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GENERAL TREES AND GENERAL LOOPS IN THE UNITARIZATION OF THE DUAL RESONANCE MODEL

Loh-ping Yu
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ABSTRACT

We present a series of beautiful, elegant, "simple" and <u>unique</u> formulas for general tree amplitudes and general loop amplitudes in the dual resonance model. In particular, new result of multiloop amplitudes with external reggeons and pure-reggeon multiloop amplitudes are derived. Various rules are given for writing down the most general tree amplitudes and loop amplitudes by simply inspecting the corresponding Feynman-like diagrams. Simple intuitive interpretations of various factors in the formulas are also discussed.

I. PHILOSOPHY AND MOTIVATION

A Veneziano formula[1] incorporates the following three properties of strong interaction:[2]

(a) All its singularities arise from narrow resonance exchanges.

(b) Regge asymptotic behavior in all channels.

(c) Exact crossing symmetry.

It follows from the properties (a), (b) that such a model must contain an infinite number of exchanged resonances instead of a finite number. Consequently, it then follows that the model necessarily possesses the duality property (hence the name "dual resonance model"), because direct channel resonance exchanges are supposed not only to dominate the low-energy region but also to describe the high-energy region, which is usually controlled by cross-channel Regge poles. Accordingly, duality is a natural consequence of the assumptions (a), (b).

However, the narrow resonance, in fact zero-width, assumption although is in consistency with the experimental fact that all Regge trajectories seem to rise linearly,[3] nevertheless clearly violates the unitarity principle of S-matrix theory[4] on the other hand; because unitarity requires branch cuts on the real axis (in the energy-squared plane) if above threshold, and all resonances, having finite widths, to lie on the second Riemann sheet beneath the cuts. However, since the dual resonance model has already incorporated almost all S-matrix principles,[4] except unitarity, it is natural to hope that suitable modifications made on the model will enable it to satisfy the unitarity principle, and will hence enable one to obtain a satisfactory theory of strong interaction. This is the motivation behind this thesis.

The idea of unitarization of the dual resonance model, as suggested by the factorization property[5] of n-point Veneziano formula, is to regard the n-point Veneziano amplitude as the Born (or tree) amplitude, which is an approximation[6] to a more exact physical scattering amplitude. One then attempts to construct, from unitarity, the higher order multiloop amplitudes, compatible[6] with the duality assumption. In other words, the unitarity constraint is imposed on the dual resonance model in a perturbative way, strictly in analogy with the perturbative field theory. But, there is one fundamental important property distinguishing the dual resonance model from the perturbative field theory. Namely, duality is assumed[6] to be true in each order of the loop diagrams; therefore all loop amplitudes related by duality are equal to one another instead of being added, as they usually are in the field theoretical model.

As we will see later, the outcome of such an approach is surprisingly elegant and simple. The results are not only mathematically beautiful but also have very simple intuitive interpretations. It might therefore be hoped that they will be useful in the framework of future theory of strong interaction.

II. OUTLINE PRINCIPLES AND SUMMARIZE THE RESULTS

The perturbative-unitarity equation can be mathematically stated as follows,

$$\mathrm{Im}\, f^{(n)} = f^{*(n-m)}\, f^{(m)}, \quad m < n, \quad m,n = 1,2,\cdots \tag{2.1}$$

where $f^{(n)}$ is the scattering amplitude to nth order in the coupling constant. Equation (2.1) means that the imaginary part of the nth order amplitude can be constructed from the mth and the (n-m)th order amplitudes. For example, the imaginary part of the planar box diagram (4th order) can be obtained from two 2nd order four-point tree diagrams. In principle, one may then use the dispersion relation to get the real part of the nth order amplitude. In practice, however, it is more convenient to use Feynman tree theorm[7] to get directly the loop amplitudes themselves. The application of the Feynman tree theorm can be carried out in two steps: (a) cuttings or factorizations, to get multiply factorized tree diagrams; (b) sewings, to get multiloop diagrams. To illustrate this theorem we again consider the planar box diagram. The theorem states that we first factorize on the n-point Veneziano tree diagram, obtaining a six-point tree diagram with two adjacent excited legs (Fig. 1), and then sew the two excited legs together by inserting a propagator and integrating over the loop momentum. We thereby obtain the planar box diagram.

Similarly, the nonplanar loop diagram can be obtained by sewing two nonadjacent excited legs together, as shown in Fig. 9. And the overlapping loop diagram can be obtained by sewing two overlapped pairs of excited legs, as shown in Fig. 11. Finally, the nonorientable loop diagram is defined by sewing an excited leg with a twisted excited leg. It has already been shown[6] that the above-mentioned four types of loops, namely, the planar loops, the nonplanar loops, the overlapping loops, and the nonorientable loops, are the complete set of loops that one can construct from the perturbative-unitarity equation, Eq. (2.1).

The construction of amplitudes from simpler amplitudes by repeatedly <u>cutting</u> and <u>sewing</u>, which represents the main part of the work of this thesis, results in a series of beautiful, elegant, and simple formulas:

(a) multiply factorized tree amplitudes,[8] Eq. (4.25),

(b) pure-reggeon tree amplitudes,[8] Eq. (4.27),

(c) multiloop amplitudes[9] (of all types), Eqs. (5.35) and (5.47),

(d) multiloop amplitudes with external reggeons, Eq. (6.5),

(e) pure-reggeon multiloop amplitudes (of all types) Eq. (6.7).

The last two classes of amplitudes, (d), (e), are the new results of this thesis report. They are presented here for the sake of completeness and for suggesting a complete theory of pure reggeon calculus.[10] A detailed mathematical calculation for the nonplanar multiloop amplitude and simple derivations therefrom of the overlapping, the nonorientable, and the planar multiloop amplitudes are given here to supplement the methods given in the former publications.[9]

Most of the formulas in the following are expressed in operator formalism.[11] We briefly review the operator formalism. An infinite set of four-dimensional harmonic oscillators $a_{\mu,n}$ $\mu = 1,2,3,4$,

$n = 1,2,\cdots\infty$ is introduced[11] to discuss the factorized excited states. They obey the commutation relations

$$[a_{\mu,n}, a^\dagger_{\nu,m}] = g_{\mu\nu}\,\delta_{nm}\,. \tag{2.2}$$

Three very useful identities are

$$\exp(fa)\, z^{a^\dagger a} = z^{a^\dagger a}\exp(fza), \tag{2.3a}$$

$$z^{a^\dagger a}\exp(fa^\dagger) = \exp(fza^\dagger)\, f^{a^\dagger a}, \tag{2.3b}$$

$$\exp(fa)\exp(ga^\dagger) = \exp(ga^\dagger)\exp(fa)\exp(f\cdot g). \tag{2.3c}$$

It turns out more convenient to introduce the coherent state, defined by

$$|z\rangle = \exp(za^\dagger)|0\rangle. \tag{2.4}$$

A number of useful properties involving coherent states are

$$a|z\rangle = z|z\rangle, \tag{2.5a}$$

$$\exp(z'a^\dagger)|z\rangle = |z' + z\rangle, \tag{2.5b}$$

$$X^{a^\dagger a}|z\rangle = |Xz\rangle. \tag{2.5c}$$

III. NOTATIONS AND THE ALGEBRA OF CROSS-RATIOS

Kikkawa and Sato's notations[12] will be used throughout the whole work. We summarize the notations as follows: Let k_μ^i be the four-momentum of the ith particle; a_n^i, $n = 1,2,\cdots\infty$ be the harmonic oscillator operators corresponding to the ith particle (if it is an excited reggeon), then

$$|k^i) \equiv \left(\frac{k^i}{1^{\frac{1}{2}}}, \frac{k^i}{2^{\frac{1}{2}}}, \cdots, \frac{k^i}{n^{\frac{1}{2}}}, \cdots\right) ,$$

$$|a^i) \equiv (a_1{}^i, a_2{}^i, \cdots, a_n{}^i, \cdots),$$

$$(M_+)_{nm} = (M_+{}^T)_{mn} \equiv \left(\frac{n}{m}\right)^{\frac{1}{2}} (-)^n \binom{-m}{n} ,$$

$$(M_-)_{mn} = (M_-{}^T)_{nm} \equiv \left(\frac{n}{m}\right)^{\frac{1}{2}} (-)^n \binom{m}{n} ,$$

$$(M_0)_{nm} \equiv \delta_{nm} ,$$

$$(a^i|x) = x|a^i) \equiv \sum_{n=1}^{\infty} a_n{}^i x^n, \tag{3.1}$$

$$(a^i|x|a^j) = \sum_{n=1}^{\infty} a_n{}^i x^n a_n{}^j ,$$

$$(a^i|xM_\pm y|a^j) = (a^j|yM_\pm{}^T x|a^i)$$

$$= \sum_{n,m=1}^{\infty} a_n{}^i x^n (M_\pm)_{nm} y^m a_m{}^j,$$

$$(a^i|M_\pm = M_\pm{}^T|a^i).$$

In Appendix A, a set of useful identities is listed.

Consider now two sets of N Koba-Nielsen variables[13] w_i and w_i', $i = 1,2,\cdots,N$; a cross-ratio defined on the w_i set is

$$P(j,k,\ell,m) \equiv P(w_j,w_k,w_\ell,w_m) \equiv \frac{(w_j - w_\ell)(w_k - w_m)}{(w_j - w_m)(w_k - w_\ell)} \,. \tag{3.2}$$

It is easy to see, from Eq. (3.2), that

(a) $0 \leq P(j,k,\ell,m) \leq 1$, except if $j = m$.

(b) $$1 - P(j,k,\ell,m) = P(j,\ell,k,m) \,, \tag{3.3a}$$

$$\frac{1}{P(j,k,\ell,m)} = P(j,k,m,\ell), \tag{3.3b}$$

$$P(j,k,\ell,m) = P(k,j,m,\ell) = P(m,\ell,k,j) = P(\ell,m,j,k), \tag{3.3c}$$

$$P(j,k,\ell,m)\, P(j,k,m,n) = P(j,k,\ell,n), \tag{3.3d}$$

$$P(j,k,\ell,m) = \frac{1}{1 - \frac{1}{P(j,m,\ell,k)}} \,. \tag{3.3e}$$

(c) If $w_j = \infty$, $w_k = 0$, $w_\ell = 1$, then

$$P(j,k,\ell,m) = w_m,$$

and we speak of w_j, w_k, w_ℓ specifying the W frame.

(d) If $w_a = \infty$, $w_b = 0$, $w_c = 1$ specifies the W frame, and $w_j' = \infty$, $w_k' = 0$, $w_\ell' = 1$ specifies the W' frame; then the projective transformation from the W frame to the W' frame is

$$w_m = P(w_a',w_b',w_c',w_m') \equiv P'(a,b,c,m), \tag{3.4}$$

and that from the W' frame to the W frame is

$$w'_m = P(w_j, w_k, w_\ell, w_m) \equiv P(j,k,\ell,m). \qquad (3.5)$$

Note that in Eq. (3.4), we use P' to define the cross ratio of the W' frame.

(e) Any cross ratio is invariant under projective transformations (duality transformations), i.e.,

$$P(p,q,r,s) \equiv P'(p,q,r,s). \qquad (3.6)$$

It is this property that greatly simplifies the calculations and frequently enables us to obtain frame-independent, projectively invariant results. As in special relativity, we can always choose a particular convenient frame, work out the answer, and then generalize it to a projectively invariant form (dual form) by forming the appropriated cross ratios.

We now come to the aglebra of cross ratios. Because, in later calculation, it is always possible to keep the momenta conserved inside the notation $|)$; we will assume, from now on, that momenta are always conserved when they appear inside the notation $|)$. Bearing this assumption in mind, we can always ignore all residue terms, i.e., terms due to the contractions of $M_\pm$ on $|k_1)$.

A typical example of cross-ratio algebra is illustrated here by considering a term describing the coupling of a^i reggeon with the a^j reggeon:[8]

$$I_{ij} = \langle 0| \exp[(a^i|P_i(j)\ M_- P(i+1,j+1,i,j)\ \ M_-^T P_j(i)|a^j)] \left| \begin{matrix}\lambda_i \\ \\ \lambda_j\end{matrix} \right\rangle, \qquad (3.7)$$

where

$$P_i(j) = P(i,i+1,i-1,j), \quad P_j(i) = P(j,j+1,j-1,i), \tag{3.8}$$

and the coherent state $|\lambda_j\rangle$ is of the form[8,14]

$$|\lambda_j\rangle = \exp\left[\sum_\ell (k_\ell|y_\ell|a^{j\dagger})\right]|0_j\rangle. \tag{3.9}$$

As we just mentioned, here we assume $\sum_\ell k_\ell = 0$. Suppose now we substitute Eq. (3.9) in Eq. (3.7) and commute $a^{j\dagger}$ to the left by using Eq. (2.3c), we obtain (omit $|\lambda_i\rangle$ for simplicity)

$$I_{ij} = \langle 0_j|\exp\left[\sum_\ell (a^i|P_i(j)M_-P(i+1,j+1,i,j)M_-^{\,T}P_j(i)y_\ell|k_\ell)\right]|0_j\rangle. \tag{3.10}$$

We can explicitly calculate Eq. (3.10), by writing

$$y_\ell = \hat{P}_j\,\hat{P}_j^{-1}(y_\ell) \equiv \hat{P}_j(z_\ell), \tag{3.11}$$

with

$$\hat{P}_j(x) \equiv \frac{1}{P_j(x)} = \frac{1}{P(j,j+1,j-1,x)} = P(j,j+1,x,j-1). \tag{3.12}$$

Substituting Eq. (3.11) in Eq. (3.10), we can explicitly simplify Eq. (3.10) as follows

$$\begin{aligned} I_{ij} &= \langle 0_j|\exp\left[\sum_\ell (a^i|P_i(j)M_-P(i+1,j+1,i,j)M_-^{\,T}P_j(i)\hat{P}_j(z_\ell)|k_\ell)\right]|0_j\rangle \\ &\qquad\qquad [\text{by Eq. (3.11)}] \\ &= \langle 0_j|\exp\left[\sum_\ell (a^i|P_i(j)M_-P(i+1,j+1,i,j)M_-^{\,T}P(j,j+1,j-1,i) \right. \\ &\qquad \left. \times\, P(j,j+1,z_\ell,j-1)|k_\ell)\right]|0_j\rangle \\ &\qquad\qquad [\text{by Eq. (3.12)}] \end{aligned}$$

Equation (3.13) continued next page

Equation (3.13) continued

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P_i(j) M_- P(i+1,j+1,i,j) M_-^T P(j,j+1,z_\ell,i) | k_\ell) \right] | 0_j \rangle$$

[by Eq. (3.30)]

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P_i(j) M_- P(i+1,j+1,i,j) \frac{1}{1 - \frac{1}{P(j,j+1,z_\ell,i)}} | k_\ell) \right] | 0_j \rangle$$

[by Eq. (3.1)]

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P_i(j) M_- P(i+1,j+1,i,j) P(j,i,z_\ell,j+1) | k_\ell) \right] | 0_j \rangle$$

[by Eq. (3.3e)]

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P_i(j) M_- P(j,i,z_\ell,i+1) | k_\ell) \right] | 0_j \rangle$$

[by Eqs. (3.3c) and (3.3d)]

$$= \langle 0_j | \exp \left\{ \sum_{\ell} (a^i | P_i(j) [1 - P(j,i,z_\ell,i+1)] | k_\ell) \right\} | 0_j \rangle$$

[by Eq. (3.1)]

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P(i,i+1,i-1,j)\, P(j,z_\ell,i,i+1) | k_\ell \right] | 0_j \rangle$$

[by Eq. (3.3a)]

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P(i,i+1,i-1,z_\ell) | k_\ell) \right] | 0_j \rangle \qquad \text{[by Eq. (3.3d)]}$$

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P_i(z_\ell) | k_\ell) \right] | 0_j \rangle \qquad \text{[by Eq. (3.8)]}$$

$$= \langle 0_j | \exp \left[\sum_{\ell} (a^i | P_i \hat{P}_j^{-1}(y_\ell) | k_\ell) \right] | 0_j \rangle . \qquad \text{[by Eq. (3.11)]} \tag{3.13}$$

This example should be enough to illustrate how the algebra of cross ratio works. However, it is more convenient to restore the coherent state $|\lambda_j\rangle$ explicitly. This can be done <u>formally</u> by inserting $\exp[\sum_\ell (k_\ell|y_\ell|a^{j\dagger})]$ to the right of $\langle 0_j|$ in Eq. (3.11), and use Eq. (2.3c) to pass it to the right, we get

$$\begin{aligned} I_{ij} &= \langle 0_j|\exp\left[\sum_\ell (k_\ell|y_\ell|a^{j\dagger})\right]\exp\left[\sum_\ell (a^i|P_i\hat{P}_j^{-1}(y_\ell)|k_\ell)\right]|0_j\rangle \\ &\cong \langle 0_j|\exp[(a^i|P_i\hat{P}_j^{-1}(\ )|a^j)]\ \exp\left[\sum_\ell (k_\ell|y_\ell|a^{j\dagger})\right]|0_j\rangle \\ &\equiv \langle 0_j|\exp[a^i|P_i\hat{P}_j^{-1}(\ )|a^j)]|\lambda_j\rangle . \end{aligned} \quad (3.14)$$

Comparing Eq. (3.14) with Eq. (3.7), we conclude the <u>symbollical</u> identity (add $|\lambda_i\rangle$ back)

$$\begin{aligned} I_{ij} &= \langle 0|\exp[(a^i|P_i\hat{P}_j^{-1}(\ )|a^j)]\left|\begin{matrix}\lambda_i \\ \lambda_j\end{matrix}\right\rangle \\ &\cong \langle 0|\exp[(a^i|P_i(j)M_-P(i+1,j+1,i,j)M_-^{T}P_j(i)|a^j)]\left|\begin{matrix}\lambda_i \\ \lambda_j\end{matrix}\right\rangle . \end{aligned} \quad (3.15)$$

Since Eq. (3.7) is symmetrical with respect to i,j indices, it follows that

$$\langle 0|\exp[(a^i|P_i\hat{P}_j^{-1}(\ )|a^j)]\left|\begin{matrix}\lambda_i \\ \lambda_j\end{matrix}\right\rangle \cong \langle 0|\exp[(a^i|(\ )\hat{P}_i^{-1}P_j|a^j)]\left|\begin{matrix}\lambda_i \\ \lambda_j\end{matrix}\right\rangle . \quad (3.16)$$

Note the two <u>symbollically</u> identities Eqs. (3.15) and (3.16) are true only if $|\lambda_i\rangle$, $|\lambda_j\rangle$ are of the form given by Eq. (3.9). As we will see later, this is indeed the way that we define[8] the factorized coherent states [see Eq. (4.20)].

But Eqs. (3.15) and (3.16) say something more, namely, it is a proof of the factorizability of the operator part of the N-reggeon tree amplitude[8] (see Appendix B also). In fact, they have already been used in the planar N-loop calculation, in Ref. 9 (UCRL-20054). Of course, Eq. (3.13) is identical to Lovelace[10] expression.

The virtue of the cross-ratio algebra is not only that it is elegant but also that it frequently yields projectively invariant expressions.

IV. MULTIPLY FACTORIZED TREE AMPLITUDES AND PURE REGGEON TREE AMPLITUDES

In Ref. 8, we have obtained a set of rules for writing down the most general formulas for these two classes of amplitudes [see Appendix B, also], by simply inspecting the corresponding tree diagrams. However, there we did not fully invoke the cross-ratio algebra; and in deriving[8] the pure-reggeon tree amplitude, we remarked that letting certain momentum go to zero by modifying the spectrum of relevant trajectories to get the asymmetrical propagator, is not a necessary procedure. Here we would like to use cross-ratio algebra to re-do the multiple factorizations and to give a proof of the remark mentioned above, by direct factorization. The reader who is interested only in the final answer may well skip to Eqs. (4.25) and (4.27).

Let us first write down the dual N-point tree amplitude in the Koba-Nielsen representation[13] (Fig. 2a)

$$G^{(0)}_{(W)} = \int \prod_{i=1}^{N} dw_i (dw_a dw_b dw_c)^{-1} \{W_N\}, \tag{4.1}$$

where $(w_i \equiv w_{i+N})$

$$\{\overline{W}_N\} = (w_a - w_b)(w_b - w_c)(w_c - w_a)$$

$$\times \prod_{i=1}^{N} (w_i - w_{i+1})^{-\alpha_0 - \frac{1}{2}(k_i^2 + k_{i+1}^2)} \prod_{i=1}^{N} (w_i - w_{i+2})^{\alpha_0 + \frac{1}{2} k_{i+1}^2}$$

$$\times \prod_{\substack{i,j=1 \\ (i \neq j)}}^{N} (w_i - w_j)^{-\frac{1}{2} k_i \cdot k_j}. \tag{4.2}$$

The three variables w_a, w_b, w_c are fixed and they specify the W frame. The set of N variables w_i, $i = 1,2,\cdots N$ are ordered, for convenience, on a circle (Fig. 2b). Figure 2(b) will hereafter be called the ordering diagram. In Eq. (4.2), we assume a trajectory function $\alpha(s) = \alpha_0 + \frac{1}{2}s$; and for N scalar external legs, the bootstrap conditions $\alpha(k_i) = \alpha_0 + \frac{1}{2}k_i^2 = 0$, $i = 1,2,\cdots N$, is satisfied.

We can interpret the factor $(w_i - w_j)^{-\frac{1}{2}k_i \cdot k_j}$ as corresponding to lines directly connecting the w_i leg to the w_j leg [if one inserts several holes inside the circle of Fig. 2(a), thus obtaining the multiloop diagram, one then naturally expects that these lines have the freedom to encircle the loops]. Since $i \neq j$ in the factors $(w_i - w_j)^{-\frac{1}{2}k_i \cdot k_j}$ the momentum is not conserved, and the factors

$$(w_i - w_{i+1})^{-\alpha_0 - \frac{1}{2}(k_i^2 + k_{i+1}^2) - 1} \quad \text{and} \quad (w_i - w_{i+2})^{\alpha_0 + \frac{1}{2}k_{i+1}^2}$$

are necessary to guarantee the projective invariance of the whole integrand in Eq. (4.1) (they can also easily be generalized to loop amplitudes).

We start by factorizing the N-point tree into M-point and (N-M+2)-point trees. We introduce[8] three frames (Fig. 3):

$$\begin{aligned}
&\text{W frame:} \quad w_1 = \infty, \quad w_2 = 1, \quad w_N = 0, \quad w_i = w_{i+N} \quad (i = 1,2,\cdots N), \\
&\text{Y frame:} \quad y_M = \infty, \quad y_{M-1} = 1, \quad y_1 = 0, \quad y_i = y_{i+M} \quad (i = 1,2,\cdots M), \\
&\text{Z frame:} \quad z_{M-1} = \infty, \quad z_M = 1, \quad z_N = 0, \quad z_j = z_{N-M+2+j} \quad (j = M-1,\cdots N),
\end{aligned} \tag{4.3}$$

the w_i's are related to the y_i's, a_j's and the factorized propagato variable t by the equation

$$\begin{aligned} w_i &= \frac{y_2}{y_i}, \quad i = 1,2,\cdots M-1, \\ t &= \frac{w_M}{w_{M-1}}, \qquad\qquad (4.4) \\ w_j &= y_2\, t\, z_j, \quad j = M,\cdots N. \end{aligned}$$

Substituting Eq. (4.4) in Eq. (4.1), we get

$$G^{(0)}_{(W)} = \int \prod_{(i\neq M,M-1,1)} dy_i \{Y_M\} \int \prod_{j\neq(M-1,M,N)} dz_j \{Z_{N-M+2}\} \times \int_0^1 dt\; t^{-\alpha(s_1)-1}\,(1-t)^{\alpha_0-1}$$

$$\times \prod_{i=1}^{M-1} \prod_{j=M}^{N} (1 - y_i\, t\, z_j)^{-k_i\cdot k_j} \qquad (4.5)$$

where $s_1 = (k_M + k_{M+1} + \cdots + k_N)^2$. Introduce[11] harmonic oscillator a^M, $a^{M\dagger}$ to factorize the last factors in Eq. (4.5):

$$\prod_{i=1}^{M-1} \prod_{j=M}^{N} (1 - y_i\, t\, z_j)^{-k_i\cdot k_j}$$

$$= \langle 0|\exp\left[\sum_{i=1}^{M-1} (k_i|y_i|a^M)\right] t^{R_a} \exp\left[\sum_{j=M}^{N} (a^{M\dagger}|z_j|k_j)\right]|0\rangle. \qquad (4.6)$$

Substituting Eq. (4.6) in Eq. (4.5), and defining

$$
\begin{aligned}
G^{(0)}_{(W)} &= \langle 0|G^{(1)}_{(Y)}\,(a^M)\,D(R,S)\,G^{(1)}_{(Z)}\,(a^{M\dagger})|0\rangle \\
&\equiv \langle 0|G^{(1)}_{(Y)}\,(a^M)|\lambda_M\rangle,
\end{aligned}
\tag{4.7}
$$

we then obtain the single-factorization result

$$
\begin{aligned}
&\langle 0|G^{(1)}_{(Y)}\,(a^M)|\lambda_M\rangle \\
&= \int \prod_{(i\neq M,M-1,1)} dy_i\,\{Y_M\}\,\langle 0|\exp\left[\sum_{i=1}^{M-1}(a^M|y_i|k_i)\right]|\lambda_M\rangle,
\end{aligned}
\tag{4.8}
$$

with $y_M = \infty$, $y_{M-1} = 1$, $y_1 = 0$.

