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HOMOGENIZATION APPROACH IN ENGINEERING

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HOMOGENIZATION APPROACH IN ENGINEERING

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1. Introduction

1.1. The Homogenization

Homogenization is an approach which studies the macrobehaviour of a medium by its microproperties. The origin of this word is related to the question of a replacement of the heterogenous material by an "equivalent" homogenous one.

Homogenization is significant in two directions:

a) as the physical and mechanical interpretation of the global behaviour, and

b) as a tool for the numerical treatment of problems with microstructures.

Problems with a microstructure play an essential role in many fields. Let us mention some:

Mechanics - e.g. in the analysis of the behaviour of composite materials,
Chemistry - e.g. in the theory of polymers,
Physics - e.g. in the problems of quantum mechanics of electrons in crystal lattices,
Reactor Engineering - e.g. in the analysis of power reactors.

For surveys and extensive references about these problems we refer the reader to [1], [2], [3], [4], and references included in this paper.

We will concentrate our attention on a simple specific model problem which will illustrate results and problems typical to the homogenization approach. We will treat here the diffusion problem only, but will sometimes make statements about the elasticity of composite materials (the diffusion equation is then replaced by the well-known elasticity equations).

1.2. The Model Problem

We are interested in the solution of the differential equation on $\Omega \subset R_2$

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i} a(\xi,\eta^H) \frac{\partial u^H}{\partial x_i} = f \tag{1}$$

with the boundary condition

$$u^H = 0 \quad \text{on} \quad \partial\Omega . \tag{2}$$

We denote $\xi = \frac{x}{H}$, $x = (x_1,x_2)$, $|x|^2 = x_1^2 + x_2^2$, $\eta^H = u_x^H = \operatorname{grad} u^H$, and assume that

$H > 0$, and $a(\xi,\eta) > 0$ is a periodic function in ξ with periodicity 1 (and will satisfy additional assumptions spelled out later). We used the notation u^H to underline the dependence on H.

Let us be more specific about our interest. We are interested in finding, with a reasonable accuracy, <u>the function</u> u^H and <u>the fluxes</u> t_i^H

$$(3) \qquad t_i^H = a(\tfrac{x}{H},\eta^H)\,\frac{\partial u^H}{\partial x_i}, \quad i = 1, 2 .$$

Fluxes t_i^H (resp. stresses in elasticity computations) are many times the primary goal in the applications.

It is necessary to underline some circumatances which are important to this problem.

a) H is a given parameter, with physical meaning which cannot be changed, e.g. cannot be made "sufficiently" small. If the diameter of Ω is about $10H - 15H$ and the function $a(\xi,\eta)$ is discontinuous with a complicated structure, a direct numerical treatment by the finite element method is virtually impossible because a reasonable accuracy can be achieved only with many elements in every cell. For more see e.g. [1] part 2, [7], [8], [9]. In three dimensions the above mentioned ratio $10H - 15H$ will be smaller. It means that H could be relatively "large".

b) There is a boundary layer behaviour (also when $\partial\Omega$ is smooth) which significantly influences the fluxes. This boundary layer was shown in [2] and [5] and explains the rather unusual failure of fibrous composite laminates, see [6].

Problems related to the media with microstructure were, and are still, studied very intensively. The first paper dealing with homogenization problems is likely [10]. [11] presents an excellent survey of ideas and results until 1925. The analysis of periodic structures plays an important role in quantum mechanics. We mention here e.g. the results related to the form of eigenfunctions in [12] and [13]. See also [14] and [15]. For additional references, see [1] and [2].

There are many different approaches for treating media with microstructures. Very typically, these approaches are leading to significantly different results. For more, see e.g. [2], [16], and [17].

To illustrate problems and results in the simplest way we will analyze separately linear and nonlinear cases and problems with and without boundaries.

2. The Linear Case without Boundary

In this section we will assume that the function $a(\xi,\eta)$ in (1) is independent of η, and is piecewise smooth, $0 < c_1 < a(\xi) < c_2 < \infty$ and f is smooth, has compact support and $\int_{R_2} f dx = 0$ [1].

We will denote $L_2(R_2)$ the usual space of square integrable functions, $L_2^{(1)}(R_2)$ Sobolev (quotient) space, with

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1) These assumptions can be made weaker.

$$\|u\|^2_{L^1_2(R_2)} = \|\frac{\partial u}{\partial x_1}\|^2_{L_2(R_2)} + \|\frac{\partial u}{\partial x_2}\|^2_{L_2(R_2)} .$$

In addition, let $K = \{x| \ |x_i| < \frac{1}{2}\}$ be a unit cube and $L^1_{2,PER}(K)$ be the quotient space of periodic functions with period 1 ,

$$\|u\|^2_{L^1_{2,PER}(K)} = \|\frac{\partial u}{\partial x_1}\|^2_{L_2(K)} + \|\frac{\partial u}{\partial x_2}\|^2_{L_2(K)} .$$

It is well-known that there exists an unique weak solution u^H of (1) for any $0 < H \leq 1$ on $\Omega = R_2$ (in $L^1_2(R_2)$. We will always normalize the free additional constant so that $\int_K u^H dx = 0$.

