Crossing Relations for Helicity Amplitudes*

T. L. Trueman and G. C. Wick

Brookhaven National Laboratory, Upton, New York

DISCLAIMER

This report was prepared as an account of work sponsored by an agency of the United States Government. Neither the United States Government nor any agency Thereof, nor any of their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by trade name, trademark, manufacturer, or otherwise does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof.

DISCLAIMER

Portions of this document may be illegible in electronic image products. Images are produced from the best available original document.

CROSSING RELATIONS

Send proofs to:

> Dr. G. C. Wick
> Department of Physics
> Brookhaven National Laboratory
> Upton, New York

Copies submitted   <u>2</u>

Manuscript pages   <u>30</u>

Figures   <u>4</u>

Tables   <u>none</u>

ABSTRACT

Crossing relations for helicity amplitudes for particles of arbitrary spin are formulated without recourse to the introduction of scalar amplitudes. The basic assumption is that the amplitudes are simply related by analytic continuation; the path of continuation is carefully specified. The relations are given a simple geometrical interpretation. The relation between $\pi N \to \pi N$ and $\pi\pi \to N\bar{N}$ obtained in this way agrees with that obtained by direct elimination of scalar amplitudes.

1. Introduction

The most common applications of crossing-relations involve particles of spin 0 or at most $\frac{1}{2}$. The customary Dirac formalism allows one to express the reaction amplitude in terms of so-called scalar amplitudes ("A" and "B" in the case of π-N scattering) and the crossing-relation then simply states that analytic continuation of a scalar amplitude from the physical region of a channel to that of a "crossed" channel yields the corresponding scalar amplitude in the crossed channel. The introduction of scalar amplitudes is not a simple matter in the general case, so that it would be technically advantageous[1] to formulate the crossing-relations in terms of some other amplitudes, which are more easily generalized, for example helicity amplitudes.(1,2)

A crossing relation for helicity amplitudes for a simple case, such as πN scattering versus $\overline{NN}$ annihilation into two pions, can, of course, be obtained indirectly by elimination of the scalar amplitudes A and B from the equations connecting A and B to the helicity amplitudes $F_{\lambda\mu}$ and $G_{\lambda\mu}$ for the two crossed-reactions. But the relation obtained is not very transparent at first sight. How can it be generalized?

Recently we arrived at a very simple geometrical interpretation of these relations, which suggests an obvious generalization. Essentially the same interpretation has been arrived at independently and apparently somewhat earlier by M. S. Marinov and V. I. Roginskii (3) and by Ja. A. Smorodinsky. (4) The results of these authors, however,

are only similar to but not identical with ours. In particular, the formulae derived by these authors do not seem to agree with those obtained by the elementary "indirect" method for πN-scattering, and must therefore be incorrect. We believe that this is due to certain complications arising from analytic continuation, complications which are not discussed in either of the above-mentioned papers. We hope, therefore, that the following remarks will contribute to the further elucidation of this problem.

2. An Example: πN Scattering

Let us first briefly recall the results of the calculation via the scalar amplitudes. The πN scattering amplitude is given by

$$\bar{u}(p_2)Tu(p_1)$$

(1)

$$T = - A + \tfrac{1}{2} \mu^{-1} i\gamma \cdot (q_1+q_2)B .$$

The notation is standard, (5) except that the usual B is replaced by B/μ. The helicity amplitudes G_{++} and G_{+-}, say, are then obtained by an appropriate choice of the Dirac spinors $u(p_1)$ and $u(p_2)$. The formulae of this section are based on the phase conventions of reference 1. Note that according to Eq. (13) of (1), there is a factor $(-1)^{s_2-\lambda_2}$ in the definition of the helicity state for "particle 2". In the following equations the pion in both the initial and final

state is taken as "particle 2", so that the factor is avoided. One has, ignoring an irrelevant overall phase-factor,

$$G_{++} = \cos(\theta_s/2) \left\{ A + \frac{k^2 + \epsilon\omega}{m\mu} B \right\} \ ,$$

$$G_{+-} = \sin(\theta_s/2) \left\{ \frac{\epsilon}{m} A + \frac{\omega}{\mu} B \right\} \ ,$$

(2)

where k and θ_s are c.m. momentum and scattering angle, $\epsilon = (m^2 + k^2)^{\frac{1}{2}}$ and $\omega = (\mu^2 + k^2)^{\frac{1}{2}}$ are c.m. energies of nucleon and meson respectively. For the relation of these variables to the Mandelstam variables s, t, u (or $\bar{s}$) and related notations we refer the reader to the papers of W. R. Frazer and J. R. Fulco. (6) With the abbreviation

$$S^2 = [s - (m+\mu)^2] \ [s - (m-\mu)^2]$$

(3)

we have

$$k^2 = S^2/4s \qquad \sin(\theta_s/2) = (-st)^{\frac{1}{2}}/S$$

(4)

$$\cos(\theta_s/2) = (S^2 + st)^{\frac{1}{2}}/S = [(m^2 - \mu^2)^2 - su]^{\frac{1}{2}}/S$$

(5)

$$\epsilon = (s + m^2 - \mu^2)/(4s)^{\frac{1}{2}} \ ; \quad \omega = (s - m^2 + \mu^2)/(4s)^{\frac{1}{2}}$$

(6)

