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PION PRODUCTION AND THE SECOND PION-NUCLEON RESONANCE

by

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Abstract

A model for the reaction $\pi + N \longrightarrow 2\pi + N$ at low energies, which includes pion-pion interaction and final state interactions in the (3,3) state, is discussed. The theory involves 2 parameters which are related to the S-wave and P-wave π-π scattering lengths. These parameters are chosen from a fit to the total cross-section for $\pi^- + p \rightarrow \pi^- + \pi^+ + n$. Meson production is predicted to be primarily in the T = 1/2 state. Predictions are made for the total cross-sections of the various channels (e.g. $\pi^+ + p \longrightarrow \pi^+ + \pi^+ + n$, etc.) in the energy range from threshold to $\gtrsim$ 500 Mev, in good agreement with experiments. Angular distributions are predicted. These are in qualitative agreement with the π^+ angular distribution for $\pi^- + p \longrightarrow \pi^- + \pi^+ + n$. From these data it is suggested that S-wave π-π scattering length has opposite sign to P-wave scattering length. A conjecture concerning rapidly rising inelastic cross-sections in a single partial wave is made to connect the large T = 1/2, $D_{3/2}$ production cross-sections with the T = 1/2, $D_{3/2}$ pion-nucleon resonance. The π-π scattering length found are $a_0 = -0.290\mu^{-1}$, $a_1 = 0.122\mu^{-1}$ and $\frac{a_2}{a_0} \rightarrow \frac{2}{5}$ by hypothesis.

1. Introduction

Several new features of the pion-nucleon interaction have been revealed recently by experimental investigation of the pion-nucleon total cross-section and the photoproduction of pions from protons. The dominant feature of π-N interactions in the energy range 200 to 800 Mev is the rapid rise of the cross-sections in the T = 1/2 state. The total cross-section reaches a resonance-like peak of about 43 mb ($\gtrsim 8\pi \lambda\!\!\!^{-2}$) at 610 mb[1]; while the elastic cross-section has a maximum of roughly 28 mb ($\gtrsim 4.5\pi \lambda\!\!\!^{-2}$) at 600 Mev. Evidence from photoproduction experiments indicate that this peaking is due to a resonance in the $D_{3/2}$, T = 1/2 state,[2] with a sizeable background in these cross sections due to interactions in other states.

By contrast, the π - N interaction in the T = 3/2 state is not strong in this energy region. The average π^+ - p inelastic cross-section in the range 300-500 Mev is about 2 mb, whereas the T = 1/2 inelastic cross-section rises almost linearly between these energies, from 2 to 15 mb. It is also observed that the charge-exchange scattering, $\pi^- + p \longrightarrow \pi^0 + n$, continues to decrease and becomes quite small (~8mb) in the region of the $D_{3/2}$, T = 1/2 resonance. This is considered to be preliminary evidence for a resonance in the $D_{3/2}$, T = 3/2 state[3] at an energy around 850 Mev, or greater. Finally, one observes the fact that the π^+ - p total cross section reaches a broad maximum at around 1.2 Bev.

One suspects that the two meson states play an important role in this behavior, and so one is led to examine the reaction $\pi + N \longrightarrow \pi + \pi + N$ in some detail. It is possible that a π - N(3,3) resonance in the final state ("isobar formation") contributes to some dgree in these phenomena. (One notes the lowest π-isobar state is $sp_{3/2} \longrightarrow D_{3/2}$. For notation, see Section 2). However, this in itself does not give a satisfactory explanation of these phenomena, since it does not explain why the T = 1/2 state is favored.

There are two obvious mechanisms for the interaction of a D-wave pion with a nucleon: nucleon recoil and the π-π interaction. Among the lowest order (in 1/M) "core" pion production terms, there is a diagram which contributes to $D_{3/2} \rightarrow sp_{3/2}$ and it exists only in the T = 1/2 state. Unfortunately, as discussed in Appendix A, appeal to experimental s-wave π-N scattering makes one think that this term, as well as the other core terms, in fact are negligible. One must thus consider that the π-π interaction initiates the two-pion state. The contribution of the π-π interaction to pion production was considered by Rodberg,[7] and in this paper we will make an extension of these ideas.

Ideally, if one wished to calculate meson production, one would like to apply the relativistic dispersion relations, using something similar to the Mandelstam hypothesis to examine the behavior of single partial wave amplitudes. Unfortunately these techniques are not at present sufficiently advanced to permit a calculation of the desired matrix elements. Therefore, we calculate meson production using the formalism of the static model,[8] which generates dispersion-like equations. Since one cannot apply the unitarity condition for three particles exactly, we will treat the π-π scattering only to first order in the π-π scattering amplitude. This appears to be the logical first step in the application of the π-π interaction to pion scattering in the medium energy range, if one believes with R. F. Peierls[5] that the $sp_{3/2}$ two meson state (and the $D_{3/2}$ state itself) is the most important intermediate state here. After one has calculated the matrix element for the $D_{3/2} \rightarrow sp_{3/2}$ process, one can in principle proceed to calculate iterations of this process: $D \rightarrow sp \rightarrow D$, $D \rightarrow sp \rightarrow D \rightarrow sp$, etc., perhaps by the dispersion techniques. It seems to us intuitively clear that if this cross-section, $D \rightarrow sp$, rises sharply with energy (viewing it as an effective potential), then the successive iterations add, and a resonance will result; that is, if the cross-section for $D \rightarrow sp$ calculated to lowest order in the π-π coupling looks like the dotted line of Figure 1, we might expect the iterated result to be like the solid of the same figure.

The above remark is just a conjecture. However, we note that since the size of the maximum is limited by unitarity, our assumption effectively is that the width of the maximum should be governed by causality. A similar consideration is found for a single channel process, for example, the (3,3) resonance of pion-nucleon scattering, where the width of that resonance is determined, through a sum rule, by the position of the resonance.[6] Therefore our conjecture is that causality provides a similar limitation for the two channel process

$$\pi + N \rightarrow \pi + N \atop \quad\;\; \rightarrow \pi + \pi + N$$

. It is of course obvious that one should not expect the total production cross-section to have the same shape as the final state $\pi - N$ (isobar) interaction (or a π-π resonance interaction), which is "smeared" out as a continuum state; rather, requirements of causality dictate how fast the cross-section should change.

2. Phase Space Considerations

The available kinetic energy, Q, in the center of mass system of two pions and a nucleon is related to the laboratory kinetic energy, T, of the incident pion by $Q = \frac{T - T_0}{1 + \frac{Q + 4\mu}{2M}}$, where T_0 is the threshold energy: $T_0 = \mu + \frac{3}{2}\frac{\mu^2}{M} \approx 170$ Mev. M is the nucleon mass, and μ is the pion mass. Near threshold the cross-section for the production of a pion is proportional to $Q^{2+\ell_1+\ell_2}$, where ℓ_1 and ℓ_2 are the relative orbital angular momenta of the system, which are conveniently taken to be the orbital angular momentum of each pion with respect to the nucleon, which near threshold can be considered to be at rest. The lowest states are (together with the threshold behavior of the cross-section):

$$
\begin{array}{lll}
P_{1/2} \rightarrow ss_{1/2} & & Q^2 \\
S_{1/2} \rightarrow sp_{1/2} & \rbrace & Q^3 \\
D_{3/2} \rightarrow sp_{3/2} & & \\
P \rightarrow pp & & \\
 \rightarrow sd_{3/2} & \rbrace & Q^4 \\
F_{5/2} \rightarrow sd_{5/2} & &
\end{array}
$$

The j values are listed when unique. In the discussion that follows we will limit ourselves to the ss and sp states. The transitions $P \rightarrow pp$, $\rightarrow sd$ which are omitted in the body of the discussion are estimated in Appendix D.

3. Meson Production Amplitude

Choice of Interactions. We wish to consider pion production by the processes of Figure 2 where (a) is the direct "knock-on" production due to the scattering of the incident pion by a self-field pion, and (b) is the "rescattered production," the rescattering being principally in the (3,3) state. Information about the π-π interaction is still meager, but the scattering in the T = 1, p-wave is undoubtedly attractive and probably has a resonance.[8,10] Qualitatively, from analysis of the τ -meson decay, one suspects that the s-wave interaction is fairly strong and repulsive if the s-wave interaction has the i-spin dependence of ϕ^4.[10,11] For our model of pion production, we will take the π-π scattering in S- and P-waves, characterized by two parameters λ_s and λ_p which are proportional to the scattering lengths in these partial waves. (See Appendix B for the relation of λ_s and λ_p to the π-π scattering lengths.) If λ_s is small, our scattering length approximation will not be too bad a description of the S-wave π-π scattering.

We shall take the effective S-wave π-π interaction to be $\frac{\lambda_s}{4}(\phi_\alpha\phi_\alpha)^2$, where ϕ_α is the pion field for a pion of isotopic spin α. The ratio of this interaction in the T = 0 and T = 2 states is 5:2. This leads to a ratio of the pion production cross-sections in the T = 1/2 and T = 3/2 states of 5:2. For a resonant final state, i.e., one of the pions and the nucleon is in a relative T = 3/2 state, the ratio becomes 10:1.

For the P-wave π-π scattering of mesons $(\vec{a},\alpha)$ and $(\vec{d},\delta)$ into $(\vec{b},\beta)$ and $(\vec{c},\gamma)$ we take the matrix element to be (in the π-π center of mass system)

$$\frac{2\lambda_p}{(\sqrt{2\omega V})^4}\left(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\gamma}\delta_{\beta\delta}\right)(\vec{a}-\vec{d})\cdot(\vec{b}-\vec{c}) \tag{3.1}$$

This is the scattering length approximation. Since the P-wave amplitude is expected to rise more strongly than k^2 (and in fact resonate), the value of λ_p determined by the fitting of the experimental data by our formulae will actually be some average of the P-wave scattering length over an energy interval; this value will be greater than the threshold P-wave scattering length if the P-wave resonance does not lie too low". The P-wave π-π interaction leads to a ratio of the pion production cross-sections in the T = 1/2 and T = 3/2 states of 4:1. For a resonant final state, the ratio becomes 8:5 (See also Appendix B). We will see that these ratios will be adequate to explain the smallness of interaction in the T = 3/2 state.

(a) Derivation of the Integral Equation. We write the interaction Hamiltonian as the sum of $\pi - N$ and $\pi - \pi$ interaction

$$H_I = \sum_k a_k V_k + h.c. + H^{\pi\pi} \tag{3.2}$$

where we neglect nucleon recoil; a_k is the pion annihilation operator of momentum $\vec{k}$ and isotopic spin k.

We take

$$V_k = \frac{ig}{2M} \frac{\vec{\sigma}\cdot\vec{k}\,\tau_k}{\sqrt{2\omega_k V}} \tag{3.3}$$

where

$$\frac{g^2}{4M^2} = 4\pi \frac{f^2}{\mu^2} \qquad \text{and } f^2 = 0.08$$

The S-matrix for the production of mesons p and q by a meson k is

$$\langle pq|S|k\rangle = \langle pq^{(-)}|k^{(+)}\rangle \tag{3.4}$$

Then, using standard Chew-Low-Wick algebra and notation,[4]

$$\langle pq|S|k\rangle = \langle pq^{(-)}|a_k^+|0\rangle - \langle pq^{(-)}|\frac{1}{\omega_p+\omega_q-\omega_k-i\varepsilon}[H_I, a_k^+]|0\rangle \tag{3.5}$$

Now

$$a_k|pq^{(-)}\rangle = \delta_{kp}|q^{(-)}\rangle + \delta_{kq}|p^{(-)}\rangle - \frac{1}{H-\omega_p-\omega_q-\omega_k+i\varepsilon}[H_I, a_k^+]^+|pq^{(-)}\rangle, \tag{3.6}$$

So we find

$$\langle pq|S|k\rangle = -\langle pq^{(-)}|[H_I,a_k^+]|0\rangle\left\{\frac{1}{\omega_p+\omega_q-\omega_k-i\varepsilon} - \frac{1}{\omega_p+\omega_q-\omega_k+i\varepsilon}\right\} = -\langle pq^{(-)}|[H_I,a_k^+]|0\rangle 2\pi i\delta(\omega_p+\omega_q-\omega_k) \tag{3.7}$$

This enables us to define as usual the transition matrix element

$$\langle pq|T|k\rangle = \langle pq^{(-)}|[H_I, a_k^+]|0\rangle. \tag{3.8}$$

Since we are mainly interested in the ss and sp channels (i.e., the processes $P_{1/2} \to ss$, $S_{1/2} \to sp_{1/2}$, and $D_{3/2} \to sp_{3/2}$), to a good approximation we can neglect the π-N "core" interaction except for the case of a $p_{3/2}$ final state meson. Of course, as the initial energy approaches that of the $D_{3/2}$ resonance, the initial state interaction of meson k becomes large and the neglect of this will lead to a violation of unitarity in this channel. But as described above, we are only intending to calculate an "effective potential" to first order in π-π scattering.

With this in mind, Equation (3.8) becomes

$$\langle pq|T|k\rangle \simeq \langle pq^{(-)}|\,[H_I^{\pi\pi}, a_k^+]\,|0\rangle \tag{3.9}$$

We define for convenience

$$\begin{aligned} [H_I, a_k^+] &\equiv H_k \\ [H_I^{\pi\pi}, a_k^+] &\equiv H_k^{\pi\pi} \\ [a_q, H_k^{\pi\pi}] &\equiv {}_qH_k \\ [a_p, {}_qH_k] &\equiv {}_{pq}H_k \end{aligned} \tag{3.10}$$

Then, using

$$|pq^{(-)}\rangle = a_q^+|p^{(-)}\rangle - \frac{1}{H-\omega_p-\omega_q+i\varepsilon} H_q\,|p^{(-)}\rangle, \tag{3.11}$$

we have

$$\langle pq|T|k\rangle \simeq \langle p^{(-)}|\,a_q\,H_k^{\pi\pi}\,|0\rangle - \sum_n \frac{\langle p^{(-)}|\,H_q^{\pi\pi +}\,|n\rangle\langle n|\,H_k^{\pi\pi}\,|0\rangle}{\omega_n-\omega_p-\omega_q-i\varepsilon}, \tag{3.12}$$

where we have inserted a complete set of states and taken $H_q \simeq H_q^{\pi\pi}$ since q will be taken to represent a meson in an S state with respect to the nucleon. The second term of (3.12) is immediately dropped since it is iteration of the π-π interaction. Further, in this approximation

$$\begin{aligned} a_q H_k^{\pi\pi}|0\rangle &= ({}_qH_k + H_k^{\pi\pi}a_k)\,|0\rangle \\ &\simeq {}_qH_k\,|0\rangle. \end{aligned} \tag{3.13}$$

Next, expanding the meson p (with obvious notation), which can have interactions with the core, we have

$$\begin{aligned} \langle pq|T|k\rangle \simeq{}& \langle 0|\,{}_{pq}H_k\,|0\rangle + \langle 0|\,{}_qH_k\,a_p\,|0\rangle \\ &- \langle 0|\,H_p^+\,\frac{1}{H-\omega_p-i\varepsilon}\,{}_qH_k\,|0\rangle. \end{aligned} \tag{3.14}$$

We now define

$$\langle 0 | {}_{pg} H_k | 0 \rangle \equiv \langle pg | T^B | k \rangle ; \tag{3.15}$$

this is the Born-approximation, or knock-on, matrix element. Again inserting a complete set of states, and using $H_p \simeq V_p$ in order to keep the π-π interaction only to first order

$$\langle pg | T | k \rangle \simeq \langle pg | T^B | k \rangle - \sum_n \frac{\langle 0 | V_p^+ | n \rangle \langle n | {}_g H_k | 0 \rangle}{\omega_n - \omega_p - i\varepsilon} - \sum_n \frac{\langle 0 | {}_g H_k | n \rangle \langle n | V_p^+ | 0 \rangle}{\omega_n + \omega_p} . \tag{3.16}$$

It is useful to illustrate these terms with diagrams (keeping only the one-meson intermediate states) in Figure 4.

The terms corresponding to Figures (4a) and (4b) give us an integral equation for the production amplitude. We are of course unable to treat the π-N and π-π scattering exactly when they both occur in the same channel. As discussed previously, our parameter λ_p will be equivalent to the P-wave scattering length averaged over an energy interval. Thus, if the P-wave resonance our treatment of the π-π scattering is expected to be adequate for incident meson energies between threshold (170 Mev, Lab) and ≃ 450 Mev.

To simplify our solution of the integral equation, we will drop the crossed term. Then

$$\langle pg | T | k \rangle \simeq \langle p | {}_g H_k | 0 \rangle \simeq \langle pg | T^B | k \rangle - \sum_n \frac{\langle 0 | V_p^+ | n \rangle \langle n | {}_g H_k | 0 \rangle}{\omega_n - \omega_p - i\varepsilon} . \tag{3.17}$$

We now make the one-meson approximation in the sum over intermediate states in Equation (3.17), which reduces it to a simple integral equation. The zero meson term of the sum in (3.17) describes the emission of the final meson p from the core into the T = 1/2, $p_{1/2}$ state. This contributes to the rescattering in the T = 1/2, p = 1/2 state and so we neglect it as well as the other non-resonant core terms.

The born term, according to our treatment of the π-π scattering is

$$\langle p q | T^B | k \rangle = \frac{-2 i f \sqrt{4\pi}}{\sqrt{2\omega_q \mathcal{V}\, 2\omega_p \mathcal{V}\, 2\omega_k \mathcal{V}}} \Big\{ \lambda_s \frac{\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{\omega^2_{\vec{k}-\vec{p}-\vec{q}}} (\delta_{\beta\gamma}\tau_\gamma + \delta_{\gamma\gamma}\tau_\beta + \delta_{\beta\gamma}\tau_\gamma) \tag{3.18}$$

$$+ \lambda_p \frac{\vec{\sigma}\cdot(\vec{k}-\vec{p}-\vec{q})}{\omega^2_{k-p-q}} \left[2\vec{k}\cdot(\vec{p}-\vec{q}) - 2\omega_k(\omega_p - \omega_q) \right] \times (\delta_{\alpha\beta}\tau_\gamma - \delta_{\alpha\gamma}\tau_\beta) \Big\}$$

in the over-all center of mass system, where $(\alpha, \vec{k})$ is the incoming meson;

$(\beta, \vec{p})$ is the rescattered final meson;

$(\gamma, \vec{q})$ is the other final meson.

We can write equation (3.17) in the form

$$T_{ab}(\vec{p}) = T_{ab}^B(\vec{p}) + \sum_{p'} \frac{f_{\vec{p}}^+(\vec{p'})\, T_{ab}(\vec{p'})}{\omega_p - \omega_{p'} + i\epsilon}, \tag{3.19}$$

which is the form of the familiar "production amplitude dispersion relation," where

$$\langle 0 | V_p^+ | p' \rangle \equiv f_{\vec{p}}^+(\vec{p'}) .$$

It is shown in Appendix C that the solution of such an equation can be simplified to yield Chew and Low's solution.[4] In the present notation, this solution of (3.19) is:

$$T_{ab}(\vec{p}) = T_{ab}^B(\vec{p}) + \sum_{p'} \frac{\omega_p}{\omega_{p'}} \frac{f_{\vec{p'}}(\vec{p})\, T_{ab}^B(\vec{p'})}{\omega_p - \omega_{p'} + i\epsilon} . \tag{3.20}$$

If we neglect all but the (3,3) rescattering, then

$$f_{\vec{p'}}(\vec{p}) = -\frac{4}{3}(4\pi)\left(\frac{f}{\mu}\right)^2 \frac{h(\omega_p)}{\omega_p} \frac{p\, p'}{\sqrt{2\omega_p \mathcal{V}\, 2\omega_{p'} \mathcal{V}}}\; {}_p\mathcal{P}_{p'}^{33} , \tag{3.21}$$

where ${}_p\mathcal{P}_{p'}^{33}$ is the projection operator for the (3,3) state,

$${}_p\mathcal{P}_{p'}^{33} = {}_pP_{p'}^{3/2} \times {}_pI_{p'}^{3/2} = \left[\frac{3\vec{p}\cdot\vec{p}' - \vec{\sigma}\cdot\vec{p}\,\vec{\sigma}\cdot\vec{p}'}{p\, p'} \right] \times \left(\delta_{pp'} - \frac{1}{3}\tau_p \tau_{p'}\right), \tag{3.22}$$

and

$$h(\omega_p) = f^{33}(\omega_p) / f_B^{33}(\omega_p) , \tag{3.23}$$

i.e., the ratio of the (3,3) scattering amplitude to its Born approximation. We note that $h(\omega_p)$ can be written in the form

$$h(\omega_p) = \frac{1}{R(\omega_p) - i\frac{4}{3}\frac{f^2}{\mu^2}\frac{p^3}{\omega_p}} \tag{3.24}$$

where $R(\omega_p)$ is a real function, having the particular values $R(0) = 1$ and at resonance $R = 0$. In the "effective-range" approximation $R(\omega_p) = 1 - \frac{\omega_p}{\omega_3}$ where ω_3 is the resonance energy.