2.1. A Typical Homogenization Result and Its Application for the Numerical Solution

<u>Theorem 1</u>. 1) There exists $U \in L_2^{(1)}(R_2)$, $\int_K U\,dx = 0$ such that

$$\|u^H - U\|_{L_2(\tilde{\Omega})} \leq CH \tag{3}$$

where $\tilde{\Omega}$ is any bounded domain [C depends on $\tilde{\Omega}$, f, a(x), but is independent of H]

2) Function U satisfies the (elliptic) differential equation

$$\sum_{i,j=1}^{2} A_{i,j} \frac{\partial^2 U}{\partial x_i \, x_j} = f \tag{4}$$

where

$$A_{m,n} = \int_K \left(\sum_{i=1}^{2} a(x) \frac{\partial W_m}{\partial x_i} \frac{\partial W_n}{\partial x_i} \right) dx \qquad m, n = 1, 2 \tag{5}$$

with $W_k = x_k + \eta_k(x)$, $\eta_k(x) \in L^1_{2,PER}(K)$, $\int_K \eta_k(x)dx = 0$, $k = 1, 2$ and $\eta_k(x)$ is such that

$$\int_K \left[\sum_{i=1}^{2} a(x) \frac{\partial W_k}{\partial x_i} \frac{\partial \zeta}{\partial x_i} \right] dx = 0 \tag{6}$$

for every $\zeta \in L^1_{2,PER}(K)$ and

$$\|u^H - U - H \sum_{i=1}^{2} \frac{\partial U}{\partial x_i} \eta_i(\xi)\|_{L^1_2(R_2)} \leq CH . \tag{7}$$

3) Denoting

$$T_i = \sum_{j=1}^{2} A_{i,j} \frac{\partial U}{\partial x_j} \qquad i = 1, 2 \tag{8}$$

then there exist periodic functions $\chi_j^{[i]}(\xi)$ such that

$$\| t_i^H - \sum_{j=1}^{2} T_j \chi_j^{[i]}(\xi) \|_{L_2(\tilde{\Omega})} \le CH \qquad i = 1, 2 \tag{9}$$

$$\int_{R_2} \left[\sum_{i=1}^{2} a\,(\xi) \left(\frac{\partial u^H}{\partial x_i} \right)^2 - \sum_{i,j=1}^{2} A_{i,j} \frac{\partial U}{\partial x_i} \frac{\partial U}{\partial x_j} \right] dx \le CH \tag{10}$$

5) Defining

$$Z_i^H(x) = \frac{1}{H^2} \int_{y-x/H \in K} a(\tfrac{y}{H}) \frac{\partial u^H}{\partial x_i}(y)\, dy \tag{11}$$

we have

$$\| T_i - Z_i^H \|_{L_2(\tilde{\Omega})} \le CH \tag{12}$$

For proof and more see [5], and also [1] and [2]. Theorem 1 can be generalized so that the error of the order H^j is obtained.

Theorem 1 can be used for numerical treatment of (1) directly or indirectly.

1) Direct way

We compute first the function $\eta_k(x)$, W_k and $\chi_j^{[i]}$ (which are independent of f) by solving particular (periodic) problems on K . Using those functions we determine the coefficients $A_{i,j}$ in (4). Then we solve in the usual manner (e.g. by the finite element method) this differential equation, and find U and T_i . Using function $\chi_j^{[i]}$ we get the approximate values for t_i^H . A particular numerical example of this apporach for elasticity equations is shown in [2].

2) Indirect way

In the spirit of the finite element method we use trial functions

$$v(x) = u^{(0)} + \sum_{i=1}^{2} \eta_i(\xi) u_i^{(1)}(x) \tag{13}$$

where $u_i^{(i)}$ are (say) bilinear functions on squares with side length $L > H$, i.e. we are using theorem 1 for the construction of special "super elements". Then we use these elements in the usual way to solve the original equation, i.e. we construct the stiffness matrix, etc. and compute the (micro) stiffness matrix simultaneously with η_i . Another possibility is to take

(14) $$v(x) = u(x) + H \sum_{i=1}^{2} \frac{\partial u}{\partial x_i} \eta_i(\xi)$$

where u is (say) a bicubic function on squares ($u \in C^1$). We underline here that it is essential in (13) to have all three terms. If we would force $u_i^{(1)} = 0$, then the answer would be completely wrong. It is easy to show that for $L \geq H$ we would get an approximate solution of a different equation.

We will show an example. Let

$$a(\xi) = \Phi \quad \text{for} \quad \xi = \hat{K} = \{\xi \,|\, |\xi_i| \leq \tfrac{1}{4}\}$$

$$a(\xi) = 1 \quad \text{for} \quad \xi \in K - \hat{K}$$

In this case $A_{11} = A = A_{22}$, $A_{12} = A_{21} = 0$, i.e. U solves equation

(15) $$A\Delta U = f$$

On the other hand, using only the function $u^{(0)}$ in (13) we will get the approximate solution of

(16) $$\tilde{A}\Delta v = f$$

Table 1 shows the coefficients A and $\tilde{A}$ for different Φ.

Φ	A	$\tilde{A}$
1	1	1
2	1.1822	1.25
4	1.3622	1.75
6	1.4522	2.25
10	1.5434	3.25
100	1.7084	25.75

Table 1

The direct and indirect methods yield some numerical results, but the question of how reliable (accurate) these results are is still open. The estimates, of course, show that the results are accurate enough when H is "sufficiently" small. But as we said earlier, H is physically given and cannot be changed. Therefore the question arises whether a given H is sufficiently small with respect to the desired accuracy (obviously the ratio of the scale of f and H is relevant), and whether the definition of the bulk coefficients stemming from the limiting behaviour $H \to 0$ is the optimal one.