One can immediately generalize Eq. (4.8) into a general frame $y_a = \infty$, $y_b = 1$, $y_c = 0$ by simply putting

$$
y_i = P(M,1,M-1,i) \equiv P_M(i),
\tag{4.9}
$$

hence the single-factorized tree amplitude (Fig. 4) in a general frame is

$$
\langle 0|G^{(1)}_{(Y)}(a^M)|\lambda_M\rangle = \int \prod_{(i\neq a,b,c)} dy_i\,\{Y_M\}\langle 0|\exp\left[\sum_{i=1}^{M-1}(a^M|P_M(i)|k_i)\right]|\lambda_M\rangle.
\tag{4.10}
$$

We now proceed to obtain the asymmetrical propagator by direct factorization on Eq. (4.10). Since we are considering the case of two dots facing each other (see Fig. 5), we relabel y_i $i = 1,2,\cdots M$ by y_i', $i = 1,2,\cdots M$, shown in Fig. 4. Then Eq. (4.10) becomes

$$\langle 0|G^{(1)}_{(Y')}(a^S)|\lambda_S\rangle = \int \prod_{(i\neq 1,2,M)} dy'_i \{Y'_M\} \langle 0|\exp\left[\sum_{\substack{i=1\\(i\neq S)}}^{M} (a^S|P_S(i)|k_i)\right]|\lambda_S\rangle, \qquad (4.11)$$

where $P_S(i) \equiv P(S,S-1,S+1,i)$. We then choose the frame $y'_1 = \infty$, $y'_2 = 1$, $y'_M = 0$ to factorize Eq. (4.11), as indicated by the dotted line in Fig. 4. We divide y'_i, $i = 1,2,\cdots,M$ into a W frame and a V frame by choosing $w_{S+1} = \infty$, $w_S = 1$, $w_1 = 0$ and $v_S = \infty$, $v_{S+1} = 1$, $v_M = 0$, so that

$$\begin{aligned} y'_i &= \frac{w_2}{w_i}, \qquad i = 1,2,\cdots S, \\ t &= \frac{y_{S+1}}{y_S}, \qquad\qquad\qquad\qquad (4.12) \\ y'_j &= w_2\, t\, v_j, \qquad j = S+1,\cdots M. \end{aligned}$$

Substituting Eq. (4.12) in (4.11), we find[15]

$$\begin{aligned} \langle 0|G^{(1)}_{(Y')}(a^S)|\lambda_S\rangle &= \int \prod_{(i\neq S+1,S,1)} dw_i \{W_{S+1}\} \int \prod_{(j\neq S+1,S+2,M)} dv_j \{V_{M-S+1}\} \\ &\times \int_0^1 dt\, t^{-\alpha(s_2)} (1-t)^{\alpha_0-1} \left(\frac{1-t}{1-tw_{S-1}}\right)^{-\alpha(k_S)} \\ &\times \langle 0|\exp\left\{\sum_{\substack{i=1\\(i\neq S)}}^{S+1} (a^S|P'_S(i)|k_i) + \sum_{\substack{i=1\\(i\neq S+1)}}^{S+1} (a^{S+1}|P'_{S+1}(i)|k_i)\right. \end{aligned}$$

Equation (4.13) continued next page

Equation (4.13) continued

$$+ \sum_{j=S+2}^{M} (a^S|M_- P'(S-1,1,S,S+1)M_-^T \; t \; P''_{S+1}(j)|k_j)$$

$$\times \left\{ \left(\frac{1-t}{1-tw_{S-1}} \right)^{R_S} t^{R_{S+1}} \right\} \exp\left\{ \sum_{j=S+2}^{M} (a^{S+1\dagger}|P''_{S+1}(j)|k_j) \right\} \left| \begin{matrix} \lambda_S \\ 0_{S+1} \end{matrix} \right\rangle , \qquad (4.13)$$

where P' refers to W frame, P" refers to V frame, i.e.,

$$P'_S(i) = P'(S,S-1,S+1,i) \equiv \frac{(w_S - w_{S+1})(w_{S-1} - w_i)}{(w_S - w_i)(w_{S-1} - w_{S+1})} ,$$

$$P'_{S+1}(i) = P'(S+1,1,S,i), \qquad (4.14)$$

$$P''_{S+1}(j) = P''(S+1,S+2,M,j) \equiv \frac{(v_{S+1} - v_M)(v_{S+2} - v_j)}{(v_{S+i} - v_M)(v_{S+1} - v_j)} ,$$

and

$$s_2 = (k_{S+2} + k_{S+3} + \cdots + k_M)^2,$$

$$R_S = \sum_{n=1}^{\infty} n \, a_n^{S\dagger} a_n^S, \qquad (4.15)$$

$$R_{S+1} = \sum_{m=1}^{\infty} m \, a_m^{S+1\dagger} a_m^{S+1} .$$

One then defines

$$\langle 0|G^{(1)}_{(Y')}(a^S)|\lambda_S\rangle \equiv \langle 0|G^{(2)}_{(W)}(a^S,a^{S+1}) \, D'(R_S,R_{S+1},s_2) \, G^{(1)}_{(V)}(a^{S+1\dagger}) \left| \begin{matrix} \lambda_S \\ 0_{S+1} \end{matrix} \right\rangle$$

$$\equiv \langle 0|G^{(2)}_{(W)}(a^S,a^{S+1}) \left| \begin{matrix} \lambda_S \\ \lambda_{S+1} \end{matrix} \right\rangle , \qquad (4.16)$$

with

$$|\lambda_{S+1}\rangle \equiv D'(R_S, R_{S+1}, s_2)\, G^{(1)}_{(V)}(a^{S+1\dagger})|0_{S+1}\rangle, \tag{4.17}$$

$$D'(R_S, R_{S+1}, s_2) = \int_0^1 dt\, t^{-\alpha(s_2)-1}(1-t)^{\alpha_0-1}\left(\frac{1-t}{1-tw_{S-1}}\right)^{R_S-\alpha(k_S)} \times t^{R_{S+1}}\,. \tag{4.18}$$

We thus obtain the second factorized tree amplitude (Fig. 5)

$$\begin{aligned} &\langle 0|G^{(2)}_{(W)}(a^S, a^{S+1})\left|\begin{matrix}\lambda_S\\ \lambda_{S+1}\end{matrix}\right\rangle \\ &= \int \prod_{(i\neq a,b,c)} dw_i \{W_{S+1}\}\langle 0|\exp\left\{\sum_{\substack{i=1\\(i\neq S)}}^{S+1}(a^S|P'_S(i)|k_i)\right. \\ &\left. + \sum_{\substack{i=1\\(i\neq S+1)}}^{S+1}(a^{S+1}|P'_{S+1}(i)|k_i) + (a^S|M_-P'(S-1,1,S,S+1)M_-{}^T|a^{S+1})\right\}\left|\begin{matrix}\lambda_S\\ \lambda_{S+1}\end{matrix}\right\rangle . \end{aligned} \tag{4.19}$$

Note that in obtaining Eq. (4.19), we have put,[14] in Eq. (4.13),

$$\sum_{j=S+2}^{M} t\, P''_{S+1}(j)|k_j) \rightarrow |a^{S+1}). \tag{4.20}$$

As demonstrated in Eq. (3.13), the coupling of a^S with a^{S+1} term in the exponent of Eq. (4.19), can be symbolically simplified to

$$(a^S|P'_S\, \hat{P}'^{-1}_{S+1}(\;)|a^{S+1}) \quad \text{or} \quad (a^S|(\;)\hat{P}'^{-1}_S\, P'_{S+1}|a^{S+1}). \tag{4.21}$$

Equations (4.18) and (4.19) are the desired proof we mentioned earlier.

We now proceed to write down the fourth-factorized trees in the most general configuration, Fig. 6. In Ref. 8 (see Appendix B) we have derived a set of rules for writing down directly the fourth-factroized tree amplitude corresponding to Fig. 6:

$$G^{(4)}_{\{Y\}}(a^{\alpha},a^{\beta},a^{\gamma},a^{\delta})$$

$$= \int \prod_{(i\neq a,b,c)} dy_i \{Y_{S+2}\} \exp\left\{ \sum_{\sigma=(\alpha,\beta,\gamma,\delta)} \sum_{\substack{i=0\\(i\neq\sigma)}}^{S+1} (a^{\sigma}|P_{\sigma}(i)|k_i) \right.$$

$$\left. + \frac{1}{2} \sum_{\substack{\sigma,\lambda=(\alpha,\beta,\gamma,\delta)\\(\sigma\neq\lambda)}} (a^{\sigma}|P_{\sigma}(\lambda)M_{-}P(\sigma+1,\lambda+1,\sigma,\lambda)M_{-}^{T}P_{\lambda}(\sigma)|a^{\lambda}) \right\}, \quad (4.22)$$

where

$$P_{\sigma}(j) = P(\sigma,\sigma+1,\sigma-1,j), \quad \sigma = \alpha,\beta,\gamma,\delta, \quad j = 0,1,\cdots,S+1, \quad (4.23)$$

and we have omitted the coherent states $|\lambda_{\sigma}\rangle$'s, for simplicity.

One interesting interpretation emerges from the form of Eq. (4.22). Equation (4.22) contains the couplings of momentum k_i to momentum k_j, the couplings of momentum k_i to the "internal coordinate" a^{σ}, and the couplings of internal coordinate a^{σ} to the internal coordinate a^{λ}.

As remarked in Sec. III, we can keep momentum conservation inside the symbol $|)$ in Eq. (4.22), by using the identity $P_{\sigma}(\sigma + 1) = 0$ and Eq. (3.13). Hence the most general form of the fourth-factorized tree amplitude (Fig. 6) is

$$G^{(4)}_{(Y)}(a^{\sigma}\text{'s}) = \int \prod_{(i\neq a,b,c)} dy_i \{Y_{S+2}\} \exp\left\{ \sum_{\sigma=\{\alpha,\beta,\gamma,\delta\}} \sum_{\substack{i=0\\(i\neq\sigma)}}^{S+1} \left(a^{\sigma} | P_{\sigma}\begin{bmatrix} i \\ \cdot \\ \sigma+1\end{bmatrix} \left| \begin{matrix} k_i \\ k_\sigma \end{matrix} \right.\right) + \frac{1}{2} \sum_{\substack{\sigma,\lambda=\{\alpha,\beta,\gamma,\delta\}\\(\sigma\neq\lambda)}} (a^{\sigma}|P_{\sigma}\,\hat{P}_{\lambda}^{-1}(\ )|a^{\lambda}) \right\}. \tag{4.24}$$

Notice the form of the expression in Eq. (4.24) is insensitive to the dot positions. (Of course, the cross ratios themselves do depend on the dot positions.)

The related fourth-factorized tree amplitude with dot on the opposite side, say, of the a^{α} leg, is obtained by simply interchanging $y_{\alpha+1}$ with $y_{\alpha-1}$ everywhere in Eq. (4,24).

The generalization of Eq. (4.24) to the Nth-factorized tree amplitude is trivial. One simply extends the summation over σ,λ in Eq. (4.24) from $\{\alpha,\beta,\gamma,\delta\}$ to $\{\alpha,\beta,\cdots,\delta\}$ (N in number)

$$G^{(N)}_{(Y)}(a^{\sigma}\text{'s}) = \int \prod_{(i\neq a,b,c)} dy_i \{Y_{S+2}\} \exp\left\{ \sum_{\sigma=\{\alpha,\beta,\cdots,\delta\}} \sum_{\substack{i=0\\(i\neq\sigma)}}^{S+1} \left(a^{\sigma} | P_{\sigma}\begin{bmatrix} i \\ \sigma+1\end{bmatrix} \left| \begin{matrix} k_i \\ k_\sigma \end{matrix} \right.\right) + \frac{1}{2} \sum_{\substack{\sigma,\lambda=\{\alpha,\beta,\cdots,\delta\}\\(\sigma\neq\lambda)}} (a^{\sigma}|P_{\sigma}\hat{P}_{\lambda}^{-1}(\ )|a^{\lambda}) \right\}. \tag{4.25}$$

[Again, the expression in Eq. (4.25) is independent of dot positions.)

The pure reggeon tree amplitudes can be obtained in a very simple way, too. The symmetrical four-reggeon tree amplitude is obtained from Eq. (4.24) by simply setting all Koba-Nielsen variables associated with scalar external legs vanish. Namely, one simply sets $\alpha = 1$, $\beta = 2$, $\gamma = 3$, $\delta = 4$ $\sum_{i=0}^{S+1} \rightarrow \sum_{i=1}^{4}$ in Eq. (4.24), hence the pure four-reggeon tree amplitude (symmetrical) is

$$G^{(4)}_{(Y)}(a^i\text{'s}) = \int \prod_{\substack{i=1 \\ (i\neq a,b,c)}}^{4} dy_i \{Y_4\} \exp\left\{ \sum_{j=1}^{4} \sum_{\substack{i=1 \\ (i\neq j)}}^{4} \left(a^j \middle| P_j \begin{bmatrix} i \\ j+1 \end{bmatrix} \middle| \begin{matrix} k_i \\ k_j \end{matrix} \right) + \frac{1}{2} \sum_{\substack{i=1 \\ (i\neq j)}}^{4} \sum_{j=1}^{4} (a^i | P_i \, \hat{P}_j^{-1}(\ ) | a^j) \right\} . \tag{4.26}$$

Similarly, the symmetrical pure N-reggeon tree amplitude is obtained from Eq. (4.25) by letting $\alpha = 1$, $\beta = 2$, $\cdots$, $\delta = N$; $\sum_{i=0}^{S+1} \rightarrow \sum_{i=1}^{N}$. Hence the (symmetrical) N-reggeon tree formula (Fig. 8) is

$$G^{(N)}_{(Y)}(a^i\text{'s}) = \int \prod_{\substack{i=1 \\ (i\neq a,b,c)}}^{N} dy_i \{Y_N\} \exp\left\{ \sum_{j=1}^{N} \sum_{\substack{i=1 \\ (i\neq j)}}^{N} \left(a^j \middle| P_j \begin{bmatrix} i \\ j+1 \end{bmatrix} \middle| \begin{matrix} k_i \\ k_j \end{matrix} \right) + \frac{1}{2} \sum_{j=1}^{N} \sum_{\substack{i=1 \\ (i\neq j)}}^{N} (a^j | P_j \, \hat{P}_i^{-1}(\ ) | a^i) \right\} . \tag{4.27}$$

Following the interpretation after Eq. (4.23), one can regard a reggeon (or a hadron) as a composite object, which is described not only by a four-momentum k_i but also by an infinite number of internal degrees of freedom $a_n^{\,i}$. One may further regard k_i as the "zero mode" of the harmonic oscillator modes [actually $k_i \sim (a_0^{\,i} - a_0^{\,i\dagger})$], hence in the exponent of Eq. (4.27), one can combine the first term with the second term into a single term. Therefore, the form of the pure N-reggeon tree amplitude, Eq. (4.27), strongly suggests that one should regard the space-time degree of freedom as the zero mode of the infinite number internal degrees of freedom, and should treat all of them *on equal footing* as a generalized coordinate or generalized momentum to describe a reggeon (or a hadron). This indicates a theory of third quantization.[10]

V. GENERAL MULTILOOP AMPLITUDES[16]

In this section we give detailed calculations (by using cross-ratio algebra) of the nonplanar multiloop amplitude,[9] and thence present very simple derivations of planar,[9] overlapping and nonorientable[9] multiloop amplitudes. We then summarize the multiloop calculations into a set of rules, which again enable us to write down directly any arbitrary multiloop formula by simply inspecting the corresponding multiloop Feynman diagram.

A. Nonplanar Multiloop Amplitudes

According to Eq. (4.25) (or Appendix B), we can directly write down the 2Nth factorized tree amplitude corresponding to Fig. 9 [compare with Eq. (4.25)]:

$$G^{(2N)}_{(Y)}(a^{\alpha},a^{\beta};\ a^{\gamma},a^{\delta};\ \cdots;\ a^{\sigma},a^{\lambda}) = \int \prod_{i} dy_i \{Y_{S+2}\}$$

$$\times \ \exp\left\{ \sum_{\substack{\alpha\in(\mathcal{L}^*)\\ \beta\in(\mathcal{L})}} \sum_{\substack{i=0\\ (i\neq\alpha,\beta)}}^{S+1} \left(a^{\alpha} \middle| P_{\alpha} \begin{bmatrix} i \\ \alpha+1 \\ \beta \end{bmatrix} \middle| \begin{matrix} k_i \\ k_{\alpha} \\ k_{\beta} \end{matrix} \right) \right.$$

$$+ \sum_{\substack{\beta\in(\mathcal{L})\\ \alpha\in(\mathcal{L}^*)}} \sum_{\substack{i=0\\ (i\neq\alpha,\beta)}}^{S+1} \left(a^{\beta} \middle| P_{\beta} \begin{bmatrix} i \\ \alpha \\ \beta-1 \end{bmatrix} \middle| \begin{matrix} k_i \\ k_{\alpha} \\ k_{\beta} \end{matrix} \right)$$

$$+ \frac{1}{2} \sum_{\substack{\alpha,\gamma\in(\mathcal{L}^*)\\ (\alpha\neq\gamma)}} (a^{\alpha}|P_{\alpha}\, \hat{P}_{\gamma}^{-1}(\ )|a^{\gamma}) + \frac{1}{2} \sum_{\substack{\beta,\delta\in(\mathcal{L})\\ (\beta\neq\delta)}} (a^{\beta}|P_{\beta}\, \hat{P}_{\delta}^{-1}(\ )|a^{\delta})$$

$$\left. + \sum_{\substack{\alpha\in(\mathcal{L}^*)\\ \delta\in(\mathcal{L})}} (a^{\alpha}|P_{\alpha}\, \hat{P}_{\delta}^{-1}(\ )|a^{\delta}) \right\} , \qquad (5.1)$$

where

$$\begin{aligned} P_\alpha(i) &= P(\alpha,\alpha+1,\alpha-1,i), \qquad \alpha\epsilon(\mathcal{L}^*), \\ P_\beta(i) &= P(\beta,\beta-1,\beta+1,i), \qquad \beta\epsilon(\mathcal{L}) . \end{aligned} \tag{5.2}$$

The notations $(\mathcal{L}^*)$, $(\mathcal{L})$ mean the set $\{\alpha,\gamma,\cdots,\sigma\}$, the set $\{\beta,\delta,\cdots,\lambda\}$ respectively. Each loop is labeled by two indices, e.g., the $(\alpha\beta)$ loop is obtained by sewing the excited α leg with the β leg. We have grouped, in Eq. (5.1), the indices $(\alpha\beta)$, $(\gamma\delta)$, $\cdots$, $(\sigma\lambda)$ as referring to different loops, with the total number of loops equal to N. The variable $t_{\alpha\beta}$ will be defined as the propagator variable corresponding to sewing the excited α leg with the β leg.

We now apply the sewing prescriptions, discussed in Ref. 9, simultaneously on the N pairs of excited legs a^α, a^β, $(\alpha\beta)\,\epsilon$ (all N loops). The prescription is as follows:

(a) replace

$$\begin{aligned} (a^\alpha| &\rightarrow (\lambda^*_\alpha - k_\alpha|\, M_-^T\left[\frac{t_{\alpha\beta}}{t_{\alpha\beta}-1}\right] , \\ (a^\beta| &\rightarrow (\lambda_\beta| , \end{aligned} \tag{5.3}$$

(b) set

$$|\lambda_\alpha) = |\lambda_\beta) = |\lambda_{\alpha\beta}), \quad k_\alpha = -k_\beta, \quad \alpha\epsilon(\mathcal{L}^*), \quad \beta\epsilon(\mathcal{L});$$

and perform the following integrations

$$\int \prod_{\alpha\epsilon(\mathcal{L}^*)} d^4k_\alpha \int_0^1 \left[\prod_{\alpha\epsilon(\mathcal{L}^*)} dt_{\alpha\beta}\, t_{\alpha\beta}^{-\ell(k_\alpha)-1} \left(1 - t_{\alpha\beta}\right)^{\alpha_0 - 1 + \frac{1}{2}k_\alpha^2} \right]$$

$$\times \int \prod_{\alpha\epsilon(\mathcal{L}^*)} d\left|\frac{\lambda_{\alpha\beta}}{\sqrt{2}}\right) d\left|\frac{\lambda^*_{\alpha\beta}}{\sqrt{2}}\right) \exp\left\{ - \sum_{\alpha\epsilon(\mathcal{L}^*)} \left(\lambda^*_{\alpha\beta}\middle|\lambda_{\alpha\beta}\right) \right\} \qquad (5.4)$$

We then get the nonplanar N-loop amplitude, denote it by FNL(N):

FNL(N)

$$= \int \prod_{\alpha\epsilon(\mathcal{L}^*)} d^4k_\alpha \int_0^1 \left[\prod_{\alpha\epsilon(\mathcal{L}^*)} dt_{\alpha\beta}\, t_{\alpha\beta}^{-\ell(k_\alpha)-1} \left(1 - t_{\alpha\beta}\right)^{\alpha_0 - 1 + \frac{1}{2}k_\alpha^2} \right]$$

$$\times \int \prod_i dy_i \{Y_{S+2}\} I, \qquad (5.5)$$

where

$$I = \int d\left|\frac{\underset{\sim}{\lambda}}{\sqrt{2}}\right) d\left|\frac{\underset{\sim}{\lambda}}{\sqrt{2}}\right) \exp\{-(\underset{\sim}{\lambda}^*|\underset{\sim}{\lambda}) + (\underset{\sim}{\lambda}^*|[C]|\underset{\sim}{\lambda})$$

$$+ \frac{1}{2}(\underset{\sim}{\lambda}^*|[D]|\underset{\sim}{\lambda}^*) + \frac{1}{2}(\underset{\sim}{\lambda}|[A]|\underset{\sim}{\lambda}) + (\underset{\sim}{\lambda}^*|\underset{\sim}{F}) + (\underset{\sim}{\lambda}|\underset{\sim}{E})\}. \qquad (5.6)$$

In Eq. (5.6), we have introduced the N-dimensional vectors in loop number space:

$$|\underset{\sim}{\lambda}) \equiv \left(|\lambda_{\alpha\beta}),\ |\lambda_{\gamma\delta}),\cdots,|\lambda_{\sigma\lambda})\right)$$

$$|\underset{\sim}{E}) \equiv \left(|E_{\alpha\beta}),\ |E_{\gamma\delta}),\cdots,|E_{\sigma\lambda})\right) \quad \text{etc.} \qquad (5.7)$$

The definitions of [A], [C], [D], $|\underset{\sim}{E})$, $|\underset{\sim}{F})$ in Eq. (5.6) are

$$Y_{\alpha\beta} \equiv M_{-}T \frac{t_{\alpha\beta}}{t_{\alpha\beta} - 1}, \tag{5.8a}$$

$$|E_{\alpha\beta}) \equiv \sum_{\substack{i=0 \\ i\neq(\mathcal{L},\mathcal{L}^*)}}^{S+1} \sum_{(\gamma\delta)\in(\mathcal{L}*\mathcal{L})} P_\beta \left[\begin{matrix} i \\ \begin{pmatrix}\alpha \\ \beta+1\end{pmatrix} \\ \begin{pmatrix}\gamma \\ \delta\end{pmatrix}\end{matrix}\right]_{(\gamma\delta)\neq(\alpha\beta)} \left| \begin{matrix} k_i \\ k_\alpha \\ k_\beta \\ k_\gamma \\ k_\delta \end{matrix} \right), \tag{5.8b}$$

$$|F_{\alpha\beta}) \equiv \sum_{\substack{i=0 \\ [i\neq(\mathcal{L},\mathcal{L}^*)]}} \sum_{(\gamma\delta)\in(\mathcal{L}*\mathcal{L})} Y_{\alpha\beta} P_\alpha \left[\begin{matrix} i \\ \begin{pmatrix}\alpha+1 \\ \beta\end{pmatrix} \\ \begin{pmatrix}\gamma \\ \delta\end{pmatrix}\end{matrix}\right]_{(\gamma\delta)\neq(\alpha\beta)} \left| \begin{matrix} k_i \\ k_\alpha \\ k_\beta \\ k_\gamma \\ k_\delta \end{matrix} \right), \tag{5.8c}$$

$$[C]_{\alpha\beta,\gamma\delta} = Y_{\alpha\beta} P_\alpha \hat{P}_\delta^{-1}(\ ), \tag{5.8d}$$

$$[D]_{\alpha\beta,\gamma\delta} = Y_{\alpha\beta} P_\alpha \hat{P}_\gamma^{-1} Y_{\gamma\delta}^T(\ ), \qquad [D]_{\alpha\beta,\alpha\beta} \equiv 0, \tag{5.8e}$$

$$[A]_{\alpha\beta,\gamma\delta} = P_\beta \hat{P}_\delta^{-1}(\ ), \qquad [A]_{\alpha\beta,\alpha\beta} \equiv 0. \tag{5.8f}$$

We now can explicitly do the integral of Eq. (5.6) by the principal axes method.[9] We write Eq. (5.6) as

$$I = \int d\left|\frac{\underset{\sim}{\lambda}}{\sqrt{2}}\right) \; d\left|\frac{\underset{\sim}{\lambda}^*}{\sqrt{2}}\right) \exp\left\{ -\frac{1}{2}\left(\left(\underset{\sim}{\lambda}\right|, \left(\underset{\sim}{\lambda}^*\right|\right) \begin{pmatrix} -[A] & [I]-[C]^T \\ [I]-[C] & -[D] \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} |\underset{\sim}{\lambda}) \\ |\underset{\sim}{\lambda}^*) \end{pmatrix} + \begin{pmatrix} (\underset{\sim}{E}) & 0 \\ 0 & (\underset{\sim}{F}| \end{pmatrix} \begin{pmatrix} |\underset{\sim}{\lambda}) \\ |\underset{\sim}{\lambda}^*) \end{pmatrix} \right\} \tag{5.9}$$

$$= \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \exp\left\{ \frac{1}{2} \sum_{n=0}^{\infty} \left(\left(\underset{\sim}{E}\right|, \left(\underset{\sim}{F}\right|\right) [GH]^n \begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix} \right\}, \tag{5.10}$$

where

$$[\Delta] = \begin{pmatrix} -[A] & [I]-[C]^T \\ [I]-[C] & -[D] \end{pmatrix} \equiv [G]-[H] , \tag{5.11}$$

$$[GH] = \begin{pmatrix} [C] & [D] \\ [A] & [C]^T \end{pmatrix} . \tag{5.12}$$

We now calculate Eq. (5.10) order by order in the [GH] matrix. We define[9] the projective operator (note $Y_{\alpha\beta} \equiv Q^{-1}_{\beta\alpha}$)

$$Q_{\beta\alpha}(x) = \frac{1}{\left(\frac{t_{\alpha\beta}}{t_{\alpha\beta}-1}\right)\left(1-\frac{1}{x}\right)}, \qquad Q^{-1}_{\beta\alpha}(x) = \frac{1}{1-\frac{(t_{\alpha\beta}-1)}{t_{\alpha\beta}x}}, \tag{5.13}$$

and the projective operator corresponding to encircling the $(\alpha\beta)$ loop:

$$R^{-1}_{\beta\alpha} = \hat{P}_\beta^{-1} Q^{-1}_{\beta\alpha} P_\alpha, \qquad R_{\beta\alpha} = P_\alpha^{-1} Q_{\alpha\beta} \hat{P}_\beta . \tag{5.14}$$