Using the above for $f_{p'}(p)$ in (3.20), we have (suppressing the subscripts ab)

$$T(p) = T^B(p) + \sum_{p'} \frac{\omega_p}{\omega_{p'}} \frac{4}{3}(4\pi)\left(\frac{f}{\mu}\right)^2 \frac{h(\omega_p)}{\omega_p} \frac{\mathcal{P}\mathcal{P}' \{T^B(p')\}_{33}}{\omega_{p'}-\omega_p - i\epsilon} , \tag{3.25}$$

where

$$\mathcal{P}\mathcal{P}'\{T^B(p')\}_{33} \equiv \mathcal{P}\mathcal{P}' \{{}_p \mathcal{P}_{p'}^{33} T^B(p')\},$$

i.e. the $D_{3/2}$, T = 3/2 part of the Born amplitude. This may be written in the following way:

$$T(p) = T^B(p) + \frac{4}{3}\left(\frac{f}{\mu}\right)^2 h(\omega_p)\frac{1}{\pi}\int_0^\infty \frac{dp'\, p'^2}{\omega_{p'}^2(\omega_{p'}-\omega_p-i\epsilon)} \sqrt{\frac{\omega_{p'}}{\omega_p}}\, \mathcal{P}\mathcal{P}'\{T^B(p')\}_{33} \tag{3.26}$$

or alternately in a form which explicitly exhibits the phase of the channels having rescattering:

$$T(p) = T^B(p) - \{T^B(p)\} + h(\omega_p)\Big\{R(\omega_p)\{T^B(\omega_p)\}_{33} + \frac{4}{3}\left(\frac{f}{\mu}\right)^2 \frac{1}{\pi} P\int_0^\infty dp' \frac{p'^2}{\omega_{p'}^2(\omega_{p'}-\omega_p)}\sqrt{\frac{\omega_{p'}}{\omega_p}}\, \mathcal{P}\mathcal{P}'\{T^B(p')\}\Big\}. \tag{3.27}$$

In Appendix D we discuss the $D_{3/2}$ partial wave and discuss the 'enhancement' integral. A good approximation to the matrix element, which will give a reasonable estimate for the rescattering integral is

$$\{T^B(p)\}_{33} \simeq \frac{-2if\sqrt{4\pi}}{\sqrt{2\omega_q V\, 2\omega_p V\, 2\omega_k V}} K(k,q,p)\left[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p} - \frac{1}{3}k^2\vec{\sigma}\cdot\vec{p}\right] \times\Big\{\left(\frac{5}{3}\lambda_S + \frac{2}{3}\omega_k^2\lambda_P\right)\left(\delta_{\beta\gamma} - \frac{1}{3}\tau_\beta\tau_\gamma\right)\tau_\gamma + (-\lambda_S - \omega_k^2\lambda_P)\left(\frac{2}{3}\delta_{\gamma\beta}\tau_\gamma + \frac{2}{3}\delta_{\gamma\gamma}\tau_\beta - \delta_{\beta\gamma}\tau_\gamma - \frac{5}{9}\tau_\beta\tau_\gamma\tau_\gamma\right)\Big\} \tag{3.28}$$

where $K(k,q,p')$ is defined by

$$\frac{}{p}P^{3/2}_{\vec{p}'}\ \frac{\vec{\sigma}\cdot(\vec{k}-\vec{p}'-\vec{q})}{\omega^2_{\vec{k}-\vec{p}'-\vec{q}}} = \frac{p'}{p}\left[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p} - \tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}\right]\mathcal{K}(k,q,p') \qquad (3.29)$$

Using this in equation (3.26) leads to $T(p) = T^B(p)$ +

$$\left(\frac{-2if\sqrt{4\pi}}{\sqrt{2\omega_q V 2\omega_p V 2\omega_k V}}\right)\left[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p} - \tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}\right]$$

$$\times\Big\{\left(\tfrac{5}{3}\lambda_s + \tfrac{2}{3}\omega_k^2\lambda_p\right)\left(\delta_{\beta\gamma} - \tfrac{1}{3}T_\beta T_\gamma\right)T_\varphi$$

$$+\left(-\lambda_s - \omega_k^2\lambda_p\right)\left(\tfrac{2}{3}\delta_{\sigma\beta}T_\gamma + \tfrac{2}{3}\delta_{\varphi\gamma}T_\beta - \delta_{\beta\gamma}T_\varphi - \tfrac{5}{9}T_\beta T_\gamma T_\varphi\right)\Big\}$$

$$\times\frac{4}{3}\left(\frac{f}{\mu}\right)^2 h(\omega_p)\,\frac{1}{\pi}\int_0^\infty dp'\,\frac{p'^4}{\omega_{p'}^2(\omega_{p'}-\omega_p-i\varepsilon)}\,\mathcal{K}(k,q,p') \qquad (3.30)$$

If we combine the direct and rescattered parts of the amplitude leading to the (3,3) channel as in (3.27), we have

$$\langle pq|T|k\rangle = \frac{-2if\sqrt{4\pi}}{\sqrt{2\omega_p V 2\omega_q V 2\omega_k V}}\Big\{\left(\delta_{pp'} - P^{3/2}_{pp'}\right)\frac{\vec{\sigma}\cdot(\vec{k}-\vec{q}-\vec{p}')}{\omega^2_{\vec{k}-\vec{q}-\vec{p}'}}$$

$$\times\Big[\lambda_s\left(\delta_{\beta'\gamma}T_\varphi + \delta_{\gamma\varphi}T_{\beta'} + \delta_{\beta'\varphi}T_\gamma\right) + \lambda_p\left(2\vec{k}\cdot(\vec{p}-\vec{q}) - 2\omega_k(\omega_p-\omega_q)\right)$$

$$\times\left(\delta_{\varphi\beta'}T_\gamma - \delta_{\varphi\gamma}T_{\beta'}\right)\Big]$$

$$+\Big\{\left(\tfrac{5}{3}\lambda_s + \tfrac{2}{3}\omega_k^2\lambda_p\right)\Big[\left(\delta_{\beta\gamma} - \tfrac{1}{3}T_\beta T_\gamma\right)T_\varphi$$

$$-\left(\lambda_s + \omega_k^2\lambda_p\right)\left(\tfrac{2}{3}\delta_{\gamma\beta}T_\varphi + \tfrac{2}{3}\delta_{\varphi\gamma}T_\beta - \delta_{\beta\gamma}T_\varphi - \tfrac{5}{9}T_\beta T_\gamma T_\varphi\right)\Big\}$$

$$\times\left[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p} - \tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}\right]\mathcal{K}(k,q,p')$$

$$\times h(\omega_p)\Big\{R(\omega_p) + \frac{4}{3}\left(\frac{f}{\mu}\right)^2\frac{1}{\pi}\int_0^\infty dp'\,\frac{p'^4}{\omega_{p'}^2(\omega_{p'}-\omega_p-i\varepsilon)}\,\frac{\mathcal{K}(k,q,p')}{\mathcal{K}(k,q,p)}\Big\} \qquad (3.31)$$

In the absence of final state interaction the last bracket would be simply equal to one. The kernal $K(k,q,p')$ has the following behavior if q is small (which is consistent with q being the s-wave meson in the amplitude of interest, $D_{3/2} \to sP_{3/2}$): For small p', $K(k,q,p')$ has the value $\sim \frac{2}{\omega_k^4}$; it falls rather slowly for increasing p', until p' is of the order of k, when it drops rapidly, becoming proportional to $\frac{1}{p'^4}$. As a result of the discussion in

Appendix D, we approximate the factor $K(k,q,p')/K(k,q,p)$ in the integrand of (3.31) as a 'cutoff' at $p' \approx k$, so that (for small ω_p) the value of the integral in (3.31) is roughly $\frac{\omega_k^2}{2}$ (see Appendix D).

However, as the final state resonance is approached, the term $R(\omega_p)$ in (3.31) goes to zero (the direct amplitude becomes cancelled by the on-the-energy-shell rescattering) so that near the final state resonance, the total production amplitude is not necessarily increased by the rescattering. The production amplitude near the final state resonance <u>is</u> increased only if the off-the-energy-shell transitions more than compensate the loss of the direct amplitude. We see from (3.31) that the effect of rescattering is to multiply the $D_{3/2} \rightarrow sp_{3/2}$ Born amplitude by roughly

$$\frac{R(\omega_p) + \frac{4}{3}\frac{f^2}{\mu^2}\frac{\omega_k^2}{3\pi}}{R(\omega_p) - i\frac{4}{3}\frac{f^2}{\mu^2}\frac{p^3}{\omega_p}} \qquad (3.31a)$$

At resonance, this has the value $\frac{1}{2\pi}\omega_k^2\frac{\omega_3}{(p_3)^3} \approx 0.75\ \omega_k^2/\mu^2$ which is less than unity for ω_k less than 3.65μ, i.e., a lab energy of less than $\approx$700 Mev.[12]

It is interesting to compare this result with a non-relativistic model in which particles are produced by a fixed source concentric with a potential. To be specific, let us take the scattering potential to be zero-ranged. Then for a source $\rho(r) = Qe^{-\mu r}$, the ratio of the production amplitude to its Born amplitude is

$$\frac{M}{M_B} = e^{i\delta}\left[\cos\delta + \frac{\mu}{k}\sin\delta\right]$$

where δ is the (re)scattering phase shift, and k is the particle momentum. We see that $\frac{M}{M_B} \xrightarrow{\delta\to 90^\circ} \frac{\mu}{k}$. However, if $\rho(r) = Q'e^{-\mu r}r$, then

$$\frac{M}{M_B} = e^{i\delta}\left[\cos\delta + \frac{\mu^2-k^2}{2\mu k}\sin\delta\right]$$

so that

$$\frac{M}{M_B} \xrightarrow{\delta\to 90^\circ} \begin{cases} \frac{\mu}{2k} & \\ \sim 0 & \text{for} \\ -\frac{k}{2\mu} & \end{cases} \begin{matrix} k \ll \mu \\ k \approx \mu \\ k \gg \mu \end{matrix}$$

One sees that the enhancement depends both on the shape of the source and the ratio of the range of the source to the particle wave length.

4. Comparison with Experimental Results

We have derived a production amplitude, Equation (3.31), by beginning with the static model and making some reasonable approximations. Now we are in a position to make comparisons with various experimental results. The available experimental information is still sparse, but enough is available to allow us to estimate the S and P-wave π-π scattering lengths and obtain some comparisons with experiments.

Our procedure will be to expand (3.31) in partial waves, keeping only $P_{1/2} \rightarrow ss_{1/2}$, $S_{1/2} \rightarrow sp_{1/2}$, and $D_{3/2} \rightarrow sp_{3/2}$. This matrix element will be evaluated in the center of mass system. A fit to the total cross-section for the process $\pi^- + p \rightarrow \pi^- + \pi^+ + n$[13] is possible by choosing λ_s and λ_p (see also Rodberg[7]). This fixes all the parameters in the theory and allows one to make further predictions in the energy range between threshold and ~500 Mev. The predictions that will be considered are total cross-sections, charge ratios, and angular distribution of the final mesons. We also will discuss the recoil nucleon momentum spectrum, following the procedure of Goebel,[14] at low energies, where the final state interaction is not important.

Let us define a reduced matrix element for convenience.

$$\langle pq | T | k \rangle = \frac{-2i f \sqrt{4\pi}}{\sqrt{2\omega_q V 2\omega_p V 2\omega_k V}} \frac{1}{\omega_k^2} M \tag{4.1}$$

Then, in the center of mass-system, we have

$$d\sigma_{prod} = \frac{f^2}{(2\pi)^4} \left[\frac{1}{2} Tr\, |M|^2 \right] \frac{pq}{k\,\omega_k^4} \left(\frac{E_k}{\omega_k + E_k} \right) d\omega_p\, d\Omega_p\, d\Omega_q \tag{4.2}$$

We expand M into partial waves and keep only the leading terms in p/k and q/k. However, we have treated the meson labelled p and the one labelled q asymmetrically, so we properly symmetrize the amplitude. In what follows, we no longer reserve the notation p and q for the final p-wave and s-wave mesons. We will describe the final mesons as $p_1 = (\vec{p}_1, \beta)$ and $p_2 = (\vec{p}_2, \gamma)$ and the initial meson as $k = (\vec{k}, \alpha)$. The result is

$$
\begin{aligned}
M = \Big\{ & \frac{\vec{\sigma}\cdot\vec{p}_1}{3}\Big[-\lambda_s(\delta_{\beta\gamma}\tau_\alpha+\delta_{\gamma\alpha}\tau_\beta+\delta_{\beta\alpha}\tau_\gamma)-2k^2\lambda_p(\delta_{\alpha\gamma}\tau_\beta-\delta_{\gamma\beta}\tau_\gamma)\Big] \\
&+\frac{\vec{\sigma}\cdot\vec{p}_2}{3}\Big[-\lambda_s(\delta_{\beta\gamma}\tau_\alpha+\delta_{\beta\alpha}\tau_\gamma+\delta_{\gamma\alpha}\tau_\beta)-2k^2\lambda_p(\delta_{\gamma\beta}\tau_\gamma-\delta_{\gamma\gamma}\tau_\beta)\Big] \\
&+\vec{\sigma}\cdot\vec{k}\Big[\lambda_s(\delta_{\beta\gamma}\tau_\alpha+\delta_{\gamma\alpha}\tau_\beta+\delta_{\beta\alpha}\tau_\gamma)\Big]\Big\} \\
&+\Big[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p}_1-\tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}_1\Big]\Big\{J(\omega_{p_1})\,h(\omega_{p_1})\Big[\Big(\tfrac{5}{3}\lambda_s+\tfrac{2}{3}\omega_k^2\lambda_p\Big) \\
&\quad\times\Big(\delta_{\beta\gamma}-\tfrac{1}{3}\tau_\beta\tau_\gamma\Big)\tau_\alpha-(\lambda_s+\omega_k^2\lambda_p)\Big(\tfrac{2}{3}\delta_{\gamma\beta}\tau_\alpha+\tfrac{2}{3}\delta_{\alpha\gamma}\tau_\beta-\delta_{\beta\alpha}\tau_\gamma-\tfrac{5}{9}\tau_\beta\tau_\gamma\tau_\alpha\Big)\Big] \\
&\quad+\frac{1}{\omega_k^2}\Big[\Big(\tfrac{10}{9}\lambda_s-\omega_k^2\lambda_p\Big)\tau_\beta\tau_\gamma\tau_\alpha+\Big(\tfrac{10}{3}\lambda_s-\tfrac{2}{3}\omega_k^2\lambda_p\Big)\Big(\delta_{\alpha\gamma}\tau_\beta-\tfrac{1}{3}\tau_\beta\tau_\gamma\tau_\alpha\Big)\Big] \\
&+\Big[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p}_2-\tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}_2\Big]\Big\{J(\omega_{p_2})\,h(\omega_{p_2})\Big[\Big(\tfrac{5}{3}\lambda_s+\tfrac{2}{3}\omega_k^2\lambda_p\Big) \\
&\quad\times\Big(\delta_{\beta\gamma}-\tfrac{1}{3}\tau_\gamma\tau_\beta\Big)\tau_\alpha-(\lambda_s+\omega_k^2\lambda_p)\Big(\tfrac{2}{3}\delta_{\gamma\beta}\tau_\alpha+\tfrac{2}{3}\delta_{\alpha\beta}\tau_\gamma-\delta_{\gamma\alpha}\tau_\beta-\tfrac{5}{9}\tau_\gamma\tau_\beta\tau_\alpha\Big)\Big] \\
&\quad+\frac{1}{\omega_k^2}\Big[\Big(\tfrac{10}{9}\lambda_s-\omega_k^2\lambda_p\Big)\tau_\gamma\tau_\beta\tau_\alpha+\Big(\tfrac{10}{3}\lambda_s-\tfrac{2}{3}\omega_k^2\lambda_p\Big) \\
&\quad\quad\times\Big(\delta_{\alpha\beta}\tau_\gamma-\tfrac{1}{3}\tau_\gamma\tau_\beta\tau_\alpha\Big)\Big]\Big\}, \qquad (4.3)
\end{aligned}
$$

with

$$J(\omega_p) = \Big[\frac{8}{3}\frac{f^2}{\mu^2}\frac{1}{2\pi} + \frac{2R(\omega_p)}{\omega_k^2}\Big]. \qquad (4.4)$$

In general we may write

$$M = A\vec{\sigma}\cdot\vec{p}_1 + B\vec{\sigma}\cdot\vec{p}_2 + C\,\vec{\sigma}\cdot\vec{k} + D\Big[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p}_1-\tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}_1\Big] + E\Big[\vec{\sigma}\cdot\vec{k}\,\vec{k}\cdot\vec{p}_2-\tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}_2\Big] \qquad (4.5)$$

We omit from $C(p_{1/2}\rightarrow ss)$ a contribution from P-wave π-π scattering which is proportional to $(\omega_{p_1} - \omega_{p_2})$; upon multiplication by the phase space, this quantity is seen to be small compared to the main terms. The contributions from $P \rightarrow pp$, the next higher partial waves, are estimated in Appendix D. Near threshold these terms are very small, and nowhere below 500 Mev do they amount to more than 10 percent of the total cross-sections. Therefore, their omission from the main discussion should not affect the estimate of λ_s and λ_p by more than 10 percent.

Total Cross-Sections and Charge Ratios. The total production cross-section, with the matrix element evaluated in the center of mass system, is given by (see (4.2))

$$\sigma = \frac{f^2}{(2\pi)^4} \int_1^{(\omega_{p_1})_{max}} d\omega_{p_1} \int d\Omega_{p_1} \int d\Omega_{p_2} \left[\tfrac{1}{2} Tr |M|^2\right] \frac{p_1 p_2}{k\, \omega_k^4} \left(\frac{E_k}{E_k + \omega_k}\right) \tag{4.6}$$

We are led to the problem of determining ($\omega_{p_1})_{max}$, the upper limit of the phase space integration. Since we are dealing with a static theory, only the meson energies contribute to the energy-conserving delta function in Equation (3.7). But obviously it is better to take into account the initial nucleon energy in determining ($\omega_{p_1})_{max}$, whereas the final nucleon kinetic energy is rather negligible from the fact that the phase-space and p-wave π-π interaction favor large pion energies. Thus, a good approximation to the energy conservation is

$$\omega_k + (E_k - M) = \omega_k + T_k \equiv \omega_k^* = \omega_{p_1} + \omega_{p_2} \tag{4.7}$$

This will simplify our numerical calculations. For predictions which depend on an integration over all phase space, one should expect only a small error. However, the effect of this approximation on the predicted energy distributions may be important.

With the above approximation for the phase space integral, we may write

$$\sigma_{prod}^{TOT} = \frac{4 f^2}{(2\pi)^2} \int_1^{\omega_k^* - 1} d\omega_{p_1} \Big\{ A^2 p_1^2 + B^2 p_2^2 + C^2 k^2 + \frac{2}{9} k^4 \left[|D|^2 p_1^2 + |E|^2 p_2^2 \right] \Big\} \frac{p_1 p_2}{k\, \omega_k^4} \left[\left(\frac{E_k}{\omega_k + E_k} \right) \right] . \tag{4.8}$$

This is to be multiplied by 1/2 if the final mesons are identical. These integrals have been evaluated numerically, using equation (4.4). The parameters λ_s and λ_p were fitted to the energy dependence of the process $\pi^- + p \rightarrow \pi^- + \pi^+ + n$[13]. They correspond to $a_0 = -0.290\mu^{-1}$ and $a_1 = 0.122\mu^{-1}$ where the signs were chosen to give repulsive S-wave π-π scattering and attractive P-wave π-π scattering. This sign of the S-wave π-π scattering comes from analysis of the τ -decay spectrum,[9] assuming the spectrum shape is a result of s-wave π-π scattering with i-spin dependence of (roughly) ϕ^4. The P-wave π-π scattering is chosen attractive to agree with the conjectured resonance in this state,[8] required to improve the understanding of nucleon form factors. The results of these calculations are shown in Figure 5 (($\pi^- + p \rightarrow \pi + \pi^+ + n$)cross-section), Figure 6 (($\pi^- + p$)

total production cross-section), Figure 7 (($\pi^+ + p$) total production cross-section), and Figure 8 ($R\left(\dfrac{\pi^+ + p \rightarrow \pi^+ + \pi^0 + p}{\pi^+ + p \rightarrow \pi^+ + \pi^+ + n}\right)$).[13,23,24,25] The other results are tabulated in Table I.