2.2. Numerical Solution - Continuation

Coming back to (13) we may ask whether the functions $\eta_i(\xi)$ leading to the construction of the super elements are optimal; so let us have

$$(17) \qquad v(x) = \sum_{j=0}^{N} u_j(x)\kappa_j(\xi)$$

with $\kappa_j(\xi)$ periodic in ξ with period 1 , and let us ask about a good (resp. optimal) choice of κ_i .

Using the results in [12] and [13] it is not hard to show that

$$(18) \qquad u^H(x) = \int_{R_2} F(t_1,t_2)\, e^{i(x_1t_1+x_2t_2)}\, \nu(t_1H,t_2H,\xi)$$

$$H^{-2}\, \lambda^{-1}(t_1H,t_2H)dt_1\, dt_2$$

where $\nu(\tau_1,\tau_2,\xi)$ (periodic in ξ with period 1) is related to the solution of a so-called quasi-static eigenvalue problem:

$$(19) \qquad \sum_{i=1}^{2} \frac{\partial}{\partial\xi_i}\, a(\xi)\, \frac{\partial\mu(\tau_1,\tau_2)}{\partial\xi_i} = \lambda(\tau_1,\tau_2)\mu$$

where

$$\nu(\tau_1,\tau_2,\xi) = e^{-i(\tau_1\xi_1+\tau_2\xi_2)}\, \mu(\tau_1,\tau_2,\xi)$$

and μ is generally a complex ~~periodic (in ξ~~) function and λ the associated eigenvalue (real),

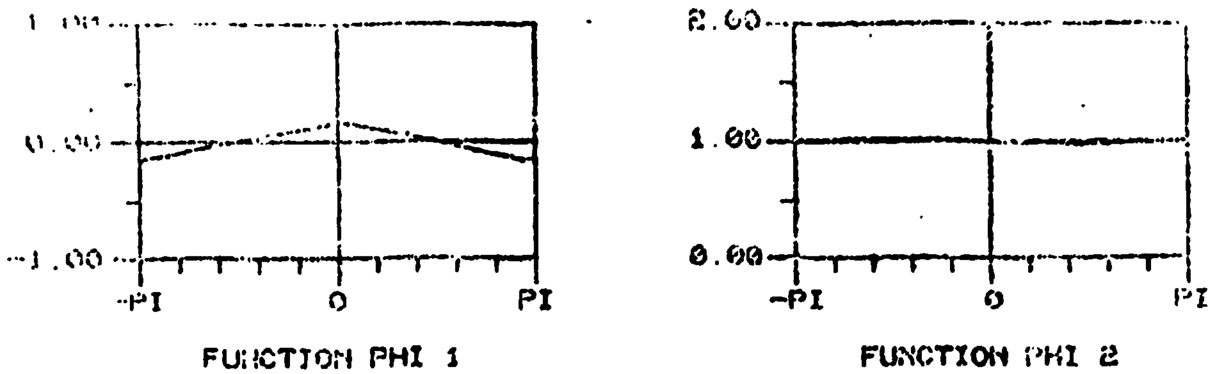
$$F(t_1,t_2) = \frac{1}{2\pi}\int_{R_2} f(x_1,x_2)\, e^{-i(t_1x_1+t_2x_2)}\, \bar{\nu}(t_1H,t_2H,\xi)dx_1\, dx_2$$

In the light of (18), it is obvious that the choice of $\kappa_j(\xi)$ in (17) is related to an approximation of the function $\nu(\tau_1,\tau_2,\xi)$. For a small H we can approximate $\nu(\tau_1,\tau_2,\xi)$ by its Taylor series (in τ_1,τ_2). Using only linear terms (in τ_1,τ_2) we will get <u>exactly</u> form (13) and in addition

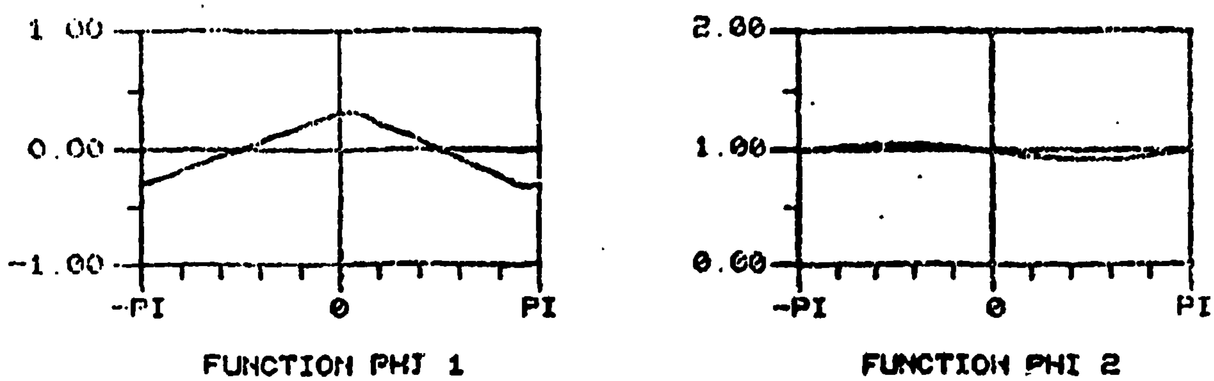
$$\lambda(\tau_1,\tau_2) = \sum_{i,j=1}^{2} A_{i,j}\tau_i\tau_j + o(\tau^2)$$

where $A_{i,j}$ are given by (5).