From Eq. (5.14), we have, for all $(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})$,

$$R^{-1}_{\beta\alpha}(y_{\alpha}) = y_{\beta+1}, \qquad R^{-1}_{\beta\alpha}(y_{\alpha+1}) = y_{\beta}. \tag{5.15}$$

With Eqs. (5.13) and (5.14), we can easily calculate, from Eq. (5.8), the following quantities and obtain [summations over $(\gamma\delta)$, $(\sigma\lambda)$ are understood]

$$|F_{\alpha\beta}) = \sum_{\substack{i=0\\ i\neq(\mathcal{L},\mathcal{L}*)}}^{S+1} \hat{P}_{\beta}\, R^{-1}_{\beta\alpha} \left[\begin{matrix} i \\ \binom{\alpha+1}{\beta} \\ \binom{\gamma}{\delta} \end{matrix}\right]_{(\gamma\delta)\neq(\alpha\beta)} \left|\begin{matrix} k_i \\ k_\alpha \\ k_\beta \\ k_\gamma \\ k_\delta \end{matrix}\right) , \tag{5.16a}$$

$$|E_{\alpha\beta}) = \sum_{\substack{i=0\\ i\neq(\mathcal{L},\mathcal{L}*)}}^{S+1} P_{\beta} \left[\begin{matrix} i \\ \binom{\alpha}{\beta+1} \\ \binom{\gamma}{\delta} \end{matrix}\right]_{(\gamma\delta)\neq(\alpha\beta)} \left|\begin{matrix} k_i \\ k_\alpha \\ k_\beta \\ k_\gamma \\ k_\delta \end{matrix}\right) , \tag{5.16b}$$

$$[C]|F)_{\alpha\beta} = \sum_{\substack{i=0\\ i\neq(\mathcal{L},\mathcal{L}*)}}^{S+1} \hat{P}_{\beta}\, R^{-1}_{\beta\alpha}\, R^{-1}_{\delta\gamma} \left[\begin{matrix} i \\ \binom{\gamma+1}{\delta} \\ \binom{\sigma}{\lambda} \end{matrix}\right]_{(\sigma\lambda)\neq(\gamma\delta)} \left|\begin{matrix} k_i \\ k_\gamma \\ k_\delta \\ k_\sigma \\ k_\lambda \end{matrix}\right) , \tag{5.16c}$$

$$[D]\,|\underset{\sim}{E})_{\alpha\beta} = \sum_{\substack{i=0\\ i\neq(\mathcal{L},\mathcal{L}^*)}}^{S+1} \hat{P}_\beta\, R^{-1}_{\beta\alpha}\, R_{\delta\gamma} \left[\begin{matrix} i \\ \binom{\gamma}{\delta+1} \\ \binom{\sigma}{\lambda}\end{matrix}\right]_{\substack{(\sigma\lambda)\neq(\gamma\delta)\\(\alpha\beta)\neq(\gamma\delta)}} \left| \begin{matrix} k_i\\ k_\gamma\\ k_\delta\\ k_\sigma\\ k_\lambda\end{matrix}\right) , \tag{5.16d}$$

$$[A]\,|\underset{\sim}{F})_{\alpha\beta} = \sum_{\substack{i=0\\ i\neq(\mathcal{L},\mathcal{L}^*)}}^{S+1} P_\beta\, R^{-1}_{\delta\gamma} \left[\begin{matrix} i \\ \binom{\gamma+1}{\delta} \\ \binom{\sigma}{\lambda}\end{matrix}\right]_{\substack{(\sigma\lambda)\neq(\gamma\delta)\\(\alpha\beta)\neq(\gamma\delta)}} \left| \begin{matrix} k_i\\ k_\gamma\\ k_\delta\\ k_\sigma\\ k_\lambda\end{matrix}\right) , \tag{5.16e}$$

$$[C]^T\,|\underset{\sim}{E})_{\alpha\beta} = \sum_{\substack{i=0\\ i\neq(\mathcal{L},\mathcal{L}^*)}}^{S+1} P_\beta\, R_{\delta\gamma} \left[\begin{matrix} i \\ \binom{\gamma}{\delta+1} \\ \binom{\sigma}{\lambda}\end{matrix}\right]_{(\sigma\lambda)\neq(\gamma\delta)} \left| \begin{matrix} k_i\\ k_\gamma\\ k_\delta\\ k_\sigma\\ k_\lambda\end{matrix}\right) . \tag{5.16f}$$

From Eqs. (5.12) and (5.16) we obtain[9] the [GH] operator, acting on the vector $\begin{pmatrix} |F) \\ |E) \end{pmatrix}$:

$$[GH]_{\alpha\beta,\gamma\delta}$$

$$= \begin{pmatrix} [C] & [D] \\ [A] & [C]^T \end{pmatrix}_{\alpha\beta,\gamma\delta} = \begin{pmatrix} \hat{P}_\beta R^{-1}_{\beta\alpha} \hat{P}_\delta{}^{-1}(\ )_{\gamma\delta} & \underset{(\alpha\beta)\neq(\gamma\delta)}{\hat{P}_\beta R^{-1}_{\beta\alpha} R_{\delta\gamma} P_\delta{}^{-1}(\ )_{\gamma\delta}} \\ \underset{(\alpha\beta)\neq(\gamma\delta)}{P_\beta \hat{P}_\delta{}^{-1}(\ )_{\gamma\delta}} & P_\beta R_{\delta\gamma} P_\delta{}^{-1}(\ )_{\gamma\delta} \end{pmatrix} . \tag{5.17}$$

This GH operator, Eq. (3.17), can also be proved to be applicable to all higher order terms. It follows that

$$[GH]^n_{\alpha\beta,\gamma\delta} = \begin{pmatrix} \hat{P}_\beta R^{-1}_{\beta\alpha} [R^\pm]^{(n-1)} \hat{P}_\delta{}^{-1}(\ )_{\gamma\delta} & \hat{P}_\beta R^{-1}_{\beta\alpha} [R^\pm]^{(n-1)} R_{\delta\gamma} P_\delta{}^{-1}(\ )_{\gamma\delta} \\ P_\beta [R^\pm]^{(n-1)} \hat{P}_\delta{}^{-1}(\ )_{\gamma\delta} & P_\beta [R^\pm]^{(n-1)} R_{\delta\gamma} P_\delta (\ )_{\gamma\delta} \end{pmatrix} \tag{5.18}$$

where $[R^\pm]^{n-1}_{\mathcal{L},\mathcal{L}''} \equiv R^\pm_{\mathcal{L}} \; R^\pm_{\mathcal{L}'} \cdots R^\pm_{\mathcal{L}''}$ with total number of $R^\pm$ equal to $(n-1)$ (summing over all intermediate $\mathcal{L}$'s is understood); and also, in the produce $R_{\beta\alpha} R^{-1}_{\delta\gamma}$ or $R^{-1}_{\beta\alpha} R_{\delta\gamma}$, it is implied that $(\alpha\beta) \neq (\gamma\delta)$.

The zeroth-order term in Eq. (5.10) can be obtained from Eqs. (5.16a) and (5.16b), with the result

$$I_0 \equiv \exp\{(\underset{\sim}{E}|\underset{\sim}{F})\} = \prod_{(\alpha\beta)\in(\mathcal{L}^*\mathcal{L})} \; \prod_{\substack{i,j=0 \\ [i,j\neq(\mathcal{L},\mathcal{L}^*)]}}^{S+1} \{y_i - R^\pm_{\beta\alpha}(y_i)\}^{-k_i \cdot k_j}$$

$$\times \prod_{\substack{i=0 \\ [i\neq(\mathcal{L},\mathcal{L}^*)]}}^{S+1} \left\{ \left[\frac{y_i - R^\pm_{\beta\alpha}(y_\gamma)}{y_i - R^\pm_{\beta\alpha}(y_\delta)}\right]^{-k_i \cdot k_\gamma}_{(\gamma\delta)\neq(\alpha\beta)} \left[\frac{y_i - y_\beta}{y_i - y_\alpha} \cdot \frac{y_i - R_{\beta\alpha}(y_\alpha)}{y_i - R^{-1}_{\beta\alpha}(y_\beta)}\right]^{-k_i \cdot k_\alpha} \right\}$$

Equation (5.19) continued

Equation (5.19) continued

$$\prod_{\substack{(\alpha\beta),(\gamma\delta),\\(\sigma\lambda)\in(\mathcal{L}^*\mathcal{L})}} \left\{ \begin{bmatrix} \begin{pmatrix} y_\alpha \\ y_{\beta+1} \end{pmatrix} \\ \begin{pmatrix} y_\gamma \\ y_\delta \end{pmatrix} \end{bmatrix}_{(\gamma\delta)\neq(\alpha\beta)} - R^{-1}_{\beta\alpha} \begin{bmatrix} \begin{pmatrix} \alpha+1 \\ \beta \end{pmatrix} \\ \begin{pmatrix} \sigma \\ \lambda \end{pmatrix} \end{bmatrix}_{(\sigma\lambda)\neq(\alpha\beta)} \right\}^{-k_{[\alpha,\beta,\gamma,\delta]}\cdot k_{[\alpha,\beta,\sigma,\lambda]}} \qquad (5.19)$$

We then calculate the nth-order term in Eq. (5.10), from Eqs. (5.16a), (5.16b), and (5.18), we get

$$I_n = \exp\left\{ \frac{1}{2} \Big((\underset{\sim}{E}|, (\underset{\sim}{F}| \Big) [GH]^n \begin{pmatrix} |\underset{\sim}{F}) \\ |\underset{\sim}{E}) \end{pmatrix} \right.$$

$$= \prod_{(\alpha\beta),\cdots,(\gamma\delta)\in(\mathcal{L}^*\mathcal{L})} \prod_{\substack{i,j=0\\ [i,j\neq(\mathcal{L},\mathcal{L}^*)]}}^{S+1} \left\{ y_i - [R^{\pm}]^{(n-1)}_{\beta\alpha,\delta\gamma}(y_j) \right\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\prod_{\substack{(\alpha\beta),\cdots,(\sigma\lambda)\\(\gamma\delta)\in(\mathcal{L}^*\mathcal{L})}} \prod_{\substack{i=0\\ [i\neq(\mathcal{L},\mathcal{L}^*)]}}^{S+1} \left\{ \left[\frac{y_i - [R^{\pm}]^{(n+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)}{y_i - [R^{\pm}]^{(n+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)} \right]^{-k_i\cdot k_\gamma}_{(\sigma\lambda)\neq(\gamma\delta)} \right.$$

Equation (5.20) continued

Equation (5.20) continued

$$\left[\frac{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(y_\delta)}{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\, R^{-1}_{\delta\gamma}(y_\delta)}\right]^{-k_i \cdot k_\gamma}_{[(\gamma\delta)\neq(\sigma\lambda),+]}$$

$$\times \left[\frac{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\, R_{\delta\gamma}(y_\gamma)}{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(y_\gamma)}\right]^{-k_i \cdot k_\gamma}_{[(\gamma\delta)\neq(\sigma\lambda),-]}$$

$$\times \prod_{\substack{(\alpha\beta),\cdots,(\sigma\lambda)\\ (\gamma'\delta'),(\gamma\delta)\in(\mathcal{L}^*,\mathcal{L})}} \left\{ \left[\begin{matrix}\dbinom{y_\alpha}{y_{\beta+1}} \\ \dbinom{y_{\gamma'}}{y_{\delta'}}\end{matrix}\right]_{(\gamma'\delta\sigma)\neq(\alpha\beta)} - R^{-1}_{\beta\alpha}[R^{\pm}]^{(n-1)}R^{-1}_{\delta\gamma}\left[\begin{matrix}\dbinom{\gamma+1}{\delta} \\ \dbinom{\sigma}{\lambda}\end{matrix}\right]_{(\sigma\lambda)\neq(\gamma\delta)} \right\}^{-\frac{1}{2}k_{[\alpha,\beta,\gamma',\delta']}\cdot k_{[\gamma,\delta,\sigma,\lambda]}}$$

$$\times \left\{ \left[\begin{matrix}\dbinom{y_\alpha}{y_{\beta+1}} \\ \dbinom{y_{\gamma'}}{y_{\delta'}}\end{matrix}\right]_{(\gamma'\delta')\neq(\alpha\beta)} \right.$$

Equation (5.20) continued

Equation (5.20) continued

$$\left. - R^{-1}_{\beta\alpha}[R^{\pm}]^{(n-1)}R_{\delta\gamma}\begin{bmatrix}\binom{\gamma}{\delta+1}\\ \binom{\sigma}{\lambda}\end{bmatrix}_{(\sigma\lambda)\neq(\gamma\delta)}\right\}^{-\frac{1}{2}k[\alpha,\beta,\gamma',\delta']\cdot k[\gamma,\delta,\sigma,\lambda]}$$

$$\times\left\{\begin{bmatrix}\binom{y_{\alpha}}{y_{\beta+1}}\\ \binom{y_{\gamma'}}{y_{\delta'}}\end{bmatrix}_{(\gamma'\delta')\neq(\alpha\beta)}\right.$$

$$\left. - R_{\beta\alpha}[R^{\pm}]^{(n-1)}R^{-1}_{\delta\gamma}\begin{bmatrix}\binom{\gamma+1}{\delta}\\ \binom{\sigma}{\lambda}\end{bmatrix}_{(\sigma\lambda)\neq(\gamma\delta)}\right\}^{-\frac{1}{2}k[\alpha,\beta,\gamma',\delta']\cdot k[L,\delta,\sigma,\lambda]}$$

$$\times\left\{\begin{bmatrix}\binom{y_{\alpha+1}}{y_{\beta}}\\ \binom{y_{\gamma'}}{y_{\delta'}}\end{bmatrix}_{(\gamma'\delta')\neq(\alpha\beta)}\right.$$

$$\left. - R_{\beta\alpha}[R^{\pm}]^{(n-1)}R_{\delta\gamma}\begin{bmatrix}\binom{\gamma}{\delta+1}\\ \binom{\sigma}{\lambda}\end{bmatrix}_{(\sigma\lambda)\neq(\gamma\delta)}\right\}^{-\frac{1}{2}k[\alpha,\beta,\gamma',\delta']\cdot k[\gamma,\delta,\sigma,\lambda]} \quad . \tag{5.20}$$

We note that, in Eq. (5.20), the operator $R^{\pm}$ occurs in different orders; hence when we take infinite product over n, there are infinite many number of cancellations[9] on the factors involving the variables associated with the excited legs, i.e., y_{α}, y_{β}, $(\alpha\beta) \in (\mathcal{L}*\mathcal{L})$. This infinite number of cancellations is necessary, because the final answer should not depend on y_{α}, y_{β}, $(\alpha\beta) \in (\mathcal{L}*\mathcal{L})$; and it is this infinite number of cancellations that lead to the invariant points of the loops.

The invariant points and the "multiplier" of the projective operator $R_{\beta\alpha}$ corresponding to $(\alpha\beta)$ loops are defined as follows:

$$R_{\beta\alpha}(z) = \frac{z[x^{(2)}_{\alpha\beta} - x^{(1)}_{\alpha\beta}X_{\alpha\beta}] - x^{(1)}_{\alpha\beta}x^{(2)}_{\alpha\beta}(1 - X_{\alpha\beta})}{z(1 - X_{\alpha\beta}) + [x^{(2)}_{\alpha\beta}X_{\alpha\beta} - x^{(1)}_{\alpha\beta}]}, \quad 0 \leq X_{\alpha\beta} < 1, \tag{5.21a}$$

$$R^{+\infty}_{\beta\alpha}(z_1) = x^{(2)}_{\alpha\beta}, \qquad z_1 \neq x^{(1)}_{\alpha\beta}, \tag{5.21b}$$

$$R^{-\infty}_{\beta\alpha}(z_2) = x^{(1)}_{\alpha\beta}, \qquad z_2 \neq x^{(2)}_{\alpha\beta}. \tag{5.21c}$$

In Appendix C, we show how the infinite number of cancellations actually lead to the invariant points of the N loops. We now combine Eqs. (B.8) and (B.9) with Eqs. (5.19) and (5.20) and hence obtain the expression for I:

$$I \equiv \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{n=0}^{\infty} I_n. \tag{5.22}$$

We then separate out all Koba-Nielsen variables y_α, y_β, $(\alpha\beta) \in (\mathcal{L}*\mathcal{L})$ in $\{Y_{S+2}\}$ of Eq. (5.5) and combine it with Eq. (5.22), we get the expression for $\{Y_{S+2}\}I$:

$$\{Y_{S+2}\}I = \frac{1}{(\det[\Delta])^{\frac{1}{2}}} \prod_{(\alpha\beta),\cdots,(\gamma\delta)\in(\mathcal{L}*\mathcal{L})} \prod_{\substack{i,j=0 \\ [i,j\neq(\mathcal{L}^*,\mathcal{L})]}}^{S+1} \prod_{n=0}^{\infty}$$

$$\times \{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j} \prod_{\substack{(\alpha\beta),\cdots,(\sigma\lambda), \\ (\gamma\delta)\in(\mathcal{L}*\mathcal{L})}} \prod_{\substack{i=0 \\ [i\neq(\mathcal{L}^*,\mathcal{L})]}}^{S+1} \prod_{n=0}^{\infty}$$

$$\times \left\{\frac{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}\right\}^{-k_i\cdot k_\gamma}_{(\sigma\lambda)\neq(\gamma\delta)} \prod_{\substack{i=0 \\ [i\neq(\mathcal{L},\mathcal{L}-1, \\ \mathcal{L}^*,\mathcal{L}^*-1)]}}^{S+1} (y_i - y_{i+1})^{\alpha_0 - 1}$$

$$\times \prod_{\substack{(\alpha\beta),(\alpha'\beta'),\cdots \\ (\sigma\lambda),(\gamma\delta)\in(\mathcal{L}*\mathcal{L})}} \prod_{n=0}^{\infty}$$

$$\times \left\{\frac{x^{(1)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}{x^{(2)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)} \cdot \frac{x^{(2)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}{x^{(1)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\gamma}_{\substack{(\alpha\beta)\neq(\alpha'\beta') \\ (\sigma\lambda)\neq(\gamma\delta)}}$$

$$\times \left\{ \prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})} (y_\alpha - y_\beta)^{k_\alpha^{\;2}} \right.$$

Equation (5.24) continued

Equation (5.24) continued

$$\times\left[\frac{R^{-1}_{\beta\alpha}(y_\alpha) - R^{-1}_{\beta\alpha}(y_\beta)}{y_\alpha - R_{\beta\alpha}(y_\beta)} \cdot \frac{R_{\beta\alpha}(y_\alpha) - R_{\beta\alpha}(y_\beta)}{y_\beta - R^{-1}_{\beta\alpha}(y_\alpha)}\right]^{-\frac{1}{2}k_\alpha^2}$$

$$\times \prod_{(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})} \frac{(y_{\alpha-1}-y_\alpha)(y_\alpha-y_{\alpha+1})(y_{\beta-1}-y_\beta)(y_\beta-y_{\beta+1})}{(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})}^{-\frac{1}{2}k_\alpha^2-1}$$

$$\times \prod_{(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})} [(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})]^{\alpha_0-1}$$

$$\left.(y_2-y_b)(y_b-y_c)(y_c-y_a)\right\} \quad . \qquad (5.24)$$

In order to express our final answer in a projective invariant form, we need to transform from the set of variables $\{t_{\alpha\beta}, y_\alpha, y_\beta, \alpha\epsilon(\mathcal{L}*), \beta\epsilon(\mathcal{L})\}$ to the new set of variables, $\{X_{\alpha\beta}, x^{(1)}_{\alpha\beta}, x^{(2)}_{\alpha\beta}, (\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})\}$. The relevant factors, in Eqs. (5.24) and (5.5), that are affected by the transformation, are

$$\prod_{\alpha\epsilon(\mathcal{L}*)}\left[dt_{\alpha\beta}\; t_{\alpha\beta}^{-\ell(k_\alpha)-1}\;(1-t_{\alpha\beta})^{\alpha_0-1+\frac{1}{2}k_\alpha^2}\right]$$

$$\times\left\{\prod_{(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})}(y_\alpha-y_\beta)^{-2}\left[\left(\frac{R^{-1}_{\beta\alpha}(y_\alpha)-R^{-1}_{\beta\alpha}(y_\beta)}{y_\alpha-R_{\beta\alpha}(y_\beta)}\cdot\frac{R_{\beta\alpha}(y_\alpha)-R_{\beta\alpha}(y_\beta)}{y_\beta-R^{-1}_{\beta\alpha}(y_\alpha)}\right)\right]^{-\frac{1}{2}k_\alpha^2}\right.$$

$$\times\prod_{(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})}\left[\frac{(y_{\alpha-1}-y_\alpha)(y_\alpha-y_{\alpha+1})(y_{\beta-1}-y_\beta)(y_\beta-y_{\beta+1})}{(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})}\right]^{-\frac{1}{2}k_\alpha^2-1}$$

$$\left.\times\prod_{(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})}[(y_{\alpha-1}-y_{\alpha+1})(y_{\beta-1}-y_{\beta+1})]^{\alpha_0-1}(y_a-y_b)(y_b-y_c)(y_c-y_a)\right\} \quad . \qquad (5.25)$$

In Appendix D, we show how the expression (5.25) is transformed into

$$\left\{\prod_{(\alpha\beta)\in(\mathcal{L}^{*}\mathcal{L})} dX_{\alpha\beta}\, X_{\alpha\beta}^{-\ell(k_\alpha)-1}\,(1-X_{\alpha\beta})^2\right\}$$

$$\times\left\{\prod_{(\alpha\beta)\in(\mathcal{L}^{*}\mathcal{L})} dx^{(1)}_{\alpha\beta}\; dx^{(2)}_{\alpha\beta}\; \frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{\left(x^{(1)}_{\alpha\beta}-x^{(2)}_{\alpha\beta}\right)^2}\right.$$

$$\left.\times\left\{\frac{[y_{\alpha-1}-R_{\beta\alpha}(y_{\beta+1})]\left(x^{(1)}_{\alpha\beta}-y_{\beta+1}\right)}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta+1})]}\right\}^{\alpha_0-1}\left\{\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})]\left(x^{(1)}_{\alpha\beta}-y_{\beta-1}\right)}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta-1})]}\right\}^{\alpha_0-1}\right\}. \qquad (5.26)$$

Combining Eqs. (5.26) and (5.24) with Eq. (5.5), we then get the nonplanar N-loop amplitude (Fig. 10a)

$$F_{NL}(N)$$

$$=\int\prod_{\alpha\in(\mathcal{L}^{*})} d^4k_\alpha \int_{U_1'}\prod_{(\alpha\beta)\in(\mathcal{L}^{*}\mathcal{L})} dX_{\alpha\beta}\, X_{\alpha\beta}^{-\ell(k_\alpha)-1}\,(1-X_{\alpha\beta})^2$$

$$\times\prod_{\{\bar{R}\}}[1-X_{\{\bar{R}\}}]^{-4}\int_{U_2'}\prod_{\substack{i=0\\ [i\neq(\mathcal{L}\mathcal{L}^{*},a,b,c)]}}^{S+1} dy_i\,[dy_a][dy_b][dy_c]$$

$$\times\prod_{(\alpha\beta)\in(\mathcal{L}^{*}\mathcal{L})}\left\{dx^{(1)}_{\alpha\beta}\; dx^{(2)}_{\alpha\beta}\; \frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{\prod_{(\alpha\beta)\in(\mathcal{L}^{*}\mathcal{L})}[x^{(1)}_{\alpha\beta}-x^{(2)}_{\alpha\beta}]^2}\right.$$

$$\left.\times\left\{\frac{[y_{\alpha-1}-R_{\beta\alpha}(y_{\beta+1})]\left(x^{(1)}_{\alpha\beta}-y_{\beta+1}\right)}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta+1})]}\cdot\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})]\left(x^{(1)}_{\alpha\beta}-y_{\beta-1}\right)}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta-1})]}\right\}^{\alpha_0-1}\right\}$$

Equation (5.27) continued

Equation (5.27) continued

$$\times \prod_{\substack{i=0 \\ [i\neq(\mathcal{L},\mathcal{L}^*,\mathcal{L}^*-1,\mathcal{L}-1)]}}^{S+1} (y_i - y_{i+1})^{\alpha_0-1}$$

$$\times \prod_{\substack{i,j=0 \\ [i,j\neq(\mathcal{L},\mathcal{L}^*)]}}^{S+1} \prod_{(\alpha\beta),\cdots,(\gamma\delta)\in(\mathcal{L}^*\mathcal{L})} \prod_{\substack{n=0 \\ [i\neq j,n=0]}}^{\infty}$$

$$\times \{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times \prod_{\substack{i=0 \\ [i\neq(\mathcal{L},\mathcal{L}^*)]}}^{S+1} \prod_{\substack{(\alpha\beta),\cdots,(\sigma\lambda), \\ (\gamma\delta)\in(\mathcal{L}^*\mathcal{L})}} \prod_{n=0}^{\infty} \left\{\frac{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda}\left(x^{(2)}_{\gamma\delta}\right)}{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}\right\}^{-k_i\cdot k_\gamma}_{(\sigma\lambda)\neq(\gamma\delta)}$$

$$\times \prod_{\substack{(\alpha\beta),(\alpha'\beta'),\cdots, \\ (\sigma\lambda),(\gamma\delta)\in(\mathcal{L}^*\mathcal{L})}} \prod_{n=0}^{\infty}$$

$$\times \left\{\frac{x^{(1)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}{x^{(2)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)} \cdot \frac{x^{(2)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}{x^{(1)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}\right\}^{-\frac{1}{2}k_\alpha\cdot k_\gamma}_{\substack{(\alpha\beta)\neq(\alpha'\beta') \\ (\sigma\lambda)\neq(\gamma\delta)}} , \tag{5.27}$$

where

$$R^{\pm}_{\beta\alpha}(z) = \frac{z[x^{(2)}_{\alpha\beta} - X^{\pm}_{\alpha\beta}\, x^{(1)}_{\alpha\beta}] - x^{(1)}_{\alpha\beta}\, x^{(2)}_{\alpha\beta}(1 - X^{\pm}_{\alpha\beta})}{z(1 - X^{\pm}_{\alpha\beta}) + x^{(2)}_{\alpha\beta}\, X^{\pm}_{\alpha\beta} - x^{(1)}_{\alpha\beta}} , \quad (\alpha\beta)\in(\mathcal{L}^*\mathcal{L}), \tag{5.28}$$

and[9,16]

$$(\det[\Delta])^{-\frac{1}{2}} = \prod_{\{\bar{R}\}} (1 - X_{\{\bar{R}\}})^{-4}; \tag{5.29}$$

$\{\bar{R}\}$ contains all projective group elements generated by $R^{\pm}_{\beta\alpha}$, $(\alpha\beta)\epsilon(\mathcal{L} * \mathcal{L})$, i.e., it contains terms like $R^{\pm n}_{\beta\alpha} R^{\pm m}_{\delta\gamma} \cdots$, $n,m = 0,1,2,\cdots,\infty$ $X_{\{\bar{R}\}}$ is the multiplier of $\{\bar{R}\}$. The symbol $[R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}$ is denoted for $R^{\pm}_{\beta\alpha} R^{\pm}_{\lambda\sigma} \cdots R^{\pm}_{\delta\gamma}$, with total number of $R^{\pm}$'s equal to n. The ordering of the variables will be mentioned after the following interpretations.