These formulae are to be used in the physical region for the

s-channel (πN scattering). In this region the square roots in (4)-(6) are all positive by definition. In the t-channel ($\pi\pi \rightarrow N\bar{N}$) the helicity amplitudes are given (6) by

$$F_{++} = -(p/m)A + (q/\mu)\cos\theta\ B$$

$$\tag{7}$$

$$F_{+-} = (Eq/m\mu)\cos\theta\ B$$

where:

$$p = (\tfrac{1}{4}t-m^2)^{\frac{1}{2}}\ ,\quad q = (\tfrac{1}{4}t-\mu^2)^{\frac{1}{2}}\ ,\quad \cos\theta = (s-u)/4pq\ ,\tag{8}$$

and the functions $A(s,t)$, $B(s,t)$ in the t-channel region are analytic continuations of the corresponding functions in the s-channel region. In these equations, the nucleon in the final state is taken as "particle 2". If A and B have the kind of singularities that are postulated in the Mandelstam representation, the analytic continuation has to go from a point say $s = s_i+i\epsilon$, $t = t_i-i\epsilon$, where ϵ is infinitesimal and positive, $s_i > (m+\mu)^2$, $t_i < 0$, $s_i u_i < (m^2-\mu^2)^2$ to a point $s = s_f-i\epsilon$, $t = t_f+i\epsilon$ where $s_f < 0$ and $s_f u_f > (m^2-\mu^2)^2$. In these equations the variable u is, of course, $u = 2(m^2+\mu^2)-s-t$. If we assume, for the sake of simplicity, that u stays real along the trajectory, then in order to reverse the sign of the imaginary parts of s and t, the trajectory must go through a point of the real s,t-plane.

If this is the only real point of the trajectory, and if the real

point lies within the triangle defined by the inequalities:

$$s < (m+\mu)^2 , \quad u < (m+\mu)^2 , \quad t < 4\mu^2 , \tag{9}$$

then we will have insured that the endpoint is still on the "first

sheet" where A and B are given by the Mandelstam formulae and have

the correct values for the t-channel.

The coefficients of A and B in Eq. (2) have certain singula-

rities (branch-singularities), namely, as can be seen from Eqs. (3)-

(6) at $s = 0$, $s = (m\pm\mu)^2$, $t = 0$ and $su = (m^2-\mu^2)^2$. The trajectory

should, of course, avoid these singularities. We may assume that the

imaginary part of s(-t) remains small throughout so that the tra-

jectory may be specified for our purposes by drawing a line in the

s,t-plane, indicating by a cross the point where the trajectory

crosses the real s,t plane. The lines $su = (m^2-\mu^2)^2$, $s = (m-\mu)^2$

and $t = 0$ divide the triangle (9) into five pieces, see Fig. 1, and

depending on where the crossing-point lies, one will get different

determinations of the coefficients in Eq. (2) at the final point in

the t-channel. Since an overall change in sign of G_{++} and G_{+-} is

unimportant, we have to distinguish only two cases: if the crossing

point lies within the hyperbolic segment delimited by $t = 0$ and

$su = (m^2-\mu^2)^2$, then the final values of $\cos(\theta_s/2)$ and $s^{-\frac{1}{2}}\sin(\theta_s/2)$

are pure imaginary and of the same sign. In all other cases, they

are pure imaginary and of opposite sign. Thus if we adopt the first alternative, we have.

$$\cos(\theta_s/2) = -2ipq\ \sin\theta/S\ , \qquad s^{-\frac{1}{2}}\sin(\theta_s/2) = 2iE/S\ , \tag{10}$$

where $2E = t^{\frac{1}{2}}$ is the total energy in the c.m. system. The minus sign in the expression for $\cos(\theta_s/2)$ is explained in the Appendix. After introducing these values into Eq. (2), we may eliminate A and B from (2) and (7), obtaining

$$G_{++} = (2i/S)\ [mq\ \sin\theta\ F_{++} + E(p-q\cos\theta)\ F_{+-}]\ ,$$

$$\tag{11}$$

$$G_{+-} = (2i/S)\ [\ E(p-q\ \cos\theta)F_{++} - mq\sin\theta\ F_{+-}]\ ,$$

where, as pointed out before, it is irrelevant which of the two determinations for the square root of (3) is used, provided it is the same in the two equations.

It is easy to verify the identity

$$(mq\ \sin\theta)^2 + E^2(p-q\ \cos\theta)^2 = \tfrac{1}{4}\ S^2 \tag{12}$$

so that, apart from the uninteresting factor i, the transformation matrix between the two sets of helicity amplitudes is an orthogonal matrix. But the meaning of this transformation is not immediately apparent.

Owing to the orthogonal nature of the transformation we are, however, tempted to write Eq. (10), disregarding the factor i, in the form

$$G_{++} = \sin X \, F_{++} + \cos X \, F_{+-}$$

$$\text{(11')}$$

$$G_{+-} = \cos X \, F_{++} - \sin X \, F_{+-}$$

where X is determined by

$$\tan X = \frac{mq \, \sin\theta}{E(p-q \, \cos\theta)} \quad . \tag{13}$$

In order to interpret this formula, we must reexamine carefully the process of analytic continuation.

3. Geometrical Interpretation

In the customary presentation of the crossing relations one rewrites the conservation law for πN scattering

$$q_1 + p_1 = q_2 + p_2 \tag{14}$$

in the form

$$q_1 - q_2 = -p_1 + p_2 \tag{15}$$

and reinterprets $-q_2$ and $-p_1$ as initial four-momentum of a pion and

final four-momentum of an antinucleon respectively. It is clear that in this interpretation the values of the four-momenta are <u>not the same</u> in Eqs. (14) and (15), since q_2 and p_1 are positive timelike in Eq. (14) and negative timelike in Eq. (15). In fact, the values of Eq. (14) correspond to the initial point of the trajectory of Fig. 1, those of (15) to the final point. Moreover one sees that also the values of q_1 and p_2 have to vary along the trajectory since

$$u = (q_1 - p_2)^2 \tag{16}$$

cannot remain constant. However, this variation of q_1 and p_2 is often disregarded, since in the end the two four-vectors revert to the real positive timelike mass-shell. In fact, if the initial point of the trajectory of Fig. 1 lies in the $u < 0$ part of the physical region, we may indeed assume that the final values of q_1 and p_2 are identical with the initial ones. Henceforward we shall make this assumption for simplicity.

