdots \frac{\delta}{\delta J(i_s i_s')} \log Z_L(J) \Big|_{J=0} \qquad 2.15$$

or- what amounts to the same – we have to compute the connected part of :

$$\frac{1}{Z(0)} \int \prod_{(ii')} DU(ii') \prod_i dq^+(i)\, dq(i)\; q^+(i_1)\gamma_{i_1 i_1'} U(i_1 i_1') q(i_1') \ldots$$

$$\ldots q^+(i_s)\gamma_{i_s i_s'} U(i_s i_s') q(i_s') \exp -[S_L] \qquad 2.16$$

The integrand of 2.16 can be written as:

$$\exp\Big[\frac{1}{2g^2}\sum_{C_e}\mathrm{Tr}\,U(C_e)\Big]\prod_{r=1}^{S}\gamma_{i_r i'_r}U(i_r i'_r)\frac{\partial\ \det M}{\partial(\gamma_{i_r i'_r}U(i_r i'_r))} \qquad 2.17$$

and the integration over the quark degrees of freedom gives[10)]

$$\int\prod_i dq^\dagger(i)\prod_i dq(i')\exp\Big[\sum_{(k,k')}q^\dagger(k)M(kk')q(k)\Big]=\det M \qquad 2.18$$

The normalization constant in front of the determinant is chosen to be one

The general form of Greens function with sources coupled to colorless quark-antiquark pairs is therefore :

$$\int DU\,\exp\Big[\frac{1}{2g^2}\sum_{C_e}\mathrm{Tr}\,U(C_e)\Big]F(U)\frac{\partial}{\partial U}\ \det M \qquad 2.19$$

The symbol $F(U)\frac{\partial}{\partial U}$ is just a shorthand for operations of the hind mentioned in eq. 2.17 . The g^{-2} dependence is <u>only</u> in the exponent associated with the elementary squares C_e Expanding the exponential gives us a power series in g^{-2} :

$$\exp\Big[\frac{1}{2g^2}\sum_{C_e}\mathrm{Tr}\,U(C_e)\Big]=\sum_{k=0}^{\infty}\Big(\frac{1}{2g^2}\Big)^k\frac{1}{k!}\Big[\sum_{C_e}\mathrm{Tr}\,U(C_e)\Big]^k \qquad 2.20$$

We now turn to an analysis of the quark determinant $\det M$

This quantity is non-negative and gauge invariant as can be seen from calculation of its logarithm :

$$\log\det M = \mathrm{Tr}\log M \equiv \mathrm{Tr}\log m\mathbb{1} + \mathrm{Tr}\log(\mathbb{1}-\tilde{\mathcal{U}}) \qquad 2.21$$

The second term in 2.21 gives:

$$-\mathrm{Tr}\sum_{k=1}^{\infty}\frac{1}{k}(\tilde{\mathcal{U}})^k = -\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{Tr}(\tilde{\mathcal{U}})^k$$

$$= -\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{(2m)^k}\sum_{C(k)}\frac{k}{\mu(C(k))}\mathrm{Tr}(\gamma U)(C(k)) \qquad \mathbf{2.22}$$

Here C(k) means a closed circuit of k steps, $\mu(C(k))$ its periodicity , the $4n\times 4n$ matrix $(\gamma U)(C(k))$ the product of the matrices $\gamma_{ii'}U(ii')$ encountered on the circuit $C(k)$.

Rearrangement of the summation in 2-22 and use of $\mathrm{Tr}(\gamma U)(C(k)) = \mathrm{Tr}\big[(\gamma U)(C^*(\frac{k}{\mu}))\big]^\mu$ where $C^*(\frac{k}{\mu})$ is the non-periodic circuit corresponding to C(k) leads to:

$$\mathrm{Tr}\log(\mathbb{1}-\tilde{\mathcal{U}}) = \sum_{C^*}\mathrm{Tr}\log\big[\mathbb{1}-(\gamma U)(C^*)(2m)^{-k(C^*)}\big]$$

So : $$\det(\mathbb{1}-\tilde{\mathcal{U}}) = \prod_{C^*}\det\big(\mathbb{1}-(\gamma U)(C^*)(2m)^{-k(C^*)}\big) \qquad 2.23$$

The $4n\times 4n$ matrix $(\gamma U)(C^*)$ is the product of matrices $\gamma_{ii'}U(ii')$ met along the non periodic circuit C^* . Clearly each factor in the product is gauge invariant . Since to every closed circuit C^* corresponds the reverse circuit C^{*R} we find that the product consists of pairs with non negative value . So the quark determinant is non-negative .

The expression 2-23 is <u>not</u> useful in connection with the Greens functions we want to evaluate (see eq.2.19) . This is so because the family of non-periodic circuits C^* contains <u>self intersecting</u> ones; on the other hand , for finite N (finite numbers of lattice points) we have a finite number of factors $(4nN)$ in a term of the determinant so there is a redundancy in formula 2.23 for the determinant .

For the calculation of Greens functions we will use the well known[5)] formula for the determinant:

$$\det M = \sum_S (-)^{\nu(S)}Z(S) \qquad 2.24$$

This formula is most easily explained if the quarks have <u>no</u> internal degrees of freedom (i.e $\gamma_{ii'}U(ii')$ is a <u>number</u> , not a matrix) . In that case S is <u>any</u> family of disjoint, <u>not</u> self-intersecting circuits on the lattice , and $\nu(S)$ is the number of circuits in the family S. $Z(S)$ is the product of the weights encountered on the circuits in S .

In the interesting case where there are non trivial quark degrees of freedom (i.e $\gamma_{ii'}U(ii')$ are matrices) formula 2-24 remains the same ; only the lattice is changed : To each lattice point we attribute $4n$ (number of quark degrees of freedom) levels , labelled by $(a\,\alpha)$; $a=1,2,3,4$; $\alpha=1,\dots,n^2$

There are <u>only</u> bonds between levels $(a\,\alpha)$ and $(a'\alpha')$ at adjacent lattice points i and i' , with weight $(\ldots)_{\ldots}\,[\mathcal{U}(i,i')]_{\ldots}$. In fig 2 and 3 we have illustrated the situation

Now we are in a position to do the final integrations over the unitary matrices $\mathcal{U}(ii')$ (the gauge field integrations) . Normalizing the volume of $\mathcal{U}(n)$ to one, we have :

$$\int D\mathcal{U}(ii')\, [\mathcal{U}(ii')]_{\kappa\kappa'}\, [\bar{\mathcal{U}}(ii')]_{\lambda\lambda'} = \frac{1}{n}\,\delta_{\kappa\lambda}\,\delta_{\kappa'\lambda'} \qquad 2.25\ (a)$$

$$\int D\mathcal{U}(ii')\, [\mathcal{U}(ii')]_{\kappa\kappa'}\, [\mathcal{U}(ii')]_{\lambda\lambda'}\, [\bar{\mathcal{U}}(ii')]_{\mu\mu'}\, [\bar{\mathcal{U}}(ii')]_{\rho\rho'} =$$

$$= \frac{1}{n^2-1}\left(\delta_{\kappa\mu}\,\delta_{\lambda\rho}\,\delta_{\kappa'\mu'}\,\delta_{\lambda'\rho'} + \kappa\leftrightarrow\lambda\,,\ \kappa'\leftrightarrow\lambda'\right) - \frac{1}{n(n^2-1)}\left(\delta_{\kappa\mu}\,\delta_{\mu'\lambda'}\,\delta_{\lambda\rho}\,\delta_{\rho'\kappa'} + \kappa\leftrightarrow\lambda\,,\ \kappa'\leftrightarrow\lambda'\right) \qquad 2\ 25(b)$$

etc.

At every bond (ii') we have a group integration as in 2 25 Note the inverse powers of n at the right hand sides , becoming higher, the higher the degree of the monomial in the $\mathcal{U}(ii')$ matrix elements is .

3- The approximated dual vector amplitude

a) The graphical rules for calculating a connected n-point function will now be stated The sources J are supposed to be coupled to "vectorial" quark-antiquark pairs :

$$q^{+}(i)\,\gamma_{ii'}\,\mathcal{U}(ii')\,q(i') \equiv m(ii') \qquad 3.1$$

Thus the source $J(ii')$ is defined on a bond rather then in a lattice point The generating functional $Z_c(J)$ is defined as

$$Z_c(J) = \log Z(J) \qquad 3.2$$

and a connected n-point function is given by

$$\sigma(m(1)\,\ldots\ldots\, m(s)) = \frac{\delta^s}{\delta J(B_1)\cdots\delta J(B_s)}\, Z_c(J)\Big|_{J=0} \qquad 3\ 3$$

We have implicitely the rules for constructing n-point functions embodied in equations 2.24 and 2.25 . The expansion parameter is g^{-2} . What we will do here is state the rules for constructing any dual * connected contribution With dual we mean a contribution that is built up by using $\mathrm{Tr}\,\mathcal{U}(C_e)$ k times , with the k elementary circuits C_e adjacent, forming a (multiply) connected surface (no two C_e 's the same and not more then two C_e's join at any bond) the sources will lie on the boundaries of the connected surfaces and the boundaries are formed by a quark line (see fig. 4)

Therefore in calculating the connected s-point Green function 3.3 we draw any quarkloop going through all s bonds (i_1,i_1'), $\cdots(i_s,i_s')$ [3] and construct all connected dual graphs possible . They might contain other quarkloops (holes) and wormholes (handles)

The rules then are

1° For every surface element a factor

2° For every vertex a factor n

3° For every bond a factor n^{-1}

4° For every quarkloop a trace over the γ - matrices attached to the $|P|$ bonds on the loop , and a factor $(-)^{\frac{|P|}{2}+1}$ Fermi statistics forces the loop to be self avoiding

5° A normalization factor , which is the same for a given Green function .