We interpret[9,17] various factors in Eq. (5.27):

(a) The divergent determinant factor

$$\prod_{\{\bar{R}\}} [1 - X_{\{\bar{R}\}}]^{-4}$$

corresponds to all "closed lines" going around the loops. The lines are not distinguished by their overall directions or by the points at which they begin.

(b) The factors raised to the power $\alpha_0 - 1$ connect successively the adjacent pairs of external legs, including the pairs across the sewed positions. These factors, together with

$$\frac{(y_a - y_b)(y_b - y_c)(y_c - y_a)}{\prod_{(\alpha\beta)\epsilon(\mathcal{L} * \mathcal{L})} [x^{(1)}_{\alpha\beta} - x^{(2)}_{\alpha\beta}]^2} \prod_{(i \neq j)} (y_i - y_j)^{-\frac{1}{2} k_i \cdot k_j}$$

are invariant under projective transformation.

(c) The factor $\{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}(y_j)\}^{-\frac{1}{2} k_i \cdot k_j}$ corresponds to all lines connecting the external leg y_i with the external leg y_j

and which go round the N loops a total number of n times (in either direction). The restriction that $\mathcal{L} \neq \mathcal{L}'$ in the product $R_{\mathcal{L}} R^{-1}_{\mathcal{L}'}$, or $R^{-1}_{\mathcal{L}} R_{\mathcal{L}'}$, implies that a line does not go successively round the same loop in opposite directions. The $n = 0$ component describes the line connecting the external legs y_i with y_j without surrounding any of the N loops.

(d) The factor

$$\left\{\frac{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}{y_i - [R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}\right\}^{-k_i \cdot k_\gamma}_{(\sigma\lambda)\neq(\gamma\delta)}$$

corresponds to all lines connecting the external leg y_i with the "center" points $x^{(1)}_{\gamma\delta}$, $x^{(2)}_{\gamma\delta}$ of the $(\gamma\delta)$ loop and which go round the other loops a total number of n times. The final loop surrounded must not be the $(\gamma\delta)$ loop.

(e) The last factor in Eq. (5.27)

$$\left\{\frac{x^{(1)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}{x^{(2)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)} \cdot \frac{x^{(2)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}{x^{(1)}_{\alpha\beta} - [R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}\right\}^{-\frac{1}{2}k_\alpha \cdot k_\gamma}_{\substack{(\alpha\beta)\neq(\alpha'\beta') \\ (\sigma\lambda)=(\gamma\delta)}}$$

corresponds to the lines connecting the $(\alpha\beta)$ loop with the $(\gamma\delta)$ loop, and going around the loops a total number of n times. The first loop surrounded must not be the $(\alpha\beta)$ loop, and the last must not be the $(\gamma\delta)$ loop. The $n = 0$ component describes the lines directly connecting the $(\alpha\beta)$ loop with the $(\gamma\delta)$ loop without going around any of the other N-2 loops.

Equation (5.27) is very useful in obtaining the overlapping and the nonorientable multiloop amplitudes, and also makes it easier to discuss the planar multiloops; we will show this in the next subsection.

However, the ordering of the variables in Eq. (5.27) is not transparent. Recall we started from the multiply factorized tree ordering

$$y_i \leq y_j, \quad \text{for} \quad j \leq i. \tag{5.30}$$

After sewing, we get two new identities, Eq. (5.15),

$$R^{-1}_{\beta\alpha}(y_\alpha) = y_{\beta+1}, \qquad R^{-1}_{\beta\alpha}(y_{\alpha+1}) = y_\beta, \tag{5.15}$$

which, together with the identities Eqs. (D.2c-g) in Appendix D enable us to obtain the ordering of the set of variables $\{y_i,\ i \neq (\mathcal{L}, \mathcal{L}^*);\ x^{(1)}_{\alpha\beta},\ x^{(2)}_{\alpha\beta},\ (\alpha\beta)\in(\mathcal{L}^*\mathcal{L})\}$. We phrase the ordering as following:[18] one simply replaces, from the original multiply factorized tree ordering, Eq. (5.30), all Koba-Nielsen variables associated with excited legs y_α, y_β, $(\alpha\beta)\in(\mathcal{L}^*\mathcal{L})$, by the corresponding invariant points, i.e.,

$$\begin{aligned} y_\alpha &\rightarrow x^{(2)}_{\alpha\beta}, \\ y_\beta &\rightarrow x^{(1)}_{\alpha\beta}, \quad (\alpha\beta)\in(\mathcal{L}^*\mathcal{L}). \end{aligned} \tag{5.31}$$

We thus obtain an ordering diagram Fig. 10b, which is associated with the nonplanar N-loop amplitude Eq. (5.27).

But, this ordering diagram, Fig. 10b, is incomplete in describing the range of integration (we need two more invariant points), and also suffers the shortcoming that not all external legs are treated on equal footing. Consequently, the integrand of Eq. (5.27) is slightly unsymmetrical with respect to all external legs. For instance, the external legs trapped between $x^{(1)}_{\alpha\beta}$ and $x^{(2)}_{\alpha\beta}$, $(\alpha\beta)\epsilon(\mathcal{L}*\mathcal{L})$, always remain trapped, but, by projective invariance or duality of Eq. (5.27), those external legs located between the invariant points of different but adjacent loops, e.g., $y_{\beta+1}, y_{\beta+2},\cdots,y_{\gamma-1}$ in Fig. 10b, are free to move past invariant points, i.e., the ordering $y_{\beta+1} < x^{(1)}_{\alpha\beta} < \cdots < x^{(2)}_{\alpha\beta}$ implies the ordering,[18]

$$x^{(1)}_{\alpha\beta} < \cdots < x^{(2)}_{\alpha\beta} < R_{\beta\alpha}(y_{\beta+1}). \tag{5.32}$$

In order to get the N-loop formula in a manifestly symmetrical form with respect to all external legs, and to discuss the range of integration in a transparently symmetrical way, we move, [due to the projective invariance of the integrand in Eq. (5.27)], all those external legs, located between the invariant points of different but adjacent loops, away from the region occupied by the invariant points. A simple renumbering yields the new ordering diagram shown in Fig. 10c

$$\begin{aligned} U_2 \;=\; [y_{S+1} &\equiv x^{(1)}_{\sigma\lambda} < y_S \leq \cdots \leq y_{\sigma+1} < y_\sigma \equiv x^{(2)}_{\sigma\lambda} < \cdots < y_\beta \\ &\equiv x^{(1)}_{\alpha\beta} < y_{\beta-1} \leq \cdots \leq y_{\alpha+1} < y_\alpha \equiv x^{(2)}_{\alpha\beta} < y_{\alpha-1} \leq y_{\alpha-2} \leq \\ &\leq \cdots \leq y_1 < y_0 < y_{S+1}]. \end{aligned} \tag{5.33a}$$

Since the action of $R^{-1}_{\beta\alpha}$ flips legs across the $(\alpha\beta)$-loop's invariant points [Eq. (5.32)], the action of $\hat{R} \equiv R^{-1}_{\lambda\sigma}\cdots R^{-1}_{\beta\alpha}$, with total number of R's equal to N, is to flip legs across all N loops, until we regain the original ordering. Hence if we move the external legs $y_0, \cdots y_{\alpha-1}$ (in Fig. 10c) infinite number of times past the N pairs of invariant points, due to the important inequality Eq. (5.32) and Eqs. (D.2c-g), we will asymptotically approach the two invariant points $x^{(1)}$, $x^{(2)}$ of the product projective operator $\hat{R} \equiv (R_{\beta\alpha} \cdots R_{\lambda\sigma})^{-1}$. It has been shown[18] that the invariant points $x^{(1)}$, $x^{(2)}$ lie outside the region occupied by the N pairs of invariant points, and hence $x^{(1)}$, $x^{(2)}$ divide the external legs variables $y_0, \cdots y_{\alpha-1}$ from the N pair invariant points of the individual R's. It can be shown[18] that the region occupied by the variables $y_0, \cdots y_{\alpha-1}$ and the regions occupied by $(R_{\beta\alpha} \cdots R_{\lambda\sigma})^{\pm n}(y_i)$, $i = 0, 1, \cdots, \alpha-1$ are disjoint $(n \neq 0)$. As $n \to \infty$, we merely approach the two invariant points $x^{(1)}$, $x^{(2)}$. Therefore, we can subtract the periodicities due to these disjoint regions by integrating over only one of them, i.e., integrate one of the variables $y_0, \cdots y_{\alpha-1}$, from $y_{\alpha-1}$ to $\hat{R}(y_{\alpha-1})$ for instance, where $y_{\alpha-1}$ lies between $x^{(1)}$ and $x^{(2)}$. Exactly similar arguments should be applied to all those external legs trapped between each individual pair of invariant points, i.e.,

$$x^{(1)}_{\alpha\beta} < y_{\beta-1} \leq \cdots \leq y_{\alpha+1} < R_{\beta\alpha}(y_{\beta-1}) < x^{(2)}_{\alpha\beta}, \quad \text{for} \quad (\alpha\beta) \in (\mathcal{L} * \mathcal{L}).$$

Consequently, in summary, the ranges of integration corresponding to the ordering diagram Fig. 10c, is

$$\begin{aligned} U_2 = [x^{(2)} &< y_{S+1} \equiv x^{(1)}_{\sigma\lambda} < y_S \leq \cdots \leq y_{\sigma+1} < R_{\lambda\sigma}(y_S) < y_\sigma \\ &\equiv x^{(2)}_{\sigma\lambda} < \cdots < y_\beta \equiv x^{(1)}_{\alpha\beta} < y_{\beta-1} \leq \cdots \leq y_{\alpha+1} < R_{\beta\alpha}(y_{\beta-1}) < y_\alpha \\ &\equiv x^{(2)}_{\alpha\beta} < x^{(1)} < y_{\alpha-1} \leq \cdots \leq y_1 \leq y_0 < \hat{R}(y_{\alpha-1}) < x^{(2)}]. \end{aligned} \tag{5.33b}$$

(The range of integration U_1 is always determined implicitly by U_2.)

Notice the complete symmetry between $R_{\beta\alpha}$'s and $\hat{R} \equiv (R_{\beta\alpha} \cdots R_{\lambda\sigma})^{-1}$, meaning that the distinction between outer and inner quark loops disappears. To reflect the symmetry, the factors raised to the power α_0-1 in Eq. (5.27) are formed to change to[9]

$$\left\{ \prod_{\substack{i=0 \\ [i\neq(\mathcal{L},\mathcal{L}^*,\mathcal{L}-1,\mathcal{L}^*-1)]}}^{S+1} (y_i-y_{i+1})^{\alpha_0-1} \left\{ \frac{[y_0-\hat{R}(y_{\alpha-1})](x^{(1)}-y_{\alpha-1})}{[x^{(1)}-\hat{R}(y_{\alpha-1})]} \right\}^{\alpha_0-1} \right.$$
$$\times \prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})} \left\{ \frac{[y_{\alpha+1} - R_{\beta\alpha}(y_{\beta-1})](y_\beta - y_{\beta-1})}{[y_\beta - R_{\beta\alpha}(y_{\beta-1})]} \right\}^{\alpha_0-1}$$
$$\left. \times \left[\prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})} X_{\alpha\beta}\, X_{\hat{R}} \right]^{\frac{1}{2}(\alpha_0-1)} \right\} . \qquad (5.34)$$

If we replace the expressions (5.34) and (5.33b) into Eq. (5.27), we then get the completely symmetrical nonplanar N-loop formula corresponding to the ordering diagram Fig. 10c:

$$\mathrm{FNL}(N)$$
$$= \int \prod_{\alpha\in(\mathcal{L}^*)} d^4k_\alpha \int_{U_1} \prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})} dX_{\alpha\beta}\, X_{\alpha\beta}^{-\ell(k_\alpha)-1} (1 - X_{\alpha\beta})^2$$

$$\times \int_{U_2} \prod_{\substack{i=0 \\ [i\neq(a,b,c)]}}^{S+1} dy_i \, \frac{(y_a - y_b)(y_b - y_c)(y_c - y_a)}{\prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})} (y_\alpha - y_\beta)^2}$$

Equation (5.35) continued

Equation (5.35) continued

$$\prod_{\{\bar{R}\}}(1-X_{\{\bar{R}\}})^{-4}\prod_{\substack{i,j=0\\ \left[\begin{smallmatrix} i\neq\alpha,\beta; \\ j\neq\gamma,\delta\end{smallmatrix}\right]}}^{S+1}\ \prod_{(\alpha\beta),\cdots(\gamma\delta)\in(\mathcal{L}*\mathcal{L})}\ \prod_{\substack{n=0\\ (n=0,i\neq j)}}^{\infty}$$

$$\times\left\{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\delta\gamma}(y_j)\right\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times\left\{\prod_{\substack{i=0\\ [i\neq(\mathcal{L},\mathcal{L}^*,\mathcal{L}-1,\mathcal{L}^*-1)]}}^{S+1}(y_i-y_{i+1})^{\alpha_0-1}\left\{\frac{[y_0-\hat{R}(y_{\alpha-1})](x^{(1)}-y_{\alpha-1})}{[x^{(1)}-\hat{R}(y_{\alpha-1})]}\right\}^{\alpha_0-1}\right.$$

$$\times\prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})}\left\{\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})](y_\beta-y_{\beta-1})}{[y_\beta-R_{\beta\alpha}(y_{\beta-1})]}\right\}^{\alpha_0-1}$$

$$\left.\times\left[\prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})}X_{\alpha\beta}\ X_{\hat{R}}\right]^{\frac{1}{2}(\alpha_0-1)}\right\}\qquad(5.35)$$

where $\hat{R}=(R_{\beta\alpha}\cdots R_{\lambda\sigma})^{-1}$.

Note that in Eq. (5.35), we have defined [due to Eq. (5.32)]

$$y_\alpha\equiv x^{(2)}_{\alpha\beta}\equiv R^{+\infty}_{\beta\alpha}(z_1),\quad z_1\neq y_\beta,\qquad(5.36a)$$

$$y_\beta\equiv x^{(1)}_{\alpha\beta}\equiv R^{-\infty}_{\beta\alpha}(z_2),\quad z_2\neq y_\alpha,\quad(\alpha\beta)\in(\mathcal{L}*\mathcal{L});\qquad(5.36b)$$

and

$$0\leq X_{\beta\alpha}<1.\qquad(5.36c)$$

(Though the amplitude corresponding to the ordering diagram Fig. 10c has the advantage of being symmetric in all quark loops, the actual calculation is much easier from the N-factorized tree in the configuration given by Fig. 10b, or Fig. 9.)

B. Overlapping Multiloop Formula

In this subsection we give a very simple derivation of the overlapping multiloop formula from the nonplanar multiloop formulas obtained in the previous subsection A.

If we interchange the labeling of the excited β leg with the excited γ leg in Fig. 9, we obtain Fig. 11. By sewing the $\alpha\beta$ pair and the γ,δ pair, we then obtain the overlapping multiloop diagram. However, the formulas of multiply factorized trees only depend on various cross-ratios associated with the excited legs, which in turn only depend on the positions of the dots attached to the excited legs. A comparison of Fig. 9 with Fig. 11 shows clearly that the two configurations Fig. 9 and Fig. 11 have identical expressions for the multiply factorized tree integrands, except for the ordering of the Koba-Nielsen variables. It follows that sewing the configuration in Fig. 11 gives a multiloop integrand identical to the configuration in Fig. 9. We therefore conclude that the overlapping multiloop integrand is identical to the nonplanar multiloop integrand of Eq. (5.27); but the ordering of y_i and $x_{\mathcal{L}}^{(1)}$, $x_{\mathcal{L}}^{(2)}$ are different, as already shown in Fig. 11 and Fig. 9 (the overlapping loops have overlapped invariant points.)

Then, by the projectively invariant nature of the overlapping multiloop integrand [Eq. (5.27)], we can move those legs "inside" all loops to the "outside," and so obtain, after renumbering, the ordering diagram for overlapping multiloops shown in Fig. 12. Because any multioverlapping multiloops can always be reduced to pairwise overlapping loops, due to the projective invariance of our multiloop integrand, it is sufficient in Fig. 12 only to consider one pair of overlapping loops R_1 and R_2. Corresponding to the overlapping loop ordering diagram Fig. 12, the range of integration is

$$\begin{aligned} U_2 = [x^{(2)} &< y_{S+1} \equiv x_2^{(1)} < y_S \equiv x_1^{(1)} < y_{S-1} \equiv x_2^{(2)} < y_{S-2} \\ &\equiv x_1^{(2)} < x^{(1)} < y_{S-3} \leq y_{S-2} \leq \cdots \leq y_0 < \hat{R}(y_{S-3}) < x^{(2)}], \end{aligned} \tag{5.37}$$

where $\hat{R} \equiv R_2^{-1}R_1R_2R_1^{-1}$; and $x^{(1)}$, $x^{(2)}$ are the two invariant points of $\hat{R}$; $x_1^{(1)}$, $x_1^{(2)}$ and $x_2^{(1)}$, $x_2^{(2)}$ are the two invariant points of R_1 and R_2 respectively. Notice that the product projective operator $R_2^{-1}R_1R_2R_1^{-1}$ is the correct projective operator to bring, say, y_{S-3} across the invariant points $[x_1^{(2)}, x_2^{(2)}, x_1^{(1)}, x_2^{(1)}]$ to the side adjacent to y_0. This fact also naturally determines the factors raised to the power α_0-1 which connect the successively adjacent pair of external legs, it is then given by

$$\left\{ \prod_{i=0}^{S-4} (y_i - y_{i+1})^{\alpha_0-1} \left\{ \frac{[y_0 - \hat{R}(y_{S-3})][x^{(1)} - y_{S-3}]}{[x^{(1)} - \hat{R}(y_{S-3})]} \right\}^{\alpha_0-1} [X_1{}^2 X_2{}^2 X_{\hat{R}}]^{\frac{1}{2}(\alpha_0-1)} \right\} \tag{5.38}$$

If we replace the last brace in Eq. (5.35) by the expression (5.38), Eq. (5.33b) by Eq. (5.37), and take the case $N = 2$ together with an obvious modification of the indices in Eq. (5.35), then we get the completely symmetrical overlapping double-loop formula corresponding to the overlapping ordering diagram in Fig. 12.

C. Nonorientable Multiloop Formula

The nonorientable loop diagram can be constructed from the multiply factorized tree diagram by sewing a pair of excited legs with dots on opposite sides. But two multiply factorized tree amplitudes with dots on opposite sides of one excited leg, say $y_{\mathcal{L}}$ leg, are related to each other by the twist operator, or equivalently, by interchanging, say $y_{\mathcal{L}+1}$ with $y_{\mathcal{L}-1}$, everywhere in the tree amplitude. It is then clear that the nonorientable multiloop integrand can be obtained from the nonplanar multiloop integrand in Eq. (5.27), by simply interchanging $y_{\mathcal{L}+1}$ and $y_{\mathcal{L}-1}$ everywhere in it. Under this interchangement, however, the external legs previously trapped between the two invariant points $x_{\mathcal{L}}^{(1)}$, $x_{\mathcal{L}}^{(2)}$ in the nonplanar case are no longer confined between them. The factors raised to the power α_0-1 then contain factors which connect variables associated with lines inside and outside the loop. Since we have the freedom to move out the external legs lying between $x_{\mathcal{L}}^{(1)}$, $x_{\mathcal{L}}^{(2)}$ into the complementary region, we must assign a <u>negative multiplier</u> $-X_{\mathcal{L}}$ to the nonorientable loop $R_{\mathcal{L}}$. After moving out all external legs away from the invariant points and renumbering them, we obtain the nonorientable ordering diagram shown in Fig. 13. (Again we only consider the nonorientable double-loop case.) The range of integration corresponding to the nonorientable ordering diagram Fig. 13, is

$$\begin{aligned} U_2 = [x^{(2)} &< y_{S+1} \equiv x_2^{(1)} < y_S \equiv x_2^{(2)} < y_{S-1} \equiv x_1^{(1)} < y_{S-2} \\ &\equiv x_1^{(2)} < x^{(1)} < y_{S-3} \leq y_{S-4} \leq \cdots \leq y_0 \leq \hat{R}(y_{S-3}) < x^{(2)}], \end{aligned} \tag{5.39}$$

where $\hat{R} \equiv R_2^{-2} R_1^{-2}$ is the correct projective operator taking y_{S-3} across the nonorientable loops R_1, R_2 to the side adjacent to y_0. Thus the factors raised to the power α_0-1, is

$$\Bigg\{ \widetilde{\prod_{i=0}^{S-4}} (y_i - y_{i+1})^{\alpha_0-1} \Bigg\{ \frac{[y_0 - \hat{R}(y_{S-3})]\left(x^{(1)} - y_{S-3}\right)}{[x^{(1)} - \hat{R}(y_{S-3})]} \Bigg\}^{\alpha_0-1} [X_1{}^2 X_2{}^2 X_{\hat{R}}]^{\frac{1}{2}(\alpha_0-1)} \tag{5.40}$$

where

$$R_i^{\pm}(z) = \frac{z[x_i^{(2)} + x_i^{(1)}\, X_i] - x_i^{(1)}\, x_i^{(2)}(1 + X_i)}{z(1 + X_i) - x_i^{(2)}\, X_i - x_i^{(1)}} \,, \quad i = 1,2, \tag{5.41}$$

and $X_i \geq 0$.

Hence, if we replace the last brace in Eq. (5.35) by the expression (5.40), Eq. (5.33b) by Eq. (5.39), let all $X_{\mathcal{L}}$ become $-X_{\mathcal{L}}$, and take the case $N = 2$ together with an obvious modification of the indices in Eq. (5.35), we have the completely symmetrical nonorientable double-loop formula corresponding to the ordering diagram Fig. 13.

D. Planar Multiloop Formula

The planar multiloop formula is a particular case of nonplanar multiloop formula, if we let all external legs inside loops disappear, i.e., if we set all Koba-Nielsen variables inside the loops vanish. The planar multiloop ordering diagram is shown in Fig. 14, which of course is a particular case of Fig. 10c. On examining Fig. 14, it is trivial to write down the range of integration

$$U_2 = [x^{(2)} < y_{S+1} \equiv x_N^{(1)} < y_S \equiv x_N^{(2)} < \cdots < y_{S-2N+3}$$

$$\equiv x_1^{(1)} < y_{S-2N+2} \equiv x_1^{(2)} < x^{(1)} < y_{S-2N+1} \leq \cdots \leq y_1 \leq y_0$$

$$< \hat{R}(y_{S-2N+1}) < x^{(2)}], \qquad (5.42)$$

where $\hat{R} \equiv R_N^{-1} R_{N-1}^{-1} \cdots R_1^{-1}$ and the factors raised to the power α_0-1:

$$\left\{ \prod_{i=0}^{S-2N} (y_i - y_{i+1})^{\alpha_0-1} \left\{ \frac{[y_0-\hat{R}(y_{S-2N+1})](x^{(1)}-y_{S-2N+1})}{[x^{(1)}-\hat{R}(y_{S-2N+1})]} \right\}^{\alpha_0-1} \right.$$

$$\left. \times \left[\prod_{(\alpha\beta)\in(\mathcal{L}*\mathcal{L})} X_{\alpha\beta}\, X_{\hat{R}} \right]^{\frac{1}{2}(\alpha_0-1)} \right\} . \qquad (5.43)$$

Again, replacing the last brace in Eq. (5.35) by the expression (5.43), Eq. (5.33b) by Eq. (5.42), and the factors $\prod_{\mathcal{L}} (1 - X_{\mathcal{L}})^2$ by[9,19] $\prod_{\mathcal{L}} (1 - X_{\mathcal{L}})$, then we get the symmetrical planar N-loop formula (with an obvious modification in the indices).

E. General Rules for Writing Down Any Arbitrarily Mixed Multiloop Formula

In this subsection we give a set of rules for writing down any arbitrarily mixed[20] multiloop amplitudes by simply examining the corresponding mixed multiloop Feynman-like diagrams (they are essentially equivalent to the corresponding ordering diagrams).

Rule 1

Given any mixed multiloop Feynman diagram, we appeal to duality and arrange the diagram such that

(a) no loop is allowed to be within another loop,

(b) no external legs are allowed to be within overlapping loops and nonorientable loops,

(c) all external legs not confined within nonplanar loops are bunched together,

(d) no more than two loops ever overlap.

It is because the range of integration is only transparent in this particular configuration, and the Feynman diagram simultaneously represents the ordering diagram, the we make these arrangements (a)-(d). (It is certainly not necessary to do so.)

Rule 2

We assign to each external scalar leg one Koba-Nielsen variable y_i, $i = 1,\cdots,S$, and one incoming four-momentum k_i; to each loop one loop momentum $k_{\mathcal{L}}$ and three parameters $x^{(1)}_{\mathcal{L}}$, $x^{(2)}_{\mathcal{L}}$, $X_{\mathcal{L}}$ which define a projective operator $R_{\mathcal{L}}(R^{-1}_{\mathcal{L}})$ corresponding to surrounding the loop in a clockwise (counter-clockwise) direction. We adopt the convention that $0 \leq X_{\mathcal{L}} < 1$ for planar, nonplanar and overlapping loops, and $X_{\mathcal{L}}$ is negative for nonorientable loops.