Thus in the latter case indicating by primes the values of $q_1, \dots$ etc. at the end of the trajectory, we may write

$$q_1' = q_1 , \quad p_2' = p_2 , \quad q_2' = -Q_2 , \quad p_1' = -P_1 , \tag{17}$$

where $q_1 Q_2$ are the pion momenta in the initial state, and $p_2 P_1$ are the nucleon and antinucleon momenta in the final state of the reaction in the t-channel. It should be noted that P_1 and Q_2 must

be different from p_1 and q_2 in Eq. (14). In this respect the usual notation is apt to lead to confusion.

We may clearly assume that at every stage of the analytic continuation the vectors $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$, $\underset{\sim}{q}_1$, $\underset{\sim}{q}_2$ (whether real or complex) lie in the xz plane, since this gives us sufficient freedom to vary s and t at will. This assumption avoids phase-factors connected with the azimuthal variable ϕ,[3] and furthermore it means that the y axis is not affected in any of the Lorentz transformations we shall encounter in the following discussion.

Let us now introduce, in addition to the customary helicity amplitudes $G_{\mu\lambda}(s,t)$ so far employed, the notion of "generalized" helicity amplitudes $G_{\mu\lambda}(p_2 q_2; p_1 q_1)$ by which we mean matrix elements of the scattering matrix T between helicity states satisfying condition (14) but not subject to the c.m. condition: $\underset{\sim}{p}_1 + \underset{\sim}{q}_1 = 0$. The phases of the nucleon helicity states are defined by Eq. (6) of (1), with $\phi = 0$ and θ unrestricted. The discontinuity at the "south pole" causes no trouble since the state obtained with $\theta = \pi$ is simply $(-1)^{2s}$ times the state with $\theta = -\pi$, independent of the helicity. Let $\underset{\sim}{\beta}$ be the velocity of the center of mass

$$\underset{\sim}{\beta} = (\underset{\sim}{p}_1 + \underset{\sim}{q}_1)/(\epsilon_1 + \omega_1) \tag{18}$$

where $\epsilon_1 = (m^2 + \underset{\sim}{p}_1^2)^{\frac{1}{2}}$, $\omega_1 = (\mu^2 + \underset{\sim}{q}_1^2)^{\frac{1}{2}}$, and denote by ℓ_β^{-1} the Lorentz transformation (in the xz-plane) which transforms the c.m.

to rest, so that $p_1 = \ell_\beta p_1^{\,0}$ $q_1 = \ell_\beta q_1^{\,0}$, where $p_1^{\,0} q_1^{\,0}$ is an initial state satisfying the c.m. condition. Then according to the transformation law of helicity states[4]

$$\mathscr{L}_\beta |p_1^{\,0} \lambda> \ = \sum_{\lambda'} \ U_{\lambda'\lambda}(p_1;\ell_\beta)|p_1 \lambda'> \qquad (19)$$

where U is a spin-rotation matrix corresponding to a rotation about the y-axis. The rotation angle is indicated in the diagram in velocity space, see Fig. 2. Besides the velocity-points corresponding to p_1 and q_1, the diagram has a point representing the velocity of the c.m. (a cross) and a point O representing the velocity of the arbitrary system, in which the momenta are p_1 and q_1. From Eq. (19) and the Lorentz invariance of the T-matrix one easily derives the connection

$$G_{\mu\lambda}(s,t) = \sum_{\mu'\lambda'} U_{\mu\mu'}(\ell_\beta^{-1};p_2)G_{\mu'\lambda'}(p_2 q_2;p_1 q_1)U_{\lambda'\lambda}(p_1;\ell_\beta) \qquad (20)$$

A similar formula holds for an arbitrary Lorentz transformation ℓ, except that in this case we would have on the left-hand side again a generalized amplitude for the values $\ell^{-1}p_2, \ldots$ etc. of the four-momenta.

We may perform the analytic continuations of $G_{\mu\lambda}(s,t)$ by continuing each of the three factors on the right hand side of Eq. (20). Let us assume, for the sake of simplicity, that the values (17) at the end of the trajectory satisfy the c.m. condition for the crossed reaction

$$\underset{\sim}{q}_1 + \underset{\sim}{q}_2 = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 = 0 \tag{21}$$

It then turns out that the end-value of the generalized helicity amplitude $G_{\mu'\lambda'}(p_2 q_2; p_1 q_1)$ coincides up to a phase factor (see Eq. (31)) with a helicity amplitude F for the crossed reaction. Such a relationship has in fact been conjectured by other authors,(3),(7) with the difference that here contrary to their results, the helicity does not change sign in the crossing process. Finally, the spin-rotation matrices U, or rather their analytic continuation, give rise to the orthogonal transformation Eq. (11), as has also been pointed out by Ya. Smorodinsky (4) and by Marinov and Roginskii. (3)

Let us first examine the behavior of the generalized amplitude $G_{\mu\lambda}(p_2 q_2; p_1 q_1)$. We can do this in two ways. To begin with, we may say that G is given by Eq. (1) when the spinors are chosen to be "helicity-spinors" $u_\mu(p_2)$ and $u_\lambda(p_1)$. This means that, for example, $u(p_1)$ must satisfy, in addition to the Dirac equation $(i\gamma p_1 + m)u(p_1) = 0$ also a helicity condition[5]

