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THE THEORY OF A TIME-DEPENDENT HEAT DIFFUSION DETERMINATION OF THERMAL DIFFUSIVITIES WITH A SINGLE TEMPERATURE MEASUREMENT

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OAK RIDGE NATIONAL LABORATORY
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FEBRUARY 1971

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CONTENTS

THE THEORY OF A TIME-DEPENDENT HEAT DIFFUSION DETERMINATION OF THERMAL DIFFUSIVITIES WITH A SINGLE TEMPERATURE MEASUREMENT

R. B. Perez, R. M. Carroll, and O. Sisman

ABSTRACT

A method to measure the thermal diffusivity of reactor fuels during irradiation is developed, based on a time-dependent heat diffusion equation. With this technique the temperature is measured at only one point in the fuel specimen. This method has the advantage that it is not necessary to know the heat generation (a difficult evaluation during irradiation).

The theory includes realistic boundary conditions, applicable to actual experimental systems. The parameters are the time constants associated with the first two time modes in the temperature-vs-time curve resulting from a step change in heat input to the specimen. With the time constants and the necessary material properties and dimensions of the specimen and specimen holder, the thermal diffusivity of the specimen can be calculated.

I. INTRODUCTION

There are two well-known and reliable ways to perform measurements of the thermal diffusivity of materials by time-dependent methods. These are the Angstrom cyclic method[1] and the flash technique.[2] The former requires a relatively large sample to allow for the measurement of the temperature at various points. The flash method is a single-temperature technique requiring an instantaneous, uniform deposition of heat in the sample, such as by a laser.

The purpose of this work is to develop the theory of a single-temperature-measurement method applicable to small-size samples and utilizing any arbitrary source of heat. This last requirement is important in order to eliminate the effect of inaccuracy in the power determination. The motivation for the present study arises from the problems encountered in trying to measure the thermal diffusivity of fissile materials under irradiation conditions. In this instance the determination of an accurate measure of the effective power input was hindered by the addition of nonfission heating[3] and by the release or accumulation of stored energy during changes of irradiation conditions.[4] Moreover, the necessity of performing the experiment inside the core of the reactor (in this particular case the Oak Ridge Research Reactor) and the prevention of thermal stresses require the use of relatively small samples.

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[1] A. J. Angstrom, *Ann. Physik Chem.* **114**, 513 (1861).

[2] W. J. Parker *et al.*, *J. Appl. Phys.* **32**, 1679 (1961).

[3] R. M. Carroll, R. B. Perez, and O. Sisman, *Nucl. Sci. Eng.* **36**, 232 (1969).

[4] R. M. Carroll, R. B. Perez, and O. Sisman, "The Effect of Stored Energy on Measurements of Thermal Diffusivity During Irradiation," *Nucl. Sci. Eng.* (in press).

In Sect. II below, the theory is developed for the simple one-region case where one considers a heated sample exchanging heat with a bulk coolant. In Sects. III, IV, and V, we consider a three-region problem which corresponds to the experimental setup presently utilized for in-pile measurements at the ORR.[3]

II. THE SINGLE-REGION PROBLEM

The geometry of this physical situation is shown in Fig. 1*a*. The following assumptions are pertinent to the model.

1. The thermal properties of the sample remain constant during the heating pulse.
2. The heat generation is assumed to be uniformly distributed and to be negligible in the surroundings.
3. The axial flow is neglected as well as the heat transport by radiation.
4. The sample loses heat to a bulk temperature T_b.

The approximation involved in assumption 1 is usually made in heat transport calculations, and it is reasonable for limited temperature increments of the sample. Assumption 2 does not affect the results because in this technique the power input does not enter into the calculations of the thermal diffusivity. Its introduction, however, eliminates the consideration of matrix elements of the power distribution inside the sample. In contrast, the assumption made in 3 is a serious one. It will be investigated in a forthcoming paper. The hypothesis set forth in assumption 4 is plausible for a large body of well "stirred" fluid coolant. On the basis of the above set of conditions, the following heat balance and boundary conditions apply:

$$\left(\nabla_r^2 - \frac{1}{\kappa_1}\frac{\partial}{\partial t}\right) T(r,\ t) = -\ p(t)/K\ , \tag{II.1}$$

$$\frac{\partial}{\partial r} T(r,\ t)\,\Big|_{r=b_1} = 0\ , \tag{II.2}$$

$$-\ b_2 \frac{\partial}{\partial r} T(r,\ t)\Big|_{r=b_2} = B_0[T(b_2,\ t) - T_b]\ , \tag{II.3}$$

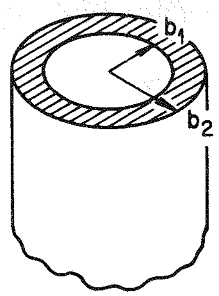
where $T(r,\ t)$ = sample temperature distribution (°C),