Rules 1° and 4° are clear from the form of the action . Rule 3° follows from the gauge field integration 2.25 a) . Rule 2° is explained in fig 5, where it is shown that for the dual graphs there is only one index cycle ,i.e a cyclic product of Kronecher delta's giving rise to a factor n The trace over γ - matrices gives for a dual graph a factor $4\times(-)^{S}$, where S is the (minimal) area enclosed by the loop . It is simple to show that $(-)^{S+\frac{P}{2}+1} = 1$ for any closed loop . (both facts concerning the - sign's are shown by inductive arguments) .

The rules for non-dual graphs (where at least in one bond more then two squares join) are more complicated , not only because of 2.25 b) , but also because a

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* We apologize for this terminology , which is due to wishful thinking.

vertex can carry several index cycles Note also that a square could be used an unlimited number of times . In using Ising type models this can be avoided by the tgh transformation .

b) The connection with dual amplitudes becomes clear when we distinguish for a given Green function the different topologies a dual surface Σ can have A surface can have: Q quarkloops and H wormholes
Independent of this let it have B bonds , V vertices , and S surfaces , where S is counted as follows; first we have A elementary surfaces used in the perturbation series , and furthermore Q surface elements corresponding to the Q quarkloops, (so $S = A+Q$)

Then we find for the contribution from ,
using rules 1° to 5°:

$$\sigma_{\Sigma}(m(1) \ldots m(s)) = 4^{Q}\left(\frac{1}{2g^2}\right)^{A} n^{V-B} \left(\frac{1}{2m\delta}\right)^{P_{tot}} \qquad P_{tot} = \text{total number of quark bonds} .$$

Use now Eulers formula :

$$V - B + S = 2 - 2H$$

and find

$$\sigma_{\Sigma}(m(1) \ldots m(s)) = n^{2-2H-Q}\, 4^{Q} \left(\frac{1}{2g^2 n}\right)^{A}\left(\frac{1}{2m\delta}\right)^{P_{tot}} \qquad 3\ 5$$

Thus we find that merely the topology of a surface (H and Q) does define a perturbation series in n^{-1}, when we keep $g^2 n$ finite . For example we recover for $Q = 1$, $H = 0$ the string type surfaces [4,12], of order n [11] Raising the number of quarklines and wormholes gives lower order contributions in , as assumed ad hoc in dual theory .
The question remains how the non-dual graphs do behave for large n and g^2 n fixed . Although the bond integrations 2.25(b) , where more then two surfaces meet , are suppressed by factor n^{-1}, there can be several index cycles at a vertex . We have no quantitative idea of what happens .

c) We briefly discuss the dynamical features of the planar diagrams ($Q = 1$, $H = 0$) . We write the contribution which is of order n :

$$\sigma(m(1) - m(s)) = 4n \sum_{P} \sum_{(\Sigma_P)} \left(\frac{1}{2g^2 n}\right)^{A_P} \left(\frac{1}{2m\delta}\right)^{|P|} \qquad 3.6$$

where Σ_P is any surface with perimeter P (A_P is its area) .
First we note that 3.6 can be written ($\hat{A}_P = A_P \delta^2$):

$$\sigma(m(1) \ldots m(s)) = 4n \sum_{P} \exp[-|P| \log 2m\delta] \sum_{(\Sigma_P)} \exp\left[-\frac{\hat{A}_P \log 2g^2 n}{\delta^2}\right] \qquad 3.7$$

Call the second sum in 3.7 $\sigma_P(m(1) \ldots)$; it is reminiscent of the quantized (Euclidean) string[4,12] with fixed boundary P:

$$\sigma_P(\text{string}) = \sum_{(\Sigma_P)} \exp\left[-\frac{\hat{A}_P}{2\pi\alpha'}\right] \qquad 3.8$$

with $\alpha' \sim 1(\text{GeV})^{-2}$. Since the cut-off $\delta^2 << 1(\text{GeV})^{-2}$ we should expect

$$0 < \log 2g^2 n << 1 \qquad 3.9$$

On the other hand a very generous bound on the number of possible surfaces with total area A and boundary P can be easily constructed in any dimension[7]
Be $A_{min}(P)$ the minimal area enclosed by P; it leads to an upper bound for 3.7:

$$\sigma(m(1) \ldots m(s)) < \frac{4n}{1 - \frac{2d-3}{2g^2 n}} \sum_{P} \exp\left[-|P|\left\{\log 2m\delta - \log\frac{2d-2}{2d-3}\right\}\right] \exp\left[-A_{min}(P) \log\frac{2g^2 n}{2d-3}\right] \qquad 3.10$$

3.10 is valid if

$$2g^2 n > 2d - 3 \; [4] \qquad 3.11$$

Thus we find for the quantity:

$$X_P = \frac{-\log \sigma_P (m(1) \dots m(s))}{A_{min}(P)}$$

the bound :

$$\log\left(\frac{2g^2n}{2d-3}\right) < X_P$$

for these perimeters P with

$$|P| \, A_{min}(P)^{-1} \to 0 \qquad 3\,12$$

Some remarks are in order :

1° The inequality 3 12 is the basis for the considerations of K-Wilson [5] and Kogut and Susskind [8] for quarkbinding

2° The validity of 3-12 is only shown for $2g^2n > 2d-3$ For d = 4 and reasonable values of n this is still away from the region where our description might be useful (see 3 9) We expect however that the r h.s. of 3.11 can be made much smaller, by taking into account that the surface Σ is closed and simply connected (this did not go into the derivation of 3.10 see ref. 17) Note that for d = 2 both 3.9 and 3.12 are fulfilled !

3° It is not hard to find conditions on the quarkmass , that make the summation over all perimeters P in the l h s.of 3.10 convergent [5]

4. Behaviour of the free energy as function of the mass , when $g^2 \to \infty$

In this section we will show that the free energy for $g^{-2}=0$ is analytic in the mass of the fermion . We have made a simplification : the quarks are spinless and the gauge-group is $U(1)$. The quarks are still represented by an anticommuting variable in the path integral .
We have also looked at the same problem , where the fermions are replaced by bosons . This case seems to be different , though we cannot give a solution

when going to the thermodynamic limit $N \to \infty$ This is so because in the fermionic case we are helped by a theorem of Heilman and Lieb [13]. This theorem does not apply in the boson case .

a) The fermion case

We take the partition function Z_N as in section 2; but we set $g^{-2}=0$ from the outset :

$$Z_N(g^{-2}=0, m\delta) = \int \mathcal{D}U \, \mathcal{D}q^+ \mathcal{D}q \, \exp \sum_{(i,j)} q^+(i) M(ij) q(j) \qquad 4.1$$

The quark integration gives us the usual determinant , which can be analysed as in section 2 in terms of closed non-intersecting disjoint tracks ; each bond B of a track has weight $\frac{e^{i\varphi(B)}}{2}$, each point not lying on a track has weight $m\delta$.

Now we do the $\mathcal{D}U = \frac{d\varphi}{2\pi}$ integrations, and it is clear that only those tracks will contribute where the weight appears twice with opposite phase . These are the dimers . The result is :

$$Z_N(0, m\delta) = \left\{ \begin{array}{l} \text{Sum of all configurations of dimers} \\ \text{and monomers . Dimers have weight } \frac{1}{4}, \\ \text{monomers weight } m\delta \end{array} \right. \qquad 4\,2$$

" Monomers " are by definition the points not lying on any track . Now we can use the above-mentioned theorem [13)]

$$\text{If} \quad F(0, m\delta) = \lim_{N\to\infty} \frac{1}{N} \log Z_N(0, m\delta) \qquad 4\,3$$

then F is analytic in $m\delta$. (Actually the theorem is much stronger and could be used for the case where sources coupled to the q^+Uq terms are introduced). Therefore any transition at $g^{-2}=0$ in the variable $m\delta$ is excluded . The result is true in any dimension . Only for d=2 , and $m\delta=0$ the result is known analytically . [14)]

b) <u>the boson case</u>

Let the boson variables be b(i) and $b^{\dagger}(i)$. Then the partition function reads (see section 2)

$$Z_N(o, m\delta) = \int \mathcal{D}U\, \mathcal{D}b^{\dagger}\, \mathcal{D}b \exp\left[b^{\dagger}(i)\, e^{i\varphi(ii')}\, b(i') - \bar{m}^2 \sum_i b^{\dagger}(i) b(i)\right] \qquad 4.4$$

$(\bar{m} = m\delta)$

Doing the boson integrations gives the <u>inverse</u> of the determinant of the boson matrix . So even for finite N Z_N is no longer a polynomial in the (reciprocal) mass . Therefore the question of convergence poses itself already for finite N . This problem is attacked best by first integrating the gauge field variables . We get then :

$$Z_N(o, \bar{m}) = \int \mathcal{D}b^{\dagger}\, \mathcal{D}b \exp\left[-\bar{m}^2 \sum_i b^{\dagger}(i) b(i)\right] \prod_{\langle ii'\rangle} J_0\left(2i\, |b(i)\, b(i')|\right)$$

Clearly if $\mathrm{Re}\,\bar{m}^2$ is large enough , 4-5 will converge . Since

$$J_0(-2i\,|b(i)\, b(i')|) \sim \sqrt{\frac{\pi}{|b(i)\, b(i')|}}\, \exp|b(i)\, b(i')| \qquad 4.6$$

for large $|b(i)|$ we find that the integrand in 4 5 will behave as:

$$\prod_i |b(i)|^{-d} \exp\left[-\bar{m}^2 \sum_i |b(i)|^2 + \sum_{\langle ii'\rangle} |b(i)\, b(i')|\right]$$