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{p}_1 \, u_\lambda(p_1) = 2\lambda(\underset{\sim}{p}_1^2)^{\frac{1}{2}} \, u_\lambda(p_1) \tag{22}$$

where $\lambda = \pm \frac{1}{2}$

Since we have made the customary assumption that there is no problem in continuing the matrix T, Eq. (1) along a path such as that of Fig. 1, the whole question reduces to the behavior of the

helicity spinors. Since p_2 reverts to the positive _real_ (i.e.
p_2 real) mass shell in the end, it is not hard to see that the
helicity μ remains unchanged in the process. The case of p_1 requires
more care. At the end of the process, u will, of course, become a
negative energy spinor, $v(P_1)$ satisfying a Dirac equation
$(i\gamma P_1 - m)v(P_1) = 0$. If we assume, moreover, that along the trajectory
$p_1{}^2$ never becomes zero, then since Eq. (22) is always satisfied by
analytic continuation, it follows that the final $v(P_1)$ must satisfy

$$-\underset{\sim}{\sigma} \cdot \underset{\sim}{P}_1 v_\lambda(P_1) = 2\lambda(\underset{\sim}{P}_1{}^2)^{\frac{1}{2}} v_\lambda(P_1) \qquad (22')$$

where the sign of the square-root on the right hand side is determined
unambiguously by continuity along the path. As we shall see
presently, the sign of the square-root is determined to be positive,
whereupon it follows that the _antinucleon_ state described by v_λ has
helicity $+\lambda$, as one can see immediately by examining the charge-
conjugate spinor $\overline{Cv}$ (or by the more elementary hole-theory argument:
a missing particle of spin $-\lambda$ in the $\underset{\sim}{P}_1$-direction, corresponds to
an antiparticle of spin $+\lambda$ in the same direction). Thus the helicity
does not change sign in the analytic continuation.

The crux of the argument, it will be seen, is the behavior
of the square-root of $\underset{\sim}{P}_1{}^2$. This is not trivial, since the vector
$\underset{\sim}{P}_1$ becomes complex along the path. Let, however, ϵ_1 be the time-
component of p_1 at any point of the trajectory so that $(\underset{\sim}{P}_1{}^2)^{\frac{1}{2}}$ is the

end value of

$$(\epsilon_1^2 - m^2)^{\frac{1}{2}} \quad . \tag{23}$$

The whole question hinges on the path of the representative point for ϵ_1 in the complex ϵ_1-plane, Fig. 3. Now assume for simplicity that q_1 and p_2 are exactly real at the endpoint. For the initial value of p_1 we write instead $p_1 + i\eta$ where p_1 and η are real, and $\eta = (\eta_0, \underset{\sim}{\eta})$ is infinitesimal. Thus $p_1^2 = m^2$ and $p_1 \cdot \eta = 0$. Calculating in the c.m. system for $p_1 + q_1$, i.e. assuming $\underset{\sim}{p_1} = -\underset{\sim}{q_1}$, the condition for $s = (p_1 + i\eta + q_1)^2$ to have a positive imaginary part is

$$0 < \eta \cdot q_1 = \eta_0 \omega_1 - \underset{\sim}{\eta} \cdot \underset{\sim}{q_1} = \eta_0 \omega_1 + \underset{\sim}{\eta} \cdot \underset{\sim}{p_1} = \eta_0 (\omega_1 + \epsilon_1) .$$

This implies $\eta_0 > 0$, i.e. the initial value of ϵ_1 is $\epsilon_1 + i\eta_0$; it has a small positive imaginary part.

Similarly the condition that $t = (p_1 - p_2)^2$ has a small positive imaginary part at the endpoint of the trajectory implies that the end-value of ϵ_1 has a negative imaginary part. Thus the endpoints are as indicated in Fig. 3, and the remaining question is whether the ϵ_1-trajectory cuts the real axis between $-m$ and $+m$ (as indicated in the figure) or not. If it does, the end-value of (23) is positive, as we have assumed. Now this can certainly be arranged, if we assume that the time components of the four-vectors

p_1 and q_2 are everywhere complex along the trajectory except at the crossing point. Since the latter is inside the triangle (9) we may assume that at this point the four-vectors $p_1, \ldots, q_2$ are all "Euclidean" (i.e. have a real time component and a pure imaginary space component). In this case the nucleon time components ϵ, and the meson time-components ω satisfy the inequalities

$$-m < \epsilon < m , \qquad -\mu < \omega < +\mu , \qquad (24)$$

so that all the energy trajectories satisfy the requirements (for mesons this would be of interest if they also had a spin).

In conclusion, the discussion above shows that owing to the large number of variables, when analytic continuation in the four-momenta is involved, the answer is by no means unique,[6] but if one chooses the convention which can be stated in general in the most simple and natural way, then the result is the one we have indicated.

If one wishes to extend this conclusion to general spins and masses, one can resort to the second method we mentioned, namely one generalizes (20) to an arbitrary Lorentz transformation ℓ:

$$G_{\mu'\lambda'}(\ell^{-1}p_2 \ell^{-1}q_2 ; \ell^{-1}p_1 \ell^{-1}q_1) = \sum_{\mu\lambda} U_{\mu'\mu}(\ell^{-1};p_2) G_{\mu\lambda}(p_2 q_2; p_1 q_1) U_{\lambda\lambda'}(p_1;\ell) \qquad (25)$$