∇_r^2 = radial Laplacian = $(d^2/dr^2) + (1/r)\ d/dr$,

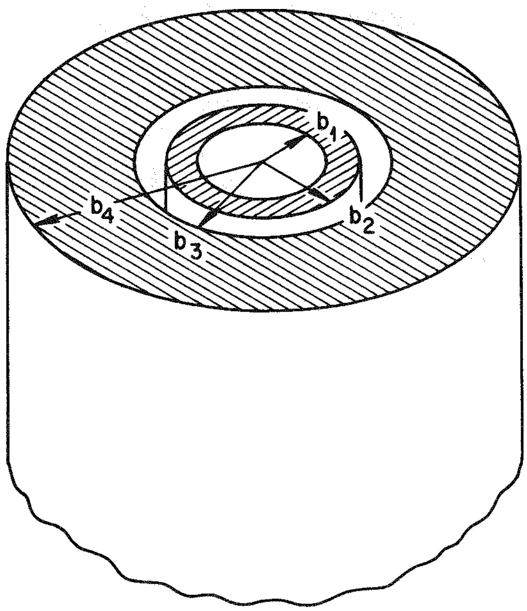
B_0 = Biot number = $b_2 H/K$ (dimensionless), (II.4)

K = thermal conductivity of the sample (cal cm^{-1} °C^{-1} sec^{-1}),

H = conductance between the sample and the coolant (cal cm^{-2} °C^{-1} sec^{-1}), ORNL-DWG 69-12572



(a)



(b)

Fig. 1 – Geometry of the One- and Three-Region Problem. (a) Cross-section dimensions of the UO_2 specimen. (b) Cross-section dimensions of the specimen holder.

$p(t)$ = heat pulse (cal cm^{-3} sec^{-1}) = $p_0\ U(t)$,

$U(t)$ = step function,

κ_1 = thermal diffusivity (cm^2/sec) = $(K/\rho c)$, where ρ is density in grams per cubic centimeter and c is specific heat in calories per gram per degree centigrade,

T_b = bulk temperature.

To solve this time-dependent heat diffusion problem, we expand the temperature in the form

$$T(r,\ t) = T_b + \sum_n \frac{1}{Y_1(\alpha_n b_1)} A_n(t)\ U_0\ (\alpha_n r)\ , \qquad \text{(II.5)}$$

where the eigenfunctions

$$U_0(\alpha_n r) = Y_1(\alpha_n b_1)\, J_0\,(\alpha_n r) - J_1\,(\alpha_n b_1)\, Y_0\,(\alpha_n r) \tag{II.6}$$

satisfy the boundary conditions (II.2) and the following orthogonality and normalization conditions (see Appendix A):

$$\int_{b_1}^{b_2} U_0\,(\alpha_p r)\, U_0\,(\alpha_n r) r\, dr = 0 \qquad (p \neq n)\,, \tag{II.7}$$

$$\int_{b_1}^{b_2} U_0^2\,(\alpha_p r) r\, dr = N_{pp} = \frac{1}{2}\{b_2^2[U_0^2\,(\alpha_p b_2) + U_1^2\,(\alpha_p b_2)] - b_1^2 U_0^2\,(\alpha_p b_1)\}\,, \tag{II.8}$$

with

$$U_1\,(\alpha_p b_2) = Y_1\,(\alpha_p b_1)\, J_1\,(\alpha_p b_2) - J_1\,(\alpha_p b_1)\, Y_1(\alpha_p b_2)\,. \tag{II.9}$$

Introduction of the expansion (II.5) into Eqs. (II.1) and (II.3) yields a set of differential equations for the modal amplitudes $A_n(t)$:

$$\tau_p \frac{d}{dt} A_p(t) + A_p(t) = a_p \frac{p(t)}{K}\,, \tag{II.10}$$

where $\tau_p = \dfrac{1}{\alpha_p^2 \kappa}$ is a characteristic time constant of the sample for the pth mode, with

$$a_p = \frac{Y_1(\alpha_p b_1)}{\alpha_p^2 N_{pp}} \int_{b_1}^{b_2} r\, dr\, U_0(\alpha_p r) \qquad (\mathrm{cm}^2)\,. \tag{II.11}$$

Moreover, the following secular equation for the special eigenvalues α_p is also obtained:

$$\alpha_p b_2\, U_1(\alpha_p b_2) - B_0\, U_0(\alpha_p b_2) = 0\,. \tag{II.12}$$

The differential equation (II.10) is easily integrated, yielding

$$A_p(t) = a_p \frac{p_0}{K} - \left[a_p \frac{p_0}{K} - A_p(0)\right] e^{-s_p t}\,, \tag{II.13}$$

$$s_p = \frac{1}{\tau_p} \qquad (\mathrm{sec}^{-1})\,, \tag{II.14}$$

whence from Eqs. (II.13) and (II.5) one obtains for the temperature distribution:

$$T(r,\,t) = T_b + \sum_n \Theta_{0n}\, U_0(\alpha_n r) - \sum_n \Theta_{1n} e^{-s_n t}\,,\ U_0(\alpha_n r) \tag{II.15}$$

with

$$\Theta_{0n} = \frac{1}{Y_1(\alpha_n b_1)} a_n \frac{p_0}{K} , \qquad \text{(II.16)}$$

$$\Theta_{1n} = \frac{1}{Y_1(\alpha_n b_1)} \left[a_n \frac{p_0}{K} - A_n(0) \right] . \qquad \text{(II.17)}$$

Experimental data (Fig. 2) show the presence of two distinct modal components in the temperature distribution of the sample. The time eigenvalues s_1 and s_2 ($s_2 > s_1$) exhibit a ratio $s_2/s_1 = 9$ and are consequently easily separated either by peeling-off techniques or by application of the least-squares method to the experimental temperature vs time curves.