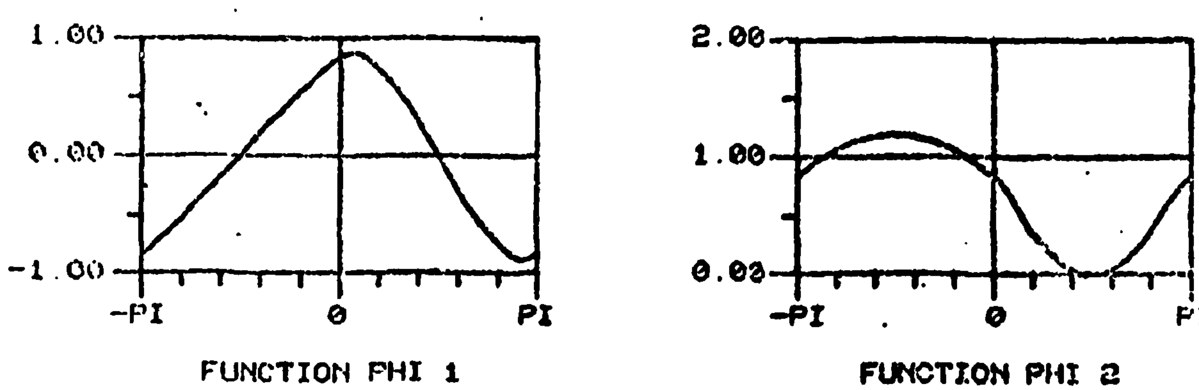
As an example let us compute a one dimensional problem with $a(\xi) = 100$ for $-\frac{1}{2} < \xi \leq 0$ and $a(\xi) = 1$ for $0 \leq \xi \leq \frac{1}{2}$. Figure 1 shows $\varphi_1 = \mathrm{Re}\,\nu(\tau,2\pi\xi)$ $\varphi_2 = \mathrm{Im}\ (\tau,2\pi\xi)$ for $\tau/2\pi = .1, .2, .5$.



$\tau/2\pi = 0.1$



$\tau/2\pi = 0.2$



$\tau/2\pi = 0.5$

FIGURE 1

Formula (18) shows more precisely the meaning of the statement that H has to be small with respect to the scale of f when the direct use of Theorem 1 is possible. It is only necessary to realize that the functions $\eta_i(\xi)$ are the derivatives of $v(\tau_1,\tau_2,\xi)$ for $\tau_1 = \tau_2 = 0$ as mentioned above. The analysis we have shown clearly hints to the way of a possible selfadaptive approach one has to follow. We have to simultaneously refine the mesh and construct proper "super elements" by eigenfunctions of quasi static problems. Therefore, an effective computation of those are of great importance. For some aspects related to this problem, see [18].

2.3. Some Additional Connections and References

The theoretical use of formula (17),together with the variational principles, can be traced (although derived from a different point of view), e.g. [19] and [20]. The functions $\kappa_j(\xi)$ are obtained intuitively in [19] and [20] but $\kappa_j(\xi) \notin L^1_{2,PER}(K)$, and they are treated similarly as nonconforming elements in the finite element method. See also [21].

As we said, the quasi static case, and its effective treatment, plays an important role. Together with [18], we also refer the reader to [22] and [23]. For a survey of problems mentioned in this paper in relation to composite materials, we refer e.g. to [24], [25], and [26]. We mentioned the same essential questions only in connection with e.g. (1). Similar problems are related to other equations, too. We mention here e.g. the case of the neutron transport equations [27], [28]. So far we have referred the reader mostly to nonmathematical papers. For a survey of recent mathematical results, we refer e.g. to survey papers [1], [2], [29], and [30].

3. Linear Case with Boundary

Let the assumptions about function a be the same as in section 2. First, we will assume that Ω is a bounded Lipschitz domain with smooth or unsmooth boundary. We are interested in this solution of (1) and (2). The solution obviously exists and is uniquely defined. The first principal question is how will Theorem 1 change in this case. The answer is given in

Theorem 2. Let U be the solution of (4) with boundary condition $U = 0$ on $\partial\Omega$. In addition, let U be smooth. Then

$$(20) \qquad \|u^H - U - H\sum_{i=1}^{2}\frac{\partial U}{\partial x_i}\eta_i(\xi)\|_{L^1_2(\Omega)} \le CH^{1/2}$$

$$(21) \qquad \|t_i^H - \sum_{j=1}^{2} T_j \chi_j^{[i]}(\xi)\|_{L_2(\Omega)} \le CH^{1/2}$$

The rate $H^{1/2}$ is optimal.

Theorem 2 shows that the presence of the boundary influences the accuracy of the homogenized solution significantly. Let us assume now that Ω is a halfplane, namely $\Omega = R^+ = \{x | x_2 > 0\}$ and denote $R^+(z) = \{x | x_2 > z\}$, $z > 0$. Then the next theorem can be proven.