The eigenvalues of the adjacency matrix appearing in the exponential should all be smaller then $\bar{m}^2$ in order to have 4.5 convergent . The eigenvalues are [9] :

$$\alpha(k_1 \cdots k_d) = 2\left(\cos \frac{k_1}{M} 2\pi + \cdots + \cos \frac{k_d}{M} 2\pi\right) \qquad 4.7$$

where each k_j runs from 1 to $N^{1/d}$ M.Therefore , independent of N, we find that 4.5 is convergent and analytic if $\mathrm{Re}\,\bar{m}^2 \geq 2d$ for $d \geq 3$. For d = 2 we have a singularity at $\bar{m}^2 = 2d$. We can push the domain of analyticity by noting that 4.5 is absolutely monotone in the variable $\bar{m}^{-2}$

This follows from the expansion

$$J_0(-2i\,|b(i)\, b(i')|) = \sum_{n=0}^{\infty} \frac{|b(i)|^{2n}}{n!} \cdot \frac{|b(i')|^{2n}}{n!} \qquad 4.8$$

and by integrating term by term . The ensuing series in $\bar{m}^{-2}$

$$Z_N(o, \bar{m}^2) = \sum_{\ell=0}^{\infty} a_\ell(N)\, \bar{m}^{-2\ell} \qquad 4.9$$

has clearly only positive coefficients , and we infer absolute monotony We know that 4.9 converges on the real m^2 axis from o to $\frac{1}{2d}$ Bernstein's theorem [15] tells us that 4.9 will converge everywhere in the circle It is easy to see that $Z_N(o, \bar{m}^2)$ must have singularities outside the circle for general dimension . What the implications are for the free energy in the thermodynamic limit we have not been able to calculate . The important difference with the fermion case is that Z_N is not a polynomial in $\bar{m}$, but a function with already singularities(which have nothing to do with the thermodynamic limit)

CONCLUSION

The long distance properties of Yang-Mills fields coupled to quarks are studied . The possibility of having a simple linear system (the string) as the leading contribution is demonstrated in an approximation scheme where the dimension n of the gauge group is the expansion parameter and where the coupling constant g^2 is proportional to $\frac{1}{n}$. This should guide us when looking for the "mass-shell" of the theory . Furthermore we have investigated for very large coupling constant what happens to the free energy as a function of the fermion mass : In order to have a critical point we should consider values of g^2 not too large . The point that should be emphasized in this approach is the following :If we want to make the lattice length δ small, much smaller then say $1\ (\mathrm{GeV})^{-1}$, then comparison of our formula for the leading contribution with the " string " formula yields $2g^2 n \approx 1$
On the other hand we would like to maintain the " area rule " eqn 3.12 , which we derived under the condition $2g^2 n > 2d-3$; this condition , trivially satisfied for d = 2 , can be certainly relaxed down to some critical value for d = 4 . We find it however difficult to do with our present means (taking into account closedness of surfaces etc) . We envisage a renormalisation transformation in configuration space that should determine the corresponding fixed point [18] , in the $n \to \infty$, $g^2 n$ fixed limit .

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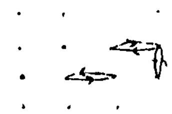
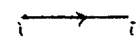
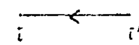
I am indebted to many . In particular I thank Dr. G 't Hooft , and Professor K.G. Wilson for useful discussions and advice . To Dr. C. Becchi and Dr S. Miracle-Sole go my thanks for their interest and patience as well as to J. Bellissard and A. Messager .

Fig. 1: The lattice with the directed bonds between any two points :



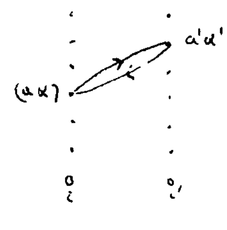


weight associated to the bond (i,i') is the matrix $\gamma(i\,i')\ U(i\,i')$



weight associated to the bond (i',i) is the matrix $\gamma(i'\,i)\ U(i'\,i)$

Fig.2 To each point i in Euclidean 4- space a number of 4n levels is attributed : each level corresponds to a quark degree of freedom; the labeling is given by (a,α) , $a = 1,2,3,4$, $\alpha = 1 \dots n^2$

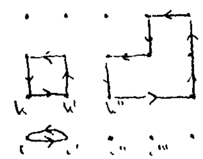


A Bond directed from i to i' and connecting levels $a\alpha$ with $a'\alpha'$ carries a weight $\gamma_{aa'}\ U_{\alpha\alpha'}$

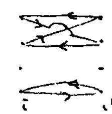
A bond directed from i'to i and connecting levels with $a\alpha$ carries a weight $\gamma_{a'a}\ U_{\alpha'\alpha}$

There are no bonds between levels at the same site .

Fig.3 Possible terms in det M



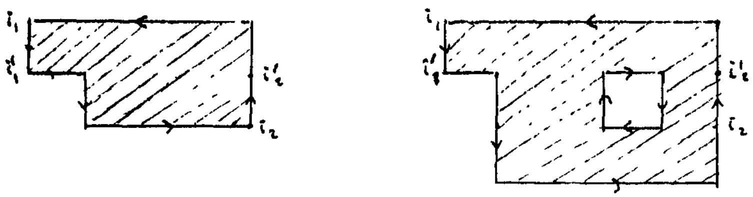
a) one quark degrees of freedom



b) 4 quark degrees of freedom

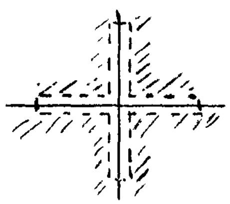
Fig.4:

Some connected planar gauge invariant quantities ($s=1$):

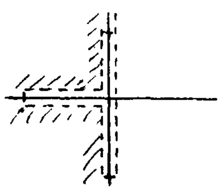


Cross-hatched regions are filled with squares .

Fig.5:



The broken lines parallel to the bonds correspond to the unitary index of the matrices involved . They are connected by a broken line crossing the bond by means of the integration 2.25 (a) . It is easily seen that also vertices on the borders of a planar surface carry an index cycle as in fig. 5 (b) .



FOOTNOTES

[1] The gauge group in question is the colorgauge group .

[2] 't Hooft mentions this possibility in ref.3

[3] If more than one quarkloop is present the sources may be coupled to different quarkloops These contributions are all of the order determined by Q , the number of quarkloops (see section 3b)

[4] The bound 2d-3 stems from the observation that one can couple an elementary square to a given one in a given bond in only 2d-3 ways , if we want to avoid overlapping squares

[5] A generous bound is got by putting :

$$R_{min}(P) > \frac{|P|}{4}$$

and by noting that:
number of closed self avoiding quarkloops going through bonds $1, \ldots, s$ with fixed length $|P|$ is smaller then

$$2d(2d-1)^{|P|-1}$$

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Quark Confinement*

1115

Kenneth G. Wilson

Laboratory of Nuclear Studies, Cornell University,

Ithaca, New York 14850

A model of confined quarks is presented. It involves an abelian (or non-abelian) gauge field coupled to a quark field. The theory is quantized on a discrete space-time lattice. It shows quark confinement for strong coupling. Specifically, for strong coupling isolated quarks have infinite mass and quark-antiquark pairs with a large separation r have an energy proportional to r. The theory has local gauge invariance as an exact symmetry in strong coupling. The strong coupling theory is far from covariant due to the lattice.

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Quark confinement is of great interest at present. It is important that the confinement mechanism be soft. That is, the potential energy of two quarks should be small when they are close together so that quarks can respond as free particles in deep inelastic electron scattering. The potential energy should become large when the two quarks are well separated, to prevent any quarks from escaping a nucleon and becoming observable quark-like particles. Kogut and Susskind[1] have speculated that the potential energy of two quarks (to be precise, a quark and an antiquark) will be proportional to the distance r between them when r is large. This is the result found in one-dimensional quantum electrodynamics for the potential energy of an electron-positron pair.

A model which exhibits quark confinement and a linear potential between quarks will be described in this paper. The model is a gauge field coupled to a quark field, quantized on a discrete space-time lattice.[2,3] The quarks of the model are confined for sufficiently strong gluon-quark coupling constant g. (For weak coupling the lattice theory behaves like a conventional continuum theory.) This is true for any space dimension.

The model has one major drawback: It is noncovariant due to the lattice. The lattice provides an ultraviolet momentum cutoff. The model can be solved explicitly only in a limit where all physical masses are much larger than this cutoff, so the explicit solution is far from covariant. The problem of achieving covariance will be discussed further at the end of this paper. The properties of the model are sufficiently remarkable to make it worth study despite the lack of covariance.

A detailed description of the model is given in two preprints.[2,3] Only some of the principal features will be described here.

The Lagrangian of the model depends on three parameters: the bare coupling constant g, the bare quark mass m_o and the lattice spacing a. The model can be solved explicitly in two limits. One limit is the weak coupling limit (g small). The other limit is g and $m_o a$ both large (but not infinite). The principal features of the large g,$m_o a$ limit are as follows:

1) An isolated quark has infinite mass.

2) A well-separated quark-antiquark pair has a finite energy proportional to their separation r, namely proportional to $(\ln g^2)r/a^2$.

3) The theory has local gauge invariance as an exact symmetry of the quantized theory (for weak coupling the local gauge symmetry is spontaneously broken).

4) The infinite quark mass is a consequence of exact local gauge invariance.