Knowing the two time eigenvalues s_1 and s_2, one proceeds in the following manner:

1. Write the secular equation (II.12) for the two space eigenvalues α_1 and α_2 and eliminate among them the Biot number B_0; we obtain

$$U_1(\alpha_1 b_2)\, U_0 \left[\alpha_1 \left(\frac{s_2}{s_1} \right)^{1/2} b_2 \right] - \left(\frac{s_2}{s_1} \right)^{1/2} U_1 \left[\alpha_1 \left(\frac{s_2}{s_1} \right)^{1/2} b_2 \right] U_0(\alpha_1 b_2) = 0 , \qquad \text{(II.18)}$$

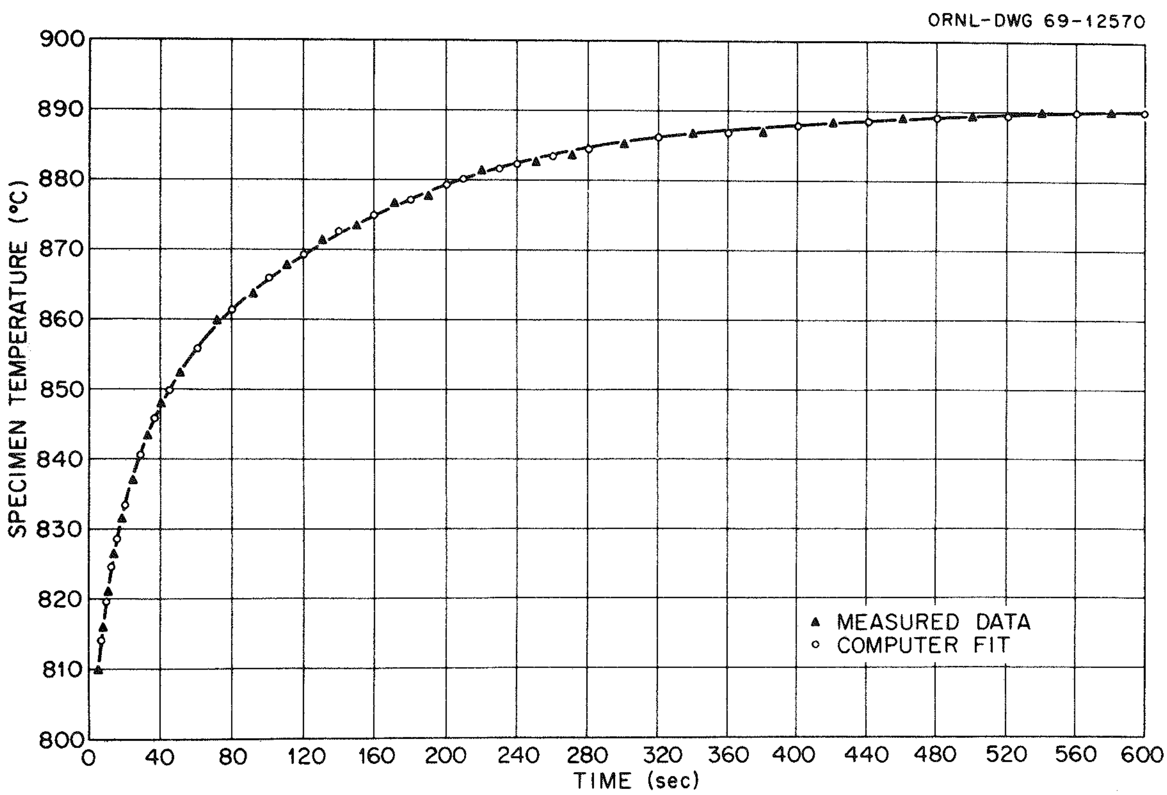


Fig. 2 – Typical Computer Fit to Heating Curve.

where we utilized the relation [from Eq. (II.14)]

$$(\alpha_2/\alpha_1)^2 = s_2/s_1 \,. \tag{II.19}$$

2. Solve the transcendental equation (II.18) using (II.19). This yields the two space eigenvalues α_1 and α_2, from which the thermal diffusivity is obtained by either of the equations

$$\kappa_1 = \frac{s_1}{\alpha_1^2} = \frac{s_2}{\alpha_2^2} \,. \tag{II.20}$$

A particularly simple expression for the thermal diffusivity is obtained for very thin samples, that is, when $\delta = b_2 - b_1$ is such that the value of $\alpha\delta$ is very small compared with unity. Then (see Appendix B) one can write

$$U_0(\alpha b_2) = -\frac{2}{\pi\alpha b_2}\,(f_0 - f_1\alpha^2\delta^2)\,, \tag{II.21}$$

$$U_1(\alpha b_2) = -\frac{2}{\pi}\frac{\delta}{b_2}\left(g_0 - \frac{1}{6}\alpha^2\delta^2\right)\,, \tag{II.22}$$

with

$$f_0 = 1 + \frac{\delta}{b_2}\left[1 + \frac{\delta}{b_2} + \left(\frac{\delta}{b_2}\right)^2\right]\,, \tag{II.23}$$

$$f_1 = \frac{1}{2} + \frac{1}{3}\frac{\delta}{b_2}\,, \tag{II.24}$$

$$g_0 = 1 + \frac{1}{2}\frac{\delta}{b_2}\left(1 + \frac{\delta}{b_2}\right). \tag{II.25}$$