Rule 3

We integrate over all loop momenta $k_{\mathcal{L}}$, all multipliers $X_{\mathcal{L}}$, all invariant points $x^{(1)}_{\mathcal{L}}$, $x^{(2)}_{\mathcal{L}}$ and all external leg variables y_i, $i = 1,\cdots,S$; i.e., we perform the integrations

$$\int \prod_{(\mathcal{L})} d^4k \int_{U_1} \prod_{(\mathcal{L})} dX_{\mathcal{L}}\, X_{\mathcal{L}}^{-\alpha(k_{\mathcal{L}})-1} \int_{U_2} \prod_{\substack{i=1\\(i\neq a,b,c)}}^{S} dy_i$$

$$\times \prod_{(\mathcal{L})} dx^{(1)}_{\mathcal{L}}\, dx^{(2)}_{\mathcal{L}} (y_a - y_b)(y_b - y_c)(y_c - y_a), \qquad (5.44)$$

2

where y_a, y_b, y_c are any three fixed variables taken out from the set of variables $[y_i,\ i = 1,\cdots,S;\ x_{\mathcal{L}}^{(1)},\ x_{\mathcal{L}}^{(2)}$ all $\mathcal{L}]$. The ordering diagram, which determines U_2 in the expression (5.44), is similar to the rearranged multiloop Feynman diagram, provided we assign $x_{\mathcal{L}}^{(1)}$, $x_{\mathcal{L}}^{(2)}$ to the points forming the loop $R_{\mathcal{L}}$ with the main tree, and $x^{(1)}$, $x^{(2)}$ to the points separating the "outside" external legs from the region occupied by the loops, as shown in Fig. 15 for example, (U_1 is implicitly determined from U_2).

Rule 4

The mixed multiloop integrand consists of

(a) the linear dependence factors (see Rule 5),

(b) the factors raised to the power α_0-1, which connect the successively adjacent pairs of external legs, and their forms depend on the ordering diagrams (see Rule 6),

(c) the divergent determinantal factors corresponding to the closed curves surrounding the loops, see Rule 7, and

(d) the momentum-dependent factors corresponding to open curves surrounding the loops (see Rule 8).

Rule 5

The linear dependence factors are: for each planar loop $\mathcal{L}$, a factor $(1 - X_{\mathcal{L}})$; for each nonplanar or overlapping loop, a factor $(1 - X_{\mathcal{L}})^2$; and for each nonorientable loop $\mathcal{L}$, a factor $(1 + |X_{\mathcal{L}}|)^2$.

Rule 6

The adjacent pair of external legs y_i and y_{i+1} contributes a factor $(y_i - y_{i+1})^{\alpha_0-1}$. Each pair of external legs, for example, the $y_{\ell+1}$ and the y_m legs in Fig. 15 that are adjacent to $x_3^{(1)}$ and $x_3^{(2)}$, contributes a factor

$$\left\{\frac{[y_{\ell+1} - R_3(y_m)](x_3^{(1)} - y_m)}{[x_3^{(1)} - R_3(y_m)]}\right\}^{\alpha_0-1} .$$

Similar factors correspond to the pair adjacent to the invariant points $x^{(1)}$, $x^{(2)}$ of the product projective operator $\hat{R}$. The product projective operator $\hat{R}$ is obtained by examining how one leg of the adjacent pair moves past all loops. When it passes (in counterclockwise direction) a planar or a nonplanar loop $\mathcal{L}$, it acquire a factor $R_{\mathcal{L}}^{-1}$; when it passes an overlapping pair of loops in the sequence $\mathcal{L}$, $\mathcal{L}'$, then it acquires a factor $R_{\mathcal{L}'}^{-1} R_{\mathcal{L}} R_{\mathcal{L}'} R_{\mathcal{L}}^{-1}$; and when it passes (in counterclockwise direction) a nonorientable loop $\mathcal{L}$, it acquires a factor $R_{\mathcal{L}}^{-2}$. For example, the product projective operator $\hat{R}$ in Fig. 15 is $\hat{R} = R_7^{-2} R_6^{-1} R_5 R_6 R_5^{-1} R_4^{-1} R_3^{-1} R_2^{-1} R_1^{-1}$. There also occur extra projectively invariant factors which come from the Jacobian transformation[9] from the sewing configuration (e.g., Fig. 10b) to the symmetrical configuration (e.g., Fig. 10c). They are, for each planar or nonplanar loop $\mathcal{L}$, a factor $X_{\mathcal{L}}^{\frac{1}{2}(\alpha_0-1)}$, for each nonorientable or overlapping loop $\mathcal{L}$ a factor $X_{\mathcal{L}}^{\alpha_0-1}$, and for (each) product projective operator $\hat{R}$ a factor $(X_{\hat{R}})^{\frac{1}{2}(\alpha_0-1)}$. Hence, for the example corresponding to Fig. 15, the factors are $(X_1X_2X_3X_4X_5^2X_6^2X_7^2X_{\hat{R}})^{\frac{1}{2}(\alpha_0-1)}$.

Rule 7

The divergent factors, corresponding to the set of closed curves surrounding the loops, are

$$\prod_{\{\bar{R}\}} (1 - X_{\{\bar{R}\}})^{-4}, \tag{5.45}$$

where $X_{\{\bar{R}\}}$ are the multipliers of the projective group elements generated by $R^{\pm}$ (for all $\mathcal{L}$) such that cyclic permutations and inverse of cyclic permutations of the elements are omitted.

Rule 8

The momentum-dependent factors, corresponding to the set of open curves surrounding the loops, i.e., the curves connecting the external leg y_i to the external leg y_j; the curves connecting the external leg y_i to the "centers" $x^{(1)}_{\mathcal{L}}$, $x^{(2)}_{\mathcal{L}}$ of the loops $\mathcal{L}$'s; and the curves connecting the "center" $x^{(1)}_{\mathcal{L}}$, $x^{(2)}_{\mathcal{L}}$ of the loop $\mathcal{L}$ to the "center" $x^{(1)}_{\mathcal{L}'}$, $x^{(2)}_{\mathcal{L}'}$ of the loop $\mathcal{L}'$. The factors are

$$\prod_{i,j\in \text{ all externals}} \; \prod_{\mathcal{L},\cdots,\mathcal{L}'\in(\mathcal{L})} \; \prod_{\substack{n=0 \\ (n=0,i\neq j)}}^{\infty} \{y_i - [R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times \prod_{i\in \text{ all externals}} \; \prod_{\mathcal{L},\cdots,\mathcal{L}'\mathcal{L}''\in(\mathcal{L})} \; \prod_{n=0}^{\infty}$$

$$\left\{\frac{y_i - [R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(x^{(2)}_{\mathcal{L}''})}{y_i - [R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(x^{(1)}_{\mathcal{L}''})}\right\}^{-k_i\cdot k_{\mathcal{L}''}}_{(\mathcal{L}'\neq\mathcal{L}'')}$$

Equation (5.46) continued

$$\prod_{\mathcal{L},\mathcal{L}',\cdots,\mathcal{L}''\mathcal{L}'''} \prod_{n=0}^{\infty}$$

$$\times \left\{ \frac{x^{(1)} - [R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(1)}_{\mathcal{L}'''}\right)}{x^{(1)} - [R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(2)}_{\mathcal{L}'''}\right)} \cdot \frac{x^{(2)} - [R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(2)}_{\mathcal{L}'''}\right)}{x^{(2)} - [R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(1)}_{\mathcal{L}'''}\right)} \right]^{-\frac{1}{2}k_{\mathcal{L}}\cdot k_{\mathcal{L}'''}}_{\substack{\mathcal{L}\neq\mathcal{L}' \\ \mathcal{L}''\neq\mathcal{L}'''}} \tag{5.46}$$

where $[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'} \equiv R^{\pm}_{\mathcal{L}} \cdots R^{\pm}_{\mathcal{L}'}$, with total number of $R^{\pm}_{\mathcal{L}}$ equal to n, and it is implied that $\mathcal{L} \neq \mathcal{L}'$ in the product $R_{\mathcal{L}} R^{-1}_{\mathcal{L}'}$ or $R^{-1}_{\mathcal{L}} R_{\mathcal{L}'}$.

We end this section by illustrating the rules 1-8 in a general example consisting of two planar loops R_1 and R_2, two nonplanar loops R_3 and R_4, one pair of overlapping loops R_5 and R_6, and one nonorientable loop R_7, as shown in Fig. 15. We then write down the answer

FML(7)

$$= \int \prod_{i=1}^{7} d^4k_i \int_{U_1} \left(\prod_{i=1}^{7} dX_i \, X_i^{-\alpha(k_i)-1} \right)$$

$$\times \int_{U_2} \prod_{\substack{i=1 \\ (i\neq a,b,c)}}^{S} dy_i \prod_{\mathcal{L}=1}^{7} dx^{(1)}_{\mathcal{L}} \, dx^{(2)}_{\mathcal{L}}$$

$$(y_a - y_b)(y_b - y_c)(y_c - y_a) \prod_{\{\bar{R}\}} [1 - X_{\{\bar{R}\}}]^{-4}$$

Equation (5.47) continued

$$\times\ (1 - X_1)(1 - X_2)(1 - X_3)^2(1 - X_4)^2(1 - X_5)^2(1 - X_6)^2(1 + |X|_7)^2$$

$$\times \prod_{\substack{i=1\\(i\neq\ell,m,S)}}^{S} (y_i - y_{i+1})^{\alpha_0-1}\left\{\frac{[y_{\ell+1}-R_3(y_m)](x_3^{(1)}-y_m)}{[x_3^{(1)}-R_3(y_m)]}\right\}^{\alpha_0-1}$$

$$\times\ \{X_1X_2X_3X_4X_5{}^2X_6{}^2X_7{}^2X_{\hat{R}}\}^{\frac{1}{2}(\alpha_0-1)}$$

$$\times \left\{\frac{[y_{m+1}-R_4(y_S)](x_4^{(1)}-y_S)}{[x_4^{(1)}-R_4(y_S)]}\right\}^{\alpha_0-1} \left\{\frac{[y_1-\hat{R}(y_\ell)](x^{(1)}-y_\ell)}{[x^{(1)}-\hat{R}(y_\ell)]}\right\}^{\alpha_0-1}$$

$$\times \prod_{i,j=1}^{S} \prod_{\substack{\mathcal{L},\cdots,\mathcal{L}'\\ \in 1,2,\cdots,7}} \prod_{\substack{n=0\\(n=0,i\neq j)}}^{\infty} \{y_i - [R^{\pm}]^{(n)}_{\mathcal{L};\mathcal{L}'}(y_j)\}^{-\frac{1}{2}k_i\cdot k_j}$$

$$\times \prod_{i=1}^{S} \prod_{\substack{\mathcal{L},\cdots,\mathcal{L}',\mathcal{L}''\\ \in 1,2,\cdots 7\\ (\mathcal{L}'\neq\mathcal{L}'')}} \prod_{n=0}^{\infty} \left\{\frac{y_i - [R^{\pm}]^{(n)}_{\mathcal{L}\mathcal{L}'}\left(x^{(2)}_{\mathcal{L}''}\right)}{y_i - [R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}\left(x^{(1)}_{\mathcal{L}''}\right)}\right\}^{-\frac{1}{2}k_i\cdot k_{\mathcal{L}''}}$$

$$\prod_{\substack{\mathcal{L},\mathcal{L}',\cdots,\mathcal{L}''\mathcal{L}'''\\ \in 1,2,\cdots,7\\ (\mathcal{L}\neq\mathcal{L}',\mathcal{L}''\neq\mathcal{L}''')}} \prod_{n=0}^{\infty}$$

$$\left\{\frac{x^{(1)}_{\mathcal{L}}-[R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(1)}_{\mathcal{L}'''}\right)}{x^{(1)}_{\mathcal{L}}-[R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(2)}_{\mathcal{L}'''}\right)} \cdot \frac{x^{(2)}_{\mathcal{L}}-[R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(2)}_{\mathcal{L}'''}\right)}{x^{(2)}_{\mathcal{L}}-[R^{\pm}]^{(n)}_{\mathcal{L}',\mathcal{L}''}\left(x^{(1)}_{\mathcal{L}'''}\right)}\right\}^{-\frac{1}{2}k_{\mathcal{L}}\cdot k_{\mathcal{L}'''}} \tag{5.47}$$

with

$$\begin{aligned} U_2 = [x^{(2)} &< x_7^{(1)} < x_7^{(2)} < x_6^{(1)} < x_5^{(1)} < x_6^{(2)} < x_5^{(2)} < x_4^{(1)} < \\ &< y_S \leq y_{S-1} \leq \cdots \leq y_{m+1} < R_4(y_S) < x_4^{(2)} < x_3^{(1)} < \\ &< y_m \leq y_{m-1} \leq \cdots \leq y_{\ell+1} < R_3(y_m) < x_3^{(2)} < x_2^{(1)} < x_2^{(2)} < \\ &< x_1^{(1)} < x_1^{(2)} < x^{(1)} < y_\ell \leq y_{\ell-1} \leq \cdots \leq y_1 < \hat{R}(y_\ell) < x^{(2)}]. \end{aligned} \tag{5.48}$$

The formula, Eq. (5.47), is manifestly projectively invariant, hence, any formulas related by duality to the Feynman diagram of Fig. 15, can also be directly written down. In fact, the only modifications are the ranges of integrations and the factors raised to the power $\alpha_0 - 1$.

VI. MULTIFACTORIZATIONS ON THE MULTILOOPS AND THE PURE REGGEON MULTILOOP AMPLITUDES

These two classes of amplitudes are the new results of this thesis work.[21] Since their derivations are very similar to Sec. V, we will not give the detailed calculations but shall only outline briefly how various new factors are obtained. Again, in the end one can intuitively interpret these new factors in a similar way to Sec. V.

A. Loop Amplitudes With External Reggeons

One simply way to obtain the multiloop amplitudes with some of external legs excited, is to sew, in Eq. (5.1), M pairs of excited legs with $M < N$. We therefore separate out, in Eq. (5.1), those factors which eventually correspond to multiloops from those which represent the excited legs. Hence we write Eq. (5.1) as

$$\begin{aligned} G^{(2N)}_{(Y)}(a's) &= \int \prod_i dy_i [L] \left\{ \prod_{t\epsilon(T)} \left[\frac{(y_{t-1}-y_t)(y_t-y_{t+1})}{(y_{t-1}-y_{t+1})} \right]^{-\alpha(k_t)} \right. \\ &\times \exp\left\{ \sum_{t\epsilon(T)} \sum_{\substack{i=0 \\ (i\neq t)}}^{S+1} (a^t|P_t(i)|k_i) + \frac{1}{2} \sum_{\substack{t,t'\epsilon(T) \\ (t\neq t')}} (a^t|P_t \hat{P}_{t'}^{-1}(\ )|a^{t'}) \right\} \\ &\times \exp\left\{ \sum_{\substack{\alpha\epsilon(\mathcal{L}^*) \\ t\epsilon(T)}} (a^\alpha|P_\alpha \hat{P}_t^{-1}(\ )|a^t) + \sum_{\substack{\beta\epsilon(\mathcal{L}) \\ t\epsilon(T)}} (a^\beta|P_\beta \hat{P}_t^{-1}(\ )|a^t) \right\}, \quad (6.1) \end{aligned}$$

where [L] is identical to the integrand of Eq. (5.1), except N is replaced by M. The notation (T) indicates all unsewed excited legs. We note that the last exponent factors in Eq. (6.1) simply contribute extra terms of $\sum_{t\in(T)} P_\beta[\hat{P}_t^{-1}(\ )]|a^t)$ to the $|E_{\alpha\beta})$ of Eq. (5.8b), and extra terms of $\sum_{t\in(T)} Y_{\alpha\beta}\ P_\alpha[\hat{P}_t^{-1}(\ )]|a^t)$ to the $|F_{\alpha\beta})$ of Eq. (5.8c). Consequently, all calculations from Eq. (5.8) go through exactly to Eq. (5.18). From Eq. (6.1), we calculate the integrand after sewing, i.e., $I \equiv \prod_{n=0}^{\infty} I_n$, and, using the following <u>symbolic</u> identities

$$P_t\ R_{\beta\alpha}\ \hat{P}_s^{-1}(\ ) \ = \ (\ )\hat{P}_t^{-1}\ R_{\beta\alpha}^{-1}\ P_s, \tag{6.2a}$$

$$\hat{P}_t\ R_{\beta\alpha}\ \hat{P}_s^{-1}(\ ) \ = \ (\ )P_t^{-1}\ R_{\beta\alpha}^{-1}\ P_s, \tag{6.2b}$$

$$y_i\ R_{\beta\alpha}^{\pm}\ \hat{P}_t^{-1}(\ )|) \ = \ (|P_t\ R_{\beta\alpha}^{\pm}(y_i), \tag{6.2c}$$

where y_i is the Koba-Nielsen variable, and Eqs. (6.2a) and (6.2b) are understood to be sandwiched between $(a^t|$ and $|a^s)$. We find again that two sets of an infinite number of cancellations occur beautifully in the operator part; and result in

$$I = [I_{\rm loop}]\ \exp\left\{\frac{1}{2}\sum_{n=1}^{\infty}\ \sum_{t,t'\in(T)}\ \sum_{\mathcal{L},\cdots,\mathcal{L}'\in(\mathcal{L})} \times\ (a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}\ \hat{P}^{-1}_{t'}(\ )|a^{t'})\right.$$

$$+ \sum_{t\in(T)}\ \sum_{n=1}^{\infty}\ \sum_{\substack{\mathcal{L},\cdots,\mathcal{L}',\mathcal{L}''\\ \in(\mathcal{L})\\ (\mathcal{L}'\neq\mathcal{L}'')}}\ \sum_{i\neq(\mathcal{L},\mathcal{L}^*)}$$

Equation (6.3) continued

$$
(a^t|P_t \begin{bmatrix} [R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(i) \\ [R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'} \begin{pmatrix} x^{(2)}_{\mathcal{L}''} \\ x^{(1)}_{\mathcal{L}''} \end{pmatrix} \\ y_{\mathcal{L}} \\ y_{\mathcal{L}*} \end{bmatrix} \left| \begin{matrix} k_i \\ k_{\mathcal{L}''*} \\ -k_{\mathcal{L}''*} \\ k_{\mathcal{L}*} \\ -k_{\mathcal{L}*} \end{matrix} \right) \Bigg\} \tag{6.3}
$$

where $[I_{\text{loop}}]$ is given by Eq. (5.22), with $N \to M$. Combining Eq. (6.3) with the first brace in Eq. (6.1), and then combining it with [L], we finally get the answer for the multiloop amplitude with T excited reggeons, corresponding to the ordering diagram Fig. 10b

$$
\begin{aligned}
FNL^{(T)}(M) = \int \{F\}_{(5.27)} \exp\Bigg\{ & \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t,t' \in (T) \\ (n=0,\, t \neq t')}} \sum_{\substack{\text{all } \mathcal{L} \\ \in (\mathcal{L})}} \\
& \times (a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'} \hat{P}^{-1}_{t'}(\ )|a^{t'}) \\
& + \sum_{t \in (T)} \sum_{n=0}^{\infty} \sum_{\substack{\text{all } \mathcal{L} \\ \in (\mathcal{L})}} \sum_{\substack{i \neq (\mathcal{L},\mathcal{L}*) \\ (i \neq t,\ n=0)}} (a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(i)|k_i) \\
& + \sum_{t \in (T)} \sum_{n=0}^{\infty} \sum_{\substack{\mathcal{L},\ldots,\mathcal{L}',\mathcal{L}'' \in (\mathcal{L}) \\ (\mathcal{L}' \neq \mathcal{L}'')}} (a^t|P_t[R]^{(n)}_{\mathcal{L},\mathcal{L}'} \begin{bmatrix} x^{(2)}_{\mathcal{L}''} \\ x^{(1)}_{\mathcal{L}''} \end{bmatrix} \left| \begin{matrix} k_{\mathcal{L}''*} \\ -k_{\mathcal{L}''*} \end{matrix} \right) \\
& \times \prod_{t \in (T)} \left[\frac{(y_{t-1} - y_t)(y_t - y_{t+1})}{(y_{t-1} - y_{t+1})} \right]^{-\alpha(k_t)} \Bigg\} ,
\end{aligned} \tag{6.4}
$$

where $\{F\}_{(5.27)}$ is given by Eq. (5.27), with M loops, and $x^{(1)}_{\mathcal{L}}$, $x^{(2)}_{\mathcal{L}}$ are the two invariant points of the loop $R_{\mathcal{L}}$. It should be pointed out that only the last factors raised to the power $-\alpha(k_t)$ and the factors raised to the power α_0-1 in $\{F\}_{(5.27)}$ are sensitive to the ordering of the external legs. Again, it is trivial to move all legs away from the region occupied by the invariant points, thereby obtaining an amplitude corresponding to the ordering diagram Fig. 10c. For simplicity, we first assume that all external legs adjacent to all invariant points in Fig. 10c are <u>not</u> excited, then the amplitude is

$$\begin{aligned} FNL^{(T)}(M) &= \int \{F\}_{(5.35)} \left\{ \prod_{t\in(T)} \left[\frac{(y_{t-1}-y_t)(y_t-y_{t+1})}{(y_{t-1}-y_{t+1})} \right]^{-\alpha(k_t)} \right\} \\ &\times \left\{ \exp\left\{ \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t,t'\in(T)\\(n=0,t\neq t')}} \sum_{\substack{\text{all } \mathcal{L}\\ \in(\mathcal{L})}} (a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}\, \hat{P}^{-1}_{t'}(\ )|a^{t'}) \right.\right. \\ &\quad + \sum_{n=0}^{\infty} \sum_{t\in(T)} \sum_{\substack{\text{all } \mathcal{L}\\ \in(\mathcal{L})}} \sum_{\substack{i\neq(\mathcal{L},\mathcal{L}^*)\\(n=0,i\neq t)}} (a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(i)|k_i) \\ &\quad \left.\left. + \sum_{t\in(T)} \sum_{n=0}^{\infty} \sum_{\substack{\mathcal{L},\cdots,\mathcal{L}',\mathcal{L}''\\ \in(\mathcal{L})\\ (\mathcal{L}'\neq\mathcal{L}'')}} (a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'} \begin{bmatrix} y_{\mathcal{L}''*} \\ y_{\mathcal{L}''} \end{bmatrix} \middle| \begin{matrix} k_{\mathcal{L}''*} \\ -k_{\mathcal{L}''*} \end{matrix}) \right\}\right\}, \end{aligned} \tag{6.5}$$

where $\{F\}_{(5.35)}$ is given by Eq. (5.35), with M loops, and $y_{\mathcal{L}}$, $y_{\mathcal{L}*}$ are the two invariant points of the loop $\mathcal{L}$.

A simple interpretation of the last brace in Eq. (6.5) again emerges:

(a) The first terms (in the exponent)

$$(a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}\ \hat{P}^{-1}_{t'}(\ )|a^{t'})$$

describes all lines which connect the reggeon a^t with the reggeon $a^{t'}$ (notice t could be equal to t', for $n \neq 0$) and which encircle the M loops a total number of n times.

(b) The second terms

$$(a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(i)|k_i)$$

describe all lines which connect the reggeon a^t with the external scalar leg y_i and which encircle the M loops a total number of n times.

(c) The last terms

$$(a^t|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}\begin{bmatrix} y_{\mathcal{L}''*} \\ y_{\mathcal{L}''} \end{bmatrix}\left|\begin{matrix} k_{\mathcal{L}''*} \\ -k_{\mathcal{L}''*} \end{matrix}\right)_{(\mathcal{L}'\neq\mathcal{L}'')}$$

describe the lines connecting the reggeon a^t with the "center" of the loop $\mathcal{L}''$ and which encircle the M loops a total number of n times.

(d) The factors raised to the power $-\alpha(k_t)$ depend on the ordering of the particles. They appear here because, for excited legs, $\alpha(k_t) \equiv \alpha_0 + \frac{1}{2}k_t^2 \neq 0$. In the case that two excited legs are adjacent to the two invariant points; for instance, if $y_{\alpha+1}$ and $y_{\beta-1}$ are excited and adjacent to $x^{(1)}_{\alpha\beta}$, $x^{(2)}_{\alpha\beta}$ in Fig. 10c, then the corresponding factor should be

$$\left\{\frac{[R_{\beta\alpha}(y_{\beta-1})-y_{\alpha+1}](y_{\alpha+1}-y_{\alpha+2})}{[R_{\beta\alpha}(y_{\beta-1})-y_{\alpha+2}]}\right\}^{-\alpha(k_{\alpha+1})}\left\{\frac{(y_{\beta-2}-y_{\beta-1})(y_{\beta-1}-x^{(1)}_{\alpha\beta})}{(y_{\beta-2}-x^{(1)}_{\alpha\beta})}\right\}^{-\alpha(k_{\beta-1})} \quad (6.6)$$

Furthermore, if the two dots associated with two excited legs "face" each other, then we need an asymmetrical propagator, similar to Eq. (4.18).

It should be observed that all the new factors appearing in Eq. (6.5), as compared with Eq. (5.35), are insensitive to the ordering diagrams. Consequently, the planar, overlapping and nonorientable multiloop diagrams with arbitrary number of excited external legs can again be discussed as in the previous section. In short, they are distinguished only by the expressions for $\{F\}_{(5.35)}$, which have already been explained in Sec. V (A-E).