Local gauge invariance means the following. For example consider the quark propagator on the lattice. This is

$$S(n) = \langle\Omega|T\psi_n\bar{\psi}_o|\Omega\rangle \tag{1}$$

where $|\Omega\rangle$ is the vacuum state and n is a four-vector with integral components [e.g. n = (0,1,2,1)] labelling a lattice site. A local gauge transformation is a transformation

$$\psi_n \rightarrow e^{i\phi_n}\psi_n \tag{2}$$

$$\bar{\psi}_n \rightarrow e^{-i\phi_n}\bar{\psi}_n \tag{3}$$

Exact symmetry under this transformation means that

$$S(n) = e^{i(\phi_n-\phi_o)}S(n) \tag{4}$$

which is possible only if

$$S(n) \propto \delta_{no} \tag{5}$$

This means quarks cannot propagate, i.e. they have infinite mass. To be precise the model will be quantized in a Euclidean metric (imaginary time). In the Schrödinger representation the propagator, for $n_o > 0$, is

$$S(n) = \langle\Omega|\psi_n e^{-Hn_o a}\bar{\psi}_o|\Omega\rangle . \tag{6}$$

The vanishing of S(n) for $n_o > 0$ is the symptom of a more general result:

$$e^{-Hn_o a}\bar{\psi}_o|\Omega\rangle = 0 \tag{7}$$

which means $\bar{\psi}_o|\Omega\rangle$ is a state with infinite energy.

The model will be quantized using the Feynman path integral framework, hence the quantity of interest is the action. The first step in defining the model is to construct the action on a discrete space-time lattice. The action will be defined so that it is exactly gauge invariant on the lattice. The crucial part of the lattice action will be the free gauge field piece, but it is convenient to consider first the quark part. Consider the free quark continuum action

$$A = \int d^4x\, \bar{\psi}(x)(i\gamma^{\mu}\nabla_{\mu}-m_o)\psi(x) \tag{8}$$

Converting this to a lattice action in the most naive way one makes the replacements

$$\int d^4x \rightarrow a^4\Sigma_n \tag{9}$$

$$\bar{\psi}(x),\psi(x) \rightarrow \bar{\psi}_n,\psi_n \tag{10}$$

$$\nabla_{\mu}\psi(x) \rightarrow (\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}})/2a \tag{11}$$

where $\hat{\mu}$ is a unit lattice vector (length a) along the axis μ. This gives

$$A = \Sigma_n \left\{ \frac{a^3}{2}\, \bar{\psi}_n i\gamma_{\mu}(\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}) - m_o a^4 \bar{\psi}_n\psi_n \right. \tag{12}$$

The product $\bar{\psi}_n\psi_{n+\hat{\mu}}$ is not gauge invariant. To make it gauge invariant one uses the standard procedure: one inserts an exponential of the line integral of the gauge field. A lattice approximation to the line integral from n to $n+\hat{\mu}$ is just $gaA_{n\mu}$ where $A_{n\mu}$ is the gauge field at lattice site n.

The expression

$$\bar{\psi}_n\gamma_{\mu}\psi_{n+\hat{\mu}}e^{igaA_{n\mu}}$$

is gauge invariant provided that the gauge transformation law for $A_{n\mu}$ is

$$A_{n\mu} \rightarrow A_{n\mu} - (\phi_{n+\hat{\mu}} - \phi_n)/ga \tag{13}$$

Thus a gauge-invariant action for the quark field is

$$A = \Sigma_n \left\{ \Sigma_{\mu} \frac{a^3}{2}\, (\bar{\psi}_n i\gamma_{\mu}\psi_{n+\hat{\mu}}e^{igaA_{n\mu}} - \bar{\psi}_{n+\hat{\mu}} i\gamma_{\mu}\psi_n e^{-igaA_{n\mu}}) + m_o a^4 \bar{\psi}_n\psi_n \right\} \tag{14}$$

In this action the variable $gaA_{n\mu}$ appears as an angle. That is, the action is periodic in the variable gaA_{μ} with period 2π. This is no accident. The gauge transformation variable ϕ_n is an angle so to preserve gauge invariance $gaA_{n\mu}$ is an angle, too.

It is natural to try to write the free gauge field action to preserve periodicity in $gaA_{n\mu}$. First we need a lattice approximation to $F_{\mu\nu}(x)$, namely

$$F_{n\mu\nu} = \frac{1}{a}\left\{A_{n+\hat{\mu},\nu} - A_{n\nu} - A_{n+\hat{\nu},\mu} + A_{n\mu}\right\} \tag{15}$$

Let

$$B_{n\mu} = gaA_{n\mu} \tag{16}$$

$$f_{n\mu\nu} = ga^2F_{n\mu\nu} \tag{17}$$

$$f_{n\mu\nu} = B_{n+\hat{\mu},\nu} - B_{n\nu} - B_{n+\hat{\nu},\mu} + B_{n\mu} \tag{18}$$

Then a gauge field action which is gauge invariant and periodic in the variables $B_{n\mu}$ is (A_{GF} means gauge field action)

$$A_{GF} = \frac{1}{2g^2} \Sigma_n \Sigma_\mu \Sigma_\nu e^{if_{n\mu\nu}} \qquad (19)$$

It is evident that this action is periodic. To show that it is gauge invariant, it is useful to represent $B_{n\mu}$ graphically by a directed line from the lattice site n to the site $n+\hat{\mu}$ (Fig. 1). A gauge transformation acts as follows: at the site $n+\hat{\mu}$, where the line is incoming, one has $B_{n\mu} \to B_{n\mu} - \phi_{n+\hat{\mu}}$. At the site n where the line is outgoing the transformation has the opposite sign: $B_{n\mu} \to B_{n\mu} + \phi_n$. Then $f_{n\mu\nu}$ is represented by a closed square (Fig. 2) with the four corners of the square being the sites n, $n+\hat{\mu}$, $n+\hat{\mu}+\hat{\nu}$, and $n+\hat{\nu}$. (Backwards lines represent a - B, e.g. from $n+\hat{\nu}$ to n represents $-B_{n\nu}$.) It is easily seen that the gauge transformations cancel at each corner of the square. Thus $f_{n\mu\nu}$ is gauge invariant.

In the limit $a \to 0$ the exponential (19) reduces to the usual continuum action. This is seen as follows. For a small, $f_{n\mu\nu}$ is also small [from Eq. (17)], hence

$$A_{GF} = \frac{1}{2g^2} \Sigma_n \Sigma_\mu \Sigma_\nu \left\{ 1 + iga^2 F_{n\mu\nu} - \frac{1}{2} g^2 a^4 F_{n\mu\nu}^2 - \ldots \right\} \qquad (20)$$

A constant term has no effect on physics; it can be ignored. The term linear in $F_{n\mu\nu}$ is zero because $F_{n\mu\nu}$ is odd under the interchange $\mu \to \nu$. Hence the double sum over μ and ν of $F_{n\mu\nu}$

gives zero. Thus the dominant term is the $F_{n\mu\nu}^2$ term. The factors of g^2 cancel, and the factor a^4 converts the sum over n to an integral. Higher order terms ($F_{n\mu\nu}^3$, etc.) have higher powers of a and therefore vanish in the $a \to 0$ limit.

The above analysis of the $a \to 0$ limit is only valid classically: It was assumed in this analysis that $F_{n\mu\nu}$ has a limit (i.e. $F_{\mu\nu}(x)$) for $a \to 0$ and this is true classically but not quantum mechanically. This will be discussed further below.

The complete action of the quark-gluon lattice theory is

$$A = \Sigma_n \Big\{ \Sigma_\mu (a^3 \bar{\psi}_n \, i\gamma_\mu \psi_{n+\hat{\mu}} e^{iB_{n\mu}} - a^3 \bar{\psi}_{n+\hat{\mu}} \, i\gamma_\mu \psi_n e^{-iB_{n\mu}}) + m_o a^4 \bar{\psi}_n \psi_n$$
$$+ \frac{1}{2g^2} \Sigma_\mu \Sigma_\nu \, e^{if_{n\mu\nu}} \Big\} \qquad (21)$$

This choice for the action is not unique. One could have used $f_{n\mu\nu}^2$ for $e^{if_{n\mu\nu}}$, for example; this would violate only the ad hoc requirement of periodicity. However only an exponential form such as given above generalizes easily to nonabelian gauge theories. In the nonabelian case I cannot find a gauge invariant form on the lattice for $f_{n\mu\nu}$, but $e^{if_{n\mu\nu}}$ is easily generalized. In the nonabelian case one has a set of variables $B_{n\mu}^\alpha$. For each choice of n and μ, the variables $B_{n\mu}^\alpha$ parameterize a finite element of the gauge group (not an infinitesimal element).

One replaces $e^{iB_{n\mu}}$ by a unitary transformation $U[B^{\alpha}_{n\mu}]$ representing this element in the quark representation. In place of $e^{if_{n\mu\nu}}$ one substitutes the trace of a product of four unitary transformations:

$$e^{if_{n\mu\nu}} \rightarrow \mathrm{Tr}\, U[B^{\alpha}_{n\mu}]U[B^{\alpha}_{n+\hat{\mu},\nu}]U^{-1}[B^{\alpha}_{n+\hat{\nu},\mu}]U^{-1}[B^{\alpha}_{n\nu}] \qquad (22)$$

This trace is gauge invariant under the nonabelian gauge group.