(if all four particles have spins, there will be two more indices to G, and two more U matrices in the product). One then keeps ℓ fixed,

and continues analytically to the values (17). This requires

analytic continuation of $U_{\lambda\lambda'}(p_1;\ell)$ to $U_{\lambda\lambda'}(-P_1;\ell)$. Let us write

$$U(p;\ell) = \mathscr{D}\{h^{-1}(p)\ell\ h(\ell^{-1}p)\} \tag{26}$$

where $\mathscr{D} = \mathscr{D}^{(s)}$ is the representation of the rotation group

pertaining to the spin s of the particle. We have to write explicitly

the components of the Lorentz transformations h(p) and h(p') as

functions of the four components of p and $p' = \ell^{-1}p$, where h(p) is

defined according to the helicity convention

$$h(p) = r_{\phi,\theta,-\phi}\ z(p) \tag{27}$$

where θ,ϕ are the polar angles of $\underline{p}$; i.e. h(p) is an ordinary Lorentz

transformation of velocity $|\underline{p}|/p_0$ in the z-direction followed by

rotations through Euler angles $-\phi,\theta,\phi$. One then performs the analytic

continuation. In order to determine the transformation properties

of the continuation of $G_{\mu\lambda}$, it is sufficient to consider small ℓ in

Eq. (25). In particular, we assume that p_{1_0} and, since ℓ is small,

also $(\ell p_1)_0$ cross the real axis between +m and -m as indicated in

Fig. 3. With this assumption, it is straightforward to show that

$$(-1)^{\lambda-\lambda'}U_{\lambda\lambda'}(-P;\ell) = U_{\lambda'\lambda}(\ell^{-1};P) = U_{\lambda'\lambda}^{-1}(P;\ell) = U_{\lambda\lambda'}^{*}(P;\ell). \tag{28}$$

The appearance of the complex conjugate is to be expected since the

particle changes side of the reactions.

Equation (25) now becomes, for the end-values (17),

$$G_{\mu'\lambda'}(\ell^{-1}p_2, -\ell^{-1}Q_2; -\ell^{-1}P_1; \ell^{-1}q_1) = \sum_{\mu\lambda} (-1)^{\lambda-\lambda'} U_{\mu'\mu}(\ell^{-1}; p_2) \times$$

$$\times \; U_{\lambda'\lambda}(\ell^{-1}; P_1) G_{\mu\lambda}(p_2, -Q_2; -P_1, q_1) \quad . \tag{29}$$

This transformation law is to be compared with that for the generalized helicity amplitude $F_{\mu\lambda}(p_2 P_1; q_1 Q_2)$. In order that this coincides with $F_{\mu\lambda}(s,t)$ as defined in (7) when $q_1 + Q_2 = 0$, the helicity state $|p_2, \mu>$ must be defined with the additional factor $(-1)^{\frac{1}{2}-\mu}$. The transformation law is then

$$F_{\mu\lambda}(\ell^{-1}p_2, \ell^{-1}P_1; \ell^{-1}q_1, \ell^{-1}Q_2) = \sum_{\mu'\lambda'} (-1)^{\mu'-\mu} F_{\mu'\lambda'}(p_2, P_1; q_1, Q_2)$$

$$\times \; U_{\mu'\mu}(\ell^{-1}; p_2) U_{\lambda'\lambda}(\ell^{-1}; P_1) \tag{30}$$

where the factor $(-1)^{\mu'-\mu}$ results from defining the nucleon helicity state with the factor $(-1)^{\frac{1}{2}-\mu}$. Thus, if there is a direct connection between F and G, it must be (apart from an overall phase factor)

$$G_{\mu\lambda}(p_2, -Q_2; -P_1, q_1) = (-1)^{\mu-\lambda} F_{\mu\lambda}(p_2 P_1; q_1 Q_2) \quad . \tag{31}$$

(Note that if the antinucleon had been taken as "particle 2" there

would be no factor $(-1)^{\mu-\lambda}$ in Eq. (31).) Notice also that no reversal of the sign of the helicity occurs; this circumstance is again strictly connected with a path such as indicated in Fig. 3. If we now specialize to a c.m. system $\underline{q}_1 + \underline{q}_2 = 0$, the right hand side becomes $(-1)^{\mu-\lambda}F_{\mu\lambda}(s,t)$.

We now come to the final step, the analytic continuation of the U-factors in Eq. (20), which is different from the preceding case, because the Lorentz transformation ℓ_β also varies along the path. We notice in fact that, along the path, ℓ_β becomes a complex Lorentz transformation and at the end it becomes $\ell_{\beta'}$, where the velocity

$$\beta' = \frac{\underline{q}_1 - \underline{p}_1}{q_{10}-p_{10}} \tag{32}$$

may be greater than unity (i.e. than the velocity of light) so that $\ell_{\beta'}$ may also be complex. In particular, when the transformation is from an annihilation c.m. to an elastic scattering c.m., β' is infinite. Since the vectors are assumed to remain in the y-z plane, $U(p_1;\ell_\beta)$ and $U(\ell_\beta^{-1};p_2)$ can each be expressed in terms of a single angle, X_1 and X_2, respectively. For the continuation process, we choose as the arbitrary point O of Fig. 2 the velocity point of the center of mass of p_2, P_1 or q_1, q_2 reached at the end of the contin-uation. Let C denote the velocity point of the center of mass of p_1, q_1 or p_2,q_2. The angles may be determined from the cosein theorem (2) applied to the triangles OCp_1 and OCp_2 respectively, see Fig. 4a.

$$\cos X_1 = \frac{\cosh \rho_{C1} \cosh \rho_{01} - \cosh \rho_{0C}}{\sinh \rho_{C1} \sinh \rho_{01}} \quad ,$$

$$\cos X_2 = \frac{\cosh \rho_{C2} \cosh \rho_{02} - \cosh \rho_{0C}}{\sinh \rho_{C2} \sinh \rho_{02}} \quad ,$$