B. The Pure Reggeon Multiloop Amplitudes

The way to obtain this class of amplitudes is simple. One simply puts all Koba-Nielsen variables, in Eq. (6.5), associated with external <u>scalar</u> legs vanish. In this limit, however, there are extra factors similar to Eq. (6.6) appear for each loop, in addition to the factors raised to the power $-\alpha(k_t)$. Hence, the pure T-reggeon nonplanar multiloop amplitude corresponding to the ordering diagram Fig. 10c is (Fig. 16)

$$\mathrm{FNL}^{(T)}(M) = \int \{F'\}_{(5.35)} \prod_{\substack{t\in(T)\\ t\neq(\mathcal{L}^{*}+1,\mathcal{L}-1)}} \left[\frac{(y_{t-1}-y_t)(y_t-y_{t+1})}{(y_{t-1}-y_{t+1})}\right]^{-\alpha(k_t)}$$

Equation (6.7) continued

Equation (6.7) continued

$$\times \prod_{\substack{\alpha\epsilon(\mathcal{L}^{*})\\ \beta\epsilon(\mathcal{L})}} \left\{ \left\{ \frac{[R_{\beta\alpha}(y_{\beta-1})-y_{\alpha+1}](y_{\alpha+1}-y_{\alpha+\alpha})}{[R_{\beta\alpha}(y_{\beta-1}) - y_{\alpha+2}]} \right\}^{-\alpha(k_{\alpha+1})} \right.$$

$$\left. \times \left\{ \frac{(y_{\beta-2}-y_{\beta-1})(y_{\beta-1}-x^{(1)}_{\alpha\beta})}{(y_{\beta-2}-x^{(1)}_{\alpha\beta})} \right\}^{-\alpha(k_{\beta-1})} \right\}.$$

$$\times \left\{ \frac{[\hat{R}(y_{\alpha-1})-y_0](y_0-y_1)}{[\hat{R}(y_{\alpha-1})-y_1]} \right\}^{-\alpha(k_0)} \left\{ \frac{(y_{\alpha-2}-y_{\alpha-1})(y_{\alpha-1}-x^{(1)})}{(y_{\alpha-2}-x^{(1)})} \right\}^{-\alpha(k_{\alpha-1})}$$

$$\times \exp\left\{ \frac{1}{2} \sum_{n=0}^{\infty} \sum_{\substack{t,t'\epsilon(T)\\ (n=0,t\neq t')}} \sum_{\substack{\text{all }\mathcal{L}\\ \epsilon(\mathcal{L})}} (a^{t}|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}\ \hat{P}^{-1}_{t'}(\ )|a^{t'}) \right.$$

$$+ \sum_{n=0}^{\infty} \sum_{t\epsilon(T)} \sum_{\substack{\text{all}\mathcal{L}\\ \epsilon(\mathcal{L})}} \sum_{\substack{i\neq(\mathcal{L},\mathcal{L}^{*})\\ (n=0,i\neq t)}}^{T} (a^{t}|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'}(i)|k_i)$$

$$\left. + \sum_{t\epsilon(T)} \sum_{\substack{\mathcal{L},\cdots,\mathcal{L}',\mathcal{L}''\\ \epsilon(\mathcal{L})\\ (\mathcal{L}'\neq\mathcal{L}'')}} \sum_{n=0}^{\infty} (a^{t}|P_t[R^{\pm}]^{(n)}_{\mathcal{L},\mathcal{L}'} \begin{bmatrix} y_{\mathcal{L}''*} \\ y_{\mathcal{L}''} \end{bmatrix} \left| \begin{matrix} k_{\mathcal{L}''*} \\ -k_{\mathcal{L}''*} \end{matrix} \right) \right\}, \tag{6.7}$$

where $\{F'\}_{(5.35)}$ is the integrand of Eq. (5.35), but with no external scalar legs. Again, the forms of the expression of the whole integrand in Eq. (6.7) is independent of dot positions. When two dots of adjacent legs face each other, an asymmetrical propagator similar to Eq. (4.18) is required.

Equation (6.7) applies equally well to planar, overlapping, and nonorientable multiloops, provided we modify the expression of $\{F'\}_{(5.35)}$ according to previous Sec. V (A-E). [The factors $\prod_{\substack{\alpha\in(\mathcal{L}^*)\\ \beta\in(\mathcal{L})}}\{\cdots\}$ in Eq. (5.7) should be omitted, however.] Again, one can give a set of rules for writing down any mixed multiloop amplitudes with any number of excited external legs. Since this is fairly obvious, we will not elaborate here.

VII. CONCLUSIONS

In this thesis we have presented a series of beautiful, elegant, and "simple" formulas of general tree amplitudes and general multiloop amplitudes in the dual resonance model. Various rules have been given for writing down directly the formulas corresponding to the Feynman-like tree or loop diagrams. Interpretations of the formulas also emerge in very simple ways. And the essential point of this approach is that the formulas are mathematically unique and exact.

The remaining problems are:

(a) elimination of the divergent determinant factors,

(b) explicit summation of subsets of multiloop diagrams to see if it is indeed unitarized,

(c) incorporation of internal symmetries beyong the Born terms (after ghosts are eliminated).

None of these problems have been solved, however.

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APPENDIX A. USEFUL IDENTITIES

We denote

$$(xM_{\pm}yM_{\pm}z)_{nm} = \sum_{\ell=1}^{\infty} x^{n}(M_{\pm})_{n\ell}\, z^{\ell}(M_{\pm})_{\ell m}\, z^{m}, \tag{A.1}$$

then we have the following set of identities

$$M_{-}M_{-}^{T} = M_{+}\,, \tag{A.2}$$

$$M_{+}M_{-}^{T} = M_{-}, \tag{A.3}$$

$$M_{-}M_{-} = M_{0}, \tag{A.4}$$

$$M_{+}xM_{-} = \frac{1}{1-x}\, M_{+}\, \frac{1}{1-1/x}, \tag{A.5}$$

$$M_{-}xM_{-} = (1-x)\, M_{-}\, \frac{1}{1-1/x}, \tag{A.6}$$

$$M_{+}(1-x)M_{-}^{T} = M_{-}\left(1-\frac{1}{x}\right) M_{-}^{T}x. \tag{A.7}$$

The identities involving the twist operator are

$$(a|M_{-} = (\bar{a}|, \tag{A.8}$$

$$M_{-}^{T}|a) = |\bar{a}), \tag{A.9}$$

$$(k|a) = -(k|\bar{a}), \tag{A.10}$$

$$(k|x|a) = -(k|\bar{a}) + (k|(1-x)|\bar{a}) \tag{A.11}$$

$$\bar{a} = (T_{a}{}^{\dagger})^{-1}\, a\, T_{a}{}^{\dagger}, \tag{A.12}$$

$$T_{a} = \,:\exp(a^{\dagger}|M_{-}|a):, \tag{A.13}$$

$$\Omega_a(k) = \exp(a^\dagger|k) : \exp(a^\dagger|M_- - I|a) :, \tag{A.14}$$

$$\Omega_a{}^\dagger(-k) = : \exp(a^\dagger|M_-{}^T - I)|a) : \exp(a|k). \tag{A.15}$$

APPENDIX B. RULES FOR WRITING DOWN MULTIPLY FACTORIZED TREE FORMULA[8]

Rule 1

We assign to each leg (scalar or excited leg) one Koba-Nielsen variable y_i, $i = 1,2,\cdots,S$, and an incoming four-momentum k_i. Corresponding to each excited leg, we assign one destruction (creation) operator $a_{\mu,n}$ $(a^+_{\mu,n})$, where μ is space-time indices, $\mu = 1,2,3,4$; n is the excited harmonic oscillator mode in question, $n = 1,2,\cdots,\infty$, and the superscript α coincides with the labeling of the Koba-Nielsen variables, $\alpha \leq S$.

Rule 2

The scalar part of the multiply factorized tree is the ordinary Koba-Nielsen representation[13] of the S-point dual amplitude, i.e., Eqs. (4.1) and (4.2). We denote it by $\int \pi dy_i \{Y_S\}$.

Rule 3

The operator part of the multiply factorized tree consists of the scalar products $a^\alpha \cdot k_i$ and $a^\alpha \cdot a^\beta$. It is

$$\langle 0| \exp\left\{ \sum_{\substack{\alpha \\ \alpha \leq S}} \sum_{\substack{i=1 \\ (i \neq \alpha)}}^{S} \left(a^\alpha \middle| P_\alpha \begin{bmatrix} i \\ \alpha \pm 1 \end{bmatrix} \middle\| \begin{matrix} k_i \\ k_\alpha \end{matrix} \right) + \frac{1}{2} \sum_{\substack{\alpha,\beta < S \\ (\alpha \neq \beta)}} (a^\alpha | P_\alpha \hat{P}_\beta{}^{-1}(\ ) | a^\beta) \right\} |\lambda' s\rangle \quad , \tag{B.1}$$

where the cross ratio $P_\alpha(i)[P_\beta(i)]$ is defined on the $\alpha(\beta)$ excited leg, such that the first variable in the cross ratio is y_α (y_β), the second variable is the variable behind the dot attached to the α (β) leg, the third variable is the variable in front of the dot attached to the α (β) leg, and the fourth variable is y_i. The notation $P_\alpha[\alpha \pm 1]$ here means that if the second variable in the cross ratio is $y_{\alpha+1}$ ($y_{\alpha-1}$), then the fourth variable is also $y_{\alpha+1}$ ($y_{\alpha-1}$). i.e., $P_\alpha[\alpha \pm 1] \equiv 0$ for all cases. We include it to ensure conservation of momentum in |). The notation $P_\alpha \hat{P}_\beta^{-1}$ has been explained in Eqs. (3.10-14). If $P_\beta(x) \equiv P(\beta,\beta-1,\beta+1,x)$, then

$$\hat{P}_\beta^{-1}(x) = y_{\beta-1} - \frac{y_{\beta+1} - y_\beta}{1 - x\left(\frac{y_\beta - y_{\beta+1}}{y_{\beta-1} - y_{\beta+1}}\right)} , \qquad (B.2)$$

and if $P_\beta \equiv P(\beta,\beta+1,\beta-1,x)$ then

$$\hat{P}_\beta^{-1}(x) = y_{\beta+1} - \frac{y_{\beta+1} - y_\beta}{1 - x\left(\frac{y_\beta - y_{\beta-1}}{y_{\beta+1} - y_{\beta-1}}\right)} . \qquad (B.3)$$

Rule 4

When two excited legs are adjacent and the two dots attached to them face each other, e.g., Fig. 5, then we need an asymmmetrical propagator, e.g., Eq. (4.18) with $w_{S-1} = P_{S+1}(S - 1)$. In all other cases no modification of the ordinary propagation is required.

Rule 5

The two amplitudes with the dot on the opposite sides of one excited leg, say the y_α leg, are related to each other by the twist operator $\Omega^+(-k_\alpha, a^\alpha)$, or equivalently, by the interchange of $y_{\alpha+1}$ with $y_{\alpha-1}$ everywhere in Eq. (B.1). The only additional complication to this rule is the case stated in Rule 4, where we need an asymmetrical propagator.

APPENDIX C. INFINITE NUMBER OF CANCELLATIONS ON y

We show how the infinite number of cancellations involving $y_{\mathcal{L}}$ beautifully occur, and enable us to pass from Eq. (5.20) to Eq. (5.22). We first consider the factors raised to the power $-k_i \cdot k_\gamma$ in Eq. (5.20). They are

$$I_n^{(e\ell)} \equiv \left\{\frac{y_i - [R^\pm]^{(n+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)}{y_i - [R^\pm]^{(n+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)}\right\}_{(\sigma\lambda)\neq(\gamma\delta)} \left\{\frac{y_i - [R^\pm]^{(n)}_{\beta\alpha,\lambda\sigma}(y_\delta)}{y_i - [R^\pm]^{(n)}_{\beta\alpha,\lambda\sigma} R^{-1}_{\delta\gamma}(y_\delta)}\right\}_{[(\gamma\delta)\neq(\sigma\lambda),+]}$$

$$\times \left\{\frac{y_i - [R^\pm]^{(n)}_{\beta\alpha,\lambda\sigma}\, R_{\delta\gamma}(y_\gamma)}{y_i - [R^\pm]^{(n)}_{\beta\alpha,\lambda\sigma}(y_\gamma)}\right\}_{[(\gamma\delta)\neq(\sigma\lambda),-]} . \qquad \text{(C.1)}$$

Using the restriction that $\mathcal{L} \neq \mathcal{L}'$ in the product $R_{\mathcal{L}} R^{-1}_{\mathcal{L}'}$ or $R^{-1}_{\mathcal{L}} R_{\mathcal{L}'}$, we can write Eq. (C.1) as follows

$$I_n^{(e\ell)} = \left\{\frac{y_i - [R^\pm]^{(n+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)}{y_i - [R^\pm]^{(n+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)} \cdot \frac{y_i - [R^\pm]^{(n)}_{\beta\alpha,\lambda\sigma}(y_\delta)}{y_i - [R^\pm]^{(n)}_{\beta\alpha,\lambda\sigma}(y_\gamma)} \cdot \frac{y_i - [R^\pm]^{(n)}\, R_{\lambda\sigma}(y_\lambda)}{y_i - [R^\pm]^{(n-1)} R_{\lambda\sigma}(y_\lambda)}\right.$$

$$\left. \cdot \; \frac{y_i - [R^\pm]^{(n-1)} R^{-1}_{\lambda\sigma}(y_\sigma)}{y_i - [R^\pm]^{(n)}\, R^{-1}_{\lambda\sigma}(y_\sigma)}\right\} \quad (n \geq 1). \qquad \text{(C.2)}$$

For $n = 0$, we have, from Eq. (5.19),

$$I_0^{(e\ell)} \equiv \left\{\frac{y_i - R^\pm_{\beta\alpha}(y_\gamma)}{y_i - R^\pm_{\beta\alpha}(y_\delta)}\right\}_{(\gamma\delta)\neq(\alpha\beta)} \left\{\frac{y_i - R_{\delta\gamma}(y_\gamma)}{y_i - R^{-1}_{\delta\gamma}(y_\delta)} \cdot \frac{y_i - y_\delta}{y_i - y_\gamma}\right\}$$

$$= \left\{\frac{y_i - R^\pm_{\beta\alpha}(y_\gamma)}{y_i - R^\pm_{\beta\alpha}(y_\delta)} \cdot \frac{y_i - R_{\delta\gamma}(y_\delta)}{y_i - R^{-1}_{\delta\gamma}(y_\gamma)} \cdot \frac{y_i - y_\delta}{y_i - y_\gamma}\right\} . \qquad \text{(C.3)}$$

Take the infinite product over n, we get, from Eqs. (C.2) and (C.3),

$$I^{(e\ell)} \equiv \prod_{n=0}^{\infty} I_n^{(e\ell)} = \left\{\frac{y_i - y_\delta}{y_i - y_\gamma}\right\}$$

$$\times \lim_{N\to\infty}\left\{\frac{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)}{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)} \cdot \frac{y_i-[R^\pm]^{(N)}R_{\lambda\sigma}(y_\lambda)}{y_i-[R^\pm]^{(N)}R^{-1}_{\lambda\sigma}(y_\sigma)}\right\}$$

$$\equiv \left\{\frac{y_i-y_\delta}{y_i-y_\gamma}\right\} \lim_{N\to\infty} \frac{\left\{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)\right\}_{[(\sigma\lambda)\neq(\gamma\delta),-]}}{\left\{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)\right\}_{[(\sigma\lambda)\neq(\gamma\delta),+]}} \quad . \qquad (C.4)$$

Again, using the restriction that $\mathcal{L} \neq \mathcal{L}'$ in the product $R_{\mathcal{L}} R^{-1}_{\mathcal{L}'}$, we can write the two limiting factors in Eq. (C.4) as

$$\left\{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)\right\}_{[\sigma\lambda\neq\gamma\delta,-]} = \left\{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\gamma)\right\}_{(\sigma\lambda)\neq(\gamma\delta)}$$

$$\times \left\{y_i-[R^\pm]^{(N)}_{\beta\alpha,\lambda\sigma}\, R_{\delta\gamma}(y_\gamma)\right\}_{(\sigma\lambda)\neq(\gamma\delta)} \cdots \left\{y_i-R^{(N+1)}_{\delta\gamma}(y_\gamma)\right\} , \qquad (C.5)$$

and

$$\left\{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)\right\}_{[(\sigma\lambda)\neq(\gamma\delta),+]} = \left\{y_i-[R^\pm]^{(N+1)}_{\beta\alpha,\lambda\sigma}(y_\delta)\right\}_{(\sigma\lambda)\neq(\gamma\delta)}$$

$$\times \left\{y_i-[R^\pm]^{(N)}_{\beta\alpha,\lambda\sigma}R^{-1}_{\delta\gamma}(y_\delta)\right\}_{(\sigma\lambda)\neq(\gamma\delta)} \cdots \left\{y_i-R^{-(N+1)}_{\delta\gamma}(y_\delta)\right\} . \qquad (C.6)$$

But, it can be proved[22] that, if $N \to \infty$, the expression

$$[R^\pm]^{(N)}_{\beta\alpha,\lambda\sigma}(z)$$

is independent of z. Hence, Eqs. (C.5) and (C.6) are equal to

$$\prod_{n=0}^{\infty}\left\{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(2)}_{\gamma\delta})\right\}_{(\sigma\lambda)\neq(\gamma\delta)} , \tag{C.7}$$

and

$$\prod_{n=0}^{\infty}\left\{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)\right\}_{(\sigma\lambda)\neq(\gamma\delta)} \tag{C.8}$$

respectively, where

$$R^{-\infty}_{\delta\gamma}(z_1) = x^{(1)}_{\gamma\delta}, \qquad z_1 \neq x^{(2)}_{\gamma\delta},$$

$$R^{+\infty}_{\delta\gamma}(z_2) = x^{(2)}_{\gamma\delta}, \qquad z_2 \neq x^{(1)}_{\gamma\delta},$$

are the two invariant points of $R_{\delta\gamma}$. Substituting (C.7) and (C.8) in Eq. (C.4), we get

$$I^{(e\ell)} = \left\{\frac{y_i-y_\delta}{y_i-y_\gamma}\right\}\prod_{n=0}^{\infty}\left\{\frac{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(2)}_{\gamma\delta})}{y_i-[R^{\pm}]^{(n)}_{\beta\alpha,\lambda\sigma}(x^{(1)}_{\gamma\delta})}\right\}_{(\sigma\lambda)\neq(\gamma\delta)} . \tag{C.9}$$

Similarly, those factors in Eq. (5.20) which are raised to the power $-k_\alpha \cdot k_\gamma$, can be shown to be equal to

$$I^{(\ell\ell)} \equiv \prod_{n=0}^{\infty} I_n^{(\ell\ell)} = \prod_{n=0}^{\infty}$$

$$\times \left\{\frac{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)}{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(1)}_{\gamma\delta}\right)} \cdot \frac{x^{(2)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}{x^{(1)}_{\alpha\beta}-[R^{\pm}]^{(n)}_{\beta'\alpha',\lambda\sigma}\left(x^{(2)}_{\gamma\delta}\right)}\right\}_{\substack{(\alpha\beta)\neq(\alpha'\beta')\\(\sigma\lambda)\neq(\gamma\delta)}}$$

Equation (C.10) continued

Equation (C.10) continued

$$\times \prod_{(\alpha\beta),(\gamma\delta)\in(\mathcal{L}\,\mathcal{L}^*)} \left[\frac{(y_\alpha-y_\delta)(y_\beta-y_\gamma)}{(y_\beta-y_\delta)(y_\alpha-y_\gamma)}\right]$$

$$\times \prod_{(\alpha\beta)\in(\mathcal{L}\,\mathcal{L}^*)} \left[\frac{R^{-1}_{\beta\alpha}(y_\alpha)-R^{-1}_{\beta\alpha}(y_\beta)}{y_\alpha-R_{\beta\alpha}(y_\beta)} \cdot \frac{R_{\beta\alpha}(y_\alpha)-R_{\beta\alpha}(y_\beta)}{y_\beta-R^{-1}_{\beta\alpha}(y_\alpha)}\right] \quad . \qquad (C.10)$$

Hence Eq. (5.22) follows.

APPENDIX D. THE JACOBIAN CALCULATIONS

The transformation from the set $\{t_{\alpha\beta}, y_\alpha, y_\beta\}$ to the new set $\{X_{\alpha\beta}, x^{(1)}_{\alpha\beta}, x^{(2)}_{\alpha\beta}\}$ is one of the most complicated calculations. We show here how Eq. (5.25) is transformed into Eq. (5.26).

We need a set of identities connecting the old set of variables to the new set of variables. We express $R^{-1}_{\beta\alpha}$, Eq. (5.14), as [using Eqs. (5.13), (5.2) and (B.2)]

$$R_{\beta\alpha}(z) = \hat{P}_\beta^{-1} Q^{-1}_{\beta\alpha} P_\alpha(z) = \frac{z(y_\beta - ay_{\beta+1}) - (y_\alpha y_\beta - y_{\alpha+1}y_{\beta+1}a)}{z(1-a) + (ay_{\alpha+1} - y_\alpha)}, \quad \text{(D.1)}$$

where

$$a = \frac{t_{\alpha\beta}}{t_{\alpha\beta} - 1} d, \qquad t = \frac{a}{a-d}, \quad \text{(D.2a)}$$

$$d = \frac{(y_\alpha - y_{\alpha-1})(y_\beta - y_{\beta-1})}{(y_{\alpha+1} - y_{\alpha-1})(y_{\beta+1} - y_{\beta-1})}. \quad \text{(D.2b)}$$

Compare Eq. (D.1) with the standard form Eq. (5.21), we obtain the set of identities

$$(1 - a) = \ell(1 - X^{-1}_{\alpha\beta}), \quad \text{(D.2c)}$$

$$y_\beta - ay_{\beta+1} = \ell\left(x^{(2)}_{\alpha\beta} - X^{-1}_{\alpha\beta}\, x^{(1)}_{\alpha\beta}\right), \quad \text{(D.2d)}$$

$$ay_{\alpha+1} - y_\alpha = \ell\left(x^{(2)}_{\alpha\beta}\, X^{-1}_{\alpha\beta} - x^{(1)}_{\alpha\beta}\right), \quad \text{(D.2e)}$$

$$y_\alpha y_\beta - y_{\alpha+1}y_{\beta+1}a = \ell\, x^{(1)}_{\alpha\beta}\, x^{(2)}_{\alpha\beta}(1 - X^{-1}_{\alpha\beta}), \quad \text{(D.2f)}$$

$$\ell = \left[\frac{\ell(y_\alpha - y_{\alpha+1})(y_{\beta+1} - y_\beta)}{X_{\alpha\beta}^{-1}\left(x_{\alpha\beta}^{(1)} - x_{\alpha\beta}^{(2)}\right)^2}\right]^{\frac{1}{2}} . \qquad \text{(D.2g)}$$

From Eqs. (D.2d) and (D.2e), we can derive the identity

$$a = \frac{y_\beta + y_\alpha\left(\dfrac{x_{\alpha\beta}^{(2)} - X_{\alpha\beta}^{-1}\, x_{\alpha\beta}^{(1)}}{x_{\alpha\beta}^{(2)}\, X_{\alpha\beta}^{-1} - x_{\alpha\beta}^{(1)}}\right)}{y_{\beta+1} + y_{\alpha+1}\left(\dfrac{x_{\alpha\beta}^{(2)} - X_{\alpha\beta}^{-1}\, x_{\alpha\beta}^{(1)}}{x_{\alpha\beta}^{(2)}\, X_{\alpha\beta}^{-1} - x_{\alpha\beta}^{(1)}}\right)} . \qquad \text{(D.3a)}$$

With the following further identities

$$y_\alpha = R_{\beta\alpha}(y_{\beta+1}) = \frac{y_{\beta+1}\left(x_{\alpha\beta}^{(2)} - x_{\alpha\beta}^{(1)}\, X_{\alpha\beta}\right) - x_{\alpha\beta}^{(1)}\, x_{\alpha\beta}^{(2)}\left(1 - X_{\alpha\beta}\right)}{y_{\beta+1}(1 - X_{\alpha\beta}) + x_{\alpha\beta}^{(2)}\, X_{\alpha\beta} - x_{\alpha\beta}^{(1)}} , \qquad \text{(D.3b)}$$

$$y_\beta = R_{\beta\alpha}^{-1}(y_{\alpha+1}) = \frac{y_{\alpha+1}\left(x_{\alpha\beta}^{(2)} - x_{\alpha\beta}^{(1)} X_{\alpha\beta}^{-1}\right) - x_{\alpha\beta}^{(1)}\, x_{\alpha\beta}^{(2)}(1 - X_{\alpha\beta}^{-1})}{y_{\alpha+1}(1 - X_{\alpha\beta}^{-1}) + x_{\alpha\beta}^{(2)}\, X_{\alpha\beta}^{-1} - x_{\alpha\beta}^{(1)}} , \qquad \text{(D.3c)}$$

$$R_{\beta\alpha}^{\pm}(z) = \frac{z\left(x_{\alpha\beta}^{(2)} - X_{\alpha\beta}^{\pm}\, x_{\alpha\beta}^{(1)}\right) - x_{\alpha\beta}^{(1)}\, x_{\alpha\beta}^{(2)}(1 - X_{\alpha\beta}^{\pm})}{z(1 - X_{\alpha\beta}^{\pm}) + x_{\alpha\beta}^{(2)}\, X_{\alpha\beta}^{\pm} - x_{\alpha\beta}^{(1)}} , \qquad \text{(D.3d)}$$

$$\frac{R_{\beta\alpha}^{\pm}(z) - x_{\alpha\beta}^{(2)}}{R_{\beta\alpha}^{\pm}(z) - x_{\alpha\beta}^{(1)}} = X_{\alpha\beta}^{\pm}\, \frac{z - x_{\alpha\beta}^{(2)}}{z - x_{\alpha\beta}^{(1)}} , \qquad \text{(D.3e)}$$

$$\frac{R^{-1}_{\beta\alpha}(y_\alpha) - x^{(1)}_{\alpha\beta}}{\left(y_\alpha - x^{(1)}_{\alpha\beta}\right)[R^{-1}_{\beta\alpha}(y_\alpha) - y_\beta]} = \frac{R_{\beta\alpha}(y_\beta) - x^{(2)}_{\alpha\beta}}{[y_\alpha - R_{\beta\alpha}(y_\beta)]\left(y_\beta - x^{(2)}_{\alpha\beta}\right)} , \qquad \text{(D.3f)}$$

one can show that the expression (5.25) is equal to (only consider one loop):

$$dX_{\alpha\beta}\, dx^{(1)}_{\alpha\beta}\, dx^{(2)}_{\alpha\beta} \left| \frac{\partial(t_{\alpha\beta}, y_\alpha, y_\beta)}{\partial(X_{\alpha\beta}, x^{(1)}_{\alpha\beta}, x^{(2)}_{\alpha\beta})} \right| a^{-\alpha_0 - \frac{1}{2}k_\alpha^{\ 2}} \frac{(a - d)^2}{ad}$$

$$\times \; [(y_\alpha - y_{\alpha-1})(y_\beta - y_{\beta-1})]^{\alpha_0 - 1} \left[\frac{(y_a - y_b)(y_b - y_c)(y_c - y_a)}{(y_{\alpha+1} - y_\alpha)(y_{\beta+1} - y_\beta)} \right]$$

$$\times \left\{ \frac{R^{-1}_{\beta\alpha}(y_\alpha) - R^{-1}_{\beta\alpha}(y_\beta)}{y_\alpha - R_{\beta\alpha}(y_\beta)} \cdot \frac{R_{\beta\alpha}(y_\alpha) - R_{\beta\alpha}(y_\beta)}{y_\beta - R^{-1}_{\beta\alpha}(y_\alpha)} \right\}^{-\frac{1}{2}k_\alpha^{\ 2}} . \qquad \text{(D.4)}$$