Introduction of Eqs. (II.21) and (II.22) into (II.18) yields an expression for the eigenvalue α_1, which can be utilized in Eq. (II.20), yielding the following relation for the computation of κ_1:

$$\kappa_1 = s_1\delta^2\left[\frac{1+\xi^2}{6g_0}\right] \qquad (\mathrm{cm}^2\ \mathrm{sec}^{-1})\,, \tag{II.26}$$

where

$$\epsilon = (s_2/s_1)^{1/2}\,, \tag{II.27}$$

which for extremely thin samples reduces to

$$\kappa_1 = \frac{1}{2}s_1\delta^2\left[\frac{1+\xi^2}{6}\right]. \tag{II.28}$$

III. THE THREE-REGION PROBLEM

The simple geometrical arrangement assumed in the previous section is very seldom realized in actual practice. The experimental apparatus utilized for the in-pile measurements at the ORR has been described elsewhere,[3] and a schematic view is shown in Fig. 1*b*. There are three regions to be considered: the sample (region 1), the helium gas (region 2), and the isolating shell (region 3). The corresponding heat balances and boundary conditions are based on the assumptions stated in Sect. II:

$$\left(\nabla_r^2 - \frac{1}{\kappa_1}\frac{\partial}{\partial t}\right) T_1(r, t) = -p(t)/K_1 , \tag{III.1}$$

$$\left(\nabla_r^2 - \frac{1}{\kappa_2}\frac{\partial}{\partial t}\right) T_2(r, t) = 0 , \tag{III.2}$$

$$\left(\nabla_r^2 - \frac{1}{\kappa_3}\frac{\partial}{\partial t}\right) T_3(r, t) = 0 , \tag{III.3}$$

$$\frac{\partial}{\partial r} T_1(r, t)\big|_{b_1} = 0 , \tag{III.4}$$

$$T_1(b_2, t) = T_2(b_2, t) , \tag{III.5}$$

$$K_1 \frac{\partial}{\partial r} T_1(r, t)\big|_{b_2} = K_2 \frac{\partial}{\partial r} T_2(r, t)\big|_{b_2} , \tag{III.6}$$

$$T_2(b_3, t) = T_3(b_3, t) , \tag{III.7}$$

$$K_2 \frac{\partial}{\partial r} T_2(r, t)\big|_{b_3} = K_3 \frac{\partial}{\partial r} T_3(r, t)\big|_{b_3} , \tag{III.8}$$

$$- b_4 \frac{\partial}{\partial r} T_3(r, t)\big|_{b_4} = B_0(T_3(b_4, t) - T_b) , \tag{III.9}$$

$$B_0 = b_4 H/K_3 , \tag{III.10}$$

$\kappa_1, \kappa_2, \kappa_3$ = thermal diffusivity for regions 1, 2, and 3.

We try solutions of the form

$$T_1(r, t) = T_b + \sum_n \frac{A_{1n}}{Y_1(b_1)} U_0(\alpha_n r) , \tag{III.11}$$

$$T_2(r, t) = T_b + \sum_n \{A_{2n} J_0(\alpha_n r) + B_{2n} Y_0(\alpha_n r)\} , \tag{III.12}$$

$$T_3(r, t) = T_b + \sum_n \frac{A_{3n}}{\beta_{2n}} Z_0(\alpha_n r) , \tag{III.13}$$

with

$$U_0(\alpha_n r) = Y_1(\alpha_n b_1)\, J_0(\alpha_n r) - J_1(\alpha_n b_1)\, Y_0(\alpha_n r)\,, \tag{III.14}$$

$$Z_0(\alpha_n r) = \beta_{2n}\, J_0(\alpha_n r) - \beta_{1n}\, Y_0(\alpha_n r)\,, \tag{III.15}$$

$$\begin{aligned} \beta_{1n} &= \alpha_n b_4\, J_1(\alpha_n b_4) - B_0\, J_0(\alpha_n b_4)\,, \\ \beta_{2n} &= \alpha_n b_4\, Y_1(\alpha_n b_4) - B_0\, Y_0(\alpha_n b_4)\,. \end{aligned} \tag{III.16}$$