(33)

where $\tanh \rho_{C1}$ is the absolute velocity of p_1 with respect to C, etc.
For an arbitrary point on the path of continuation and using different
masses for p_1 and p_2, q_1 and q_2 for the sake of generality

$$\cos X_1 = \frac{(s+m_1{}^2-\mu_1{}^2)p_{10} - 2(p_{10}+q_{10})m_1{}^2}{S_1 (p_{10}{}^2-m_1{}^2)^{\frac{1}{2}}} \quad ,$$

$$\cos X_2 = \frac{(s+m_2{}^2-\mu_2{}^2)p_{20} - 2(p_{20}+q_{20})m_2{}^2}{S_2 (p_{20}{}^2-m_2{}^2)^{\frac{1}{2}}} \quad ,$$

(34)

evaluated in the rest system of 0. Here $S_1{}^2 = [s-(m_1-\mu_1)^2][s-(m_1+\mu_1)^2]$,
etc. At the end of the specified path, one obtains the positive
determinations of S_1 and $(p_{10}{}^2-m_i{}^2)^{\frac{1}{2}}$. Thus there are no new problems
in the continuation of $\cos X_i$. We must next determine how $\sin X_i$
continues. Consider the initial configuration when 0 and C are
connected by a real Lorentz transformation. Let θ denote the angle
from $-\underline{p}_1$ to $\underline{q}_1$ in 0. Further, define α_1 to be the angle from $\underline{p}_1 + \underline{q}_1$
to $\underline{p}_1$ in 0. (In order to determine signs correctly, it is essential
to pay attention to the directions of the angles. For example, θ
is positive if a positive rotation about the y-axis takes $\underline{p}_1$ parallel
to $\underline{q}_1$.) These two angles are related by

$$\sin \alpha_1 \;=\; \sin\theta (q_{10}^2 - \mu_1^2)^{\frac{1}{2}} / [(p_{10} + q_{10})^2 - s]^{\frac{1}{2}} \;, \tag{35}$$

where the positive determination of the square roots must be taken. The sine theorem (2) applied to triangle OCp_1, Fig. 4a, yields the relation

$$\sin X_1 \;=\; \sin \alpha_1 \; \frac{2m_1 [(p_{10} + q_{10})^2 - s]^{\frac{1}{2}}}{S_1} \;, \tag{36}$$

and again the positive determination of the square roots is to be taken. Consequently,

$$\sin X_1 \;=\; \sin\theta \; \frac{2m_1 (q_{10}^2 - \mu_1^2)^{\frac{1}{2}}}{S_1} \;; \tag{37}$$

since the continuations of S_1 and $(q_{10}^2 - \mu^2)^{\frac{1}{2}}$ have already been specified, Eq. (37) allows us to express the continuation of $\sin X_1$ unambiguously in terms of $\sin\theta$. Clearly, $\sin X_2$ can be continued in the same way. This completes the continuation of Eq. (20) and provides the relation between $G_{\mu\lambda}(s,t)$ and $F_{\mu\lambda}(s,t)$.

Let us first apply these results to the πN problem. At the end of the continuation we have

$$\cos X_1 \;=\; -\cos X_2 \;=\; -\frac{(s + m^2 - \mu^2)(t/4)^{\frac{1}{2}}}{S(t/4 - m^2)^{\frac{1}{2}}} \;=\; \frac{2E(p - q\,\cos\theta)}{S} \;,$$

$$\tag{38}$$

$$\sin X_1 \;=\; \sin X_2 \;=\; \frac{2mq\,\sin\theta}{S} \;.$$

Eq. (20) is then

$$G_{\mu\lambda}(s,t) = \sum_{\mu'\lambda'} d_{\mu'\mu}^{\frac{1}{2}}(\pi-\chi_1) d_{\lambda'\lambda}^{\frac{1}{2}}(\chi_1)(-1)^{\mu'-\lambda'} F_{\mu'\lambda'}(s,t), \qquad (39)$$

or

$$G_{++} = \sin \chi_1 \, F_{++} + \cos \chi_1 \, F_{+-} \quad,$$

$$(40)$$

$$G_{+-} = \cos \chi_1 \, F_{++} - \sin \chi_1 \, F_{+-} \quad,$$

which agrees with Eq. (11).

The formulae are readily generalized to the case where all four particles have spin. For notational convenience, we continue to write the formulae as if a baryon of mass m_1 and spin s_1 is crossed with a meson of mass μ_2 and spin σ_2. These are easily translated to other cases. The relation between the helicity amplitude is (baryon indices μ, λ, meson indices α,β):

$$G_{\mu\beta,\lambda\alpha}(s,t) = \sum_{\mu',\lambda',\beta',\alpha'} (-1)^{\eta} \, d_{\mu'\mu}^{s_2}(\chi_2) d_{\lambda'\lambda}^{s_1}(\chi_1) d_{\beta'\beta}^{\sigma_2}(\psi_2) d_{\alpha'\alpha}^{\sigma_1}(\psi_1) F_{\mu'\lambda',\beta'\alpha'}(s,t) ,$$

$$(41)$$

where

$$\cos X_1 = \frac{-(s+m_1^2-\mu_1^2)(t+m_1^2-m_2^2)-2m_1^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2p\sqrt{t}\ S_1} \quad ,$$

$$\cos X_2 = \frac{(s+m_2^2-\mu_2^2)(t+m_2^2-m_1^2)-2m_2^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2p\sqrt{t}\ S_2} \quad , \qquad (42)$$

$$\cos \psi_1 = \frac{(s+\mu_1^2-m_1^2)(t+\mu_1^2-\mu_2^2)-2\mu_1^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2q\sqrt{t}\ S_1} \quad ,$$

$$\cos \psi_2 = \frac{-(s+\mu_2^2-m_2^2)(t+\mu_2^2-\mu_1^2)-2\mu_2^2(m_2^2-m_1^2+\mu_1^2-\mu_2^2)}{2q\sqrt{t}\ S_2} \quad ,$$