Theorem 3. Assume that f is smooth, $f(x) = 0$ for $|x| > 1$. Let $\tilde{\Omega} \subset R^+$ be any bounded domain and $\tilde{\Omega}(z) = \tilde{\Omega} \cap R^+(z)$. Then

1)

$$(22) \qquad \| u^H - U - H \sum_{i=1}^{2} \frac{\partial U}{\partial x_i} \eta_i(\xi) \|_{L_2^1(\tilde{\Omega}(z))} \leq C \left[H^{\frac{1}{2}} e^{-\frac{z\tilde{C}}{H}} + H \right]$$

$$(23) \qquad \| t_i^H - \sum_{j=1}^{2} T_j \chi_j^{[i]}(\xi) \|_{L_2^3(\tilde{\Omega}(z))} \leq C \left[H^{\frac{1}{2}} e^{-\frac{z\tilde{C}}{H}} + H \right] .$$

2) There exists function $\rho(\xi)$, $\kappa_i^{(k)}(\xi)$ defined on $P = \{\xi | |\xi_1| < \frac{1}{2}, \xi_2 > 0\}$ periodic in ξ_1 with period 1 and ξ exponentially decreasing with ξ_2 so that

$$(24) \qquad \| u^H - U - H \sum_{i=1}^{2} \frac{\partial U}{\partial x_i} \eta_i(\xi) - H \frac{\partial U}{\partial x_2}(x_1, 0) \rho(\xi) \|_{L_2^1(\tilde{\Omega})} \leq CH$$

$$(25) \qquad \| t_i^H - \sum_{j=1}^{2} T_j \kappa_j^{[i]}(\xi) - \sum_{j=1}^{2} T_j(x_1, 0) \kappa_j^{(i)}(\xi) \|_{L_2(\tilde{\Omega})} \leq CH \quad .$$

For the proof, see [5].

Theorem 3 shows clearly the existence of a boundary layer which was mentioned in section 1. There is an open question whether similar behaviour holds when the domain is bounded, say, with a smooth boundary(or is also a halfplane with irrational angle between the cell orientation direction and the boundary of the halfplane). We mean e.g. the question of the validity of (22) and (23) when $\text{dist}(\tilde{\Omega}, \partial\Omega) \geq z$.

The term

$$\sum_{j=1}^{2} T_j(x_1, 0) \, \kappa_j^{[i]}(\xi)$$

is the boundary layer term, which can play an essential role in explaining failures of composite materials. In the case when the domain is a halfplane, this term is practically computable because we can compute only a string, say, of 3 cells and force $\kappa_j^{(i)}(\xi_1, 3) = 0$. For a numerical example of the boundary layer computation, see [2].

Although we discussed only the Dirichlet boundary condition, similar behaviour holds for other conditions too. In the case of the halfplane the generalization of Theorem 3 leads to higher order error estimates.

It is clear that the boundary layer is of utmost importance. There are no known results for general (say smooth) domains. But physically intuitive deliberations hint of the existence of boundary layers with the width which is not of exponential character. We expect to address this question in a future paper. So we see that the boundary creates essential difficulties which have to be treated very carefully. All questions and problems mentioned in previous sections (e.g. "large" H) are even more complicated when arising here. Although there are many papers devoted to different questions of homogenization, there are no papers addressing and treating the problem of the boundary layer.

4. The Nonlinear Problem [2)]

In this section we will discuss the nonlinear problem (1) (2). Denote by $W_p^1(\Omega)$, $1 < p < \infty$ the usual Sobolev space furnished with the norm

$$\|u\|^p_{W_p^1(\Omega)} = \|u\|^p_{L_p(\Omega)} + \sum_{i=1}^{2} \|\frac{\partial u}{\partial x_i}\|^p_{L_p(\Omega)}$$

where $L_p(\Omega)$ is the common space of functions integrable with the p-th power. By $\overset{0}{W}{}_p^1(\Omega)$ we denote the subspace with zero traces on $\partial\Omega$. Denote $W^1_{p,PER}(K)$ the space of periodic functions on K (analogously as in section 2). The existence and other properties of problem (1) (2) are studied in many papers. We refer the reader e.g. to [31], [32], and [33].

Let us list the assumed properties of the function $a(\xi,\eta)$, $\xi \in K$, $\eta \equiv (\eta_1,\eta_2) \in R_2$ used in (1).

1) For any $\eta \in R_2$, $a(\xi,\eta)$ is periodic in ξ with periodicity 1.
2) There exists a number $p \geq 2$ such that

$$\left|\frac{\partial^j}{\partial\eta_1^k\partial\eta_2^\ell} a(x,\eta)\right| \leq K(1 + |\eta|)^{p-2-j}\,, \qquad j = 0, 1, 2, \quad k + \ell = j \tag{26}$$

3) Problem (1) (2) is properly posed; let $\bar{G} \in W_p^1(\Omega)$, $\bar{w} \in \overset{0}{W}{}_p^1(\Omega)$ and $\bar{u}^H = \bar{G} + \bar{w}$ such that $(0 < \gamma \leq \gamma_0)$ for any $v \in \overset{0}{W}{}_p^1(\Omega)$

$$\left|\left(\int_\Omega \sum_{i=1}^{2} a(\xi, \bar{\eta}^H) \frac{\partial\bar{u}^H}{\partial x_i}\frac{\partial v}{\partial x_i} - fv\right) dx\right| \leq \gamma\|v\|_{W_p^1(\Omega)} \tag{27}$$

then for any $G \in W_p^1(\Omega)$, $\|G-\bar{G}\|_{W_p^1(\Omega)} \leq \gamma$ there exists at least one $u^H \in W_p^1(\Omega)$,

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2) We discuss here our special model problem. For the general case, see [34].

$u^H = G + w$, $w \in \overset{0}{W}{}^1_p(G)$ such that

$$\int_\Omega \left(\sum_{i=1}^{2} a(\xi,\eta^H) \frac{\partial u^H}{\partial x_i} \frac{\partial v}{\partial x_i} - fv \right) dx = 0$$

for any $v \in \overset{0}{W}{}^1_p(\Omega)$ and

$$\|\bar{u}-u\|_{W^1_p(\Omega)} \le C\gamma^\rho$$

where C and $0 < \rho \le 1$ are independent of γ and H.