The above expansions already satisfy the boundary conditions (III.4) and (III.9). The remaining four boundary conditions will determine the coupling coefficients

$$\eta_{1n} = A_{2n}/A_{1n}\,,$$

$$\eta_{2n} = B_{2n}/A_{1n}\,,$$

$$\eta_{3n} = A_{3n}/A_{1n}\,,$$

and the eigenvalues α_n. If we now introduce the expansions (III.11), (III.12), and (III.13) into (III.5), (III.6), (III.7), and (III.8), we obtain

$$\eta_{1n}\, J_0(\alpha_n b_2) + \eta_{2n}\, Y_0(\alpha_n b_2) = Y_1^{-1}(\alpha_n b_1)\, U_0(\alpha_n b_2)\,, \tag{III.17}$$

$$\eta_{1n}\, J_1(\alpha_n b_2) + \eta_{2n}\, Y_1(\alpha_n b_2) = \frac{K_1}{K_2}\, Y_1^{-1}(\alpha_n b_1)\, U_1(\alpha_n b_2)\,, \tag{III.18}$$

$$\eta_1\, J_0(\alpha_n b_3) + \eta_{2n}\, Y_0(\alpha_n b_3) - \beta_2^{-1}\, \eta_{3n}\, Z_0(\alpha_n b_3) = 0\,, \tag{III.19}$$

$$\eta_{1n}\, J_1(\alpha_n b_3) + \eta_{2n}\, Y_1(\alpha_n b_3) - \frac{K_3}{K_2}\, \beta_2^{-1}\, \eta_{3n}\, Z_1(\alpha_n b_3) = 0\,. \tag{III.20}$$

From Eqs. (III.17) and (III.18) we obtain

$$\eta_{1n} = \frac{1}{W_n\, Y_1(\alpha_n b_1)} \left[U_0(\alpha_n b_2)\, Y_1(\alpha_n b_2) - \frac{K_1}{K_2}\, U_1(\alpha_n b_2)\, Y_0(\alpha_n b_2) \right], \tag{III.21}$$

$$\eta_{2n} = \frac{-1}{W_n\, Y_1(\alpha_n b_1)} \left[U_0(\alpha_n b_2)\, J_1(\alpha_n b_2) - \frac{K_1}{K_2}\, U_1(\alpha_n b_2)\, J_0(\alpha_n b_2) \right] = -\,\eta_{02n} \tag{III.22}$$

with

$$W_n = -\,\frac{2}{\pi \alpha_n b_2}\,. \tag{III.23}$$

The pair of equations (III.19) and (III.20) provide two equivalent expressions for the coupling factor η_{3n}. These are

$$\eta_{3n} = \frac{K_2 \beta_{2n}}{K_3\, Z_1(\alpha_n b_3)} \left[\eta_{1n}\, J_1(\alpha_n b_3) - \eta_{02n}\, Y_1(\alpha_n b_3)\right] , \tag{III.24}$$

$$\eta_{3n} = \frac{\beta_{2n}}{Z_0(\alpha_n b_3)} \left[\eta_{1n}\, J_0(\alpha_n b_3) - \eta_{02n}\, Y_0(\alpha_n b_3)\right] . \tag{III.25}$$

Finally by equating Eqs. (III.24) and (III.25) we obtain the secular equation for the determination of the eigenvalues α_n:

$$\left(\frac{K_2}{K_3}\right) \frac{Z_0\,(\alpha_n b_3)}{Z_1\,(\alpha_n b_3)} = \frac{\eta_{1n}\, J_0(\alpha_n b_3) - \eta_{02n}\, Y_0(\alpha_n b_3)}{\eta_{1n}\, J_1(\alpha_n b_3) - \eta_{02n}\, Y_1(\alpha_n b_3)} = \frac{W_0(\alpha_n b_3)}{W_1(\alpha_n b_3)} . \tag{III.26}$$

The temperature distribution in each region can now be expressed as a function of a single set of modal amplitudes $A_{1n}(t)$:

$$T_1(r,\ t) = T_b + \sum_n \frac{1}{Y_1(\alpha_n b_1)} A_{1n}(t)\, U_0(\alpha_n r) , \tag{III.27}$$

$$T_2(r,\ t) = T_b + \sum_n A_{1n}(t)\, W_0(\alpha_n (r) , \tag{III.28}$$

$$T_3(r,\ t) = T_b + \sum_n \frac{\eta_{3n}}{\beta_{2n}} A_{1n}(t)\, Z_0(\alpha_n r) , \tag{III.29}$$

with

$$W_0(\alpha_n r) = \eta_{1n}\, J_0(\alpha_n r) - \eta_{02n}\, Y_0(\alpha_n r) , \tag{III.30}$$

where, in view of the boundary conditions, the following relations hold:

$$U_1(\alpha_n b_1) = 0 , \tag{III.31}$$

$$Y_1^{-1}(\alpha_n b_1)\, U_0(\alpha_n b_2) = W_0(\alpha_n b_2) , \tag{III.32}$$

$$\frac{K_1}{K_2} Y_1^{-1}(\alpha_n b_1)\, U_1(\alpha_n b_2) = W_1(\alpha_n b_2) , \tag{III.33}$$

$$W_0(\alpha_n b_3) = \frac{\eta_{3n}}{\beta_{2n}} Z(\alpha_n b_3) , \tag{III.34}$$

$$W_1(\alpha_n b_3) = \frac{K_3}{K_2} \frac{\eta_{3n}}{\beta_{2n}} Z_1(\alpha_n b_3) , \tag{III.35}$$

$$\alpha_n b_4\, Z_1(\alpha_n b_4) - B_0\, Z_0(\alpha_n b_4) = 0 , \tag{III.36}$$

with the functions U_1, W_1, and Z_1 defined in Appendix A.