The cross-sections for reactions which are initiated by $\pi^+ + p$ are computed to a slightly higher energy than those initiated by $\pi^- + p$, since they do not have the rapidly rising $T = 1/2$, $D_{3/2}$ channel, and therefore omission of the initial state interaction above 500 Mev for those processes is not expected to be as serious as for the $\pi^- + p$ processes. These predictions are also in excellent agreement with the $T = 1/2$ production and the inelastic charged production from $\pi^- + p$ of Crittenden et al.[1] up to an energy of about 470 Mev. Together with the agreement expressed by Figures 5-8, this indicates that the production cross-sections may be characterized by two constants with cross-sections having energy dependence and charge ratios as given by our amplitude. This principally depends only on the relative contribution of S-wave and P-wave π-π scattering. The energy dependence of individual partial waves is more critically displayed by the angular distributions of the final mesons.

Angular Distributions. Using the results of Equations (4.2) and (4.7) and (4.3) we may compute the angular distributions of the final mesons. The angular distributions of meson p in the center of mass system is given by

$$\frac{d\sigma}{d\Omega} = x + y\cos\theta + z\cos^2\theta \tag{4.9}$$

where,

$$y = \frac{2f^2}{(2\pi)^3}\int_1^{\omega_k^*-1} d\omega_{p_1}\,\frac{p_1 p_2}{k\,\omega_k^4}\left(\frac{E_k}{\omega_k+E_k}\right)\left[2AC\,p_1 k + \frac{4}{3}(\mathrm{Re}\,D)\,C\,p_1 k^3\right] \tag{4.10}$$

$$z = \frac{2f^2}{(2\pi)^3}\int_1^{\omega_k^*-1} d\omega_{p_1}\,\frac{p_1 p_2}{k\,\omega_k^4}\left(\frac{E_k}{\omega_k+E_k}\right)\left[\frac{1}{3}|D|^2 p_1^2 k^4 + 2(\mathrm{Re}\,D)A\,p_1^2 k^2\right] \tag{4.11}$$

and, of course,

$$x = \frac{\sigma}{4\pi} - \frac{1}{3}z \tag{4.12}$$

The angular distribution of meson p_2 is computed in a similar manner. These coefficients have been evaluated for the process $\pi^- + p \rightarrow \pi^- + \pi^+ + n$. Practically the only angular distribution data available in the energy region of interest are for the π^+ meson in this channel.[13]

The results are illustrated in Figures 9 and 10. The angular distributions predicted for the π^+ meson are compared with the experiments at 317, 371, and 427 Mev in Figurds 11-13. At 317 and 427 Mev, the theory is in qualitative agreement with the experimental points, while at 371 Mev, the agreement is poor. The production of the π^+ meson in the forward direction in the center of mass seems to be evidence for opposite signs of S-wave and P-wave π-π scattering, at least within the assumptions made here. For example, at 364 Mev

$$Y = \left[0.45\left(\frac{\lambda_s}{4\pi}\right)^2 - 2.74\left(\frac{\lambda_s}{4\pi}\right)\left(\frac{\lambda_p}{4\pi}\right)\right] \text{mb/st.} \qquad (4.13)$$

and at 468 Mev

$$Y = \left[0.78\left(\frac{\lambda_s}{4\pi}\right)^2 - 7.8\left(\frac{\lambda_s}{4\pi}\right)\left(\frac{\lambda_p}{4\pi}\right)\right] \text{mb/st.} \qquad (4.14)$$

Therefore, unless λ_s is at least five to ten times larger than λ_p, one cannot hope to have $Y > 0$ if λ_s and λ_p have the same sign. If λ_s is so much larger than λ_p, then one cannot reproduce the sharply rising cross-sections. Thus, λ_s and λ_p must have opposite signs.

The qualitative description of the situation is as follows: If the π-π scattering is isotropic ($\lambda_p = 0$), then both mesons in the final state tend to go forward because the favoring of small momentum transfer to the nucleon. But this effect is small compared to the effect on the angular distribution of a large anisotropy of the π-π scattering ($|\lambda_p| \gtrsim |\lambda_s|$). The p-wave π-π scattering itself gives a $\cos^2\theta$ term and a $\cos\theta$ term by interfering with the s-wave scattering. Thus, if $\lambda_p \sim \lambda_s$, so that the π^- scatters forward, the π^+ will go backwards in the center of mass system, whereas if $\lambda_p \sim -\lambda_s$, the π^+ will scatter forwards. Quantitatively we find that the predicted angular distribution of the π^+ meson has too much forward-backward asymmetry and too much $\cos^2\theta$.

From Equation (4.10), we see that contributions to Y came from $\mathrm{Re}(A^*C)$ and

Re(D*C) (note that A and C are real in our treatment), that is, from interference between $S_{1/2}$ and $P_{1/2}$ and between $D_{3/2}$ and $P_{1/2}$, with roughly 2/3 of the forward-backward asymmetry coming from the $D_{3/2} - P_{1/2}$ interference. As is consistent with the above arguments, the π^- meson is predicted to have a backward peaked angular distribution. The large forward-backward asymmetry would be reduced if the initial state interaction in the $D_{3/2}$, T = 1/2 state were considered. Let us denote δ_f as the effective final state phase; with δ_f being obtained by averaging the final state phase factor weighted by the phase space at fixed total energy. Numerical computations indicate that $\sin\delta_f / \cos\delta_f < 1/3$ below 370 Mev and reaches 1 by 470 Mev. So we may roughly set $\delta_f \approx 0$. Then $\mathrm{Re}(DA^*) \approx A\,|D|\,\cos\delta_{D_{3/2}}$ (4.15) and similarly for the other interference terms involving the $D_{3/2}$ amplitude. Since $\cos\delta_{D_{3/2}}$ becomes small as the $D_{3/2}$, T = 1/2 resonance is approaches, the contribution of the $D_{3/2} - P_{1/2}$ interference term to the forward-backward asymmetries is reduced. Similarly the contribution of the $D_{3/2} - S_{1/2}$ interference to the $\cos^2\theta$ term is reduced. The forward-backward asymmetry may be further decreased by decreasing the ratio of the $S_{1/2}$ to $D_{3/2}$ amplitudes; this ratio may have been overestimated in our calculation. Since the available π-π center of mass energy is lower than the conjectured π-π resonance energy in our energy region, the π-π phase should not have become too large, and thus would not alter the above argument appreciably.[16] In passing, we remark that the core production of $S_{1/2} \rightarrow sp_{1/2}$, with the π^+ in the p-wave, interferes destructively with the cloud term; however, the effect is probably small.

Unfortunately, the experimental data are sparse. Angular distributions of the π^- meson in the $\pi^- + p \rightarrow \pi^- + \pi^+ + n$ in the region of 430 Mev would be useful in verifying the arguments concerning the relative sign of the S-wave and P-wave π-π scattering. The difficult experiment of measuring the recoil nucleon polarization would provide information on the imaginary parts of the interference terms discussed above, and would provide a check on the arguments presented here.

Nucleon Momentum Spectrum. We now wish to examine our predictions for the nucleon spectrum in pion production with some thought being given to the consequences of a pion-pion resonance on this spectrum. The method we will follow will be similar to that of Goebel,[14] with notation similar to Figure 1 of that paper; we will label the final mesons $(\vec{p}_1, \omega_{p_1})$ and $(\vec{p}_2, \omega_{p_2})$ in keeping with the previous discussion. We will evaluate the cross-section in the laboratory system. The cross-section is then given by

$$d\sigma_{prod} = \frac{f^2}{(2\pi)^4} \frac{|M|^2}{k\omega_k^4} \left(\frac{d^3p_1}{\omega_{p_1}}\right)\left(\frac{d^3p_2}{\omega_{p_2}}\right) \delta(\omega_{p_1} + \omega_{p_2} + E_\Delta - W), \qquad (4.16)$$

where $W = \omega_k + M$ is the total energy, $E_\Delta = (\Delta^2 + m^2)^{1/2} = T_\Delta + m$ is the recoil nucleon energy, and M is defined by Equation (4.1). As in Goebel,[14] we describe the final state by the relative momentum of the final mesons <u>in their own center of mass</u> $\vec{\xi} = \frac{1}{2}(p_1 - p_2)_c$ and the total momentum of the mesons $\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{k} - \vec{\Delta}$.

It is straightforward to show that

$$\frac{d^3p_1}{\omega_{p_1}} \frac{d^3p_2}{\omega_{p_2}} = \frac{d^3\xi}{\omega_\xi} \frac{d^3P}{\Omega_\xi}, \qquad (4.17)$$

where

$$\omega_\xi^2 = \xi^2 + \mu^2, \quad \Omega_\xi^2 = \omega_\xi^2 + \frac{1}{4}P^2. \qquad (4.18)$$

The total energy of the mesons is $2\Omega_\xi$, and so energy conservation is expressed by

$$\Omega_\xi = \frac{1}{2}(W - E_\Delta). \qquad (4.19)$$

Squaring this, we find the useful relation

$$\omega_\xi^2 = \frac{1}{2}\left[\vec{k}\cdot\vec{\Delta} - WT_\Delta + \frac{1}{2}\mu^2\right]. \qquad (4.20)$$

We also introduce the velocity of the mesons in their center of mass system

$$v_\xi = \frac{\xi}{\omega_\xi} = \left(\frac{\vec{k}\cdot\vec{\Delta} - WT_\Delta - \frac{3}{2}\mu^2}{\vec{k}\cdot\vec{\Delta} - WT_\Delta + \frac{1}{2}\mu^2}\right)^{1/2} \qquad (4.21)$$

Thus, Equation (4.16) may now be written

$$d\sigma_{prod} = \frac{f^2}{(2\pi)^4} \frac{|M|^2}{k\omega_k^4} \frac{v_\xi}{2} d\phi_\xi \, d(\cos\theta_\xi) \, d^3P. \qquad (4.22)$$

In order to introduce M explicitly, we will limit ourselves to low enough energies so that final state pion-nucleon interactions are unimportant. To first order in the π-π scattering, Equation (3.18) is appropriate. In the coordinate system introduced above, in the non-relativistic approximation for the nucleon

$$M = \vec{\sigma}\cdot\vec{\Delta}\left(\frac{\omega_M}{\omega_\Delta}\right)^2\Big\{\lambda_s\left[\delta_{\beta\gamma}\tau_\gamma+\delta_{\gamma\gamma}\tau_\beta+\delta_{\beta\alpha}\tau_\gamma\right] + 2\lambda_p\left[(\vec{k}+\vec{\Delta})_c\cdot\vec{\xi}\right]\left(\delta_{\gamma\gamma}\tau_\beta-\delta_{\gamma\beta}\tau_\gamma\right)\Big\}. \tag{4.23}$$

The π-π scattering should be expressed in the final pions' own center of mass; therefore, $(\vec{k}+\vec{\Delta})_c$ is to be evaluated in the π-π center of mass. To proceed, we transform this quantity to the laboratory system. We find

$$(\vec{k}+\vec{\Delta})_c = \left[(\vec{k}+\vec{\Delta})\cdot\vec{P}\right]\frac{\vec{P}}{P^2}\left(\frac{\Omega_\xi}{\omega_\xi}-1\right) + (\vec{k}+\vec{\Delta}) - \frac{\vec{P}}{\omega_\xi}(\omega_k+\Omega_\xi). \tag{4.24}$$

When $|M|^2$ is computed and the angular integrations $\int d\phi_\xi\, d(\cos\theta)$ are performed, it is clear that the interference term proportional to $\lambda_s\,\lambda_p$ vanishes. With some algebra, we can show that

$$\int\frac{d\phi_\xi\, d(\cos\theta_\xi)}{4\pi}\left[(\vec{k}+\vec{\Delta})_c\cdot\vec{\xi}\right]^2 = \frac{(\vec{k}+\vec{\Delta})^2\xi^2}{3} + \frac{(k^2-\Delta^2)}{12}v_\xi^2 + \frac{(\vec{k}-\vec{\Delta})^2}{3}(\omega_k+\Omega_\xi)^2 v_\xi^2 - \frac{2}{3}(k^2-\Delta^2)\Omega_\xi(\omega_k+\Omega_\xi)v_\xi^2. \tag{4.25}$$

Direct numerical evaluation shows that v_ξ^2 is sharply peaked at $\approx \Delta = k$. Therefore, in the cross-section $v_\xi^3(k^2-\Delta^2)$ and $v_\xi^3(k^2-\Delta^2)^2$ can be neglected compared to $v_\xi^3(\vec{k}+\vec{\Delta})^2$. One can also neglect $v_\xi^3(\vec{k}-\vec{\Delta})^2$ when one recalls that the final nucleon goes into the forward hemisphere, and so $\vec{\Delta}\approx\vec{k}$. Thus, $(\vec{k}-\vec{\Delta})v_\xi^3$ is small. Then, to a good approximation our cross-section is (to first order in λ_s and λ_p)

$$d\sigma_{prod} = \frac{f^2}{(2\pi)^3}\frac{\Delta^2}{k\omega_\Delta^4}v_\xi\left\{\lambda_s^2A^2+\frac{4}{3}\lambda_p^2B^2(\vec{k}+\vec{\Delta})^2\xi^2\right\}d^3\Delta, \tag{4.26}$$

where A and B are the appropriate isotopic spin factors for the channel under consideration. We can perform the integral over $d\Omega_\Delta$ in Equation (4.26). For the process $\pi^- + p \rightarrow \pi^- + \pi^+ + n$, the result is

$$\frac{d\sigma_{prod}}{d\Delta} = \frac{4f^2\Delta^3}{k^2\omega_\Delta^4}\left\{16\left(\frac{\lambda_s}{4\pi}\right)^2 I_s + \frac{8}{3}\left(\frac{\lambda_p}{4\pi}\right)^2 I_p\right\}, \tag{4.27}$$

where

$$I_s = k\Delta\int_{\xi=0}^{x=1}dx\, v_\xi = \left\{2\omega_M^2 v_M - \mu^2\ln\left|\frac{1+v_M}{1-v_M}\right|\right\} \tag{4.28}$$

$$x = \cos\theta_{k\Delta}$$

$$I_p = \left[(k^2+\Delta^2)I_1 + 2k\Delta I_2\right]$$

$$I_1 = 2k\Delta \int_{\xi=0}^{x=1} dx\, \xi^2 v_\xi$$
$$= \frac{k\Delta}{4}\left\{ 20\,\omega_M^4 v_M^3 - 12\,\omega_M^4 v_M + 6\mu^4 \ln\left|\frac{1+v_M}{1-v_M}\right| \right\} \quad (4.29)$$

$$I_2 = 2k\Delta \int_{\xi=0}^{x=1} dx\, x\, \xi^2 v_\xi$$
$$= \Big\{ \Big[-\tfrac{4}{3}\omega_M^6 v_M^5 + \omega_M^4 v_M^3 (2k\Delta + \tfrac{8}{3}) + \omega_M^2 v_M (5 - 3k\Delta) \quad (4.30)$$
$$- \left(\tfrac{1}{2}\omega_M^2 v_M + \tfrac{5}{2}\omega_M^2 - \tfrac{3}{2}k\Delta\right) \ln\left|\frac{1+v_M}{1-v_M}\right| \Big\}$$

and

$$\omega_M^2 = \tfrac{1}{2}\left[k\Delta - WT_\Delta + \tfrac{1}{2}\mu^2\right] \quad (4.31)$$

while

$$\xi_M^2 = \tfrac{1}{2}\left[k\Delta - WT_\Delta + \tfrac{3}{2}\mu^2\right]. \quad (4.32)$$

Equation (4.27) may be easily modified to take into account the possible p-wave π-π resonance:[15]

$$\frac{d\sigma_{prod}}{d\Delta} = \frac{4f^2\Delta^3}{k^2\omega_\Delta^4}\left\{ 16\left(\frac{\lambda_S}{4\pi}\right)^2 I_S + \frac{8}{3}\left(\frac{\lambda_P}{4\pi}\right)^2 \frac{1}{|D(\xi)|^2} I_P \right\} \quad (4.33)$$

where $D(\xi)$ is the denominator function for P-wave π-π scattering normalized so that $D(0) = 1$. By comparison with the scattering amplitude of Frazer-Fulco,[8] and using Equation (B14), we identify $\frac{2}{3}\left(\frac{\lambda_P}{4\pi}\right) = \left(\frac{\Gamma}{\nu_\lambda}\right)$. We have evaluated Equation (4.33) numerically, using the Frazer-Fulco form for $D(\xi)$, using $\Gamma = 0.04$ and $\nu_\lambda = 1.5$, where $\nu = \xi^2$ and $t = 4(\nu + \mu^2)$, $(t_\lambda \sim 10\mu^2)$. We have chosen to compute our results for an incident meson laboratory energy of 290 Mev. We have also computed the production cross-section for the process $\pi^+ + p \to \pi^+ + \pi^+ + n$, which depends only on λ_s. For that process

$$\frac{d\sigma_{prod}}{d\Delta} = \frac{4f^2\Delta^3}{k^2\omega_\Delta^4}\, 8\left(\frac{\lambda_S}{4\pi}\right)^2 I_S \quad (4.34)$$

the nucleon spectra may be integrated to yield

$$\sigma_{\pi^-\pi^+} = \left[5.12\left(\frac{\lambda_S}{4\pi}\right)^2 + 0.09\right] mb \quad (4.35)$$

and

$$\sigma_{\pi^+\pi^+} = 2.56\left(\frac{\lambda_S}{4\pi}\right)^2 mb. \quad (4.36)$$

The interpolated value for $\sigma_{\pi^+\pi^+}$ is $\gtrsim 0.1$ mb at this energy, so we fix $\lambda_S/4\pi = 0.198$ (compare this with $\frac{\lambda_S}{4\pi} = 0.231$ of the earlier sections of this paper). We note that $\frac{\Gamma}{\nu_\lambda} = \frac{2}{3}\left(\frac{\lambda_P}{4\pi}\right)$ corresponds to $(\lambda_P/4\pi) = -0.04$, while in the other estimate we had ($\lambda_p/4\pi$) = -0.182, which agrees with our view, mentioned earlier, that the value of λ_p obtained from the experimental data should be larger than the threshold scattering length.

The results are illustrated in Figures 14 and 15. From Figure 15, the predicted value of $\sigma_{\pi^+\pi^-}$ is $\approx$ 0.29 mb which lies within experimental errors, as seen in Figure 5. The effect of the π-π resonance can be seen when set D(ξ) = 1 in Figure 14, which shows that the peaking is largely kinematical.

We see that the effect of the π-π resonance is to increase the height of the peak in $\frac{d\sigma}{d\Delta}$ for $\pi^- + p \rightarrow \pi^- + \pi^+ + n$. Equation (4.34) has not been expanded in partial waves, no approximations were made for energy conservation, and final state pion-nucleon interactions have been omitted. This will account for the slightly different estimate of ($\lambda_s/4\pi$) made on the basis of Equation (4.8).

For quantitative comparison of $d\sigma/d\Delta$ with experiment, the parameters of the π-π scattering, Γ and γ_λ , should be known more precisely. Therefore, we can only consider the results of Figure 14 to indicate qualitatively what is expected. It seems to us that it is difficult to deduce the existence of a π-π resonance by means of $d\sigma/d\Delta$ in low energy production experiments for two reasons; namely, the π-π scattering cross-section is "smeared" by the distribution of pion energies in the cloud and the π-π resonance peak will probably fall above Δ_{max} in $\frac{d\sigma}{d\Delta}$ for energies low enough for the pion-nucleon interactions to be unimportant. Finally, Equation (4.36) indicates the difficulty in extrapolating $\frac{d\sigma}{d\Delta}$ to the pole $\Delta^2 = -\mu^2$ because of the rapid variation with Δ near $\Delta = 0$ and because of the uncertain ratio of P-wave and S-wave π-π scattering.

5. Single Partial Wave Amplitudes and Conclusions

With the aid of equation (4.4a), we have evaluated the single partial wave production amplitudes. These results are shown in Table II. For comparison the estimated cloud contributions of $P_{3/2} \rightarrow pp$ and $P_{1/2} \rightarrow pp$ are also shown.