4) Finally we will specify assumptions for the solution of an associated problem. Let us define it first. This problem consists of finding a periodic (vector) function $\chi(\xi,\sigma) \equiv (\chi^{[1]},\chi^{[2]})$, $\chi^{[j]}(\xi,\sigma) \in W^1_{p,PER}(K)$ (in ξ), $j = 1,2$ so that

$$\int_K \chi^{[j]}(\xi,\sigma)d\xi = 0$$

and

$$\int_K \left(\sum_{i=1}^{2} a(\xi,\mu) \left(\delta_i^j + \frac{\partial \chi^{[j]}}{\partial x_i}\right) \frac{\partial v}{\partial x_i} \right) d\xi = 0$$

for any $v \in W^1_{p,PER}(K)$ where $\mu = (\mu_1,\mu_2)$

$$\mu_i = \sum_{k=1}^{2} \sigma_k \left(\delta_i^k + \frac{\partial \chi^{[k]}}{\partial x_i}\right)$$

and δ_i^k is the Kronecker symbol. Now we assume

4.a) For every $\sigma \in R_2$ there exists (at least one) solution $\chi(\xi,\sigma)$ of the associated problem.

4.b) Function $\chi(\xi,\sigma)$ has two derivates with resp. to σ.

4.c) Let $\sigma(x)$ be a function defined on K such that

$$\left|\frac{\partial \sigma}{\partial x_i}\right| \le \zeta \qquad 0 \le \zeta \le \zeta_0 .$$

Then functions $\psi^{j;k,\ell}(x) = \left.\frac{\partial^j \chi}{\partial \sigma_1^k \partial \sigma_2^\ell}\right/_{\sigma=\sigma(x)} \in W^1_p(K)$ $\qquad j = 0,1,2 \quad k + \ell = j$ and

$$\|\psi^{j;k,\ell}(x) - \frac{\partial^j \chi}{\partial \sigma_1^k \partial \sigma_2^\ell}(x,\sigma(0))\|_{W^1_p(K)} \le C\zeta$$

where C does not depend on ζ.

The assumptions we made about the associated problems can be analyzed e.g. with the theory developed in [31].

We define for every $\sigma \in R_2$

$$(28) \qquad A_{\ell,k}(\sigma) = \int_K \Bigg[\Bigg(\sum_{i=1}^{2} a(x,\mu)\Big(\delta_i^{\ell} + \frac{\partial}{\partial x_i}\chi^{[\ell]}(x,0)\Big)\Big(\delta_i^{k} + \frac{\partial}{\partial x_i}\chi^{[k]}(x,\sigma)\Big)\Bigg)\Bigg]dx$$

$\ell,k = 1,2$

Now we can formulate the homogenized problem of (1)(2).

$$(29) \qquad \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} A_{i,j}(U_x) \frac{\partial}{\partial x_j} U = f$$

$$(30) \qquad U = 0 \quad \text{on} \quad \partial\Omega .$$

In [34] the following theorem (more general) has been proven.

Theorem 4. Let there exists the solution of the homogenized problem (29) (30). We assume in addition that the solution U has three bounded derivatives. Then for every $0 < H < H_0$ there exists at least one solution u^H of (1) and (2) and

$$(31) \qquad \|u^H - U - H \sum_{i=1}^{2} \frac{\partial U}{\partial x_i} \chi^{(i)}(\xi,U_x)\|_{W_p^1(\Omega)} \le CH^{\rho/p}$$

Let us now make some additional comments to Theorem 4. First, similar as in section 3, the right hand side of (31) has in fact two parts, one due to the differential equation (of order H^{ρ}) and due to the boundary conditions (of order $H^{\rho/p}$). For large p the loss of accuracy is very big. So it is desirable to find ways to analyze this boundary layer behaviour which are expected to lead to an accuracy of the order H^{ρ} . Although we formulated Theorem 4 directly for a bounded domain, there is a version of it for a domain without boundary which leads to the term H^{ρ} in the right hand side. The problem of optimal treatment for large H has now beome more difficult.

We will study as an example a special case when the function $a(\xi,\eta)$ has a form stemming from the laminates and depends only on the variable ξ_1 (is independent of ξ_2). We will assume that

$$a(\xi,\eta) = a_1(|\eta|^2) > 0$$
$$-.5 < \xi_1 < -.2$$
$$.2 < \xi_1 < .5$$
$$a(\xi,\eta) = a_2(|\eta|^2) > 0$$
$$-.2 < \xi_1 < .2$$

(cont.)

$$a_i(z) = \alpha_i \qquad 0 \le z \le \mu_i$$

$$a_i(z) = \alpha_i + \beta_i(z^2-\mu_i^2)^\gamma \qquad \mu_i \le z < \infty$$

$$\alpha_1 = 1.0 \ , \ \alpha_2 = 10 \ , \ \beta_1 = 5 \ , \ \beta_2 = .1$$

$$\gamma = 2 \ , \ \mu_1 = 1.5 \ , \ \mu_2 = 10$$

Then we get $p = 10$, $\rho = \frac{1}{9}$ and the Homogenized equations has the form

$$\frac{\partial}{\partial x_1} A_{1,1}(\sigma_1,\sigma_2) \frac{\partial U}{\partial x_1} + \frac{\partial}{\partial x_2} A_{2,2}(\sigma_1,\sigma_2) \frac{\partial U}{\partial x_2} = f$$

where $\sigma_1 = \frac{\partial U}{\partial x_1}$, $\sigma_2 = \frac{\partial U}{\partial x_2}$. The graphs of functions $A_{1,1}$, $A_{2,2}$ are given in Figure 2 and Figure 3.