and

$$\sin X_1 = 2m_1 q\ \sin\theta/S_1 \quad ,$$
$$\sin X_2 = 2m_2 q\ \sin\theta/S_2 \quad , \qquad (43)$$
$$\sin \psi_1 = 2\mu_1 p\ \sin\theta/S_1 \quad ,$$
$$\sin \psi_2 = 2\mu_2 p\ \sin\theta/S_2 \quad , \quad ,$$

with

$$p = [t^2-2t(m_1^2+m_2^2)+(m_1^2-m_2^2)^2]^{\frac{1}{2}}/2\sqrt{t} \quad ,$$

$$q = [t^2-2t(\mu_1^2+\mu_2^2)+(\mu_1^2-\mu_2^2)^2]^{\frac{1}{2}}/2\sqrt{t} \quad .$$

The quantity η depends on which particles are defined as "particle 2"
in G and F. (See the discussion following Eqs. (2) and (31).) Suppose

that in G the mesons are taken to be "particle 2". The value of η

for the various possible choices in the definition of F are tabulated

below:

"particle 2"	η
p_2, q_2	$\lambda-\mu + \alpha - \beta$
p_2, q_1	$\lambda-\mu$
p_1, q_1	0
p_1, q_2	$\alpha-\beta$

Perhaps the easiest convention to remember is that $\eta = 0$ if an

uncrossed particle is "particle 1" in both G and F while a crossed

particle changes from "particle 1" to "particle 2" and vice versa.

ACKNOWLEGMENT

APPENDIX

A short explanation of the expression for $\cos(\frac{\theta_s}{2})$, Eq. (10), is given. First note that from (5) and (8)

$$\cos(\frac{\theta_s}{2}) = \pm 2ipq \, \sin\theta / S \quad , \tag{A.1}$$

or

$$\sin\theta_s = \pm 4ipq(-st)^{\frac{1}{2}} \sin\theta / S \, , \tag{A.2}$$

for an arbitrary point on the path of continuation. Consider the invariant quantity

$$\Phi = \begin{vmatrix} q_{1o} & q_{1x} & q_{1z} \\ p_{1o} & p_{1x} & p_{1z} \\ p_{2o} & p_{2x} & p_{2z} \end{vmatrix} . \tag{A.3}$$

We evaluate Φ in the initial configuration in the center of mass of p_1, q_1, continue the expression to the final configuration and evaluate in the center of mass of P_1, P_2. Initially,

$$\Phi = \sqrt{s} \, (p_{1x}p_{2z} - p_{2z}p_{1z}) = -s^2 \sin\theta_s / 4\sqrt{s} \quad ; \tag{A.4}$$

finally

$$\Phi = -\sqrt{t} \ (Q_{1x}P_{1z} - P_{1x}Q_{1z}) = -2Epq \sin\theta \quad , \tag{A.5}$$

or

$$\sin\theta_s = 8E \ pq \ \sqrt{s} \ \sin\theta/S^2$$

$$= -4ipq(-st)^{\frac{1}{2}} \sin\theta/S^2 \quad , \tag{A.6}$$

and hence one must take the minus sign in (A.1). Note that this relation is independent of which way the path of continuation circles $us = (m^2-\mu^2)^2$; in fact, the path of Fig. 1 and the condition that $\sin\theta_s > 0$ initially require $\sin\theta < 0$ in the final configuration.

REFERENCES

1.   M. Jacob and G.C. Wick, Ann. Phys. 7, 404 (1959).

2.   G.C. Wick, Ann. Phys. 18, 65 (1962).

3.   M.S. Marinov and V.I. Roginskii, preprint from Institute for Experimental and Theoretical Physics, Moscow, 1963.

4.   Ja. A. Smorodinsky, Dubna preprint, 1963.

5.   G.F. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957).

6.   W.R. Frazer and J.R. Fulco, Phys. Rev. 117, 1603 (1960); ibid 119, 1420 (1960).

7.   A.O. Barut, Phys. Rev. 130, 436 (1963).

FOOTNOTES

1. We are indebted to Prof. M. Goldberger for repeatedly drawing our attention to this question.

2. The opposite sign for S in the two equations corresponds, however, to the alternative choice of the crossing point!

3. See, for example, reference 1.

4. See, for example, reference 2, especially the Appendix.

5. Equation (22) does not, of course, determine the phase and normalization of $u_\lambda(p_1)$. We assume that the latter are chosen according to the convention explained above when p_1 is real. They are then determined for the other values by analytic continuation.

6. We are indebted to Prof. R. Cutkosky for a remark in this connection.

FIGURE CAPTIONS

Fig. 1   The s,t plane for $\pi N \rightarrow \pi N$ and $\pi\pi \rightarrow N\bar{N}$ (drawn for $m = 4\mu$).

Fig. 2   Velocity space diagram. The ends of the segments p_1 and q_1 represent the velocity points of p_1 and q_1; X represents the velocity point of the c.m. of p_1 and q_1; O is an arbitrary point. The arrows indicate the directions in which the spin components are measured.

Fig 3   The complex ϵ_1-plane.

Fig. 4a   The velocity space diagram for an arbitrary point along the path of continuation. C represents the c.m. of p_1 and q_1 at this point; O represents the c.m. of P_1 and p_2 in the final configuration.