We note that the $D_{3/2}$, $T = 1/2$ cross-section exceeds the unitarity limit at ~550 Mev; $D_{3/2}$, $T = 3/2$ at ~820 Mev; $T = 1/2$, $S = 1/2$ at ~550 Mev; and $T = 3/2$, $S = 1/2$ at ~750 Mev. According to our conjecture, one would expect the $T = 1/2$, $D_{3/2}$ amplitude to have a resonance in the neighborhood of 550 Mev. This is not in unreasonable agreement with the experimental value of 610 Mev, considering the approximations made. We would also expect a resonance in the $T = 3/2$, $D_{3/2}$ amplitude in the region of 820 Mev. This is not inconsistent with the suggestion of Carruthers[3] in his analysis of the very small cross-section $\pi^- + p \rightarrow \pi^0 + n$ in the resonance region. We note that the S-wave production becomes surprisingly large, but emphasize that it is premature to predict S-wave resonances in view of our underestimate of the $D_{3/2} - S_{1/2}$ ratio[12] and the still uncertain predictions of the S-wave scattering theory. It is a little disappointing not to find the $D_{3/2} - S_{1/2}$ ratio larger, since the attempt to make a more quantitative study of the model of R. F. Peierls[5] (with the dominance of the $D_{3/2}$ state, with an S-wave and the (3,3) pion-nucleon isobar) originally led to this investigation. Our difficulty is directly related to the smallness of the off-the-energy shell part of the rescattering integral, which does not sufficiently compensate for the loss of the direct amplitude when the rescattered pion is near resonance. The short-ranged contributions to both scattering and production also require further consideration.

Walker, et al., have made a phase shift analysis of pion-nucleon experiments in the energy region 350-600 Mev[17] including the inelastic data. Comparing Table I of that paper with Table II above leads to some interesting observations. Except for the above mentioned difficulties with S-wave production, our production cross-sections are qualitatively in agreement with the phase shift analysis. The phases δ_{13}, δ_{33}, α_1, and α_{11} are positive and rapidly rising (δ_{TJ} for D-waves and α_{TJ} for P-waves). Production from T = 3/2, $P_{3/2}$ is small as might be expected since the (3,3) total cross-section nearly "exhausts" the sum rule.[6] Meson production from T = 3/2 is small, and α_3 is negative, presumably still dominated by the short-ranged interactions although the meson production in the S-states is not negligible.

For a detailed prediction of the single partial wave π-N scattering amplitudes, one should investigate the single partial wave dispersion relations. For example, Bowcock, Cottingham, and Lurie,[18] have discussed low energy π-N scattering, taking into account π-π scattering effects in the $\pi + \pi \rightarrow N + \bar{N}$ contribution to the Mandelstam amplitude, using the approximation discussed by Cini and Fubini.[9] They find that the P-wave π-π scattering contribution to T = 1/2, $D_{3/2}$ is attractive and to the T = 3/2, $D_{3/2}$ is repulsive, while that from the "core terms," including the (3,3) resonance in the crossed-terms, as discussed by Chew, et al.,[6] is repulsive for T = 1/2, $D_{3/2}$ and attractive for T = 3/2, $D_{3/2}$. Since experimentally both $D_{3/2}$ amplitudes have positive phases, we believe that the large inelastic $D_{3/2}$ amplitude must be included in any discussion of the D phases, and furthermore, our calculations indicate that the "knock-on" production process is the dominant contributor to this amplitude and the other low-energy meson production amplitudes.

The third π-N resonance is expected to develop in a way very similar to that discussed for the second resonance. The two pion state with both mesons in p-waves with respect to the nucleon may be expected to be dominant. Quantitative calculation is expected to be even more difficult than in this case since all three pairs of final particles have strong interactions. Finally, we comment that it seems plausible that the broad maximum in the T = 3/2 total cross-section above 1 Bev is due to the superposition of many states coming from both two pion and three pion intermediate states.

To summarize, we have seen with the conjecture of a rapidly rising effective interaction leading to a resonance, that we are able to understand in some detail low energy pion production and the origin of the second pion-nucleon resonance.

APPENDIX A

Core Production into ss and sp States: The contents of this appendix are included for completeness. Let us consider the perturbation result for $\pi N \rightarrow \pi\pi N$ in the lowest order of pion coupling and of nucleon recoil (i.e., 1/M). For instance, the matrix element for $\pi^- + p \rightarrow \pi^- + \pi^+ + n$ is

$$\mathcal{M} = \left(\frac{g}{2M}\right)^3 \frac{1}{\sqrt{2\omega_k V 2\omega_{p_1} V 2\omega_{p_2} V}} 4\sqrt{2}\, i \left[\vec{\sigma}\cdot\vec{k}\left(\frac{-\omega_{p_1}}{\omega_k} + \frac{\vec{k}\cdot(\vec{p}_1+\vec{p}_2)}{\omega_k^2}\right) - \vec{\sigma}\cdot\vec{p}_2 + \frac{\omega_k}{\omega_{p_1}}\vec{\sigma}\cdot\vec{p}_1\right] \tag{A1}$$

where we have kept terms in the final meson momenta only to zero and first order and where $\vec{k}$ is the initial momentum and $\vec{p_1}$ and $\vec{p_2}$ are the momenta of the final π^+ and π^- meson, respectively. The cross-section is then

$$d\sigma = \frac{128 f^6}{\mu^6 k}\left[V^2\omega_{p_1}^2 + p_2^2\left(1 - \frac{2}{3}V^2 + \frac{V^4}{3}\right) + p_1^2\left(\frac{\omega_k^2}{\omega_{p_1}^2} + \frac{2}{3}\frac{\omega_k}{\omega_{p_1}}V^2 + \frac{1}{3}V^4\right]\right. \tag{A2}$$

where $f^2 = 0.08$ and $V = k/\omega_k$. Very roughly we can set $V \approx 1$, but consider the final mesons to be non-relativistic, $\omega_{p_1} \approx \omega_{p_2} \approx \mu$; the total cross-section thus becomes

$$\sigma \simeq 32\pi \frac{f^6}{\mu^6}\frac{Q^2}{k}\left\{1 + \left[\left(\frac{\omega_k}{\mu}\right)^2 + \frac{2}{3}\right]\frac{Q}{3\mu} + \cdots\right\} \tag{A3}$$

where Q is the kinetic energy in the final state in the center of mass system. At 300 Mev, (A3) thus becomes 0.6 mb. This is to be compared with the experimental value of ~0.55 mb!

This agreement is fortuitous. If one analyzes the terms which enter into the perturbation matrix element (A1), one finds that most of them involve an S-wave pair vertex, which can be related (more or less) to S-wave scattering.

For instance, the terms contributing to $P_{1/2} \rightarrow ss_{1/2}$ are shown in Figure 16. The first two actually contribute nothing near threshold: The contributions of the i-spin independent (non-charge exchange) pair vertex cancel (equal but opposite energy denominators); the amplitude from the i-spin dependent (charge exchange) pair vertex is proportional to the difference of the meson energies, which introduces an additional factor Q into the matrix element. In Figure 16c, if it is legitimate to approximate the pair vertex by the real T = 3/2 S-wave scattering, this term is reduced by a factor of 20 from the Born approximation. No such semi-empirical argument can be made for the fourth term, but what little is understood about π-N S-wave interactions suggests that this term too is actually much smaller than the perturbation value.

Consider now the $S_{1/2} \rightarrow sp_{1/2}$ production (with the π^+ in the p-wave), which contributes the bulk of our calculated cross-section at 300 Mev. The dominant diagrams are shown in Figure 17. The sum of these terms is proportional to the charge exchange S-wave scattering amplitude. This is smaller than the perturbation value by only a factor of 1/2 for zero momentum scattering, but this ratio to the perturbation value becomes smaller at higher energies. Just how much of this decrease exists in the off-the-energy-shell process which we have here is uncertain. However, evidence from the theory of the S-wave production in $NN \rightarrow \pi D$, which involves the same type of off-the-energy-shell scattering, seems to indicate that the decrease occurs off as well as on the energy shell.[20] Thus, we estimate that the cross-section for $S_{1/2} \rightarrow sp(\pi^+$ in the p-wave) is no larger than 1/10 of the perturbation value.

One might note the occurrence of the term $(\vec{\sigma} \cdot \vec{k})\vec{K}_{\circ} \cdot (\vec{p}_1 + \vec{p}_2)$ in M of equation of (A1); this contributes to $D_{3/2} \rightarrow sp_{3/2}$, the state which has final state enhancement. The diagram which contributes to this is that shown in figure 18, with the nucleon kinetic energy in the energy denominator expanded to first order, where the pair vertex is the non-charge dependent term. It is clear that this diagram does not exist for π^+p production, i.e., in the T = 3/2 state. Thus, in lowest order of recoil, one finds $D_{3/2} \rightarrow sp_{3/2}$ production, and it occurs only in the T = 1/2 state - a very nice result. However, since the experimental non-charge exchange scattering amplitude is 1/100 of the perturbation value, it appears that this production term is quite negligible. It is true that the S-wave pair <u>production</u> vertex is not necessarily directly connected with the scattering vertex (see next section), but it would be surprising if they differed by a factor of 100.

Thus, we conclude that the cross-section for meson production near threshold (ss and sp states) due to direct interaction with the nucleon core is quite small, and inadequate to fit the observed $\pi^- p$ results.

We conclude this section with a discussion of the relative sign of the core and pion cloud contributions to $S_{1/2} \rightarrow sp_{1/2}$ production in $\pi^- + p \rightarrow \pi^- + \pi^+ + n$. The dominant diagrams are those of Figure 17. Quantitatively, the Born approximation matrix element for the core contribution is

$$\mathcal{M}_{core} = 2\sqrt{2}\left(\frac{g}{2M}\right)^3 \frac{(\omega_k + \omega_{p_2})\, i\vec{\sigma}\cdot\vec{p}_1}{\sqrt{2\omega_k \mathcal{V}\, 2\omega_{p_1}\mathcal{V}\, 2\omega_{p_2}\mathcal{V}}\;\;\omega_{p_1}} \tag{A4}$$

where the notation of (A1) has been retained. The reduced matrix element, as defined by Equation (4.1), is

$$M_{core} = -\sqrt{2}\;(4\pi)\frac{f^2}{\mu^2}\;\frac{(\omega_k + \omega_{p_2})}{\omega_{p_1}}\;\omega_k^2\;\vec{\sigma}\cdot\vec{p}_1 \tag{A5}$$

By comparison, Equation (4.3), the cloud contribution to $S_{1/2} \rightarrow sp_{1/2}$ for this channel is

$$M_{cloud} = -\frac{2\sqrt{2}}{3}\left(\lambda_s + k^2\lambda_p\right)\vec{\sigma}\cdot\vec{p}_1 - \frac{2\sqrt{2}}{3}\left(\lambda_s - k^2\lambda_p\right)\vec{\sigma}\cdot\vec{p}_2 \tag{A6}$$

The core contribution only interferes with that cloud term which has the π^+ meson in the p-wave. This will reduce the forward peaking of the π^+, but it has no effect on the backward peaking of the π^-. However, the quantitative estimate of the core production (A5) is uncertain due to the difficulty in evaluating the S-wave pair production vertices. This is discussed in the next section.

Connection Between S-wave Pair Production and Scattering Vertices. Assume that the S-wave scattering amplitude at low energies can be expanded in powers of ω (neglecting the influence of the branch points, $\omega = \pm\mu$). Then we can write ($\mu = 1$):

$$\langle \pi^+ p | f | \pi^+ p \rangle = a + b(\omega_1 + \omega_2) + c(\omega_1^2 + \omega_2^2) + d\,\omega_1\omega_2 + \cdots \tag{A7}$$

$$\langle \pi^- p | f | \pi^- p \rangle = a - b(\omega_1 + \omega_2) + c(\omega_1^2 + \omega_2^2) + d\,\omega_1\omega_2 + \cdots \tag{A8}$$

$$\langle \pi^+\pi^- p | f | p \rangle = a + b(\omega_+ - \omega_-) + c(\omega_+^2 + \omega_-^2) - d\,\omega_+\omega_- + \cdots \tag{A9}$$

where (A8) and (A9) follow by "crossing" from (A7).

We see that (at least in principle) the coefficients a, b, 2c + d, etc., can be deduced from experimental π-N scattering; but to predict the S-wave pair production vertex near threshold, the coefficients a, 2c - d, etc., are needed. Only the <u>sum</u> of 2c and d is determined by real scattering.

On theoretical grounds one could invoke the principle that the coefficients of successively higher powers of ω should decrease by factors of the order of μ/M. This is well known to be incompatible with the experimental data.

It is not clear even what the <u>sum</u> of 2c and d actually is. Experimentally, $\langle \pi^+, p | f | \pi^+, p \rangle$ appears to be constant, but the size of the charge exchange cross-section shows that the odd coefficients (b, d, etc.) are large; therefore, so must be the even.

APPENDIX B

S-wave π-π Scattering. We take the matrix element of $H_I = \frac{\lambda_s}{4}(\phi_\sigma\phi_\sigma)^2$ between an initial state of two pions of quantum numbers $(\vec{a},\alpha),(\vec{b},\beta)$ and final state $(\vec{c},\gamma)$ and $(\vec{d},\delta)$. These quantum numbers are summarized as a, b, c, and d. Then

$$\mathcal{M} = \langle cd | \int d^3r \, \frac{\lambda_s}{4} (\phi_\sigma\phi_\sigma)^2 | ab \rangle$$

$$= \frac{2\lambda_s}{(\sqrt{2\omega V})^4} \left(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta} \right) \qquad \text{(B1)}$$

One can verify that

$$P_0 = \tfrac{1}{3}\delta_{\alpha\beta}\delta_{\gamma\delta} \qquad \text{(B2)}$$

$$P_1 = \tfrac{1}{2}\left(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}\right) \qquad \text{(B3)}$$

$$P_2 = \tfrac{1}{2}\left(-\tfrac{2}{3}\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}\right) \qquad \text{(B4)}$$

are the projection operators for π-π scattering in the T = 0, 1, and 2 states, respectively. Then

$$\mathcal{M}_s = \frac{2\lambda_s}{(\sqrt{2\omega V})^4} (2P_2 + 5P_0) \qquad \text{(B5)}$$

The cross-section for π-π scattering in an S-state, to lowest order in λ_s is

$$\sigma_s = \left(\frac{\lambda_s}{4\pi}\right)^2 \frac{8\pi}{\omega_k^2} \frac{[4P_2 + 25P_0]}{16} \qquad \text{(B6)}$$

where ω_k is the center of mass energy of each pion. From the condition that σ_s must not exceed $\frac{8\pi}{k^2}$, we see that if $(\lambda_s/4\pi)$ is as large as of the order of unity, the cross-section must, in fact, decrease faster than $1/\omega^2$ so that the effective S-wave π-π interaction cannot be well described by $\frac{\lambda_s}{4}(\phi_\sigma\phi_\sigma)^2$. Our fit of near threshold data gives $(\lambda_s/4\pi) = 0.231 < 1$ and so, in fact, does not violate this condition. This value of λ_s corresponds to $a_0 = \left(\frac{\sin\delta_0}{k}\right)_{k=0} = -0.290\mu^{-1}$ and $a_2 = \left(\frac{\sin\delta_2}{k}\right)_{k=0} \cong -0.116\mu^{-1}$ and thus, $(a_0 - a_2) = -0.174\mu^{-1}$ all in units of μ.

From analysis of the τ-decay spectrum, assuming that <u>all</u> the asymmetry comes from the scattering of pion pairs in S-waves, Khuri and Treiman find $(a_0 - a_2) \simeq -0.7$, while Sawyer and Wali find $(a_0 - a_2) \simeq -0.32$[9]; the difference in the prediction reflects different approximations made in solution of the integral equation involved. However, large uncertainties probably should be assigned all of the above values, due to the uncertainty in assumptions made although it is reassuring that our analysis and the τ -decay both seem to require repulsive S-wave π-π scattering.

In the notation of Equation (3.40), the i-spin operator occurring in the knock-on meson production via the S-wave π-π interaction is

$$\langle \beta\gamma | \mathcal{A} | \alpha \rangle = \delta_{\beta\gamma}\tau_\alpha + \delta_{\gamma\alpha}\tau_\beta + \delta_{\alpha\beta}\tau_\gamma \tag{B7}$$

The isotopic spin dependence is determined by the result

$$\sum_{\beta\gamma} \langle \alpha' | \mathcal{A}^+ | \beta\gamma \rangle \langle \beta\gamma | \mathcal{A} | \alpha \rangle = 10\, {}_{\alpha'}I_\alpha^{3/2} + 25\, {}_{\alpha'}I_\alpha^{1/2} \tag{B8}$$

where

$${}_{\alpha'}I_\alpha^{3/2} = \delta_{\alpha'\alpha} - \frac{1}{3}\tau_{\alpha'}\tau_\alpha \quad \text{and} \quad {}_{\alpha'}I_\alpha^{1/2} = \frac{1}{3}\tau_{\alpha'}\tau_\alpha$$

are the projection operators for T = 3/2 and T = 1/2, respectively.

The projection of index β into a T = 3/2 state results in

$$\langle \beta\gamma | {}_{3/2}\mathcal{A} | \alpha \rangle = \sum_{\beta'} {}_\beta I_{\beta'}^{3/2} \langle \beta'\gamma | \mathcal{A} | \alpha \rangle = \delta_{\beta\gamma}\tau_\alpha + \delta_{\beta\alpha}\tau_\gamma - \frac{2}{3}\delta_{\alpha\gamma}\tau_\beta \tag{B9}$$

(Of course, because of the symmetry of $\langle \beta\gamma | \mathcal{A} | \alpha \rangle$, we have $\langle \beta\gamma | {}_{3/2}\mathcal{A} | \alpha \rangle = \langle \alpha\gamma | \mathcal{A}_{3/2} | \beta \rangle$, etc.) Corresponding to (B8), we find

$$\sum_{\beta\gamma} \langle \alpha' | ({}_{3/2}\mathcal{A})^+ | \beta\gamma \rangle \langle \beta\gamma | {}_{3/2}\mathcal{A} | \alpha \rangle = \frac{5}{3}\, {}_{\alpha'}I_\alpha^{3/2} + \frac{50}{3}\, {}_{\alpha'}I_\alpha^{1/2} \tag{B10}$$

The squared matrix elements of (B7) and (B9) between individual charge states are given in Table III.

<u>P-wave π-π Scattering.</u> We take the P-wave matrix element for the scattering of meson a and d into mesons b and c to be equal to

$$\mathcal{M}_p = \frac{2\lambda_p}{(\sqrt{2\omega V})^4}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})(\underset{\sim}{a} - \underset{\sim}{d})\cdot(\underset{\sim}{b} - \underset{\sim}{c}) \tag{B11}$$

(where $\underset{\sim}{a}$ is a momentum four vector) in the center of mass system.

The P-wave π-π cross-section, to lowest order in λ_p is

$$\sigma_p = \frac{4}{3}\left(\frac{\lambda_p}{4\pi}\right)^2 \frac{8\pi k^4}{\omega_k^2}\rho_1 = \frac{3(8\pi)}{k^2}\sin^2\delta_1 \tag{B12}$$

and it immediately follows that

$$a_1 = \left(\frac{\sin\delta_1}{k^3}\right)_{k=0} = -\left(\frac{2}{3}\right)\left(\frac{\lambda_P}{4\pi}\right) \tag{B13}$$

and our value $(\lambda_P/4\pi) = -0.182$ means $a_1 = 0.122$ in units of μ.

The i-spin matrix element in the knock-on meson production from P-wave π-π scattering is

$$\langle\beta\gamma|\mathcal{A}|\alpha\rangle = (\delta_{\alpha\beta}T_\gamma - \delta_{\alpha\gamma}T_\beta) \tag{B14}$$

The i-spin dependence of the meson production depends on

$$\sum_{\beta\gamma}\langle\alpha'|\mathcal{A}^+|\beta\gamma\rangle\langle\beta\gamma|\mathcal{A}|\alpha\rangle = 2\,{}_{\alpha'}I_\alpha^{3/2} + 8\,{}_{\alpha'}I_\alpha^{1/2} \tag{B15}$$

Projection of meson β into a T = 3/2 state leads to

$$\langle\beta\gamma|{}_{3/2}\mathcal{A}|\alpha\rangle = \left(\delta_{\alpha\beta} - \frac{1}{3}T_\beta T_\alpha\right)T_\gamma \tag{B16}$$

and

$$\sum_{\beta\gamma}\langle\alpha'|({}_{3/2}\mathcal{A})^+|\beta\gamma\rangle\langle\beta\gamma|\mathcal{A}|\alpha\rangle = \frac{5}{3}\,{}_{\alpha'}I_\alpha^{3/2} + \frac{8}{3}\,{}_{\alpha'}I_\alpha^{1/2} \tag{B17}$$

The squared matrix elements of (B14) and (B17) between individual charge states are given in Table IV.

APPENDIX C

Solution of the Integral Equation. The integral equation obeyed by a causal production amplitude is

$$T(\omega) = T^{B}(\omega) + \frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \frac{\mathrm{Im}\, T(\omega')}{\omega' - \omega - i\varepsilon} \tag{C1}$$

where the actual production amplitude is $T(\omega)$ multiplied by appropriate factors of momentum, according to the orbital momenta of the production process. Taking all interactions to just first order, except for the final state interaction, one has $T = e^{i\delta} |T|$, where δ is the scattering phase shift of the final state interaction. We may write

$$T^{B}(\omega) = \frac{1}{\pi}\int_{-\infty}^{a} d\omega' \frac{\mathrm{Im}\, T^{B}(\omega')}{\omega' - \omega - i\varepsilon} \tag{C2}$$

where $a < \mu$; therefore, $T^{B}(\omega)$ is real in the physical region of the production process.