Before we can proceed with the solution of the time-dependent problem, we must prove the orthogonality properties of the eigenfunctions U_0, W_0, and Z_0 and obtain an expression for the normalization integral. This program is developed in Appendix A. The results are:

$$Y_1^{-1}(\alpha_p b_1)\, Y_1^{-1}(\alpha_n b_1)\, I_{1pn} + \frac{K_2}{K_1} I_{2pn} + \frac{K_3}{K_1} \frac{\eta_{3p}}{\beta_{2p}} \frac{\eta_{3n}}{\beta_{2n}} I_{3pn} = 0 , \qquad \text{(III.37)}$$

$$Y_1^{-2}(\alpha_p b_1)\, I_{1pp} + \frac{K_2}{K_1} I_{2pp} + \frac{K_3}{K_1} \left(\frac{\eta_{3p}}{\beta_{2p}}\right)^2 I_{3pp} = N_{pp} , \qquad \text{(III.38)}$$

where

$$I_{1pn} = (\alpha_n^2 - \alpha_p^2) \int_{b_1}^{b_2} r\, dr\, U_0(\alpha_p r)\, U_0(\alpha_n r), \qquad \text{(III.39)}$$

$$I_{2pn} = (\alpha_n^2 - \alpha_p^2) \int_{b_2}^{b_3} r\, dr\, W_0(\alpha_p r)\, W_0(\alpha_n r) , \qquad \text{(III.40)}$$

$$I_{3pn} = (\alpha_n^2 - \alpha_p^2) \int_{b_3}^{b_4} r\, dr\, Z_0(\alpha_p r)\, Z_0(\alpha_n r) , \qquad \text{(III.41)}$$

$$I_{1pp} = \int_{b_1}^{b_2} r\, dr\, U_0^2(\alpha_p r) , \qquad \text{(III.42)}$$

$$I_{2pp} = \int_{b_2}^{b_3} r\, dr\, W_0^2(\alpha_p r) , \qquad \text{(III.43)}$$

$$I_{3pp} = \int_{b_3}^{b_4} r\, dr\, Z_0^2(\alpha_p r) , \qquad \text{(III.44)}$$

$$N_{pp} = \frac{1}{2} \left\{ Y_1^{-2}(\alpha_p b_1)\, b_2^2 \left[\left(1 - \frac{K_2}{K_1}\right) U_0^2(\alpha_p b_2) + \left(1 - \frac{K_1}{K_2}\right) U_1^2(\alpha_n b_2) - \frac{b_1^2}{b_2^2} U_0^2(\alpha_n b_1) \right] \right.$$
$$\left. + \frac{K_2 - K_3}{K_1} b_3^2 \left[W_0^2(\alpha_p b_3) - \frac{K_2}{K_3} W_1^2(\alpha_p b_3) \right] + \frac{K_3}{K_1} \left(\frac{\eta_3 p}{\beta_2 p}\right)^2 (b_4^2 + \alpha_p^{-2} B_0^2)\, Z_0^2(\alpha_p b_4) \right\} . \qquad \text{(III.45)}$$

The above integrals are given also in Appendix A.

Introduction of the eigenfunction expansions (III.27), (III.28), and (III.29) into the heat diffusion equations and proceeding in the usual way yields:

$$\tau_{pp} \frac{d}{dt} A_{1p}(t) + \sum_{n \neq p} \tau_{pn} \frac{d}{dt} A_{1n}(t) + A_{1p}(t) = a_p \frac{p(t)}{K_1}, \qquad \text{(III.46)}$$

where the following time constants have been introduced:

$$\tau_{pp} = \tau_{1p} H_{1pp} + \frac{K_2}{K_1} \tau_{2p} H_{2pp} + \frac{K_3}{K_1} \tau_{3p} H_{3pp} \, , \tag{III.47}$$

$$\tau_{pn} = \tau_{1p} H_{1pn} + \frac{K_2}{K_1} \tau_{2p} H_{2pn} + \frac{K_3}{K_1} \tau_{3p} H_{3pn} \, , \tag{III.48}$$

with

$$\tau_{1p} = \frac{1}{\alpha_p^2 \kappa_1} \, , \; \tau_{2p} = \frac{1}{\alpha_p^2 \kappa_2} \, , \; \tau_{3p} = \frac{1}{\alpha_p^2 \kappa_3} \, , \tag{III.49}$$

$$H_{1pp} = Y_1^{-2}(\alpha_p b_1) \frac{I_{1pp}}{N_{pp}} \, , \; H_{2pp} = \frac{I_{2pp}}{N_{pp}} \, , \; H_{3pp} = \left(\frac{\eta_{3p}}{\beta_2 p} \right)^2 \frac{I_{3pp}}{N_{pp}} \, , \tag{III.50}$$