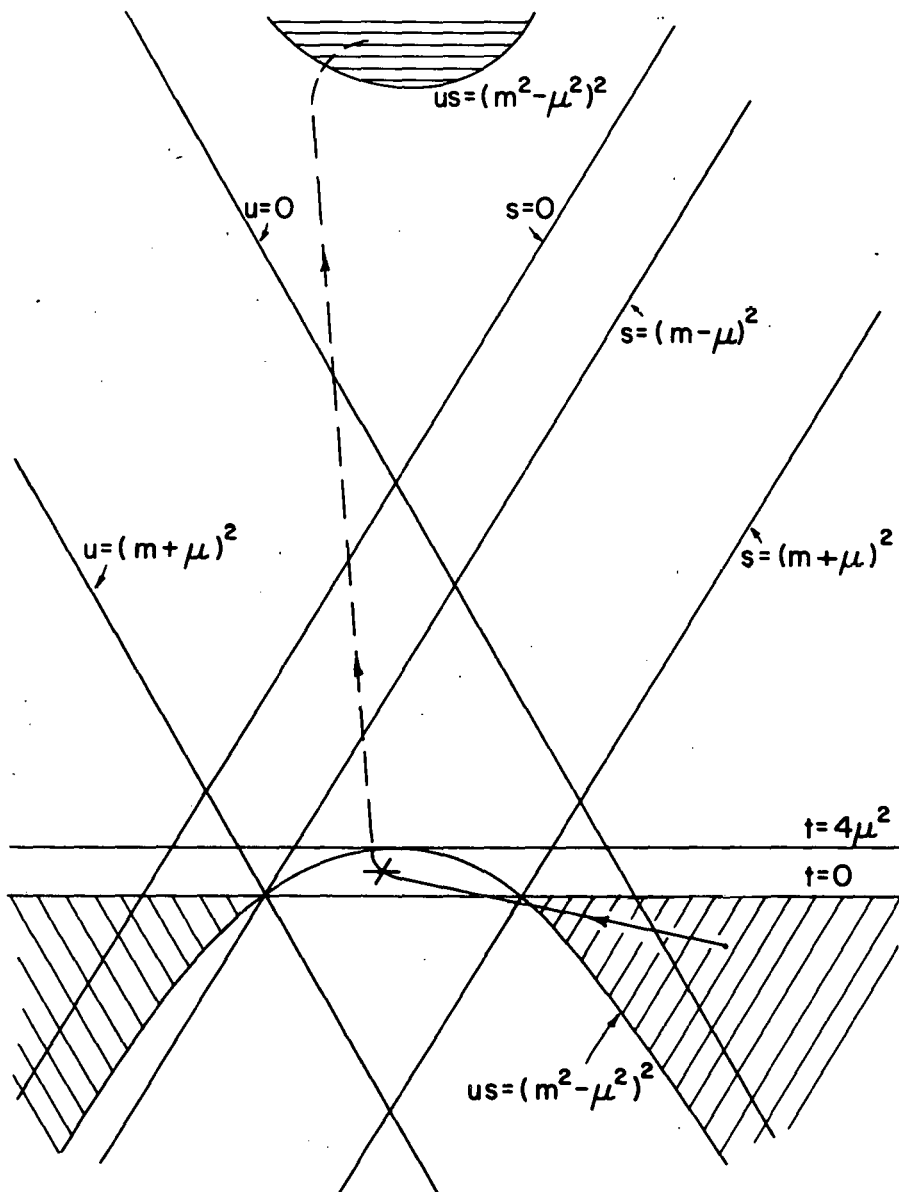Fig. 4b   The velocity space diagram for the final configuration.

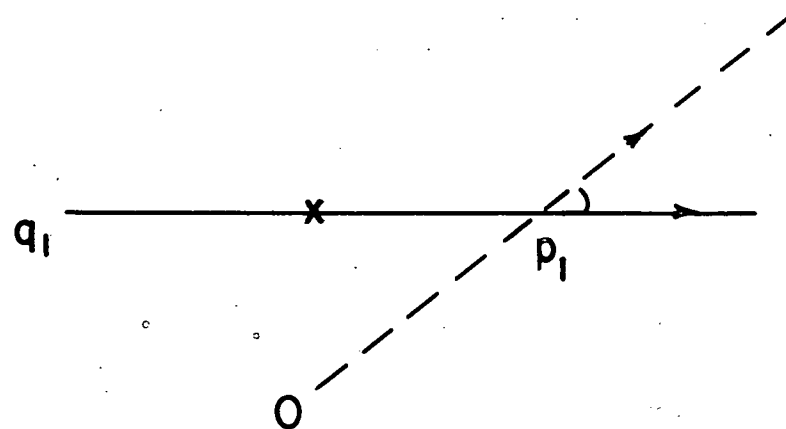$us=(m^2-\mu^2)^2$

$u=0$

$s=0$

$s=(m-\mu)^2$

$u=(m+\mu)^2$

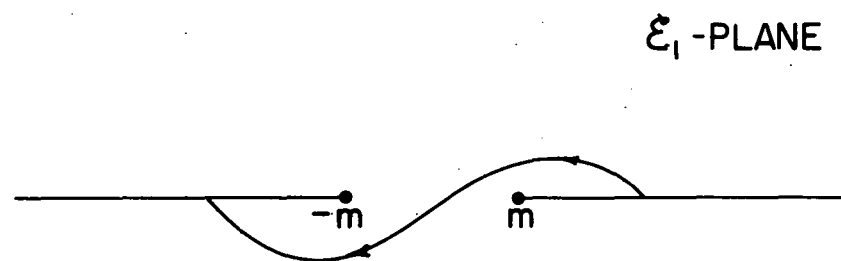$s=(m+\mu)^2$

$t=4\mu^2$

$t=0$

$us=(m^2-\mu^2)^2$

FIGURE 1

FIGURE 2

$\mathcal{E}_1$ -PLANE

FIGURE 3

FIGURE 4a

FIGURE 4b