The quantum mechanics of the lattice gauge theory can now be defined using the Feynman path integral. The action defined above is actually in a Euclidean metric[4], so the resulting path integral will define vacuum expectation values for imaginary times. One can still define the full quantum mechanics in real time using the "transfer matrix" method, which is discussed in Ref. 2. The path integral involves integrating over all values of the $B_{n\mu}$ plus an "integral" over the fermion variables ψ_n. The fermion integral is discussed in Ref. 3; denote it by < >. Then the path integral for the quark propagator is

$$S(n) = \frac{\prod_m \prod_\mu \int_{-\pi}^{\pi} dB_{m\mu} \, \langle \psi_n \bar{\psi}_o e^{A} \rangle}{\prod_m \prod_\mu \int_{-\pi}^{\pi} dB_{m\mu} \, \langle e^{A} \rangle} \qquad (23)$$

where A is the action of Eq. (23). The periodicity of A with respect to the $B_{n\mu}$ makes it unnecessary to integrate over more

than 1 period ($-\pi$ to π) of $B_{n\mu}$. The denominator is necessary to remove vacuum-to-vacuum graphs. The denominator will be denoted by Z in subsequent formulae.

In the weak coupling limit, $1/g^2$ is large and therefore the path integral will be dominated by values of $f_{n\mu\nu}$ for which $e^{if_{n\mu\nu}}/g^2$ is near its maximum. This means, essentially, small $f_{n\mu\nu}$, which allows one to make the expansion of Eq. (20), giving back the conventional gauge theory action. There are details that have to be straightened out in this limit but it appears that the small g limit of the lattice theory agrees with the usual continuum theory. (This requires that $m_o a$ also be small.)

The interesting limit is the limit g large. Calculations are simple in this limit if $m_o a$ is also large. In this case both the gauge field action and the γ_μ terms of the quark action can be treated as a perturbation. The lowest order approximation is to neglect these terms altogether, giving

$$S(n) \simeq Z^{-1} \prod_m \prod_\mu \int_{-\pi}^{\pi} dB_{m\mu} \, \langle \psi_n \bar{\psi}_o e^{m_o a^4 \Sigma_m \bar{\psi}_m \psi_m} \rangle \qquad (24)$$

Because the action in this expression is quadratic in the fields ψ_n and $\bar{\psi}_n$, the < > can be calculated by Wick's theorem. The bare propagator to be used in Wick's theorem is

$$S_o(n) = (m_o a^4)^{-1} \delta_{no} \qquad (25)$$

The approximation (24) gives $S(n) = S_o(n)$; therefore $S(n) \propto \delta_{no}$ as predicted earlier.

In higher orders it remains true that $S(n)$ is proportional to δ_{no}. To see this one first expands in powers of the γ_μ terms in the action and picks out terms which give a nonzero fermion integral < >. It is not difficult to see that the fermion integral is nonzero only if one can combine terms of the form $a^3\bar{\psi}_m i\gamma_\mu\psi_{m+\hat{\mu}}e^{+iB_{m\mu}}$ (or complex conjugate) to form a path from the origin to the lattice site n: the term $\bar{\psi}_m\gamma_\mu\psi_{m+\hat{\mu}}e^{iB_{m\mu}}$ is represented by a line element from $m+\hat{\mu}$ to m in this path. The fermion integral then replaces the ψ's and $\bar{\psi}$'s by products of $(m_o a^4)^{-1}$. One is left with the integration over B's which now has the form

$$Z^{-1}\prod_m\prod_\mu\int_{-\pi}^{\pi} dB_{m\mu}\, e^{-i\Sigma_o^n B_{m\mu}} \exp\left\{\frac{1}{2g^2}\Sigma_m\Sigma_\mu\Sigma_\nu e^{if_{m\mu\nu}}\right\} \qquad (26)$$

where $\Sigma_o^n B_{m\mu}$ means a sum over the B's on the path from 0 to n defined above.

To clarify this formula a bit consider the case that the site n is a nearest neighbor site to the origin, say $n = (0,1,0,0) = \hat{1}$. Then one needs only a single term, namely $-a^3\bar{\psi}_{\hat{1}}i\gamma_1\psi_o e^{-iB_{o1}}$ from the expansion. That is, one writes

$$\langle\psi_1\bar{\psi}_o \exp\left\{\Sigma_{n\mu}a^3(\bar{\psi}_n i\gamma_\mu\psi_{n+\hat{\mu}}e^{iB_{n\mu}} - \bar{\psi}_{n+\hat{\mu}}i\gamma_\mu\psi_n e^{-iB_{n\mu}})\right\}\rangle$$
$$\simeq \langle\psi_1\bar{\psi}_o\left\{1 - a^3\bar{\psi}_{\hat{1}}i\gamma_1\psi_o e^{-iB_{o1}} - \ldots\right\}\rangle \qquad (27)$$

where only one term from the expansion of the exponential is written out. This bracket when calculated by Wick's theorem gives

$$(m_o a^4)^{-2}a^3\,(-i\gamma_1)e^{-iB_{o1}} \qquad (28)$$

Here the sum $\Sigma_o^n B_{m\mu}$ reduces to the single term B_{o1}. When the exponential in (27) is expanded to higher orders there are additional terms contributing to the expression (28); these additional terms are of higher order in $(m_o a)^{-1}$ and contain more complicated sums of B's in the exponential.

The integral (26) is zero to all orders in g^{-2}, except when n = 0. It is zero in lowest order because $\int_{-\pi}^{\pi} dB\, e^{-iB} = 0$ and there is one such integral for each link in the path from 0 to n. Expanding in powers of g^{-2}, the next term has as an integrand

$$\Sigma_{\ell\pi\sigma}e^{-i\Sigma_o^n B_{m\mu}}e^{if_{\ell\pi\sigma}} \qquad (29)$$

The exponential can be represented by a more complicated path combining a square (for $f_{\ell\pi\sigma}$) with the original path from 0 to n. There is no way a closed square of B's can cancel an open path, so in order g^{-2} there remain some integrals of the form $\int_{-\pi}^{\pi} dB\, e^{-iB}$. This is true in all higher orders as well:

hence $S(n)$ is proportional to δ_{no}.

The underlying reason for $S(n)$ being proportional to δ_{no} is local gauge invariance. The action given in Eq. (21) is invariant to local gauge transformations. In continuum theories one has to add a gauge-fixing term, which explicitly breaks local gauge invariance. This was not necessary for the lattice theory. The reason for this is the finite range of the $B_{n\mu}$ integrations. In the continuum theory, where the path integral over $A_\mu(x)$ has infinite limits, exact local gauge invariance leads to divergent infinite integrations.[5] No such divergences appear in the lattice theory. It was shown in the introduction that local gauge invariance implies that $S(n)$ is proportional to δ_{no}.

In weak coupling the quark propagator is certainly not proportional to δ_{no}. What happens in weak coupling is that local gauge invariance is spontaneously broken. This means, as usual with spontaneous breaking, that an infinitesimal gauge breaking term must be added to the action in order to calculate $S(n)$ correctly. The details of this spontaneous breaking are complex and have not been carefully worked out as yet.[6]

Now consider a well-separated quark-antiquark pair. The quantity of interest is the minimum energy of the pair at a fixed distance r. One has to form a gauge invariant state

containing this pair -- otherwise the state will have infinite energy. A simple gauge-invariant state is

$$\bar{\psi}_n e^{i\Sigma_o^n B_{m\mu}} \psi_o|\Omega\rangle \tag{30}$$

with the lattice site n being at time 0 ($n_o = 0$) and a spatial distance $r = |\vec{n}|a$ from the origin. This state, using the shortest path from 0 to n, turns out to have the minimum energy for two quarks separated by r, in the limit of large g and large $m_o a$. To determine its energy one can study the time dependence of the Schrödinger representation expression

$$\langle\Omega|\left\{\bar{\psi}_n e^{i\Sigma_o^n B_{m\mu}} \psi_o\right\}^+ e^{-Ht}\left\{\bar{\psi}_n e^{i\Sigma_o^n B_{m\mu}} \psi_o\right\}|\Omega\rangle \tag{31}$$

With $t = n_o a$ this becomes in the Heisenberg representation

$$\langle\Omega|T\bar{\psi}_{n_o,o}\, e^{-i\Sigma_o^n B_{m\mu}}\, \psi_{n_o,\vec{n}}\, \bar{\psi}_{o,\vec{n}}\, e^{i\Sigma_o^n B_{m\mu}}\, \psi_o|\Omega\rangle \tag{32}$$

When this is computed through the Feynman path integral and the fermion integral is computed, one is left with a gauge field expectation value to compute, namely

$$Z^{-1} \prod_m \prod_\mu \int_{-\pi}^{\pi} dB_{m\mu} \exp\left\{i\, \Sigma\, B_{m\mu}\right\} \exp\left\{\frac{1}{2g^2} \Sigma_m \Sigma_\mu \Sigma_\nu e^{if_{m\mu\nu}}\right\} \tag{33}$$

where Σ is a sum over the closed path illustrated in Fig. 3. The two horizontal (equal-time) legs are the paths originally present in the vacuum expectation value; the vertical legs are generating by expanding in powers of the γ_μ terms of the complete action. The shortest vertical path is used to get the minimum power of $(m_o a)^{-1}$ from the fermion integral. (This power has been omitted in the expression (33).)

The gauge field integral is easily determined for large g. In order to eliminate integrals of the form $\int_{-\pi}^{\pi} dB\, e^{\pm iB}$, it is necessary to expand to sufficiently high order in g^{-2} so that the entire area enclosed by the path of Fig. 3 can be filled by squares representing $f_{m\mu\nu}$'s (Fig. 4). There is a factor g^{-2} for each $f_{m\mu\nu}$. Hence the expectation value (33) behaves as

$$(g^{-2})^{|\vec{n}| n_o}$$

This is proportional to $e^{-En_o a}$ where E is the energy of the quark-antiquark state. Thus

$$E \simeq \frac{1}{a} |\vec{n}| \ln g^2 = r(\ln g^2)/a^2 \qquad (34)$$

Hence the speculation of Kogut and Susskind is confirmed for this model when g and $m_o a$ are both large.