$$H_{1pn} = \frac{Y_1^{-1}(\alpha_p b_1) \, Y_1^{-1}(\alpha_n b_1)}{N_{pp}} \, \frac{I_{1pn}}{\alpha_n^2 - \alpha_p^2} \, , \; H_{2pn} = \frac{I_{2pn}}{(\alpha_n^2 - \alpha_p^2) N_{pp}} \, , \tag{III.51}$$

$$H_{3pn} = \frac{\eta_{3n}}{\beta_{2n}} \, \frac{\eta_{3p}}{\beta_{2p}} \, \frac{I_{3pn}}{(\alpha_n^2 - \alpha_p^2) N_{pp}} \, ,$$

and

$$a_p = \frac{1}{\alpha_p^2 \, Y_1(\alpha_p b_1) \, N_{pp}} \int_{b_1}^{b_2} U_0(\alpha_p r) \, r \, dr = \frac{b_2}{\alpha_p^3 \, Y_1(\alpha_p b_1)} \, \frac{U_1(\alpha_p b_2)}{N_{pp}} \; (\mathrm{cm}^2) \; . \tag{III.52}$$

From the result (III.46) one sees that the modal amplitudes are coupled through the time constants τ_{pn}. The individual time constants are weighted sums of the characteristic times of the regions involved in the problem (i.e., τ_{1p}, τ_{2p}, and τ_{3p}) with the dimensionless matrix elements H_{ipp} and H_{ipn} ($i = 1, 2, 3$) acting as the weight factors.

IV. THE TWO-MODES APPROXIMATION

In this case Laplace transformation of the set (III.46) and especialization to two modes yields

$$(1 + s\tau_{11}) A_{11}(s) + s\tau_{12} A_{12}(s) = \frac{a_1}{K_1} p(s) + \Sigma_1 \, , \tag{IV.1}$$

$$s\tau_{21} A_{11}(s) + (1 + s\tau_{22}) A_{12}(s) = \frac{a_2}{K_1} p(s) + \Sigma_2 \, , \tag{IV.2}$$

where for a step function input heat pulse one has

$$p(s) = \frac{1}{s}\, p_0 \,, \tag{IV.3}$$

and

$$\Sigma_1 = \tau_{11} A_{11}(0) + \tau_{12} A_{12}(0) \,, \tag{IV.4}$$

$$\Sigma_2 = \tau_{21} A_{11}(0) + \tau_{22} A_{12}(0) \,. \tag{IV.5}$$

This gives the following expressions for the Laplace-transformed modal amplitude:

$$A_{11}(s) = \frac{[a_1 + s(a_1\tau_{22} - \tau_{12}a_2)]\, p_0/K_1 + s[\Sigma_1(1 + s\tau_{22}) - s\tau_{12}\Sigma_2)]}{s[s^2 + s(\tau_{11} + \tau_{22}/d_{12}) + 1/d_{12}]d_{12}} \,, \tag{IV.6}$$

$$A_{12}(s) = \frac{[a_2 + s(a_2\tau_{11} - \tau_{21}a_1)]\,(p_0/K_1) + s[\Sigma_2(1 + s\tau_{11}) - \tau_{21}s\Sigma_1]}{s\{s^2 + [s(\tau_{11} + \tau_{22})/d_{12}] + (1/d_{12})\}d_{12}} \,, \tag{IV.7}$$

with

$$d_{12} = \tau_{11}\tau_{22} - \tau_{12}\tau_{21} \,. \tag{IV.8}$$

The poles are located at $s = 0$ and $s = -s_1$, $s = -s_2$, where

$$s_1 = \frac{1}{2}\left[-\frac{\tau_{11} + \tau_{22}}{d_{12}} + \sqrt{\left(\frac{\tau_{11} + \tau_{22}}{d_{12}}\right)^2 - \frac{4}{d_{12}}}\right] , \tag{IV.9}$$

$$s_2 = \frac{1}{2}\left[-\frac{\tau_{11} + \tau_{22}}{d_{12}} - \sqrt{\left(\frac{\tau_{11} + \tau_{22}}{d_{12}}\right)^2 - \frac{4}{d_{12}}}\right] , \tag{IV.10}$$

with the relations

$$|s_1| + |s_2| = \frac{\tau_{11} + \tau_{22}}{d_{12}} \,, \tag{IV.11}$$

$$s_1 s_2 = \frac{1}{d_{12}} \,. \tag{IV.12}$$

Straightforward Laplace inversions of Eqs. (IV.6) and (IV.7) yield

$$A_{11}(t) = R_{11}^{(0)} - R_{11}^{(1)} e^{-s_1 t} - R_{11}^{(2)} e^{-s_2 t} \,, \tag{IV.13}$$

$$A_{12}(t) = R_{12}^{(0)} - R_{12}^{(1)} e^{-s_1 t} - R_{12}^{(2)} e^{-s_2 t} \,, \tag{IV.14}$$