The practical implementation of the nonlinear homogenization leads to many open questions because of the size of the problem. Very likely we have to compute the functions $\chi^{[k]}(x,\sigma)$ and $A_{i,j}(\sigma_1,\sigma_j)$ simultaneously during the incremental procedure. This, of course, will be very expensive and many serious computational problems will have to be solved. There is another possibility to compute the coefficients $A_{i,j}(\sigma_1,\sigma_2)$ in advance for a small number of σ and make heavy use of the fitting procedure. This, of course, is possible only for materials of hyperelastic type where loading and unloading goes the same way. This is not true for most of the real materials. In these cases only a step by step solution is possible. We mentioned the results only for the diffusion equation. Similar results hold for the elasticity problem, too, but implementation is even more complicated. There are not too many approaches available for the nonlinear case. For some in the field of composite matrices we refer to [35], [36], [37], and [38].

5. Some Additional Remarks

So far we concentrated our attention to the problems stemming from numerical treatment (direction b) mentioned in section 1. We cannot elaborate here in the second direction although there are many interesting and deep relations with basic problems in mechanics. It is obvious that on the microscale all materials exhibit a structure which is revealed in the course of the deformation processes of a comparable scale. The classical equations of homogenous continua cannot predict certain observable phenomena when the scale of solutions (say wavelength) approaches the characteristic dimension of the heterogenity. The whole approach of the homogenization leads to the problem of higher order continuum theory and basic problems of mechanics. For more about higher order continuum theories, we refer the reader e.g. to [39], [40], [41], [42].

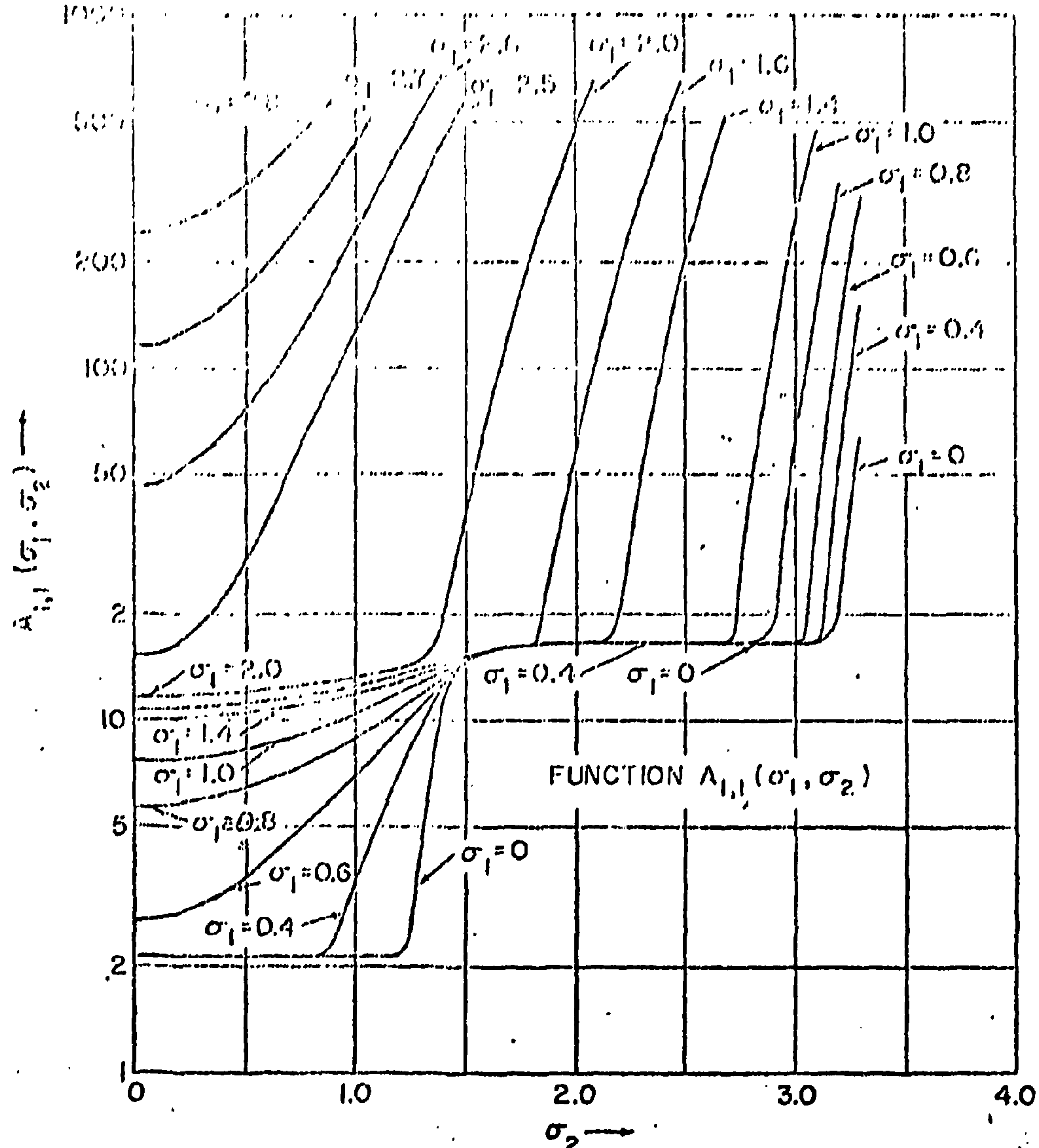
FUNCTION $\Lambda_{1,1}(\sigma_1, \sigma_2)$
$\Lambda_{1,1}(\sigma_1, \sigma_2)$
σ_2
0
1.0
2.0
3.0
4.0
1
2
5
10
20
50
100
200
$\sigma_1 = 2.0$
$\sigma_1 = 1.4$
$\sigma_1 = 1.0$
$\sigma_1 = 0.8$
$\sigma_1 = 0.6$
$\sigma_1 = 0.4$
$\sigma_1 = 0$
$\sigma_1 = 0.4$
$\sigma_1 = 0$
$\sigma_1 = 1.0$
$\sigma_1 = 0.8$
$\sigma_1 = 0.6$
$\sigma_1 = 0.4$
$\sigma_1 = 0$