Consider finally the limit $a \to 0$. One would like the energy E(r) to be finite in this limit. This is evidently not true if g^2 is held fixed (and large) as $a \to 0$; this limit gives $E(r) \to \infty$, from Eq. (34).

This is a different result than the earlier (classical) result that a limit exists for $a \to 0$. The classical argument fails for the quantum mechanics at large g because $B_{n\mu}$ ranges independently from $-\pi$ to π for each n and μ. Only for a negligible subset of this integration range is $B_{n\mu}$ a slowly varying function of the lattice label n, and hence only for this negligible subset do continuum functions $A_\mu(x)$ and $F_{\mu\nu}(x)$ exist.

The only hope to obtain a continuum limit is to let g decrease as a decreases; but this means leaving the range of g for which the large g approximation is good. Nevertheless one can discuss qualitatively the behavior of E(r) for intermediate g. First, note that in weak coupling E(r) is a constant for large r (twice the quark mass); in other words the coefficient of r is zero. For sufficiently large coupling g,

$$E(r) = rc(g^2)/a^2 \qquad (35)$$

where $c(g^2) \simeq \ln g^2$ for sufficiently large g^2 but there are additional terms of order $1/g^2$, $1/g^4$, etc. from the full $1/g^2$ expansion.

There must be a critical coupling g_c where the transition from exact gauge invariance to spontaneously broken gauge invariance takes place. If the vacuum changes continuously at g_c, then one would expect $c(g_c^2) = 0$ so that the energy law changes smoothly to the energy law for weak coupling. In this case one can easily define a limit $g \rightarrow g_c$ from above simultaneously with $a \rightarrow 0$ such that $E(r)$ has a continuum limit. Unfortunately it is also possible that there is a discontinuous jump at g_c from a symmetric vacuum to a spontaneously broken vacuum, and in this case $c(g_c^2)$ can be nonzero. This is called a "first order" transition. In this situation no continuum limit is possible.

The evidence from a rough mean field calculation[2,3] is that the transition is first order. However the analysis has so far been restricted to the case $m_0 a \gg 1$, i.e. static quarks. A much broader study needs to be made allowing for small $m_0 a$, ordinary SU(3)XSU(3), and a nonabelian gauge group. This has not been attempted yet.

The importance of the lattice gauge theory is that it provides a specific quantum field theory with confined quarks. It is therefore a model one can use to study what is reasonable in such a theory. Consider for example the potential energy law (34). This form for $E(r)$ has always seemed (to the author, at least) to be in contradiction with locality. In a theory with no zero mass particles (there are no zero mass particles in the strongly coupled gauge theory) it has

usually been the rule that the potential energy of two particles decreases exponentially at large distances, apart from the rest energy.

One can make a rough argument to show that a linear potential energy is a natural consequence of locality given that isolated quarks have infinite mass. In a theory with specific states with infinite energy, the Hamiltonian is not the most well-defined operator and it is better to study e^{-Ht}. Consider the expectation value of e^{-Ht} for a quark-antiquark ($q\bar{q}$) state with separation r:

$$\langle q\bar{q}|e^{-Ht}|q\bar{q}\rangle$$

For sufficiently distant q and $\bar{q}$, one expects e^{-Ht} to behave like a product of operators, one each for q and $\bar{q}$; in other words one expects

$$\langle q\bar{q}|e^{-Ht}|q\bar{q}\rangle \simeq \langle q|e^{-Ht}|q\rangle \langle\bar{q}|e^{-Ht}|\bar{q}\rangle + O(e^{-b(t)r}) \quad (36)$$

This is a cluster decomposition formula so one's intuition is that the error in this formula falls off exponentially in r, as indicated. The function $b(t)$ is unknown. If the quark and antiquark had a finite mass m, one would then have

$$e^{-E(r)t} = e^{-2mt} + O(e^{-b(t)r}) \quad (37)$$

i.e.

$$E(r) = 2m + O(e^{-b(t)r}) \quad (38)$$

which is the normal situation. But for infinite mass quarks the right hand side of (36) vanishes. In this case one is left with

$$E(r)t = b(t)r \tag{39}$$

Clearly in this case b(t) must be proportional to t and E(r) is proportional to r. Thus the linear form for E(r) appears reasonable, and the model provides an example to back up this general argument.

Ongoing studies of the lattice gauge theory model include extensive calculations of the strong coupling expansion[7] and a formulation of the Hamiltonian of a nonabelian gauge theory on the lattice.[8]

I have been helped by many people at Cornell, Orsay, and elsewhere to bring these ideas into focus.

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2. K. Wilson, Cornell preprint CLNS-262 (out of print; submitted to Phys. Rev. D).
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4. This can be seen from the approximation (20) which has no i multiplying $F_{n\mu\nu}^2$. The anticommutators $\{\gamma_\mu,\gamma_\nu\}$ must be $\delta_{\mu\nu}$ (in place of $g_{\mu\nu}$ or $-\delta_{\mu\nu}$). Also the quark action has unwanted symmetries: it is, for example, symmetric to $\psi_n \to (-1)^{n_1}\psi_n$, $\bar{\psi}_n \to (-1)^{n_1}\bar{\psi}_n$, $B_{n1} \to B_{n1}+\pi$ which implies that quarks with lattice momentum π have the same energy as quarks of zero momentum. This unpleasant feature can be removed by adding a (gauge invariant) lattice approximation to $\int a\bar{\psi}(x)\nabla^2\psi(x)d^4x$ to the action; the extra factor a is to make this term vanish in the continuum limit.
5. See, e.g. E. Abers and B. W. Lee, Physics Reports 9C, 1 (1974), especially p. 83.
6. See Refs. 2 and 3 for incomplete discussions of the spontaneous breaking.
7. R. Balian, J. M. Drouffe, and C. Itzykson (unpublished).
8. J. B. Kogut and L. Susskind (unpublished).

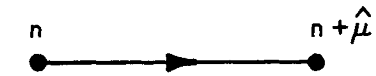
Figure Captions

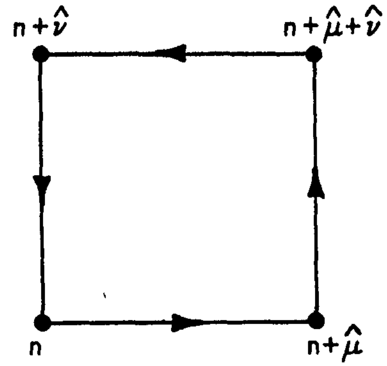
Fig. 1 Graphical representation of gauge field $B_{n\mu}$.

Fig. 2 Graphical representation of $f_{n\mu\nu}$.

Fig. 3 Closed loop sum of $B_{n\mu}$ appearing in Eq. (33).

Fig. 4 Squares enclosed by the loop of Fig. 3; the loop sum is cancelled by the sum of $f_{\ell\pi\sigma}$ over the 12 enclosed squares.





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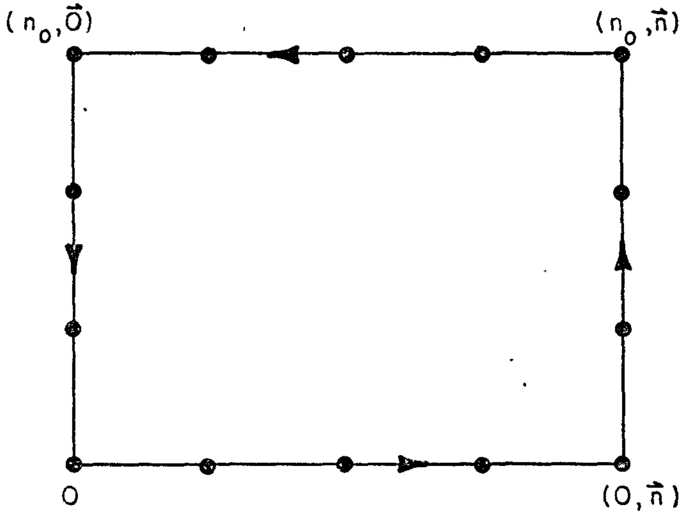

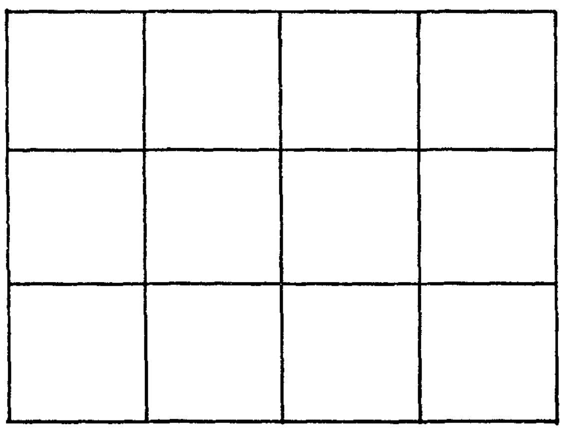

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V - CRITICAL EXPONENTS

SHORT DISTANCE EXPANSIONS AND CRITICAL PHENOMENA

by

E. BREZIN

Wilson's short-distance expansion of an operator product gives informations about the next-to-leading asymptotic behavior of correlation functions in the critical domain. It may also be extended to the region in which the symmetry is broken and more generally to a situation with an arbitrary value of the expectation value of the field.