The integral equation (C1) may be solved by the N/D method.[8] We recall that the scattering amplitude for the final state interaction can be written as $f(\omega) = N(\omega)/D(\omega)$, where $D(\omega)$ is analytic except for a cut starting from threshold, on which its phase is $\mp\delta(\omega)$, the scattering phase shift, while $N(\omega)$ is analytic except perhaps below threshold ("left hand cut"). Clearly, $D(\omega) = e^{-\frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \frac{\delta(\omega')}{\omega' - \omega - i\varepsilon}}$ [21] (In our example the physical cuts of $f(\omega)$ and $T(\omega)$ run from μ to ∞.) Therefore, we can write down the solution to (C1) at sight, which has the correct analyticity properties.

$$T(\omega) = \frac{1}{D(\omega)}\, \frac{1}{\pi}\int_{-\infty}^{a} d\omega' \frac{D(\omega')\, \mathrm{Im}\, T^{B}(\omega')}{\omega' - \omega - i\varepsilon} \tag{C3}$$

This can be transformed into a quite different looking form. We first observe

$$\frac{1}{\pi}\int_{-\infty}^{a} d\omega' \frac{D(\omega')\, \mathrm{Im}\, T^{B}(\omega')}{\omega' - \omega - i\varepsilon} = \frac{1}{2\pi i}\int_{\Gamma_1} d\omega' \frac{D(\omega')\, T^{B}(\omega')}{\omega' - \omega} \tag{C4}$$

where Γ_1 is the contour shown in Figure 19. The cuts illustrated are those of the analytic function $D(\omega)T^{B}(\omega)$. We can now "expand" the contour into Γ_2, and assume that the integral around ∞ goes to zero (subtractions should be made as necessary if this is not true). Thus

$$\begin{aligned} \frac{1}{2\pi i}\int_{\Gamma_1} d\omega' \frac{D(\omega')\, T^{B}(\omega')}{\omega' - \omega} &= D(\omega)T^{B}(\omega) + \frac{1}{2\pi i}\int_{\Gamma_2} d\omega' \frac{D(\omega')\, T^{B}(\omega')}{\omega' - \omega} \\ &= D(\omega)T^{B}(\omega) - \frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \frac{\mathrm{Im}\, D(\omega')\, T^{B}(\omega')}{\omega' - \omega - i\varepsilon} \end{aligned} \tag{C5}$$

where we made use of the fact that $T^B(\omega)$ is real for $\omega > \mu$. Now, since

$$\mathrm{Im}\, D(\omega) = N(\omega)\, \mathrm{Im}\left[\frac{1}{f(\omega)}\right] \tag{C6}$$

$\cong -kN(\omega)$ in the elastic approximation for $f(\omega)$,

for $\omega > \mu$, we can finally write

$$T(\omega) = T^B(\omega) + \frac{1}{D(\omega)}\frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \, \frac{T^B(\omega')\left[k' N(\omega')\right]}{\omega' - \omega - i\varepsilon} \tag{C7}$$

If $N(\omega)$ has only a simple pole at $\omega = 0$ (a reasonable approximation to the final state π-N interaction in the T = 3/2, J = 3/2 state), then the solution given by (C7) reduces to that of Equation (3.20) of the text. The result of (C7) is identical to that given by Omnes,[21] while that of (3.20) had been deduced directly by Chew and Low by consideration of the analytical properties of $T(\omega)$.[3]

The extension to the case of final and initial interactions is trivial. Now $T = e^{i(\delta_1+\delta_2)}\,|T|$, where δ_1 is the phase of the initial interaction and is the phase of the final interaction. For reactions where the cuts for the initial and final state are identical then a repetition of the above derivation leads to the solution

$$T(\omega) = T^B(\omega) - \frac{1}{D_1(\omega)D_2(\omega)}\frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \, \frac{T^B(\omega')\, \mathrm{Im}\left[D_1(\omega')D_2(\omega')\right]}{\omega' - \omega - i\varepsilon} \tag{C8}$$

$$\mathrm{Im}\left[D_1(\omega)D_2(\omega)\right] = -\sin(\delta_1+\delta_2)\,|D_1(\omega)|\cdot|D_2(\omega)| \tag{C9}$$

$$|D(\omega)| = N(\omega)\frac{1}{|f(\omega)|} \qquad \omega > \mu \tag{C10}$$

and

$$f(\omega) = \frac{e^{i\delta}\sin\delta}{k} \qquad \text{for } \omega > \mu \tag{C11}$$

Therefore,

$$T(\omega) = T^B(\omega) + \frac{1}{D_1(\omega)D_2(\omega)}\frac{1}{\pi}\int_{\mu}^{\infty} d\omega' \, \frac{k_1' k_2'\, T^B(\omega')N_1(\omega')N_2(\omega')\sin(\delta_1+\delta_2)}{(\omega' - \omega - i\varepsilon)\sin\delta_1 \sin\delta_2} \tag{C12}$$

Further approximations for $N(\omega)$ may be made if desired.

From (C12), we notice that the production amplitude will have a pole whenever either $D_1(\omega) = 0$ or $D_2(\omega) = 0$, i.e., at resonances of the initial or final state interaction. For a final state with three or more particles, a resonant final state interaction of two of these particles should not imply a resonance as a function of the total center of mass energy since it does not give us a pole in the total energy plane.

APPENDIX D

Partial Wave Amplitudes. The projection into single partial wave amplitudes is accomplished by elementary integration, projecting out the partial wave amplitudes of the final mesons, using the following projection operators.

$${}_{a}P_{a'}^{S} = 1 \qquad \text{for S-wave mesons}$$

$${}_{a}P_{a'}^{1/2} = \vec{\sigma}\cdot\vec{a}\ \vec{\sigma}\cdot\vec{a}' \qquad \text{for } P_{1/2} \text{ mesons} \qquad \text{(D1)}$$

$${}_{a}P_{a'}^{3/2} = 3\vec{a}\cdot\vec{a}' - \vec{\sigma}\cdot\vec{a}\ \vec{\sigma}\cdot\vec{a}' \qquad \text{for } P_{3/2} \text{ mesons}$$

These operators are sufficient to allow us to project the angular momentum states of interest. For the on-the-energy-shell transitions, on the average, both p and q are considerably smaller than k; for these transitions we have expanded in powers of q/k and p/k and kept only the leading terms. This is the origin of the terms in (4.3) and (4.4a). However, in the rescattering integral, we must consider the matrix element for large values of p.

The $D_{3/2}$ matrix element which originates from P-wave π-π scattering is found to be proportional to

$$\left[\vec{k}\cdot\vec{p}\ \vec{\sigma}\cdot\vec{k} - \tfrac{1}{3}k^2\vec{\sigma}\cdot\vec{p}\right] K(k,q,p)\left[\omega_k^2 + p^2 - 2\omega_k(\omega_p - \omega_q)\right] \qquad \text{(D2)}$$

The kernal K(k,q,p), defined by (3.28) and (3.29), which occurs in the rescattering by both the S-wave and P-wave π-π interaction, requires some further discussion.

The Kernel K(k,q,p). We first make the approximation of letting $q \to 0$ in K(k,q,p). This means the neglect of q^2 against k^2 or p^2. This is not a bad approximation for our purposes, since $q^2 << k^2$ in the region of interest; and since the important contribution to the rescattering integral in (3.31) comes from the region $p' \approx k$, we can say $q^2 << (p')^2$ also. Finally, although q is on the average not much smaller than p, they are both much smaller than k, so that (in the on-the-energy-shell matrix element) both q and p can be allowed to approach zero in K(k, q, p) without excessive error.

By elementary integration

$$\mathcal{K}(k,q,p) \equiv \mathcal{K}(k,p)$$
$$= 3\left\{\left[\frac{-1}{2k^2p^2} + \frac{3\omega^2}{8k^4p^2}\right] + \ln\left(\frac{\omega^2_{k-p}}{\omega^2_{k+p}}\right)\left[\frac{3\omega^4}{32k^5p^3} - \frac{\omega^2}{8k^3p^3} - \frac{1}{8k^3p}\right]\right\} \quad \text{(D3)}$$

where

$$\omega^2 = k^2+p^2+\mu^2 \quad \text{and} \quad \omega^2_{k\pm p} = k^2+p^2 \pm 2kp+\mu^2$$

the following properties of K(k,p) may be easily obtained:

$$\mathcal{K}(k,p) \xrightarrow[p\to 0]{} \frac{1}{\omega_k^4}\left\{\left[2 - \frac{4}{5}x^2 - \cdots\right] - \left(\frac{\mu^2}{k^2}\right)\left[4x^2 + \cdots\right] + \cdots\right\} \quad \text{(D4)}$$

where

$$x \equiv p/k$$

and

$$\mathcal{K}(k,p) \xrightarrow[p\to \infty]{} \frac{1}{k^4}\left\{\left[\frac{2}{5}\frac{1}{x^4} + \frac{4}{35}\frac{1}{x^6} + \cdots\right] + \left(\frac{\mu}{k}\right)^2\left[\frac{4}{5}\frac{1}{x^6} + \cdots\right] + \cdots\right\} \quad \text{(D5)}$$

For the region of interest $(\frac{\mu}{k})^2 << 1$, then

$$\mathcal{K}(k,p) \approx \frac{1}{k^4}\psi(x)$$

with

$$\psi(x) = \frac{3}{8}\left[3 - \frac{1}{x^2} + \frac{(1+3x^2)(1-x^2)}{2x^3}\ln\left|\frac{1+x}{1-x}\right|\right] \quad \text{(D6)}$$

where the expansions for small and large x are

$$\psi(x) \xrightarrow[x\to 0]{} \left[2 - \frac{4}{5}x^2 - \frac{6}{35}x^4 - \cdots\right] \quad \text{(D7)}$$

$$\psi(x) \xrightarrow[x\to \infty]{} \left[\frac{2}{5}\frac{1}{x^4} + \frac{4}{35}\frac{1}{x^6} + \frac{2}{35}\frac{1}{x^8} + \cdots\right]$$
$$\psi(1) = 3/4 \; ; \; \psi'(1) = -\ln\infty \quad \text{(D8)}$$

These properties are illustrated in Figure 20. Terms of order $(\mu/k)^2$, if included in $\psi(x)$ would make $\psi(x)$ decrease more rapidly for small x, and less rapidly for large x. The properties illustrated above show that the factor K(k, q, p')/K(k,q,p) is roughly the character of a cutoff at $p' \gtrsim k$.

We must consider the rescattering integral

$$\frac{1}{\pi}\int_0^\infty dp' \frac{p'^4}{\omega_{p'}^2(\omega_{p'}-\omega_p - i\varepsilon)}\frac{\mathcal{K}(k,p')}{\mathcal{K}(k,p)}\left[\omega_k^2 + p'^2 - 2\omega_k\omega_{p'} + 2\omega_k\omega_q\right] \equiv S(k,q) \quad \text{(D9)}$$

We will approximate

$$\mathcal{K}(k,p) = \frac{2}{\omega_k^4} \quad \text{for } p < k$$
$$= 2/5p^4 \quad \text{for } p > k \qquad \text{(D10)}$$

Then, roughly we find for ω_k in our range of interest

$$S(-k,q) = \frac{\omega_k^2}{\pi}\left\{\frac{\omega_k^2}{2} + \omega_k^2\left[-\frac{7}{60} + \ln\left(\frac{\Lambda}{\omega_k}\right)\right]\right\}$$
$$\approx \frac{\omega_k^2}{\pi}\left(\frac{\omega_k^2}{2}\right) \qquad \text{(D11)}$$

where Λ is a cut-off of the order of the nucleon mass. This value found (D11) is what one would obtain for a sharp cutoff at $\omega_p = \omega_k$. The evaluation of the rescattering integral for the S-wave π-π scattering is similar. These results justify equation (3.31) and (3.31a) when K(k,p')/K(k,p) is treated as a sharp out-off.

Higher Angular Momentum States. In the body of this paper, we have discussed the contribution of the threshold amplitudes $P_{1/2} \rightarrow ss$, $S_{1/2} \rightarrow sp_{1/2}$, and $D_{3/2} \rightarrow sp_{3/2}$. In this section, we wish to make an estimate of the omitted angular momentum states, principally $P_{1/2} \rightarrow pp$, $P_{3/2} \rightarrow pp$, and $P_{3/2} \rightarrow sd_{3/2}$. These amplitudes are, of course, quite small near threshold because of phase space considerations. As the available energy increases, these transitions may become important since they rise as Q^4, where Q is the available energy.

A calculation of these amplitudes is complicated by the fact that in P→ pp, both of the final mesons may have strong T = 3/2, $p_{3/2}$ interactions in addition to strong initial state interactions in the T = 3/2, $P_{3/2}$ state. Using our calculated enhancements for $D_{3/2} \rightarrow sp_{3/2}$ as a guide, we do not expect these strong rescatterings to strongly modify an estimate of these transitions based on the knock-on matrix element alone. Of course, in addition, there are purely core transitions for P → pp, which have been estimated by the various static calculations,[22] which are of the same order of magnitude as the P→ pp cloud terms in the T = 1/2 state, and a factor of 3-4 larger than the P→ pp cloud terms in the T = 3/2 state. We have estimated the matrix element for P→ pp for the knock-on process. The contribution to the matrix element from P-wave π-π scattering is expected to be $\approx \omega_k^2$ larger than that of the S-wave π-π scattering, so we have only considered the P-wave part. Using the value of λ_p found in the text, we obtain the results of Table IX. We see from Table IX that the corrections of σ_p to the total cross-sections is 10 percent or less in our energy region. For example, reference to Figure 5 indicates that the addition of the corrections of Table IXC would not alter the agreement with experiment. The omission of a detailed discussion of the $P \longrightarrow pp$ transitions thus appears to be justified. The only important correction to be noted is that the $P_{1/2} \rightarrow pp$, $T = 1/2$ transition, as estimated here, is an important fraction of the total $T = 1/2$, $P_{1/2}$ transition above $\approx$ 350 Mev.

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Our treatment of the P-wave π-π scattering in the single partial amplitudes may not be adequate if the conjectured resonance has a total energy as low as 2.3μ.

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LIST OF FIGURES

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TABLE I

Production Cross-Sections (mb)

Lab Energy-Mev	$\sigma_{1/2}$	$\sigma_{3/2}$	σ_{π^-}	$\sigma(\pi^- + p) \to$ $\pi^-\pi^+n$	$\pi^0\pi^-p$	$\pi^0\pi^0n$	$\sigma(\pi^+ + p) \to$ $\pi^+\pi^+n$	$\pi^+\pi^0p$
171	0	0	0	0	0	0	0	0
265	0.51	0.18	0.40	0.30	0.07	0.03	0.08	0.10
364	3.43	0.99	2.82	1.71	0.51	0.60	0.25	0.74
468	11.4	2.74	8.51	5.16	1.79	1.56	0.75	1.99
578	26.0	5.51	19.1				1.52	4.99

TABLE II

Single Partial Wave Cross-Sections (mb)

Lab Energy Mev	$D_{3/2}$ T=1/2	$D_{3/2}$ T=3/2	$S_{1/2}$ T=1/2	$S_{1/2}$ T=3/2	$P_{1/2} \longrightarrow ss$ T=1/2	$P_{1/2} \longrightarrow ss$ T=3/2	$P_{1/2} \longrightarrow pp$ T=1/2	$P_{1/2} \longrightarrow pp$ T=3/2	$P_{3/2} \longrightarrow pp$ T=1/2	$P_{3/2} \longrightarrow pp$ T=3/2	$2\pi \lambda\!\!\!^{-2}$
171	0	0	0	0	0	0	0	0	0	0	53.5
265	0.18	0.066	0.098	0.026	0.23	0.09	0.007	0.002	0.003	0.001	40.0
364	1.98	0.53	0.85	0.22	0.60	0.24	0.09	0.02	0.04	0.02	21.9
468	7.11	1.50	3.22	0.82	1.05	0.42	0.40	0.10	0.20	0.10	16.2
578	16.2	2.93	8.35	2.00	1.45	0.58	1.12	0.28	0.56	0.28	12.6
690		4.96		4.11	1.89	0.76					10.2
819		8.40		8.05	2.36	0.86					8.33

TABLE III

Charge Amplitudes from S-wave Pion-Pion Interaction

Process	$\lvert\vartheta\rvert^2$	$\lvert 3/2\,\vartheta\rvert^2$
$\pi^- + p \rightarrow \pi^- + \pi^0 + p$	2	13/9
$\rightarrow \pi^- + \pi^+ + n$	16	74/9
$\rightarrow \pi^0 + \pi^0 + n$	2	18/9
$\pi^+ + p \rightarrow \pi^+ + \pi^0 + p$	2	13/9
$\rightarrow \pi^+ + \pi^+ + n$	8	2/9

TABLE IV

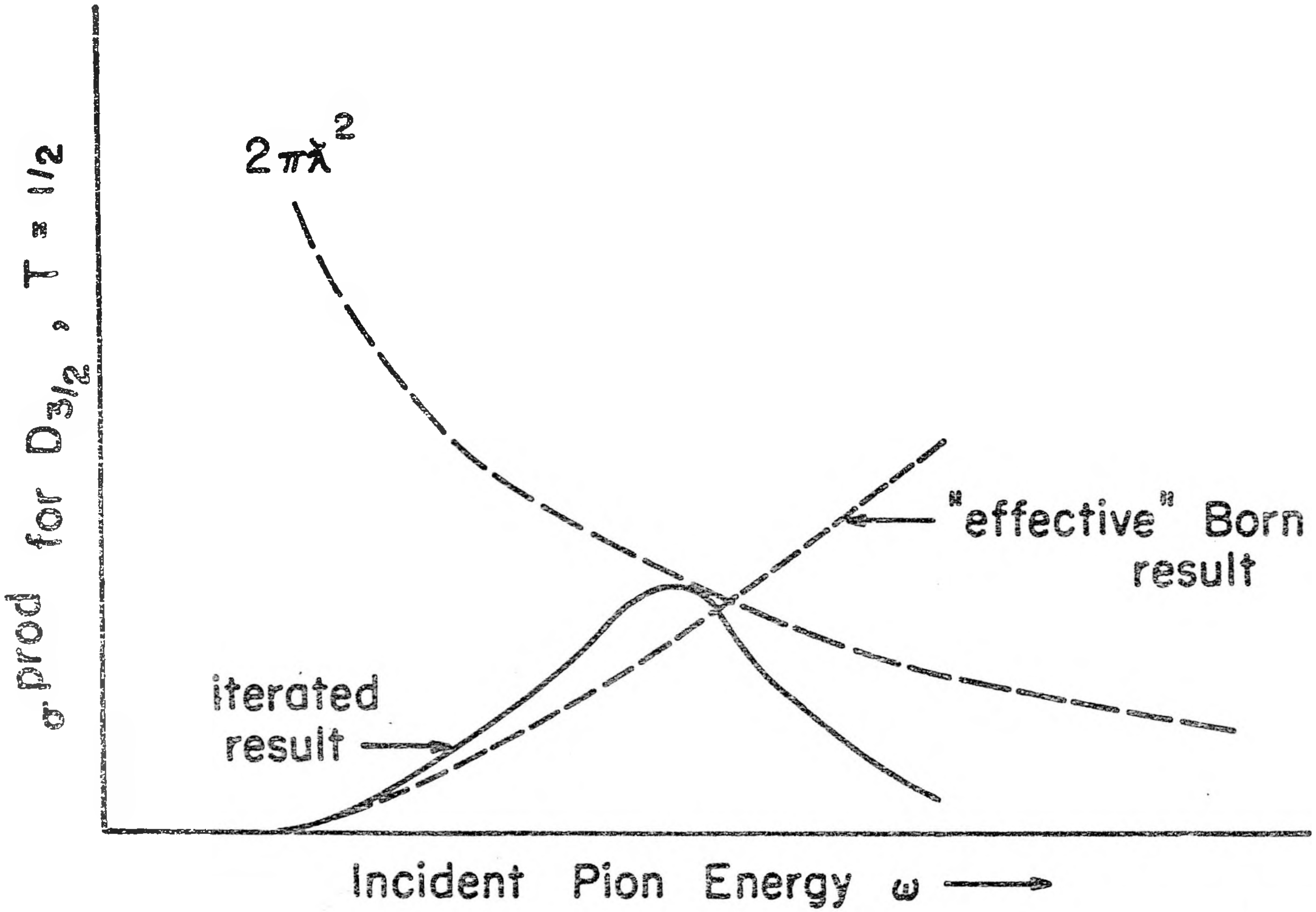
Charge Amplitudes from P-wave Pion-Pion Interaction