Now we specialie to the frame $x^{(1)}_{\alpha\beta} = \infty$, $x^{(2)}_{\alpha\beta} = 0$. Then $R^{\pm}_{\beta\alpha} \to X^{\pm}_{\alpha\beta}$ and $y_a = x^{(1)}_{\alpha\beta}$, $y_b = 1$, $y_c = x^{(2)}_{\alpha\beta}$, so that

$$a \to 1 ,$$

$$\ell \to \frac{y_{\beta+1} - X^{-1}_{\alpha\beta}\, y_{\alpha+1}}{x^{(1)}_{\alpha\beta}\, X^{-1}_{\alpha\beta}} ,$$

$$y_\alpha \to X_{\alpha\beta}\, y_{\beta+1} ,$$

$$y_\beta \to X^{-1}_{\alpha\beta}\, y_{\alpha+1} . \qquad \text{(D.5)}$$

Hence the expression (D.4) reduces to

$$dX_{\alpha\beta}[dx^{(1)}_{\alpha\beta}][dx^{(2)}_{\alpha\beta}][J]\Bigg|_{\substack{x^{(1)}_{\alpha\beta}=\infty\\ x^{(2)}_{\alpha\beta}=0}} [(y_{\alpha-1}-X_{\alpha\beta}y_{\beta+1})(y_{\alpha+1}-X_{\alpha\beta}y_{\beta-1})]^{\alpha_0-1}$$

$$\times \; X_{\alpha\beta}^{-\ell(k_\alpha)+1}\,\frac{(a-d)^2}{\ell^2 X^{-1}_{\alpha\beta}d}\,. \qquad (D.6)$$

The calculation of the Jacobian factor [J] is rather complicated. But it gives[9]

$$J \equiv \frac{\partial(t_{\alpha\beta},\, y_\alpha,\, y_\beta)}{\partial(X_{\alpha\beta},\, x^{(1)}_{\alpha\beta},\, x^{(2)}_{\alpha\beta})} = \frac{\ell^2 d}{(a-d)^2}\,\frac{(1-X_{\alpha\beta})^2}{X_{\alpha\beta}^{\;3}}\,. \qquad (D.7)$$

Substituting Eq. (D.7) in (D.6), we obtain

$$dX_{\alpha\beta}[dx^{(1)}_{\alpha\beta}][dx^{(2)}_{\alpha\beta}](1-X_{\alpha\beta})^2\, X_{\alpha\beta}^{-\ell(k_\alpha)-1}$$

$$\times \; [(y_{\alpha-1}-X_{\alpha\beta}\,y_{\beta+1})(y_{\alpha+1}-X_{\alpha\beta}\,y_{\beta-1})]^{\alpha_0-1}\,. \qquad (D.8)$$

We then uniquely generalize[9] the expression (D.8) to a general frame, it is

$$dX_{\alpha\beta}\, dx^{(1)}_{\alpha\beta}\, dx^{(2)}_{\alpha\beta}\, X_{\alpha\beta}^{-\ell(k_\alpha)-1}(1-X_{\alpha\beta})^2\,\frac{(y_a-y_b)(y_b-y_c)(y_c-y_a)}{\left(x^{(1)}_{\alpha\beta}-x^{(2)}_{\alpha\beta}\right)^2}$$

$$\times \left\{\frac{[y_{\alpha-1}-R_{\beta\alpha}(y_{\beta+1})][x^{(1)}_{\alpha\beta}-y_{\beta+1}]}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta+1})]}\right\}^{\alpha_0-1}\left\{\frac{[y_{\alpha+1}-R_{\beta\alpha}(y_{\beta-1})][x^{(1)}_{\alpha\beta}-y_{\beta-1}]}{[x^{(1)}_{\alpha\beta}-R_{\beta\alpha}(y_{\beta-1})]}\right\}^{\alpha_0-1}. \qquad (D.9)$$

Hence the expression (5.26) follows.

FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

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14. This is always possible in later calculations.

15. It is because $\alpha(k_S) \neq 0$ for the a^S leg that we get the extra factor $\left(\frac{1 - t}{1 - tw_{S-1}}\right)^{-\alpha(k_S)}$.

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17. This interpretation is identical to KSV interpretation, provided we shrink the loop to a point and expand the external meson quark-antiquark lines, so that the Feynman diagram simultaneously represents the dual diagram.

18. We state this without proofs.

19. One needs a similar calculation of the Jacobian factors to show this, see Ref. 9, UCRL-20054.

20. Here we mean mixing of planar, nonplanar, overlapping, and nonorientable loops in an arbitrary way.

21. We understand that D. Collop obtains similar results independently.

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FIGURE CAPTIONS

Fig. 1. Feynman tree theorem for planar box diagram.

Fig. 2. (a) N-point amplitude with scalar legs.

(b) N-point tree ordering diagram.

Fig. 3. Single factorization.

Fig. 4. Change the Y frame to the Y' frame for second factorization.

Fig. 5. Asymptotic propagator case (for a^{S+1} leg).

Fig. 6. Fourth-factorized tree diagram.

Fig. 7. Four-reggeon tree amplitude (symmetrical).

Fig. 8. N-reggeon tree amplitude (symmetrical).

Fig. 9. $\underline{2}$Nth-factorized tree diagram (for nonplanar loops).

Fig. 10. (a) Nonplanar N-loop amplitude with external scalar legs.

(b) Partial ordering diagram, which is mimic to $\underline{2}$Nth-factorized tree ordering.

(c) Nonplanar N-loop ordering diagram.

Fig. 11. $\underline{2}$Nth-factorized tree diagram (for overlapping loops).

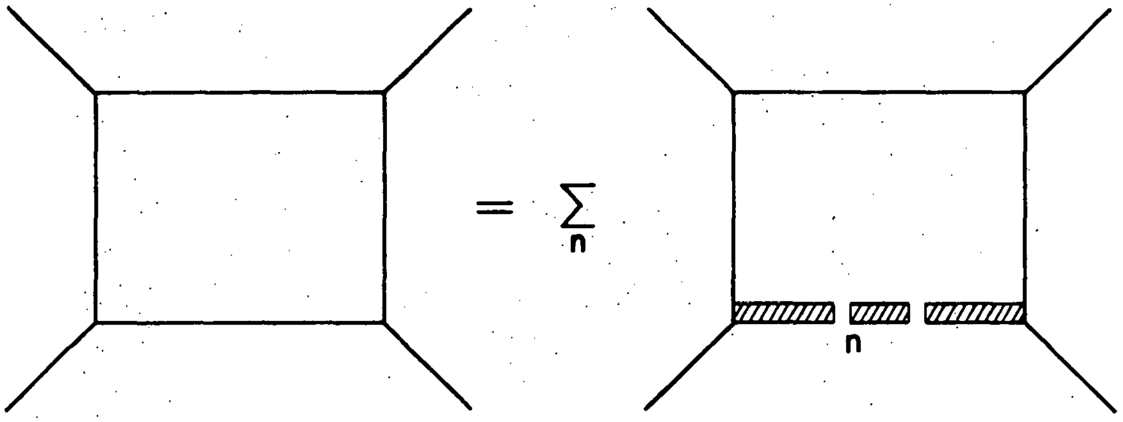
Fig. 12. Overlapping double-loop ordering diagram.

Fig. 13. Nonorientable double-loop ordering diagram.

Fig. 14. Planar N-loop ordering diagram.

Fig. 15. Mixed multiloop Feynman diagram, also simultaneously represents mixed multiloop ordering diagram.

Fig. 16. Nonplanar M-loop diagram with all external legs excited.



(Fig. 1)

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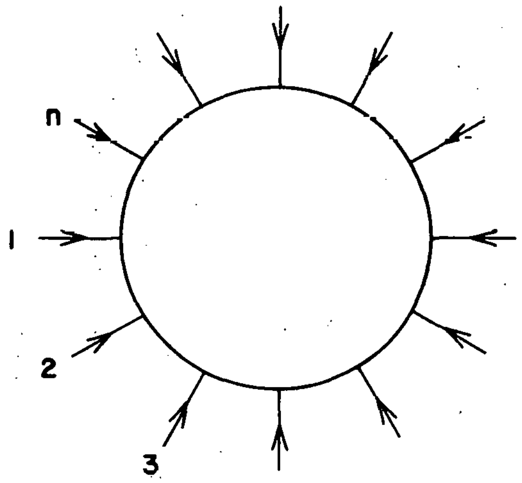

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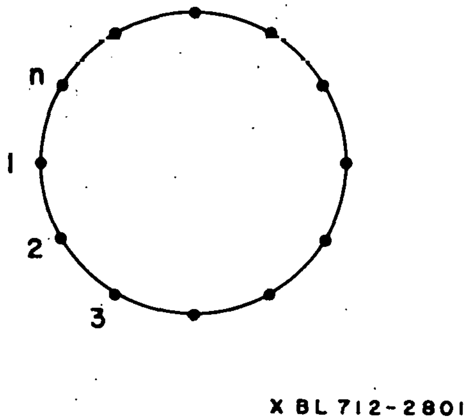

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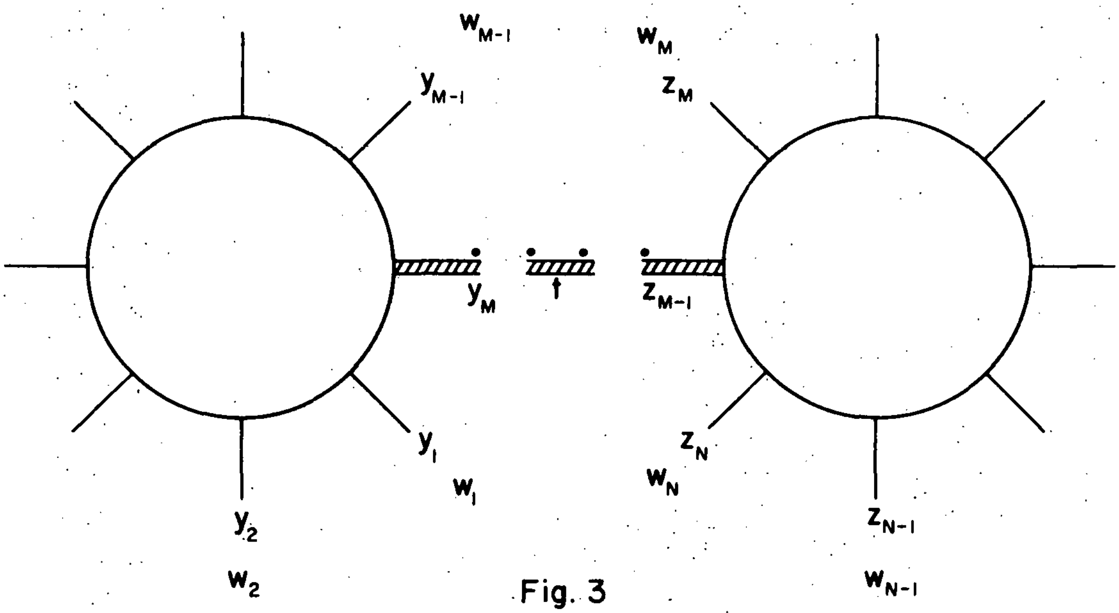

Fig. 3

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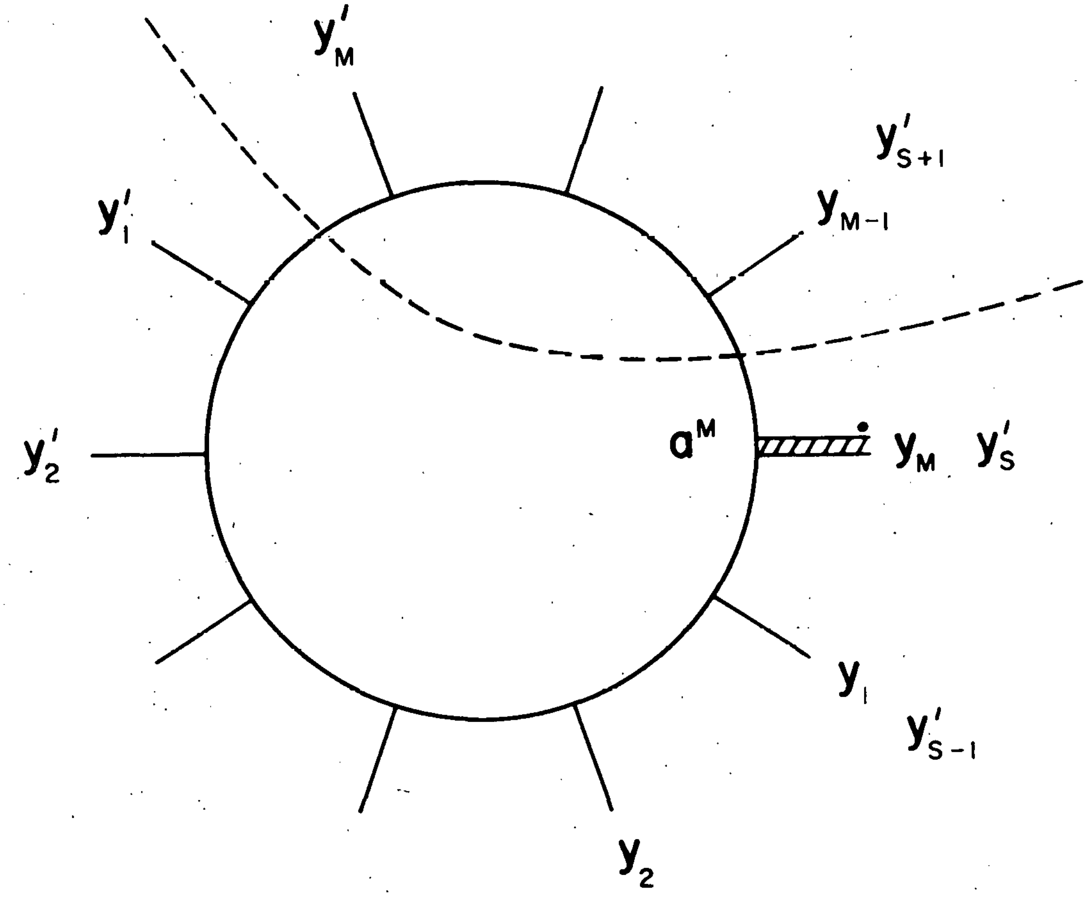

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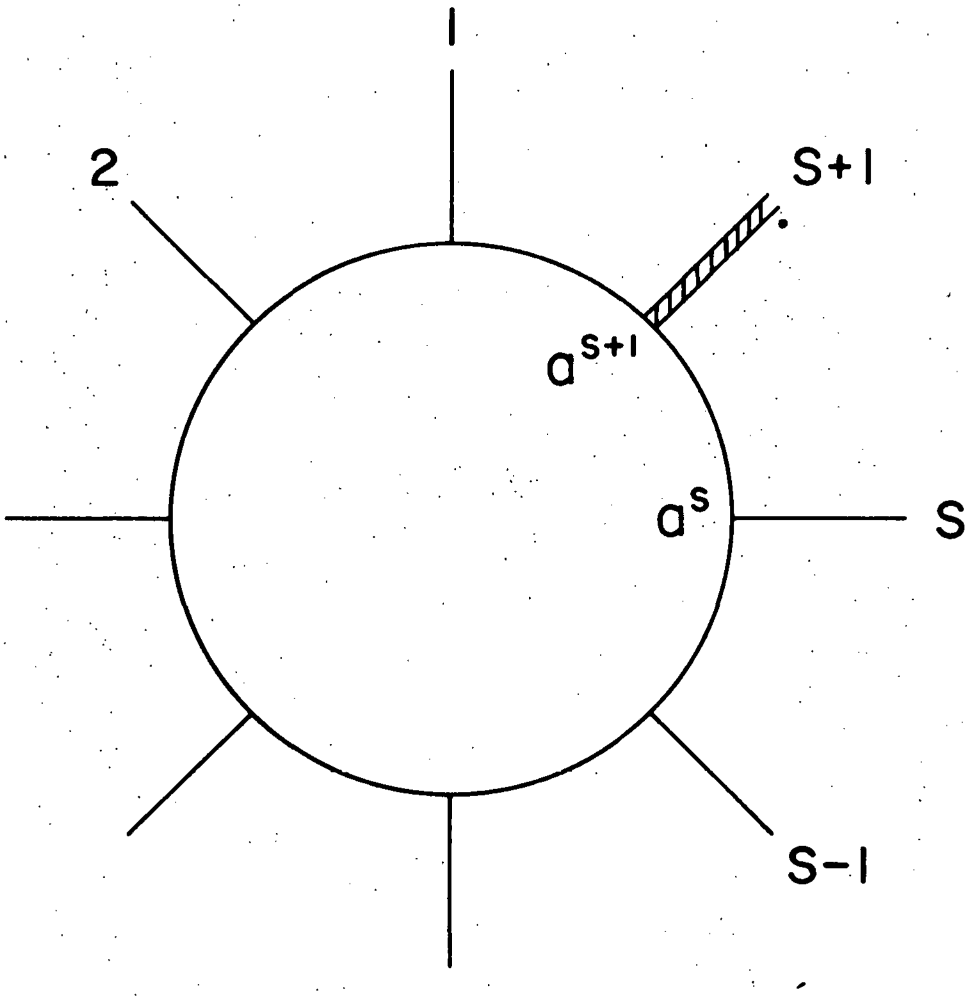

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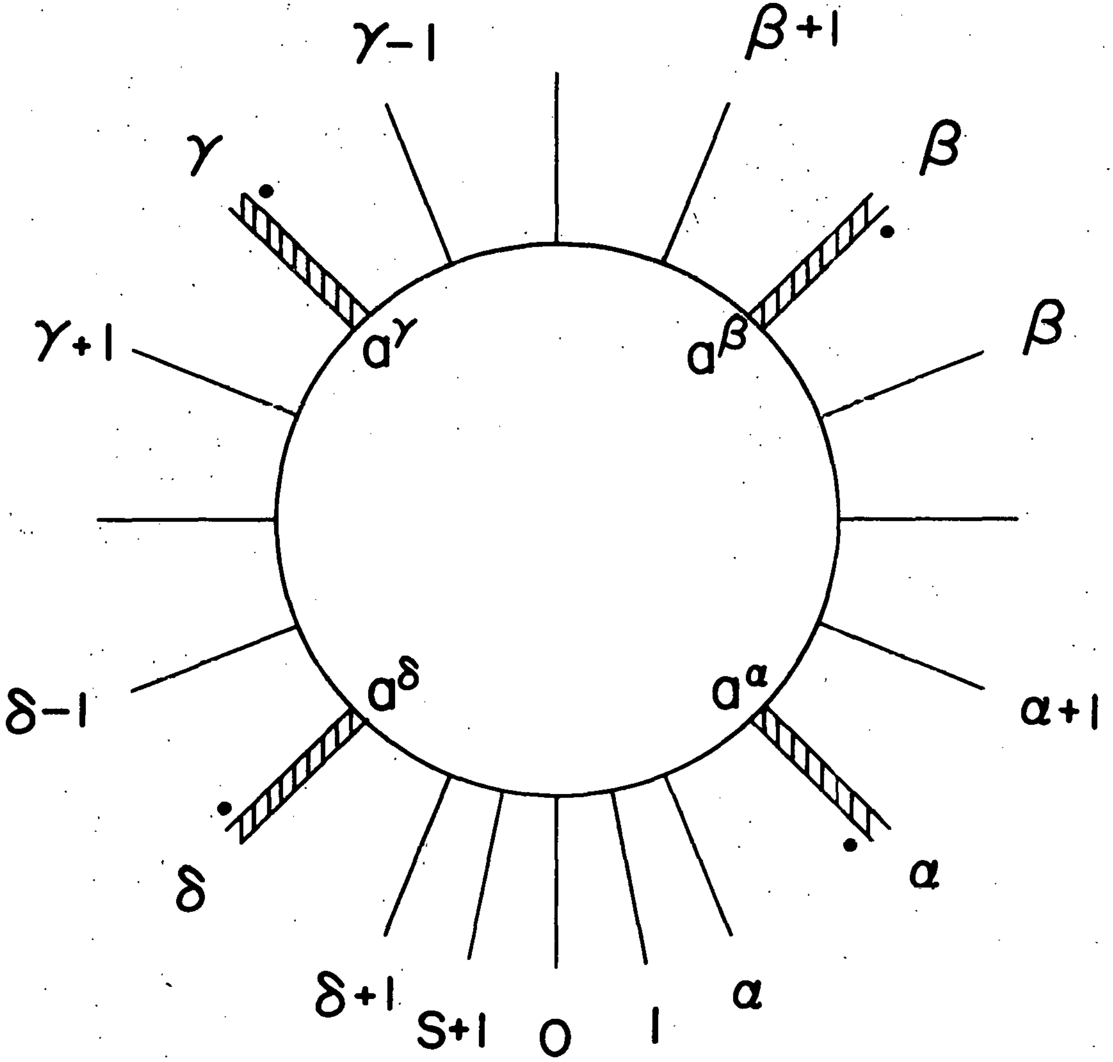

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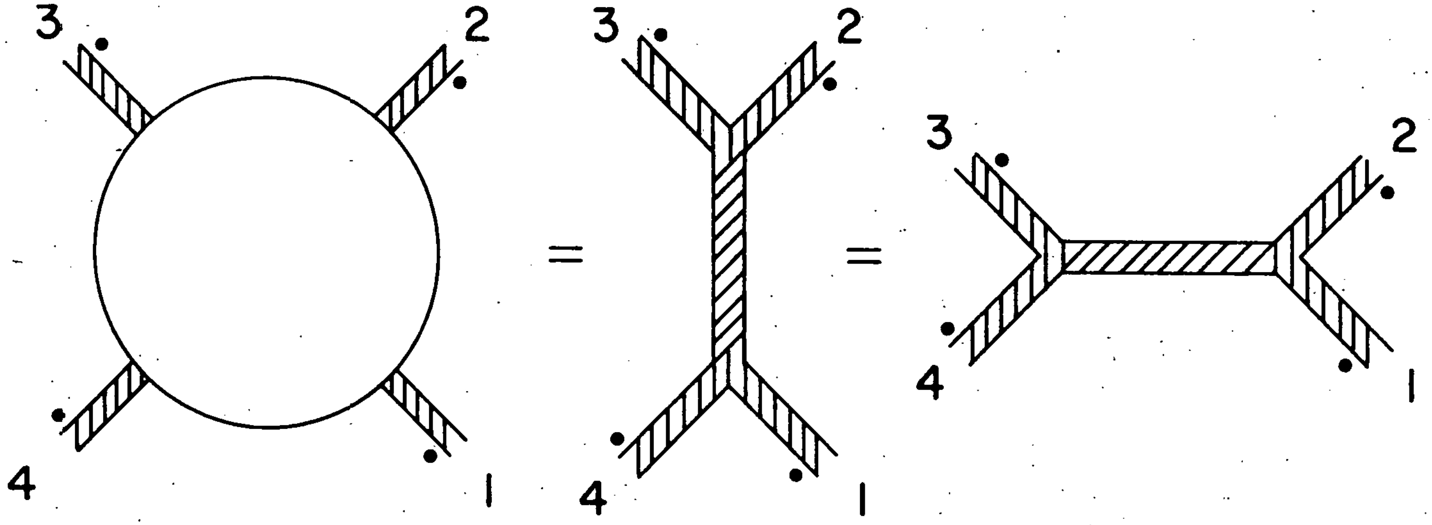

(Fig. 7)

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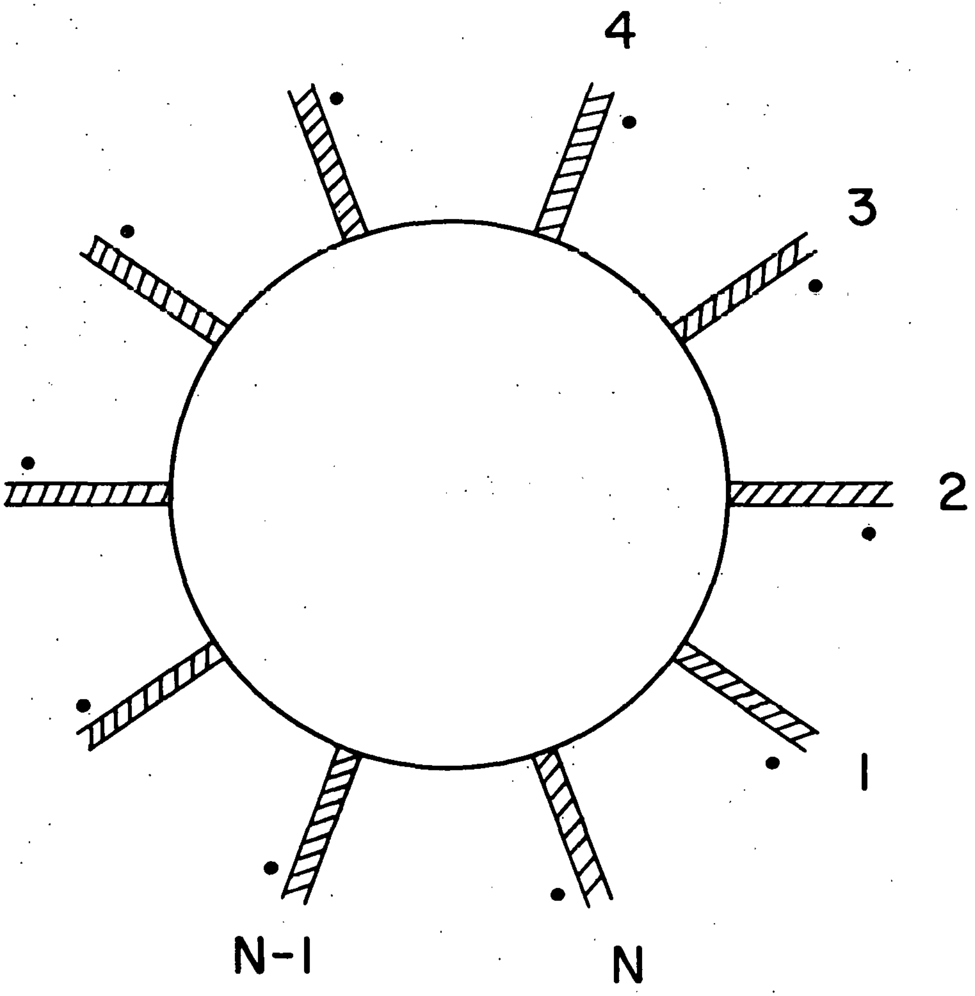