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Ref.TH.1943-CERN

CRITICAL PHENOMENA IN FERROMAGNETIC SPIN SYSTEMS ON LATTICES

J. Zinn-Justin
CERN -- Geneva
and
Service de Physique Théorique, CEN, Saclay

ABSTRACT

A perturbation expansion of the partition function for a spin system on a lattice, is used in order to justify the renormalization group equations satisfied by the correlation functions in the critical domain of a second-order phase transition. These renormalization group equations correspond to a field theoretical formulation of Wilson's theory of critical phenomena.

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the Colloquium on Lagrangian Field Theories,
Marseille, June 1974

Ref.TH.1943-CERN
17 October 1974

We shall describe how renormalization group equations appear naturally in the analysis of the structure of the correlation functions near the critical temperature for a spin system on a lattice. These renormalization group equations allow us then to discuss, in a field theoretical language, Wilson's theory of critical phenomena [1]. A more detailed exposal of these results can be found in a review article by Brezin, Le Guillou and Zinn-Justin [2].

1. MEAN FIELD THEORY

For simplicity we shall only consider two-spin interactions, and we shall often specialize the formulae to the Ising model.

The partition function $Z(H)$ in presence of a magnetic field H reads:

$$Z(H) = \sum_{\{S_i\}} \exp\left[\sum_{i,j} V_{ij} S_i S_j + \sum_i H_i S_i\right] \tag{1}$$

where the index i corresponds to the lattice site, and where we have included the factor $\beta = 1/kT$ in the interaction. We assume that V_{ij} is positive (the interaction is ferromagnetic), short-range, and satisfies the hypercubic symmetry and the properties:

$$V_{ij} = V(|\vec{r}_i - \vec{r}_j|) \tag{2}$$

$$V_{ii} = 0 \tag{3}$$

The last condition is purely technical and can be removed in a more general treatment [2].

A way of deriving the mean field approximation is to use the convexity inequality

$$\langle \exp A \rangle \geq \exp \langle A \rangle \tag{4}$$

where the mean value is defined in terms of a one-body density:

$$\exp - \beta \mathcal{H}_0 = \exp \sum_i X_i S_i \tag{5}$$

The X_i are then chosen to maximize $\exp \langle A \rangle$. Taking for A:

$$A = -\beta(\mathcal{H} - \mathcal{H}_0) = \sum_{i,j} V_{ij} S_i S_j + \sum_i H_i S_i - X_i S_i \tag{6}$$

one obtains Peierls variational principle:

$$\ln Z \geq \ln Z_0 - \langle \beta(\mathcal{H} - \mathcal{H}_0) \rangle_0 \tag{7}$$

We now introduce the free energy $W(H)$:

$$W(H) = \ln Z(H) \tag{8}$$

and the function $A(X)$:

$$A(X) = \ln \sum_{\{S\}} \exp SX \tag{9}$$

After maximization on the X_i, the right-hand side of inequality (7) gives the Weiss mean-field approximation for W:

$$W = \sum_{i,j} V_{ij} A'(X_i) A'(X_j) + \sum_i A(X_i) + (H_i - X_i) A'(X_i) \tag{10}$$

where the X_i are given in terms of H by:

$$H_i - X_i + 2 \sum_j V_{ij} A'(X_j) = 0 \tag{11}$$

It is more convenient to discuss the problem in terms of the magnetization M_i:

$$M_i = \frac{\partial W}{\partial H_i} = A'(X_i) \tag{12}$$

The corresponding potential $\Gamma(M)$ is given by the Legendre transformation:

$$\Gamma(M) + W(H) = \sum_i H_i M_i \tag{13}$$

Here we obtain:

$$\Gamma(M) = -\sum_{i,j} V_{ij} M_i M_j + \sum_i B(M_i) \tag{14}$$

with:

$$\sum_i B(M_i) = \sum_i M_i X_i - A(X_i) \tag{15}$$

In the Ising model this gives

$$\Gamma(M) = -\sum_{i,j} V_{ij} M_i M_j + \sum_i \left[M_i \tanh^{(-1)} M_i + \tfrac{1}{2} \ln(1-M_i^2) \right] \quad (16)$$

The magnetization, in the absence of external field, is given by the minimum of Γ. By translation invariance M_i is independent of i, and satisfies:

$$-2VM + \frac{\partial B}{\partial M} = 0 \quad (17)$$

where we have defined

$$V = \sum_i V_{ij} \quad (18)$$

It is easy to see from (16) that for the Ising model, the minimum of Γ stays at the origin as long as V is smaller than a critical value $V_c = \frac{1}{2}$, i.e., as long as the temperature T is larger than a critical value T_c, and then continuously leaves the origin. This continuity of the magnetization is the characteristic of a second-order phase transition. For $V - V_c$ small, i.e., $T - T_c$ small, the magnetization is small and all the properties of the system are determined by the first terms of the expansion of $\Gamma(M)$ around $M \sim 0$, and $T = T_c$

$$\Gamma(M) = -\sum_{i,j} V_{ij} M_i M_j + \sum_i \frac{a}{2!} M_i^2 + \frac{b}{4!} M_i^4 \quad (19)$$

where a, b, in the situation that we describe, are positive ($a = 1$, $b = 2$ for the Ising model). The critical temperature is given here by:

$$V_c = \frac{a}{2} \quad (20)$$

In particular, the equation of state

$$H = \frac{\partial \Gamma}{\partial M} \quad (21)$$

can be written, for H, M and $T - T_c$ small, and after a rescaling of T and M, in a universal scaling form:

$$H/M^{\delta} = f\left((T-T_c)/M^{1/\beta}\right) \quad (22)$$

with

$$f(x) = 1 + x \quad (23)$$

and the critical exponents β and δ have the value

$$\beta = 1/2$$
$$\delta = 3 \quad (24)$$

From the functional (19), one can also derive the two-point correlation function G_{ij}:

$$G_{ij}^{(-1)} = \frac{\partial^2 \Gamma}{\partial M_i \partial M_j} \quad (25)$$

In zero magnetization the Fourier transform $G(\vec{q})$ of G_{ij} has the form:

$$G(\vec{q}) = \sum_j G_{ij} \exp i \vec{q}\cdot\vec{r}_{ij} = \left[a - 2V(\vec{q})\right]^{(-1)} \quad (26)$$

with

$$V(\vec{q}) = \sum_j V_{ij} \exp i \vec{q}\cdot\vec{r}_{ij} \qquad |q_\alpha| \leq \pi \quad (27)$$

From the assumption that V_{ij} is short-range, we know that $V(\vec{q})$ is analytic in q_α for small q. By cubic symmetry it has the expansion

$$V(\vec{q}) = V(1 - \rho^2 \vec{q}^2) + O(q_\alpha^4) \quad (28)$$

$G(\vec{q})$ can then be written:

$$G(\vec{q}) = G(0)\left[1 - \xi^2 \vec{q}^2\right] + O(q_\alpha^4) \quad (29)$$

where ξ is the correlation length.

It diverges with $T - T_c$ as

$$\xi \sim (T-T_c)^{-\nu} \quad (30)$$

with

$$\nu = 1/2 \quad (31)$$

At T_c, $G(q)$ behaves for small q like:

$$G(\vec{q}) \sim q^{-2} \tag{32}$$

All these properties are universal in the sense that they depend only on a few general features of the system.

These predictions can be compared to what we know for the Ising model with next-neighbour interaction. The important feature that emerges from this comparison is that the critical properties of the Ising model depend on the dimension d of the lattice space, unlike the mean-field approximation.

For d infinite, mean-field theory is exact. However, for $d = 1$ we know that no phase transition exists at finite temperature. For $d = 2$ the Ising model can be solved exactly in zero external field. It exhibits a second-order phase transition, various physical quantities like the spontaneous magnetization, the correlation length, etc., have a power law behaviour near T_c, but with different critical exponents

$$\begin{cases} \beta = 1/8 \\ \delta = 15 \\ \nu = 1 \\ \eta = 1/4 \end{cases} \tag{33}$$

where η characterizes the behaviour of $G(q)$ for small q at T_c:

$$G(\vec{q}) \sim q^{\eta-2} \tag{34}$$

These reasons, as well as some phenomenological observations lead various authors like Widom, Domb and Hunter, Kadanoff to postulate, in general, scaling properties and power law behaviour of types (22), (30) and (34), but with universal functions and critical exponents varying with the dimension d of the lattice space.

In order to show why the mean approximation fails, we shall first construct a perturbation expansion, whose first term is precisely the mean-field theory result. Again this can be done for general spin systems [2], but we shall restrict ourselves to the special case that we have considered before.

2. PERTURBATION EXPANSION ON A LATTICE

The main idea is to write the density matrix $\exp - \beta\mathcal{H}(S)$ as an integral over one-body densities, in order to be able to sum over spin configurations.

In the special case considered previously, this is achieved by the transformation [3]

$$\exp \sum_{i,j} V_{ij} S_i S_j = \int d\alpha_i \exp - \sum_{i,j} \alpha_i V_{ij}^{(-1)} \alpha_j + \sum_i 2\alpha_i S_i \tag{35}$$

which in the case of the Ising model yields:

$$Z(H) = \int d\alpha_i \exp - \sum_{i,j} \alpha_i V_{ij}^{(-1)} \alpha_j + \sum_i \ln \mathrm{ch}(2\alpha_i + H_i) \tag{36}$$

If one now computes $Z(H)$ using the steepest descent method, one obtains an expansion similar to the loop-wise expansion of quantum field theory. The first term of this expansion, corresponding to the tree approximation in field theory, yields the mean-field approximation. The main difference with quantum field theory, is that the lattice structure provides a natural cut-off in momentum space to the theory:

$$|q_\alpha| \leq \pi$$

From this expansion, it is easy to understand when and why mean-field approximation fails. The free propagator in zero field is

$$G(\vec{q}) = V(\vec{q}) / [1 - 2V(\vec{q})]^{(-1)} \tag{37}$$

which at T_c behaves like $1/\vec{q}^2$.