Process	$\|\mathcal{A}\|^2$	$\|\mathcal{A}_{3/2}\|^2$
$\pi^- + p \rightarrow \pi^- + \pi^0 + n$	2	1/9
$\rightarrow \pi^- + \pi^+ + n$	4	2
$\rightarrow \pi^0 + \pi^0 + n$	0	2/9
$\pi^+ + p \rightarrow \pi^+ + \pi^0 + p$	2	13/9
$\rightarrow \pi^+ + \pi^+ + n$	0	2/9

TABLE V

Cross-Section for Higher Angular Momentum States (mb)

Mev	$\pi^- + p \rightarrow \pi^- + \pi^+ + n$		$\sigma_{1/2}$	T = 1/2	$P_{1/2} \rightarrow S_{31/2}$	$P_{1/2} \rightarrow pp$	$P_{3/2} \rightarrow pp$	$P_{3/2} \rightarrow pp$	$P_{1/2} \rightarrow pp$
	σ (without $P \rightarrow p\gamma$)	$P \rightarrow p\gamma$	(without $P \rightarrow pp$)	$P \rightarrow pp$	T=1/2	T = 1/2	T=1/2	T=3/2	T=3/2
171	0	0	0	0	0	0	0	0	0
265	0.30	0.0050	0.51	0.010	0.23	0.007	0.003	0.001	0.002
364	1.71	0.06	3.43	0.13	0.60	0.09	0.04	0.01	0.02
468	5.16	0.30	11.4	0.59	1.05	0.40	0.20	0.05	0.10
578					1.45	1.12	0.56	0.14	0.28



σ^{prod} for $D_{3/2}$, T = 1/2
$2\pi\lambda\!\!\!^{-2}$
"effective" Born result
iterated result
Incident Pion Energy ω ⟶

Fig. 1

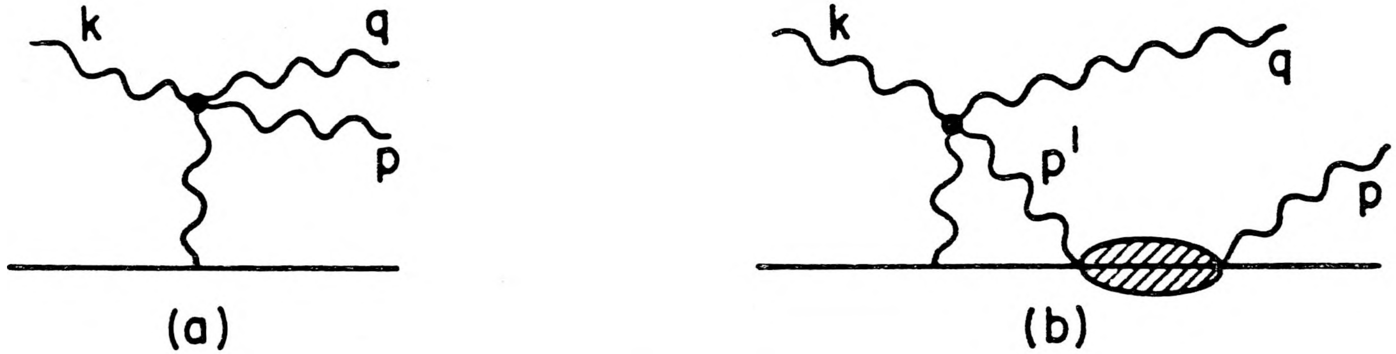


Fig. 2

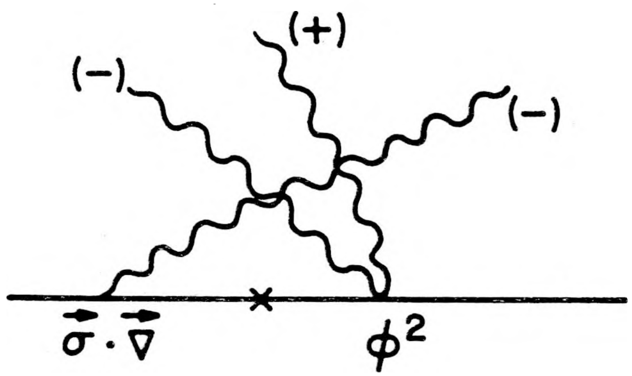


Fig. 18 $D_{3/2}$ Core Diagrams

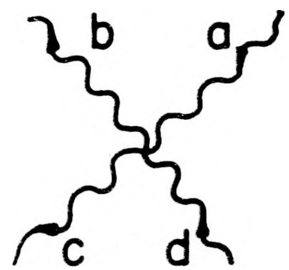


Fig. 3 Pion - Pion Scattering

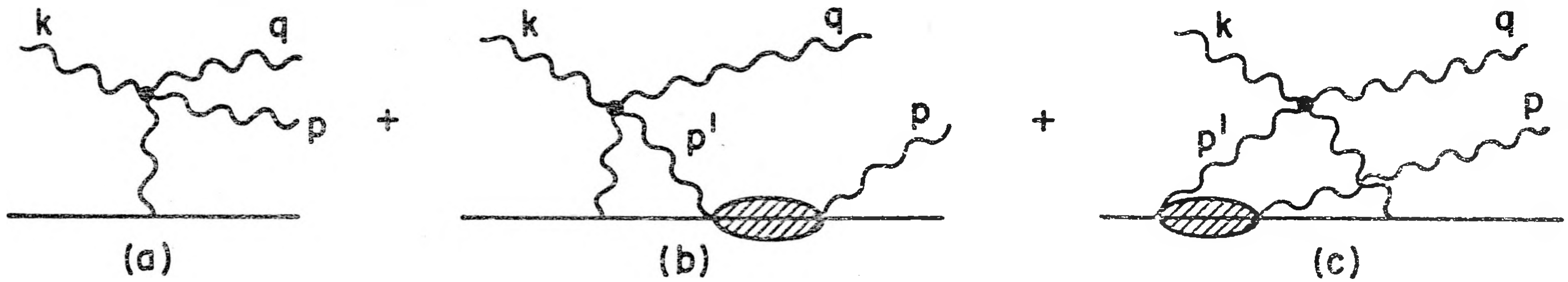
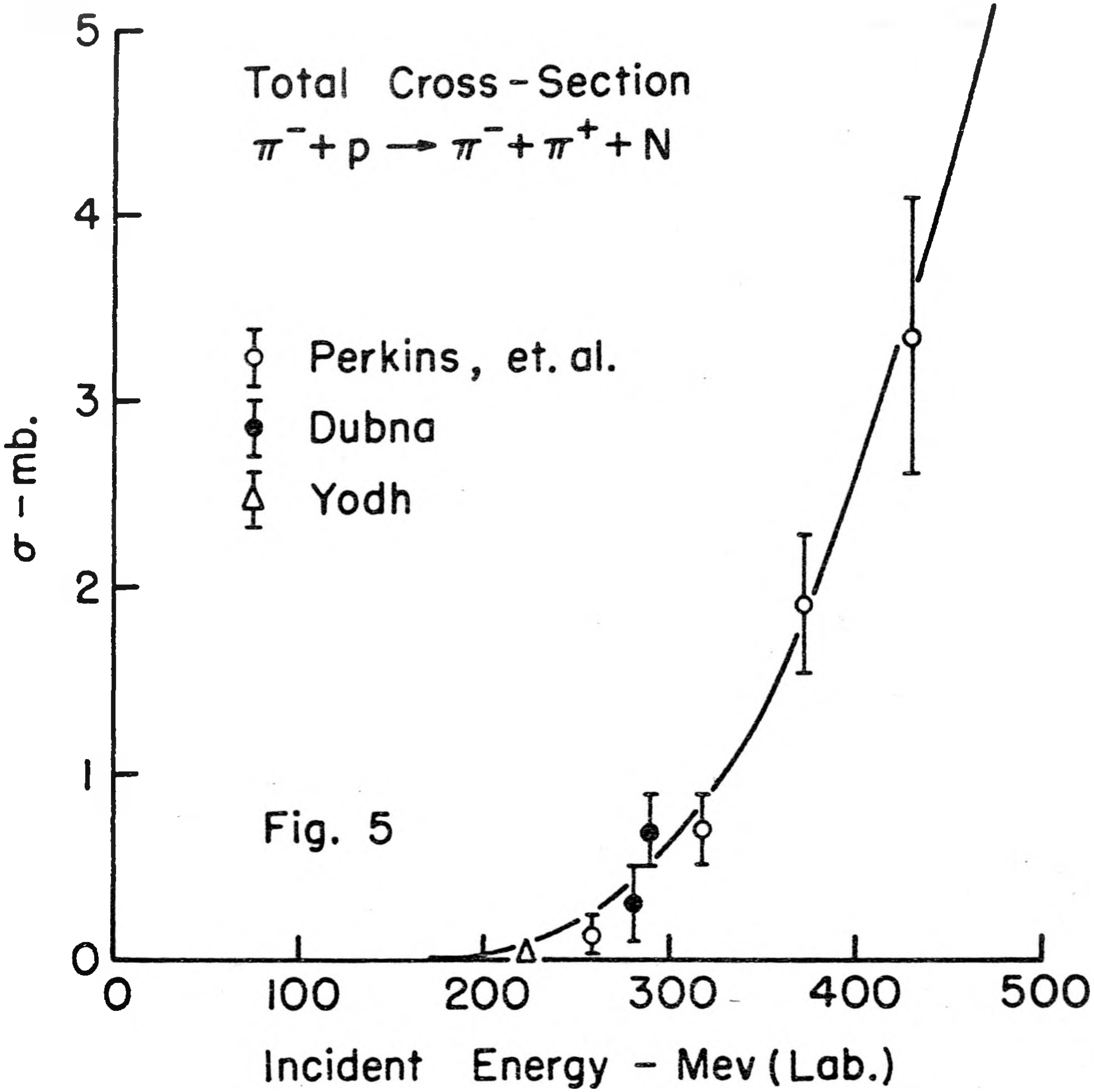


Fig. 4 Pion Production Diagrams



Total Cross-Section
$\pi^- + p \rightarrow \pi^- + \pi^+ + N$
Perkins, et. al.
Dubna
Yodh
Fig. 5
σ - mb.
0
1
2
3
4
5
0
100
200
300
400
500
Incident Energy - Mev (Lab.)

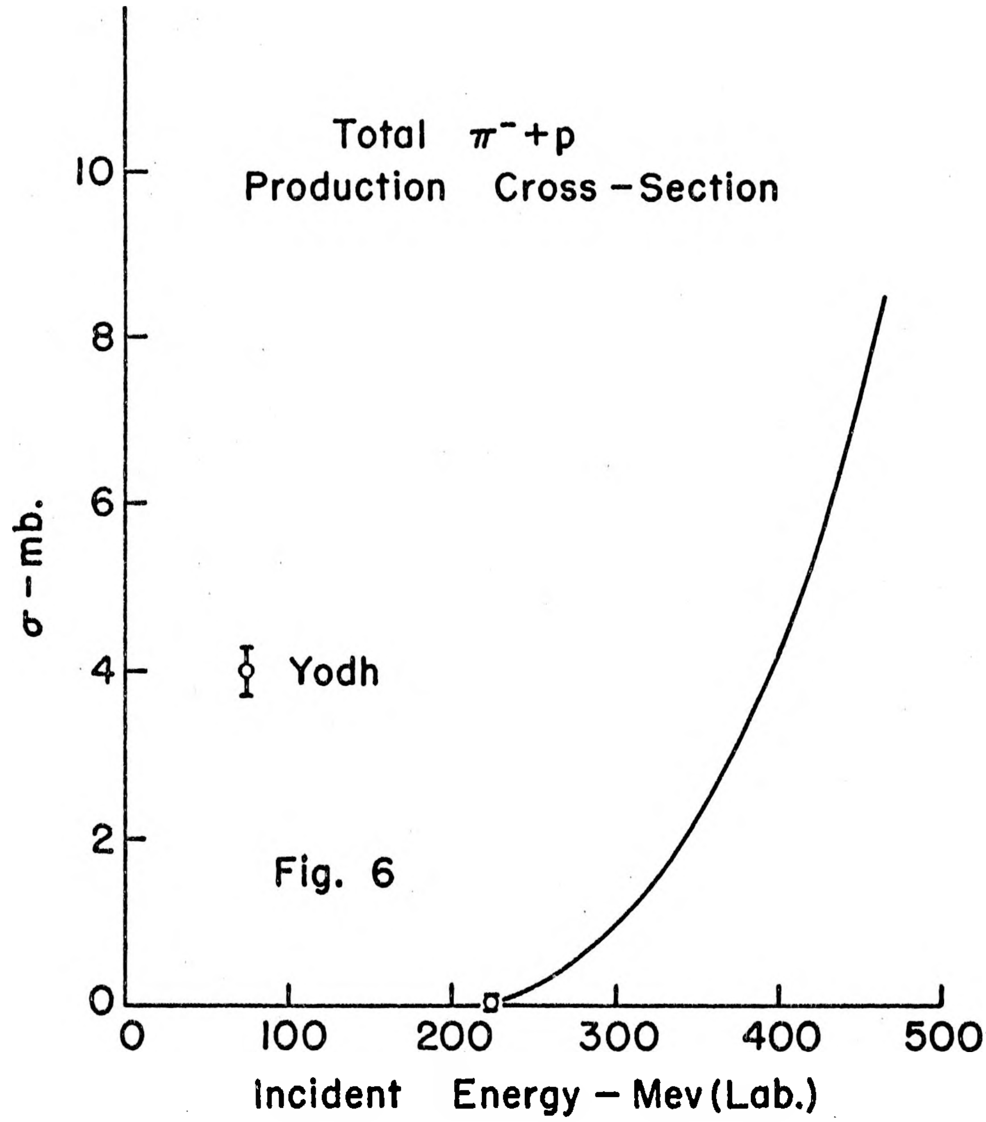
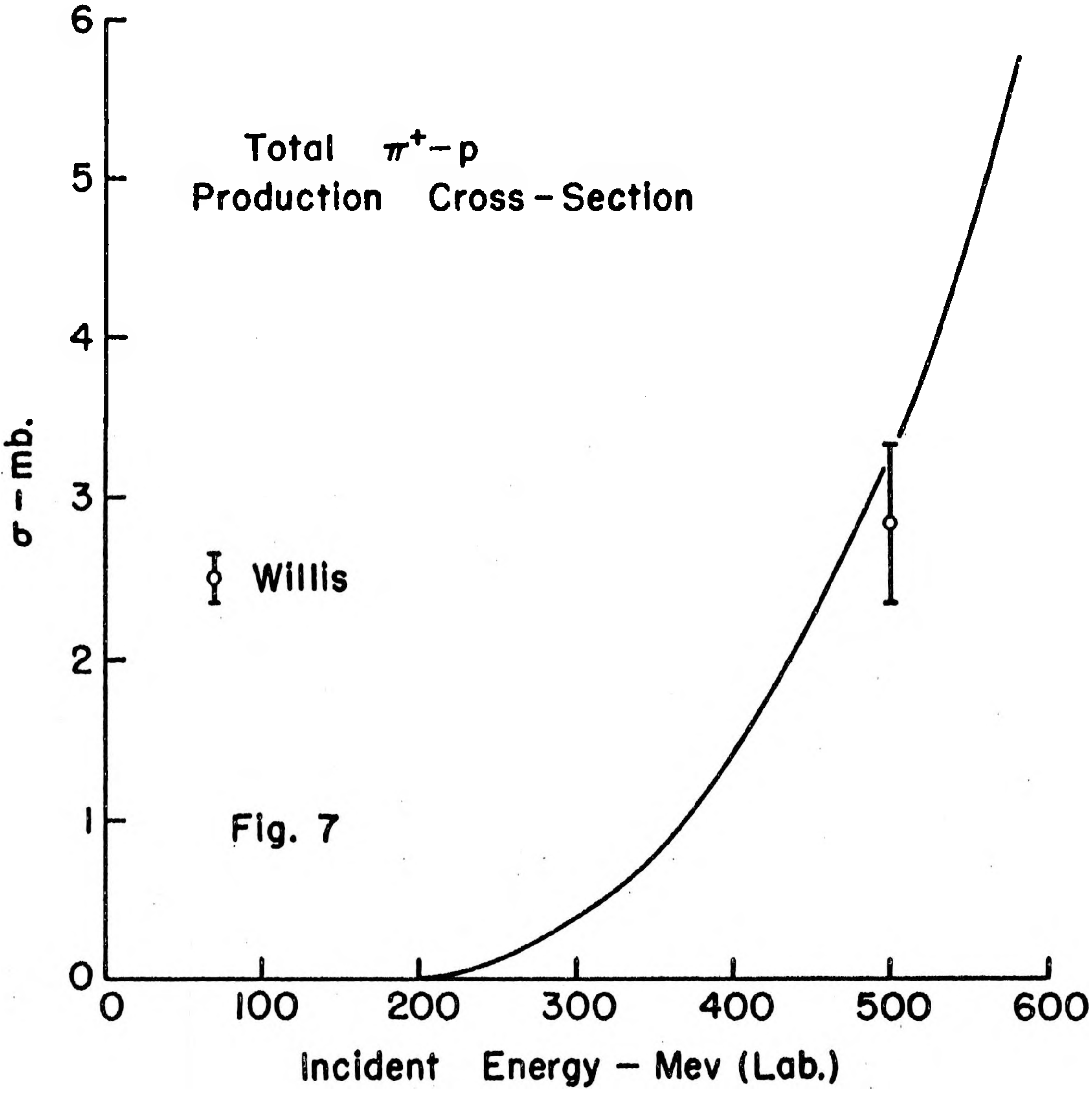

Total $\pi^- + p$
Production Cross-Section
σ - mb.
Yodh
0
2
4
6
8
10
0
100
200
300
400
500
Incident Energy - Mev (Lab.)

Fig. 6



Total $\pi^+ - p$
Production Cross-Section
Willis
Fig. 7
σ - mb.
0
1
2
3
4
5
6
0
100
200
300
400
500
600
Incident Energy - Mev (Lab.)