FIGURE 2

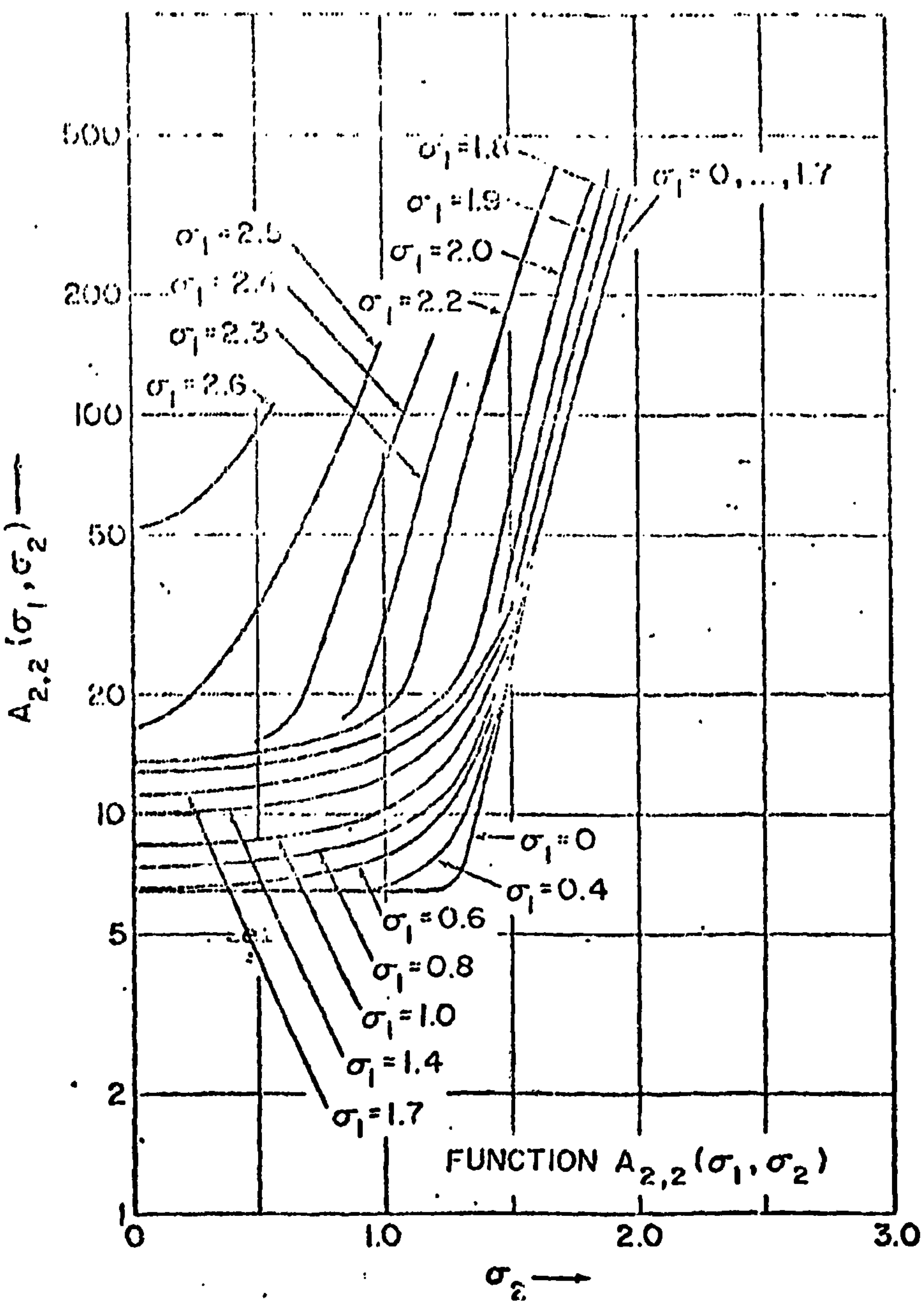
500
200
100
50
20
10
5
2
1
0
1.0
2.0
3.0
$A_{2,2}(\sigma_1, \sigma_2)$
σ_2
$\sigma_1 = 1.8$
$\sigma_1 = 0, \ldots, 1.7$
$\sigma_1 = 1.9$
$\sigma_1 = 2.5$
$\sigma_1 = 2.0$
$\sigma_1 = 2.4$
$\sigma_1 = 2.2$
$\sigma_1 = 2.3$
$\sigma_1 = 2.6$
$\sigma_1 = 0$
$\sigma_1 = 0.4$
$\sigma_1 = 0.6$
$\sigma_1 = 0.8$
$\sigma_1 = 1.0$
$\sigma_1 = 1.4$
$\sigma_1 = 1.7$
FUNCTION $A_{2,2}(\sigma_1, \sigma_2)$

FIGURE 3

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