A typical one-loop correction to the inverse correlation length squared ξ^{-2} will involve an integral of the form:

$$\delta(\xi^{(-2)}) \sim \int d^d q \, G(\vec{q}) G_c(\vec{q}) \sim (T - T_c) \int_{|q| \leq \pi} \frac{d^d \vec{q}}{\vec{q}^2 (\vec{q}^2 + T - T_c)} \tag{38}$$

For d larger than four, the correction to ξ^{-2} behaves like $T - T_c$ and will not modify the power law behaviour of ξ^{-2} near T_c, but only the coefficient in front of the power. On the contrary, for $d < 4$, this correction will dominate the first term for T close to T_c. This suggests that this approximation can no longer be valid for $d < 4$. In order to evaluate the infra-red singularities of all terms of the perturbation expansion, one can use power counting arguments.

The same arguments which tell us that the term proportional to M_i^4 is the least singular in the large momentum region of all even interactions, show that it is the most infra-red singular. Also, as long as an interaction is not renormalizable, it is not infra-red singular. This explains the role of dimension four. Above four dimensions, no interaction, even in the field, is renormalizable. Therefore, no infra-red singularities will appear for instance in the two-point function. The result is that the coefficients which appear in the mean-field approximation will be modified, but not the general scaling and behaviours. Below four dimensions, the perturbation expansion is useless and the local four-point interaction gives the most singular contribution. Wilson and Fischer [4] have first remarked that if one expands the terms in the perturbation expansion in the parameter $\epsilon = 4 - d$, then order by order the mean-field behaviour is only modified by power of logarithms which can be summed by renormalization group techniques.

3. RENORMALIZATION GROUP EQUATIONS

For the reasons given above, one can in a first step study the critical behaviour of a local renormalizable field theory whose Hamiltonian density reads:

$$\mathcal{H}(x) = \frac{c}{2!}\left(\partial_\mu S(x)\right)^2 + \frac{a}{2!} S^2(x) + \frac{b}{4!} S^4(x) \tag{39}$$

but in which the diagrams of the perturbation series are computed with a cut-off of order one in momentum space. The quantities a, b, c are, near T_c, regular in the temperature.

Let us now make the following remark. As we are only interested in the critical domain defined by

$$\begin{cases} \text{momenta} & |p_i| \ll 1 \\ \text{magnetization} & M \ll 1 \\ t = T - T_c \ll 1 & \end{cases}$$

it is convenient to perform a change of scale which gives a large value Λ to the cut-off

$$\begin{cases} p_i \longrightarrow p_i/\Lambda\,, \quad x_i \longrightarrow \Lambda x_i \\ S, M \longrightarrow \zeta S, \zeta M \end{cases} \tag{40}$$

with

$$\zeta = c^{-1/2}\,\Lambda^{1-d/2}$$

The Hamiltonian then becomes:

$$\mathcal{H}(x) = \frac{1}{2!}\left(\partial_\mu S(x)\right)^2 + \frac{1}{2!} r_0 S^2(x) + \frac{1}{4!} g_0 \Lambda^\epsilon S^4(x) \tag{41}$$

with

$$\epsilon = 4 - d$$

We have introduced g_0 the dimensionless coupling constant.

If r_{0c} is the critical value of r_0, the critical domain is now defined by

$$\begin{cases} |t| \sim |r_0 - r_{0c}| \ll \Lambda^2 \\ \text{momenta} \quad \ll \Lambda \\ \text{magnetization} \quad \Lambda^{1-d/2} \end{cases}$$

Instead of considering the limit where all the quantities t, p_i, M become small, we can now consider the limit in which Λ, the cut-off, becomes large. The behaviour of the theory in the critical domain can therefore be derived from renormalization theory [5].

It is convenient to study first the critical theory $T = T_c$, which corresponds to a massless quantum field theory. Deviation from the critical temperature will be obtained then by adding to the Hamiltonian a small term proportional to $t\,S^2(x)$.

In order to define renormalized correlation functions for a massless theory, we have to introduce a mass scale parameter μ. The renormalized proper vertices [correlation function moments of $\Gamma(M_i)$] are defined in terms of μ by the renormalization conditions:

$$\begin{aligned} &\Gamma_R^{(2)}(p;\mu,g)\big|_{p^2=0} = 0 \\ &\frac{\partial}{\partial p^2}\Gamma_R^{(2)}(p;\mu,g)\big|_{p^2=\mu^2} = 1 \\ &\Gamma_R^{(4)}(p_i)\big|_{S.P.(\mu)} = \mu^\epsilon g \end{aligned} \tag{42}$$

where the symmetry point $SP(\mu)$ is given by:

$$p_i \cdot p_j = \mu^2/3 \; (4\,\delta_{ij} - 1)$$

The original (bare) correlation functions and the renormalized one are related by:

$$\Gamma^{(N)}(p_i, g_0, \Lambda) = Z^{-N/2}(g_0, \Lambda/\mu)\, \Gamma_R^{(N)}(p_i, g, \mu) \qquad (43)$$

This equation actually means that at g and μ fixed, $Z^{N/2}\,\Gamma^{(N)}$ has a limit for large Λ. This implies:

$$\Lambda \frac{d}{d\Lambda}\bigg|_{g,\mu} \left[Z^{N/2}(g_0, \Lambda/\mu)\, \Gamma^{(N)}(p_i, g_0, \Lambda) \right] \underset{|p_i| \ll \Lambda}{\approx} 0 \qquad (44)$$

This yields [6] the renormalization group equation for $\Gamma^{(N)}$

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + W\!\left(g_0, \frac{\Lambda}{\mu}\right) \frac{\partial}{\partial g_0} - \frac{N}{2}\, \eta\!\left(g_0, \frac{\Lambda}{\mu}\right) \right] \Gamma^{(N)}(p_i, g_0, \Lambda) \approx 0 \qquad (45)$$

with

$$\begin{cases} W\!\left(g_0, \frac{\Lambda}{\mu}\right) = \Lambda \frac{d}{d\Lambda}\Big|_{g,\mu} g_0 \\ \eta\!\left(g_0, \frac{\Lambda}{\mu}\right) = -\Lambda \frac{d}{d\Lambda}\Big|_{g,\mu} \ln Z\!\left(g_0, \frac{\Lambda}{\mu}\right) \end{cases} \qquad (46)$$

But $\Gamma^{(N)}(p_i, g_0, \Lambda)$ does not depend on μ, therefore $W(g_0, \Lambda/\mu)$ and $\eta(g_0, \Lambda/\mu)$ are also independent of μ

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + W(g_0) \frac{\partial}{\partial g_0} - \frac{N}{2}\, \eta(g_0) \right] \Gamma^{(N)}(p_i, g_0, \Lambda) = 0 \qquad (47)$$

We are now interested in the large Λ limit of the $\Gamma^{(N)}$. This limit is dominated by the infra-red stable zeros of $W(g_0)$. It is easy to verify that $W(g_0)$ has the form:

$$W(g_0) = -\varepsilon\, g_0 + a(\varepsilon)\, g_0^2 + O(g_0^3) \qquad (48)$$

It has therefore an infra-red stable zero g_0^* of order ε for small ε

$$g_0^* = \frac{\varepsilon}{a(0)} + O(\varepsilon^2) \qquad (49)$$

Therefore the $\Gamma^{(N)}$ behave like:

$$\Gamma^{(N)}(p_i, \Lambda, g_0) \underset{|p_i| \ll \Lambda}{\sim} \Lambda^{N\eta/2} f_N(p_i) \qquad (50)$$

with

$$\eta = \eta(g_0^*) \qquad (51)$$

This gives, in particular, for $\Gamma^{(2)}(p, \Lambda, g_0)$:

$$\Gamma^{(2)}(p, \Lambda, g_0) = G^{(-1)}(p) \sim \Lambda^{\eta}\, p^{2-\eta}$$

This shows that the correlation function $G(p)$ has a power behaviour of the form defined in Eq. (34).

From Eq. (47) one can derive all the correlation function scaling in zero magnetization at T_c. In order to study small deviations from the critical temperature, one has to add to the critical Hamiltonian a term of the form $t\, S^2(x)$.

But one can express the correlation functions with $t \neq 0$ in terms of the correlation functions of the critical theory with an arbitrary number of S^2 insertions

$$\Gamma^{(L,N)}(q_i, p_j, \Lambda, g_0) = \langle \underbrace{S^2 \cdots S^2}_{L},\ \underbrace{S \cdots S}_{N} \rangle_{1PI} \qquad (52)$$

Each S^2 insertion needs a new renormalization factor $Z_2(\Lambda/\mu, g_0)$. This leads to renormalization group equations for the $\Gamma^{(L,N)}$ which can be summed and yield finally renormalization group equations for $\Gamma^{(N)}(p_i, g_0, \Lambda, t)$. It is also possible to expand the correlation functions with non-zero magnetization in terms of the magnetization. The coefficients are correlation functions with zero magnetization whose renormalization group equations are known. Summation of all these equations yields finally:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + W(g_0)\frac{\partial}{\partial g_0} - \left(\frac{1}{\nu(g_0)} - 2\right) t\frac{\partial}{\partial t} - \frac{\eta(g_0)}{2}\left(H\frac{\partial}{\partial H} + \frac{N}{2}\right)\right] \Gamma^{(N)}(p_i, g_0, \Lambda, t, H) = 0 \qquad (5')$$

From this equation, one can derive all the scaling laws.

* * *

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