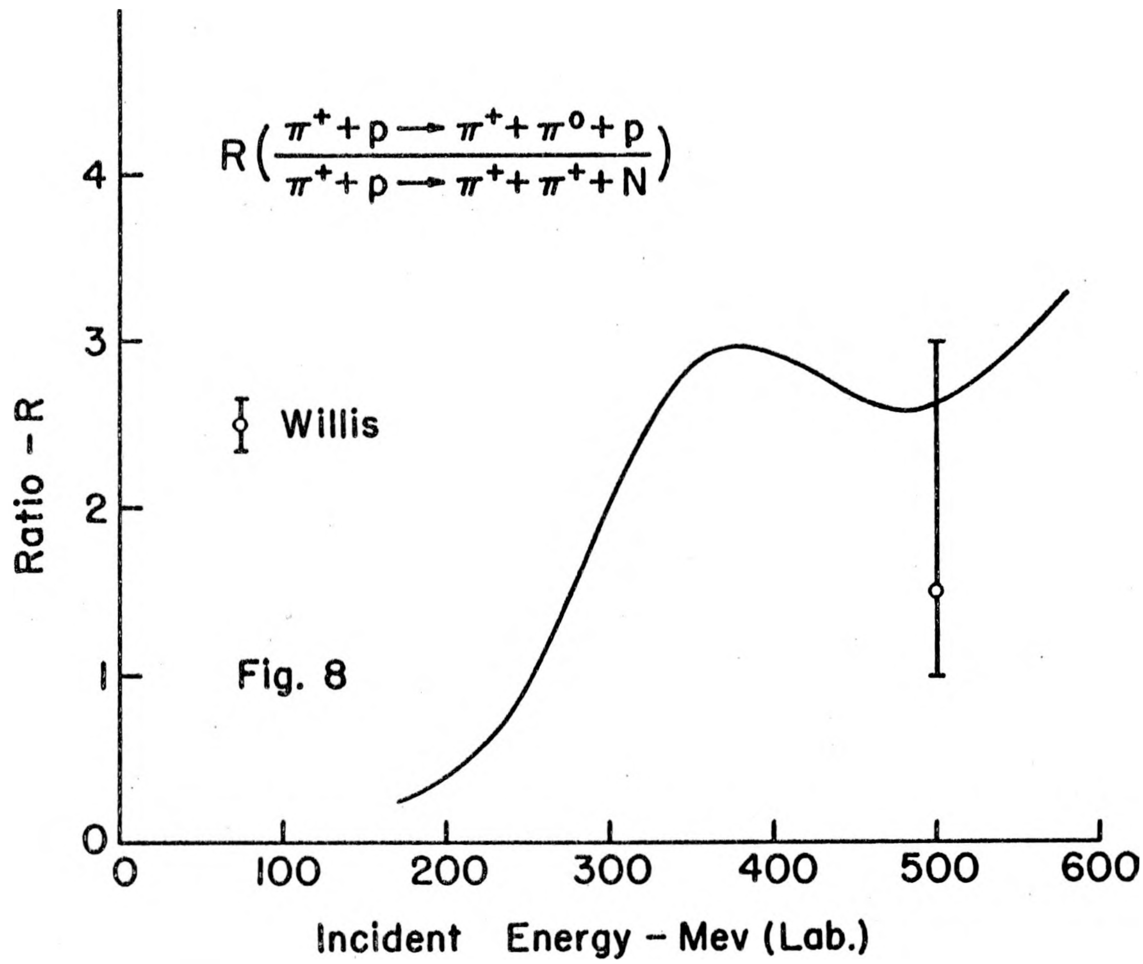
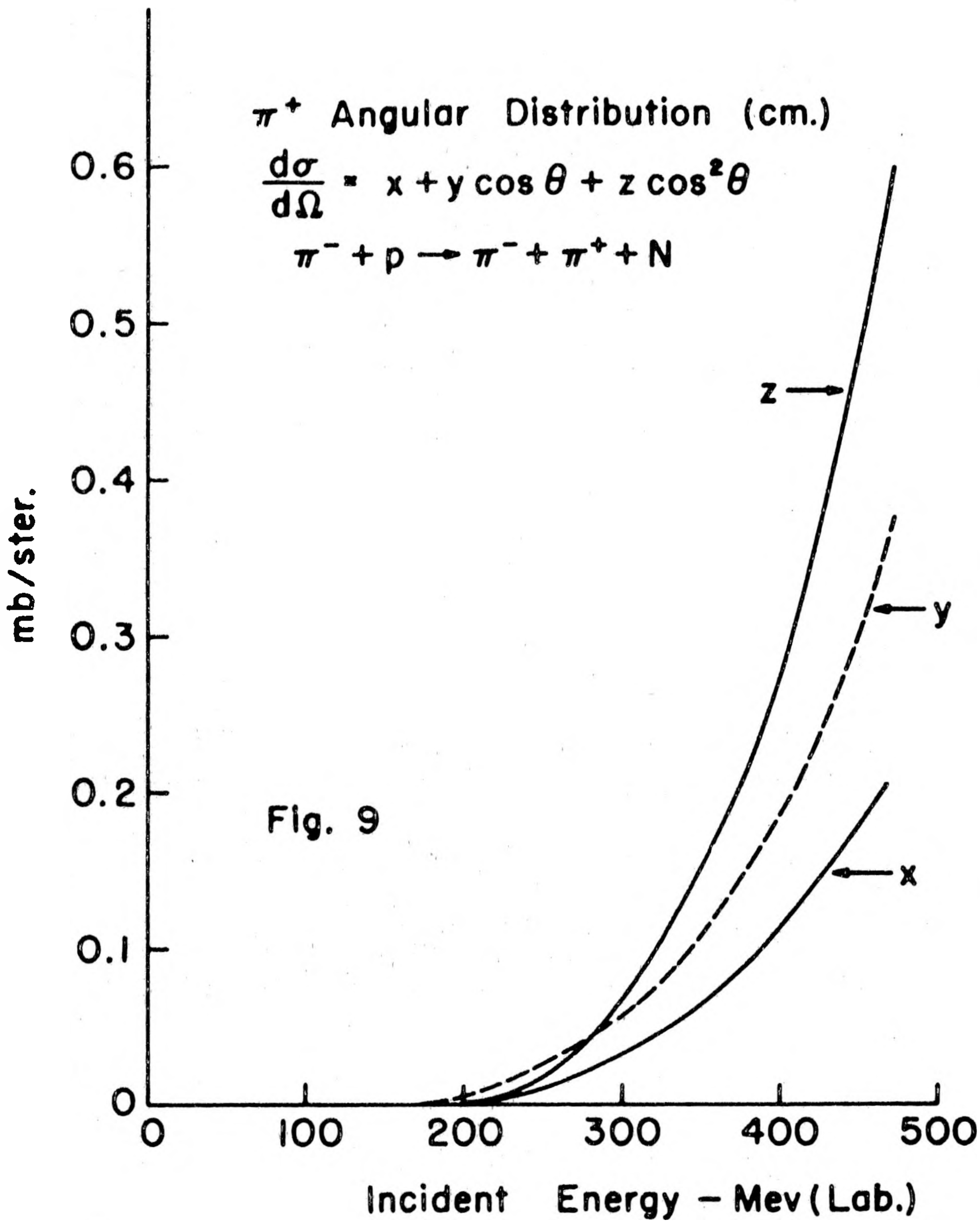

$R\left(\frac{\pi^+ + p \rightarrow \pi^+ + \pi^0 + p}{\pi^+ + p \rightarrow \pi^+ + \pi^+ + N}\right)$
Willis
Ratio – R
0
1
2
3
4
0
100
200
300
400
500
600
Incident Energy – Mev (Lab.)

Fig. 8



π^+ Angular Distribution (cm.)
$\frac{d\sigma}{d\Omega} = x + y\cos\theta + z\cos^2\theta$
$\pi^- + p \rightarrow \pi^- + \pi^+ + N$
Fig. 9
z
y
x
0.6
0.5
0.4
0.3
0.2
0.1
0
mb/ster.
0
100
200
300
400
500
Incident Energy – Mev (Lab.)

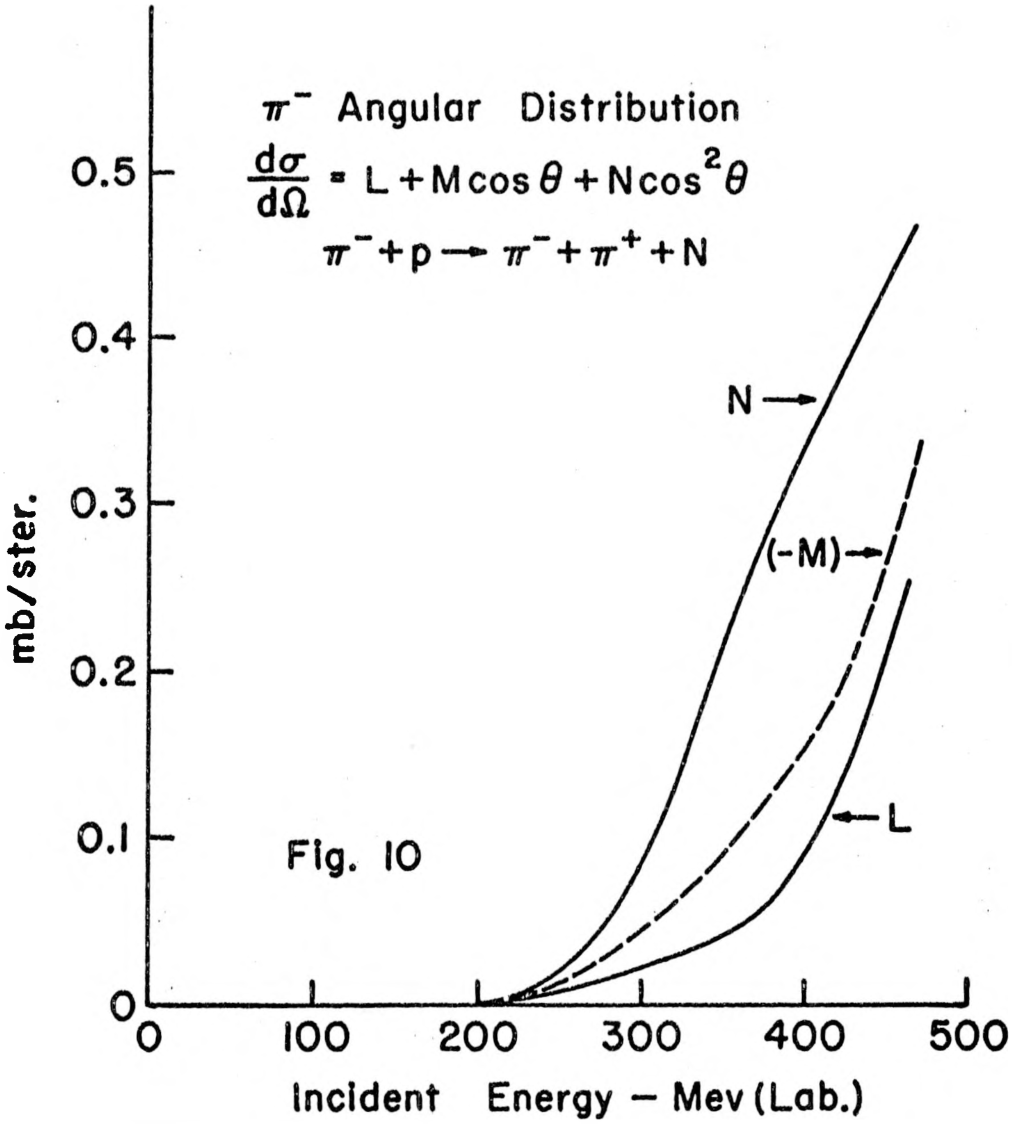

π^- Angular Distribution
$\frac{d\sigma}{d\Omega} = L + M\cos\theta + N\cos^2\theta$
$\pi^- + p \rightarrow \pi^- + \pi^+ + N$
N
(-M)
L
0.5
0.4
0.3
0.2
0.1
0
mb/ster.
0
100
200
300
400
500
Incident Energy — Mev (Lab.)

Fig. 10

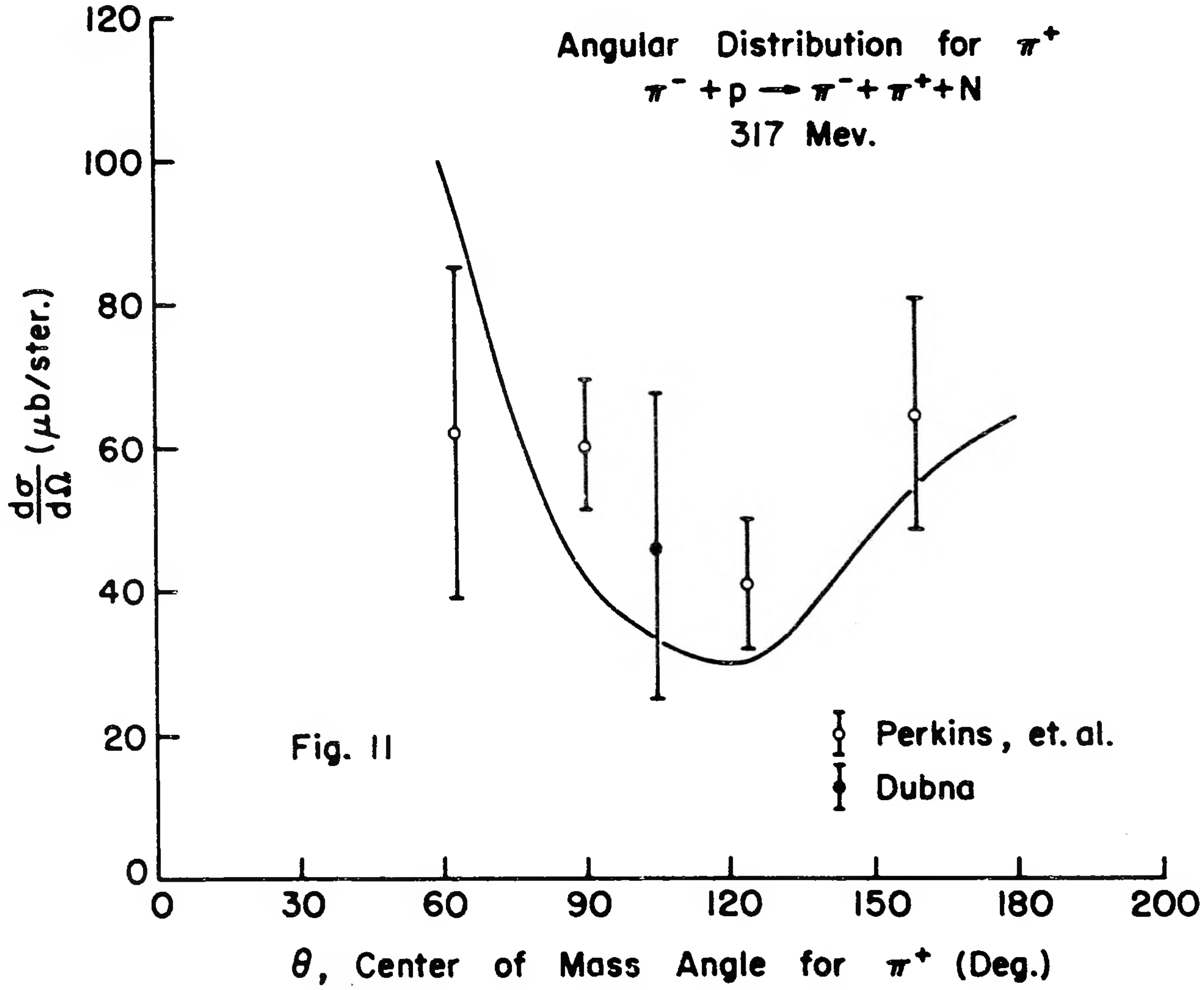
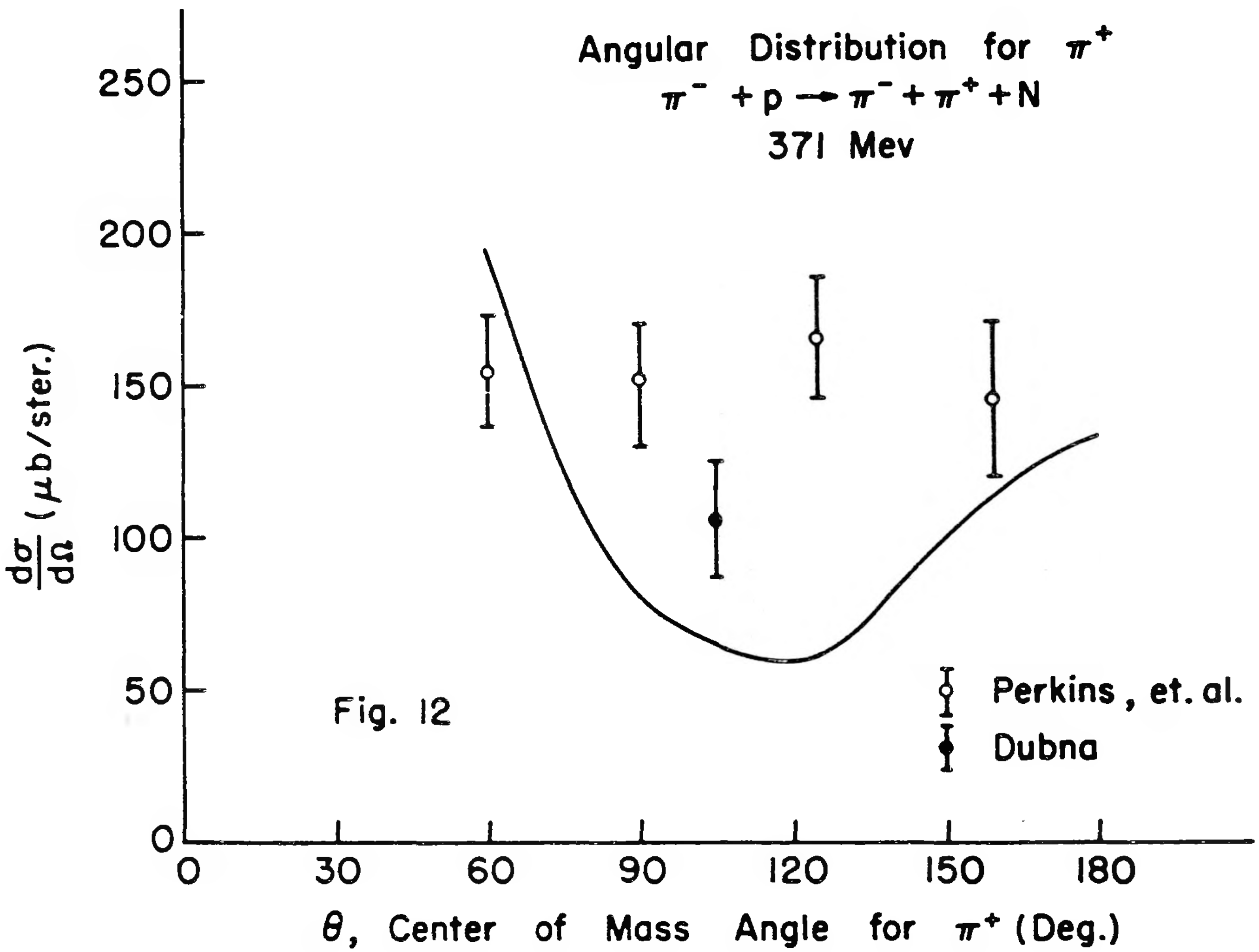
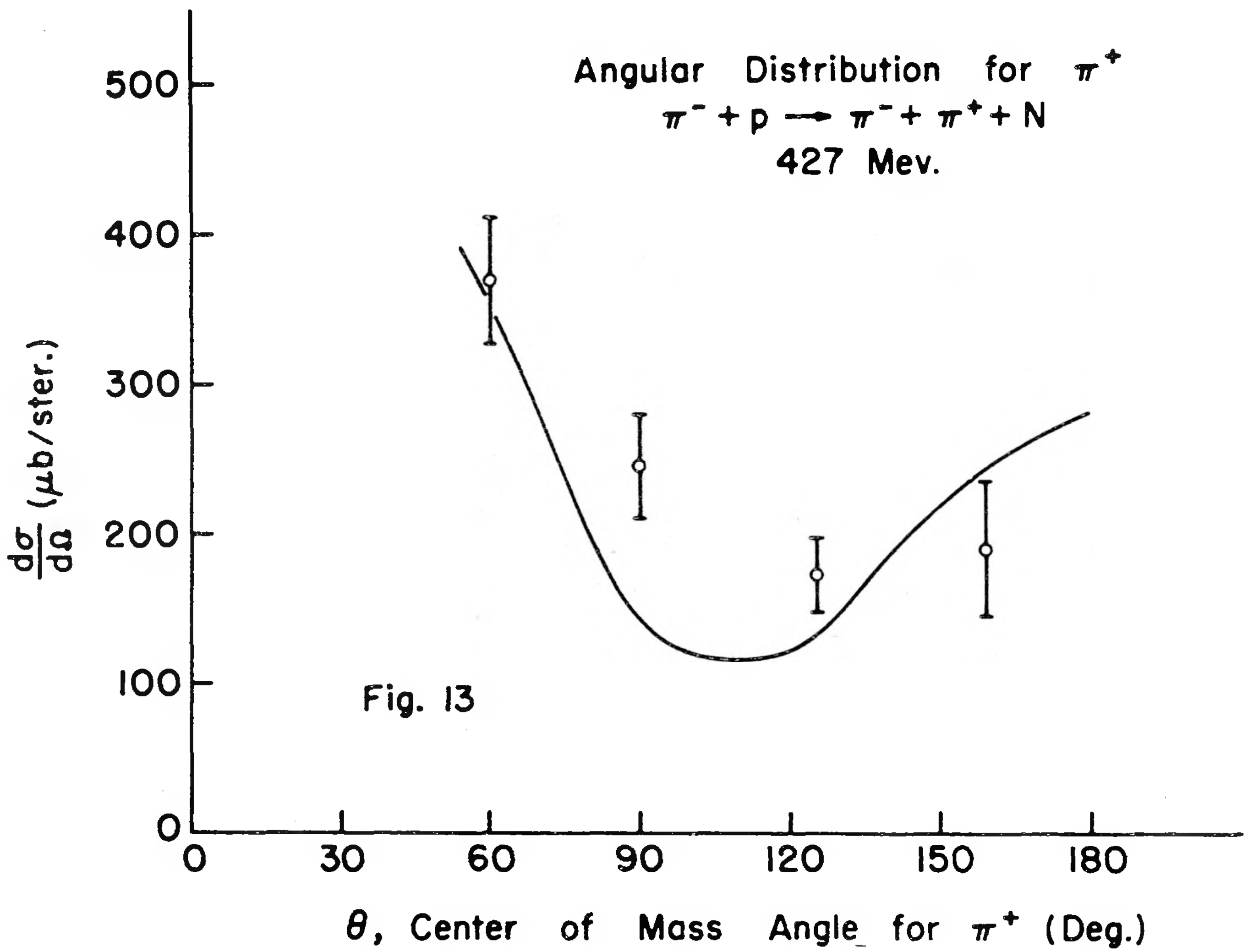

Angular Distribution for π^+
$\pi^- + p \rightarrow \pi^- + \pi^+ + N$
317 Mev.
$\frac{d\sigma}{d\Omega}$ (μb/ster.)
120
100
80
60
40
20
0
0
30
60
90
120
150
180
200
Perkins, et. al.
Dubna
θ, Center of Mass Angle for π^+ (Deg.)