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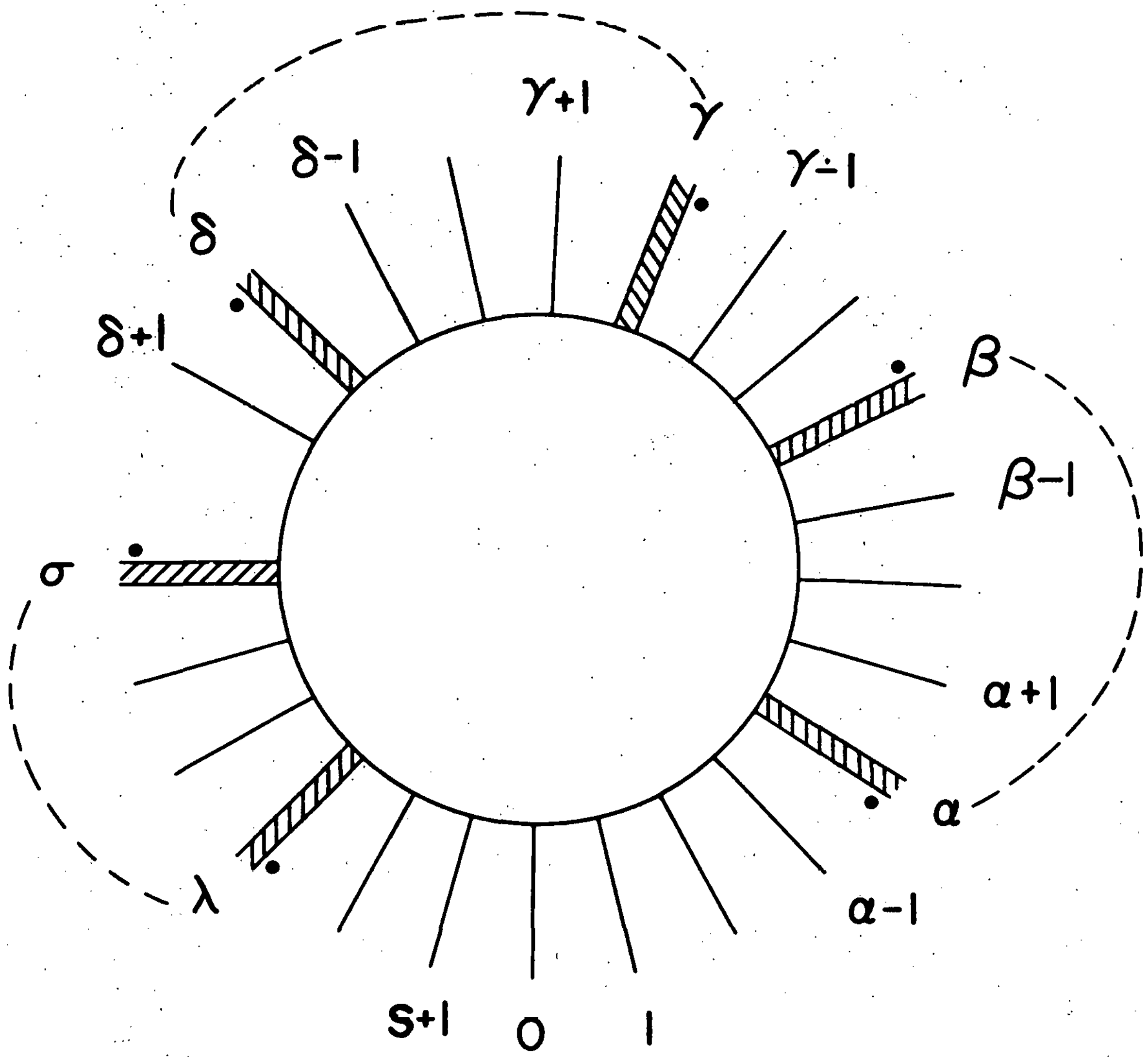

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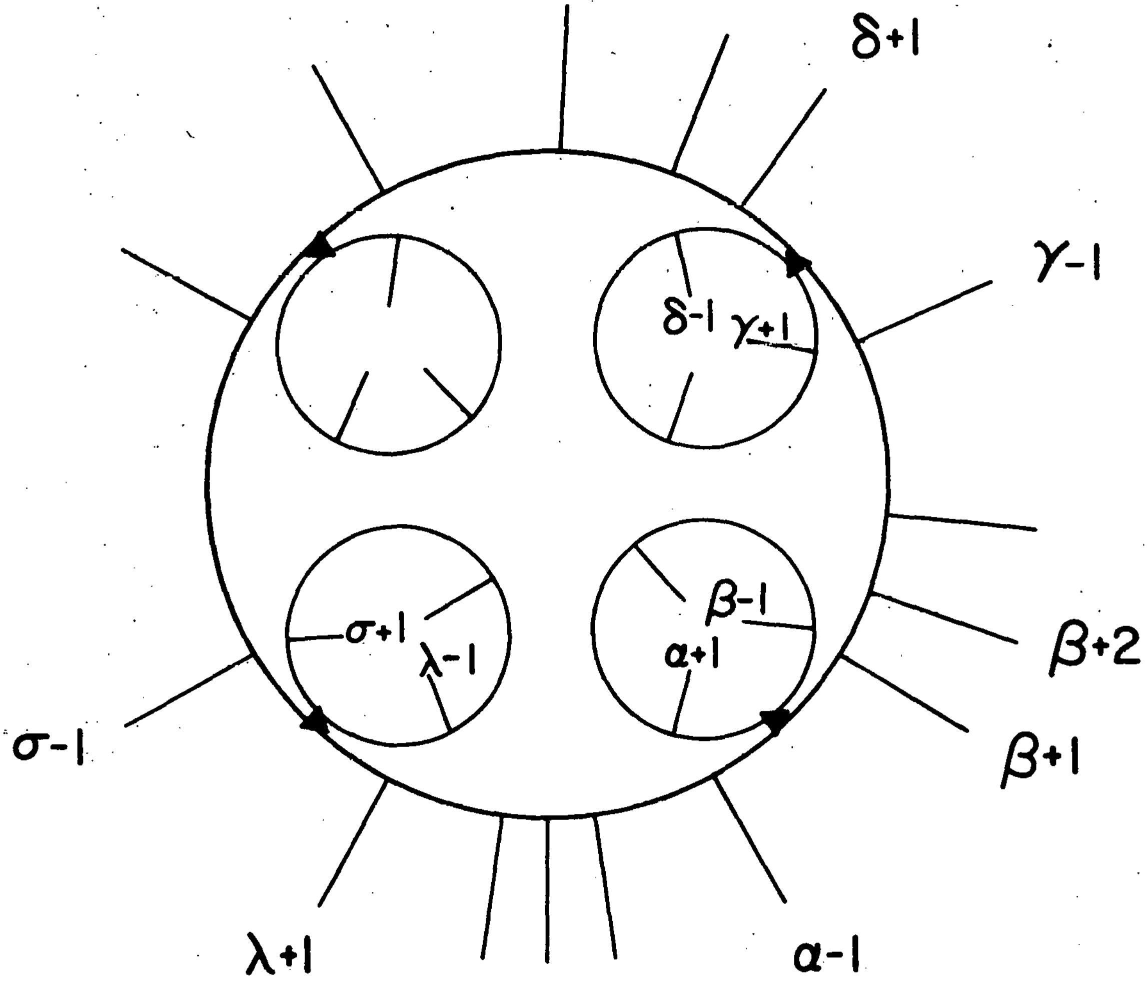

(Fig. 10a)

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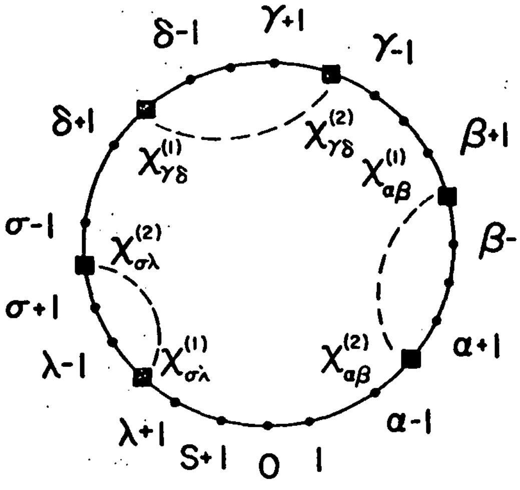
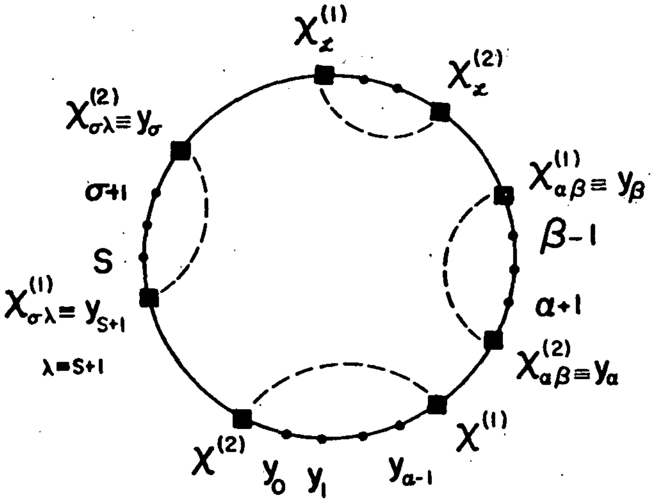


(Fig. 10b)



(Fig. 10c)

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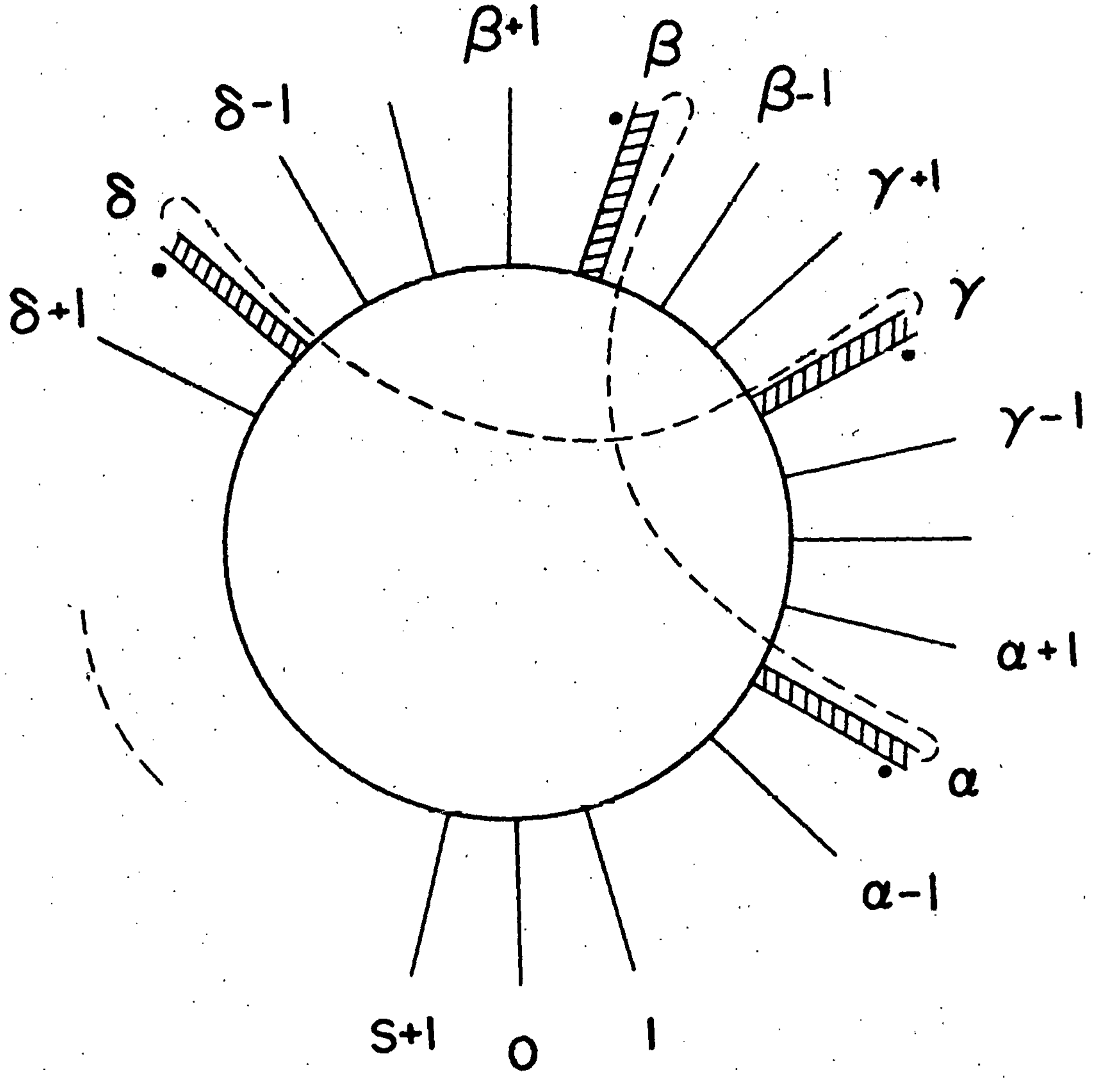

(Fig. II)

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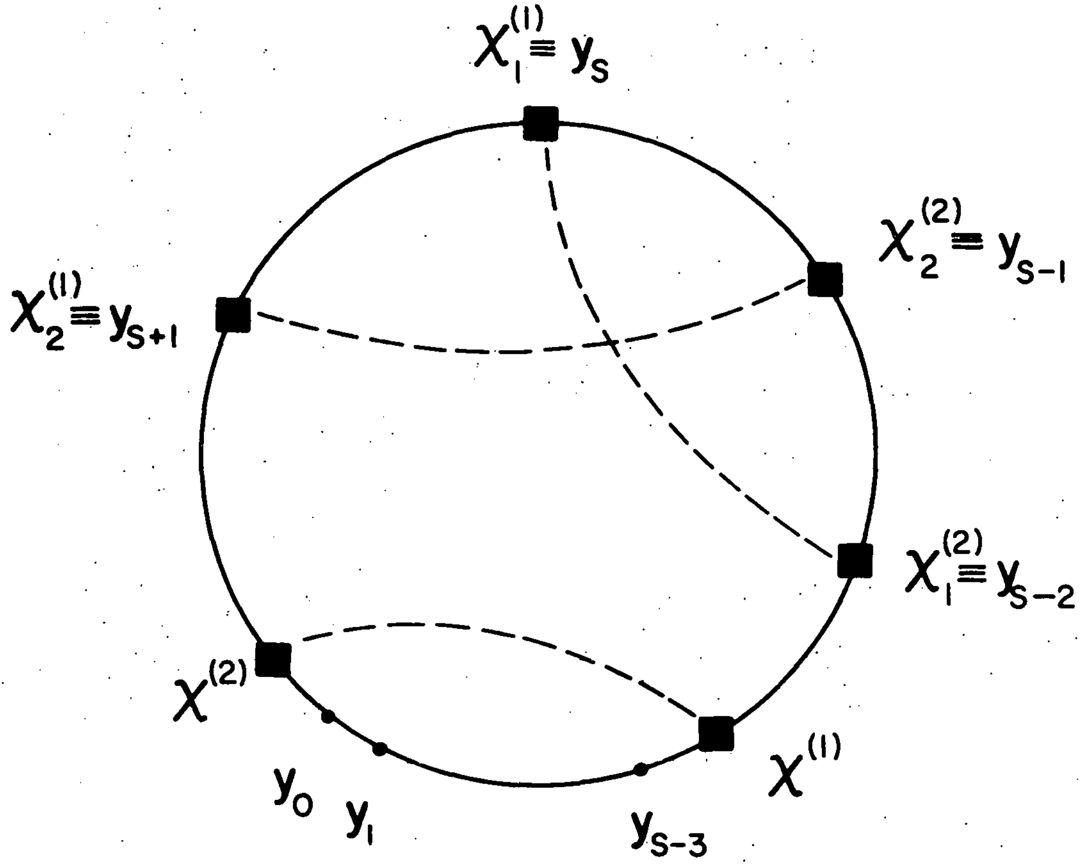

(Fig. 12)

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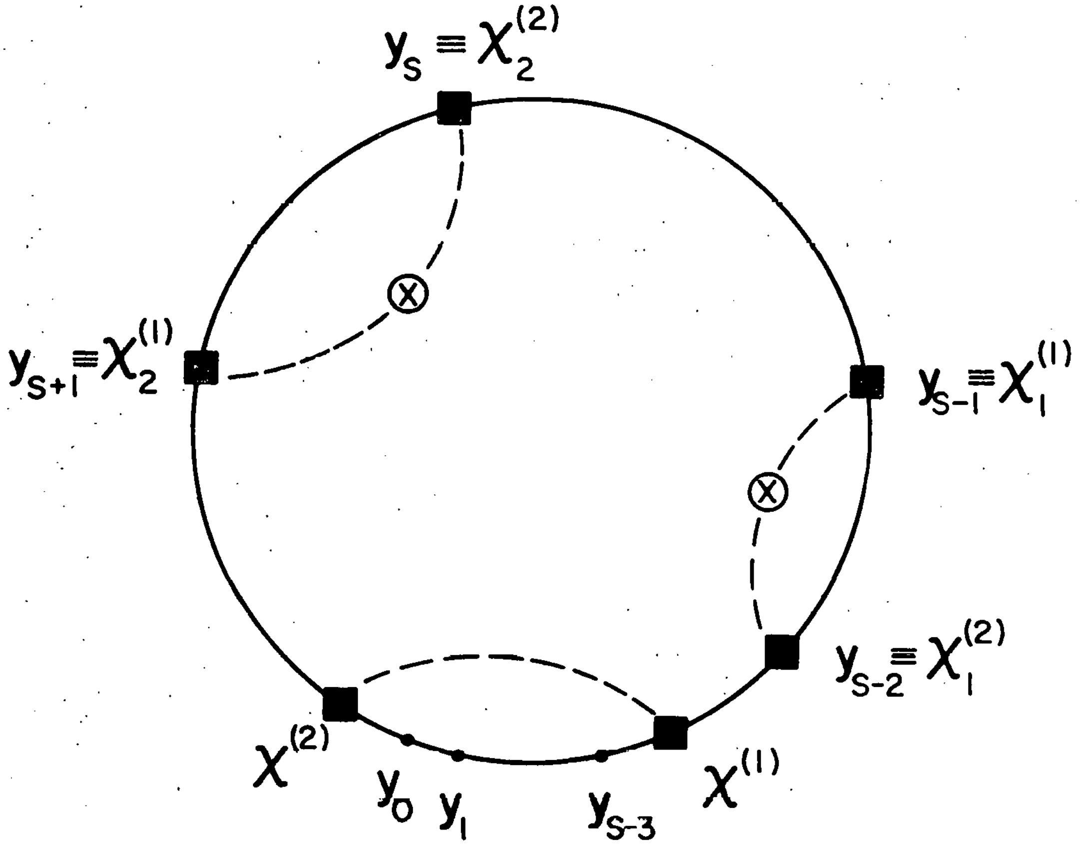

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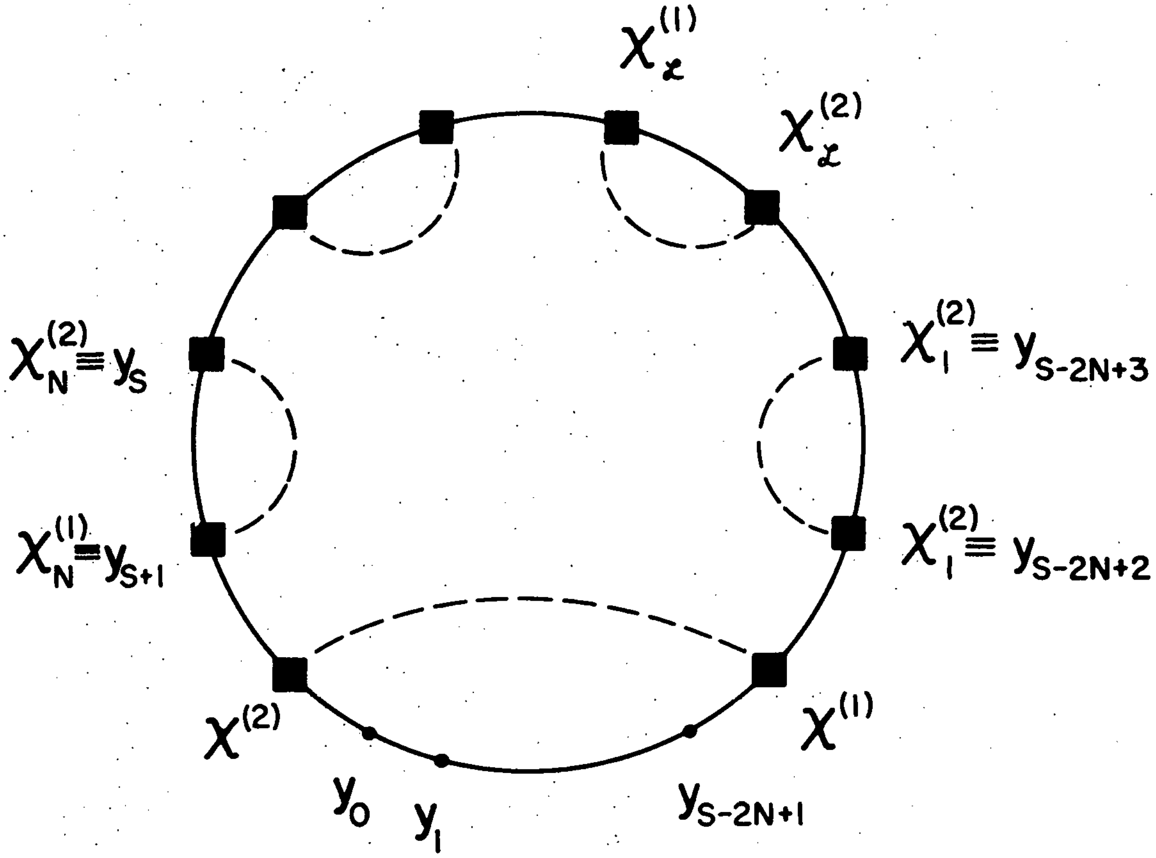
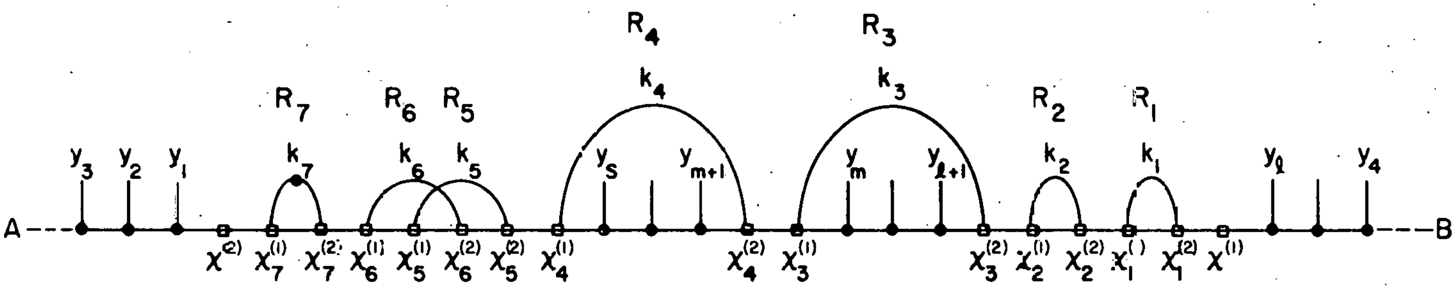


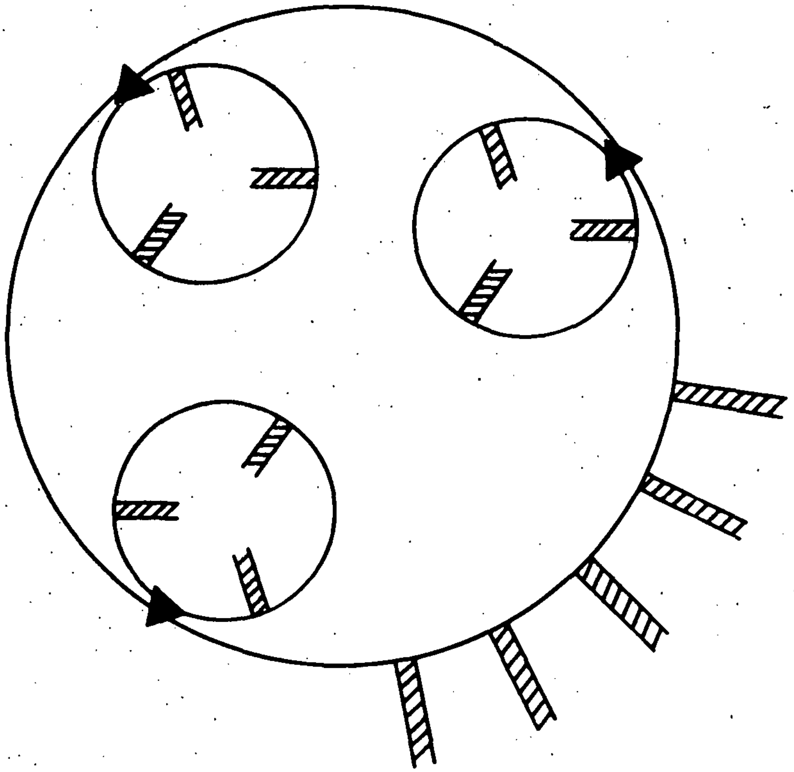
(Fig. 14)

XBL 712-2788



XBL712-2786

(Fig. 15)



(Fig. 16)

XBL712-2789

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