Fig. 11



Angular Distribution for π^+
$\pi^- + p \rightarrow \pi^- + \pi^+ + N$
371 Mev
250
200
150
100
50
0
$\frac{d\sigma}{d\Omega}$ (μb/ster.)
0
30
60
90
120
150
180
θ, Center of Mass Angle for π^+ (Deg.)
Perkins, et. al.
Dubna

Fig. 12



Angular Distribution for π⁺
π⁻ + p → π⁻ + π⁺ + N
427 Mev.
500
400
300
200
100
0
dσ/dΩ (μb/ster.)
0
30
60
90
120
150
180
θ, Center of Mass Angle for π⁺ (Deg.)

Fig. 13

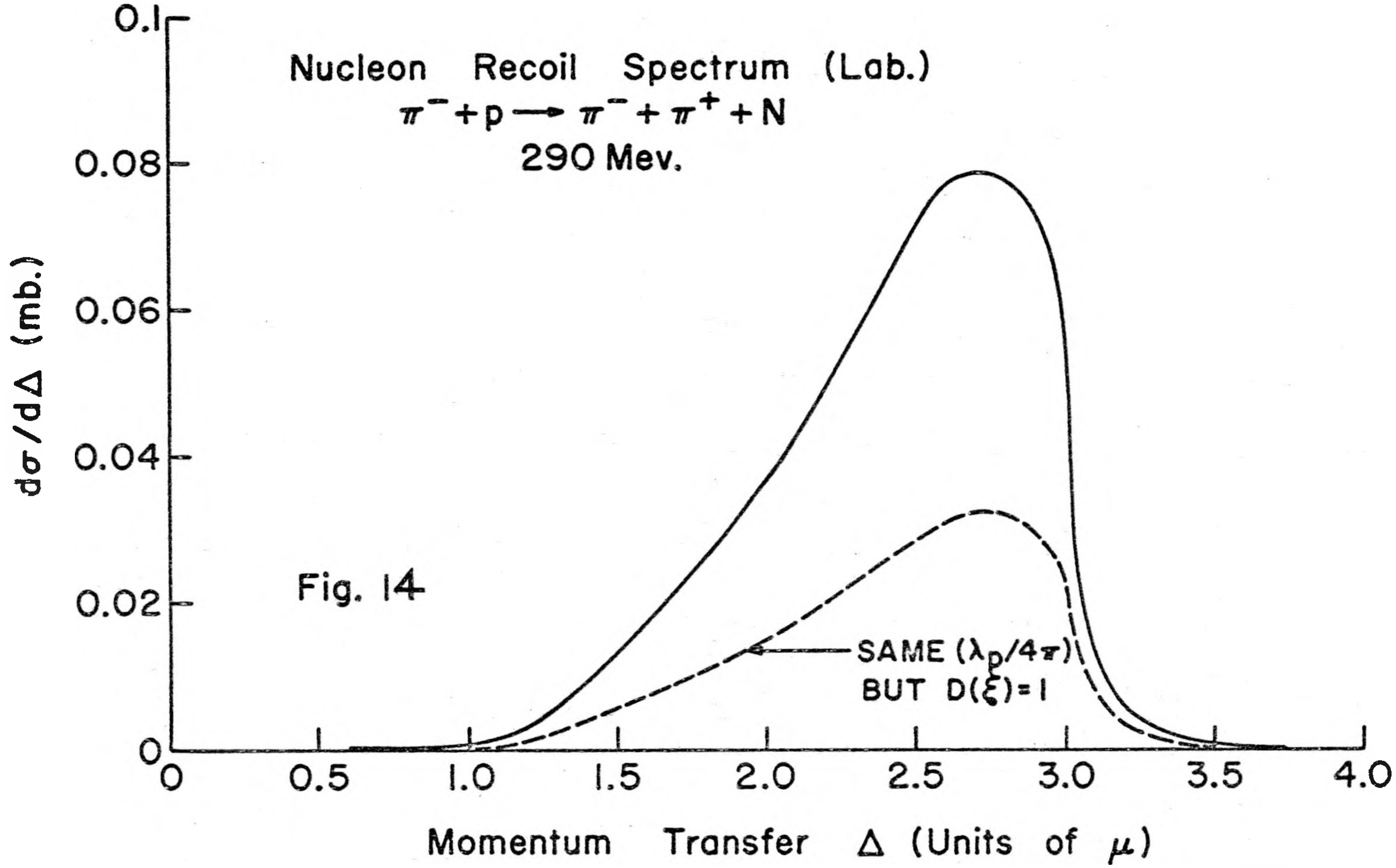

Nucleon Recoil Spectrum (Lab.)
$\pi^- + p \rightarrow \pi^- + \pi^+ + N$
290 Mev.
SAME ($\lambda_p/4\pi$)
BUT D(ξ)=1
$d\sigma/d\Delta$ (mb.)
0.1
0.08
0.06
0.04
0.02
0
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Momentum Transfer Δ (Units of μ)

Fig. 14

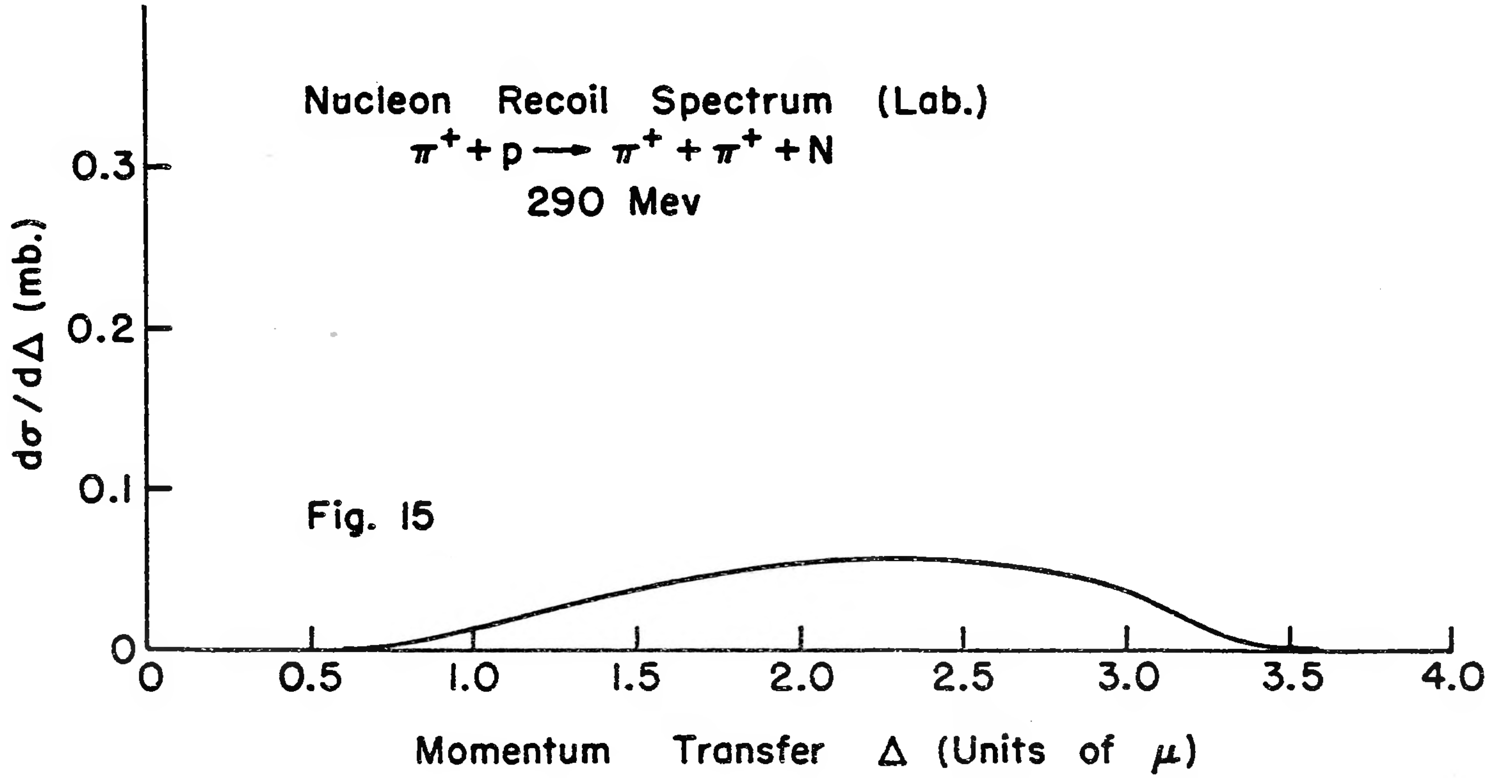

Nucleon Recoil Spectrum (Lab.)
$\pi^+ + p \rightarrow \pi^+ + \pi^+ + N$
290 Mev
$d\sigma/d\Delta$ (mb.)
0.3
0.2
0.1
0
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
Momentum Transfer Δ (Units of μ)

Fig. 15

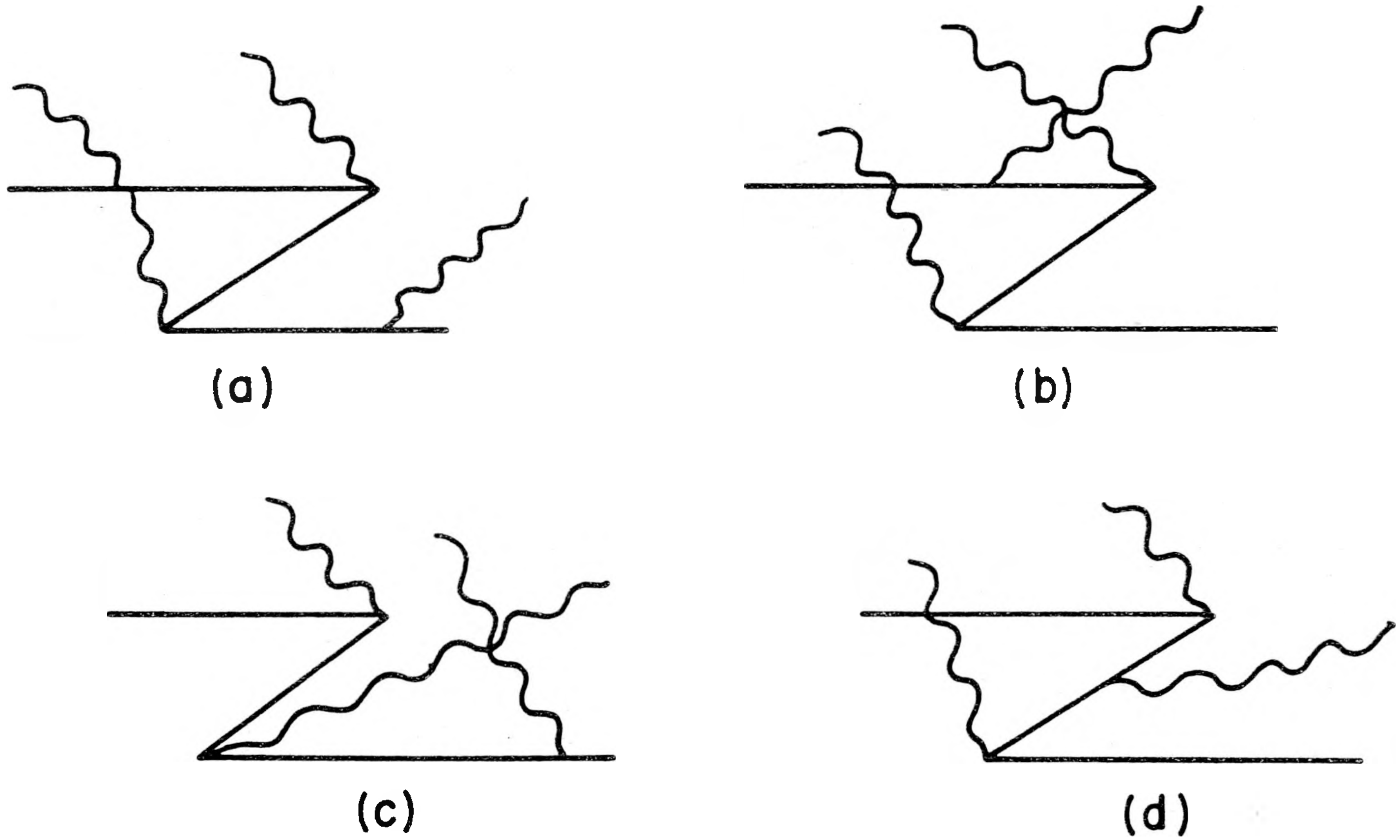


Fig. 16

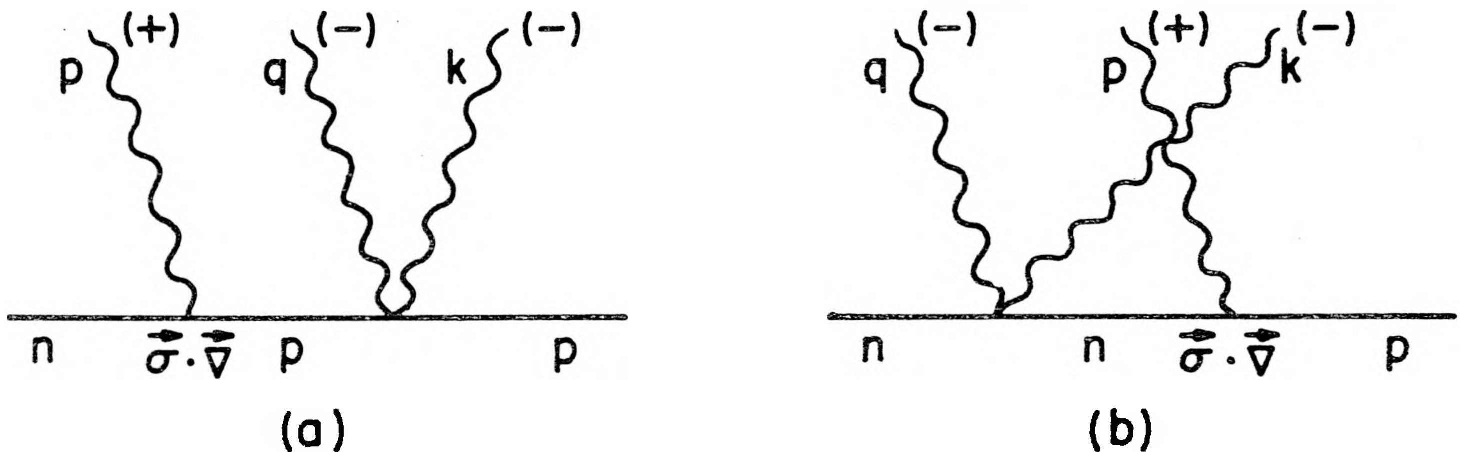


Fig. 17 Dominant Core Diagrams

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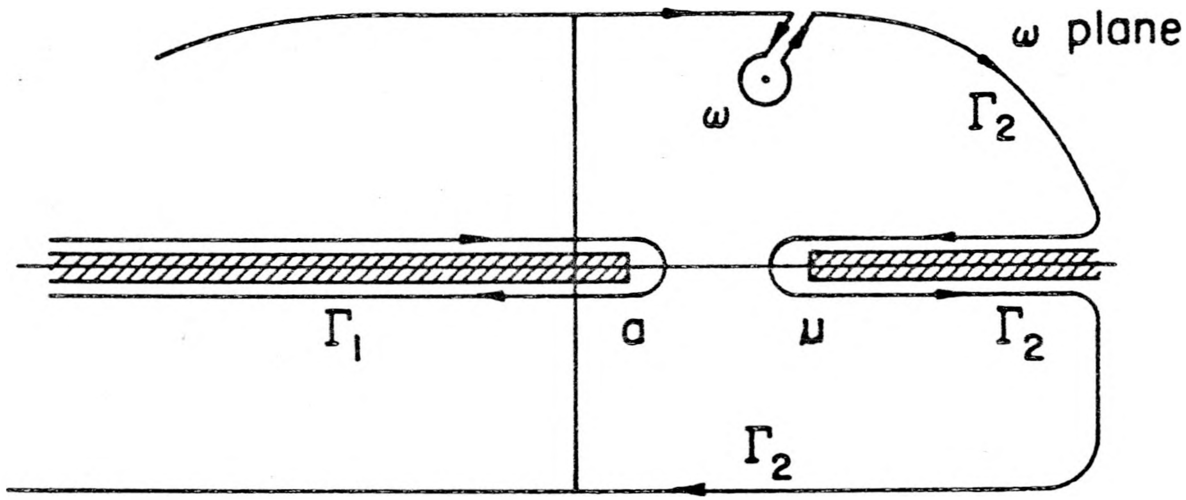

Fig. 19 Contours for Integral Equation

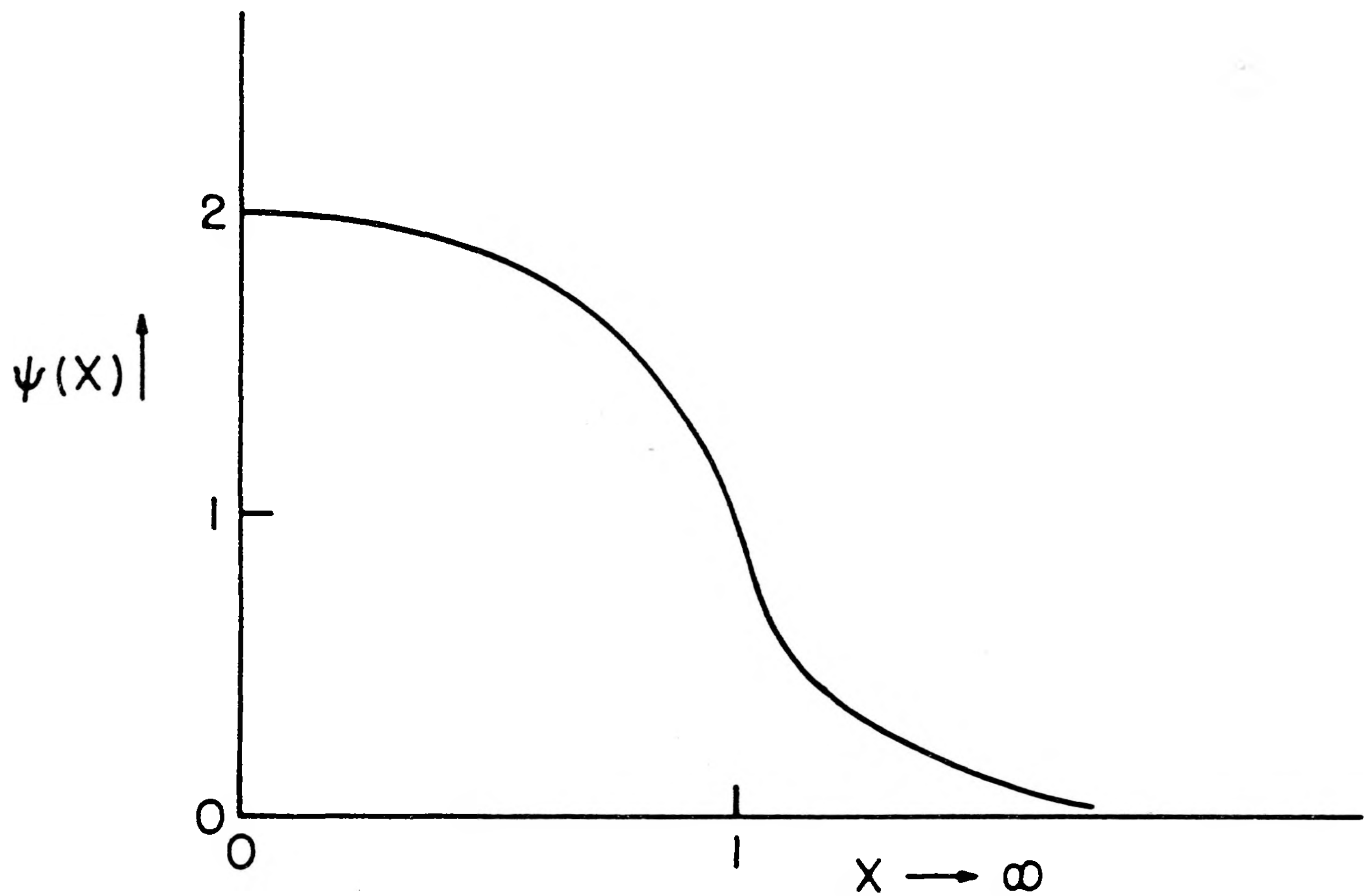

Fig. 20 The Kernel ψ(X)