where

$$R_{11}^{(0)} = a_1 p_0/K_1 \; , \tag{IV.15}$$

$$R_{12}^{(0)} = a_2 \frac{p_0}{K_1} \; , \tag{IV.16}$$

$$R_{11}^{(1)} = \frac{1}{1-(s_1/s_2)} \left\{ [a_1 - s_1(a_1\tau_{22} - a_2\tau_{12})] \frac{p_0}{K_1} - s_1[\Sigma_1(1 - s_1\tau_{22}) + \Sigma_2\tau_{12}s_1] \right\} \; , \tag{IV.17}$$

$$R_{11}^{(2)} = \frac{1}{(s_2/s_1)-1} \left\{ s_2[\Sigma_1(1 - s_2\tau_{22}) + s_2\tau_{12}\Sigma_2] - [a_1 - s_2(a_1\tau_{22} - \tau_{12}a_2)] \frac{p_0}{K_1} \right\} \; , \tag{IV.18}$$

$$R_{12}^{(1)} = \frac{1}{1-(s_1/s_2)} \left\{ [a_2 - s_1(a_2\tau_{11} - a_1\tau_{21})] \frac{p_0}{K_1} - s_1[\Sigma_2(1 - s_1\tau_{11}) + \Sigma_1\tau_{21}s_1] \right\} \; , \tag{IV.19}$$

$$R_{12}^{(2)} = \frac{1}{(s_2/s_1)-1} \left\{ s_2[\Sigma_2(1 - s_2\tau_{11}) + s_2\tau_{21}\Sigma_1] - [a_2 - s_2(a_2\tau_{11} - a_1\tau_{21})] \frac{p_0}{K_1} \right\} . \tag{IV.20}$$

Therefore within the two-modes approximation one has, from Eqs. (III.27), (III.28), (III.29), (IV.13), and (IV.14), the following expressions for the temperature distribution:

$$T_1(r,\, t) = T_b + R_{11}^{(0)}\, U_{10}(\alpha_1 r) + R_{12}^{(0)}\, U_{10}(\alpha_2 r) - [R_{11}^{(1)}\, U_{10}(\alpha_1 r) + R_{12}^{(1)}\, U_{10}(\alpha_2 r)]e^{-s_1 t}$$
$$- [R_{11}^{(2)}\, U_{10}(\alpha_1 r) + R_{12}^{(2)}\, U_{10}(\alpha_2 r)]e^{-s_2 t} \; , \tag{IV.21}$$

$$U_{10}(\alpha_n r) = Y^{-1}(\alpha_n b_1)\, U_0(a_n r) \; , \tag{IV.22}$$

$$T_2(r,\, t) = T_b + R_{11}^{(0)}\, W_0(\alpha_1 r) + R_{12}^{(0)}\, W_0(\alpha_2 r) - [R_{11}^{(1)}\, W_0\,(\alpha_1 r) + R_{12}^{(1)}\, W_0\,(\alpha_2 r)]e^{-s_1 t}$$
$$- [R_{11}^{(2)}\, W_0(\alpha_1 r) + R_{12}^{(2)}\, W_0(\alpha_2 r)]e^{-s_2 t} \; , \tag{IV.23}$$

$$T_3(r,\, t) = T_b + R_{11}^{(0)} \frac{\eta_{31}}{\beta_{21}} Z_0(\alpha_1 r) + R_{12}^{(0)} \frac{\eta_{32}}{\beta_{22}} Z_0(\alpha_2 r) - \left[R_{11}^{(1)} \frac{\eta_{31}}{\beta_{21}} Z_0(\alpha_1 r) \right.$$
$$\left. + R_{12}^{(1)} \frac{\eta_{32}}{\beta_{22}} Z_0(\alpha_2 r) \right] e^{-s_1 t} - \left[R_{11}^{(2)} \frac{\eta_{31}}{\beta_{21}} Z_0(\alpha_1 r) + R_{12}^{(2)} \frac{\eta_{32}}{\beta_{22}} Z_0(\alpha_2 r) \right] e^{-s_2 t} . \tag{IV.24}$$

V. DETERMINATION OF THE THERMAL DIFFUSIVITY FOR THE CENTRAL REGION

This section will be devoted to the problem of determining κ_1, assuming that the thermal diffusivities of the other regions are known. The unknowns to be determined are κ_1, B_0, and the two spatial eigenvalues α_1 and α_2. To this end we have at our disposal Eqs. (IV.11) and (IV.12), arising from the knowledge of the two time eigenvalues s_1 and s_2, and the secular equation (III.26) written for α equal to α_1 and α_2. The first step is to rewrite those equations showing explicitly their dependence on κ_1, K_2, K_3, and B_0. We define the following functions:

$$L_{0n}(b_2, b_3) = Y_1(\alpha_n b_3)\, J_0(\alpha_n b_2) - J_1(\alpha_n b_3)\, Y_0(\alpha_n b_2)\,, \tag{V.1}$$

$$L_{1n}(b_2, b_3) = Y_1(\alpha_n b_3)\, J_1(\alpha_n b_2) - J_1(\alpha_n b_3)\, Y_1(\alpha_n b_2)\,, \tag{V.2}$$

$$Q_{0n}(b_2, b_3) = Y_0(\alpha_n b_3)\, J_0(\alpha_n b_2) - J_0(\alpha_n b_3)\, Y_0(\alpha_n b_2)\,, \tag{V.3}$$

$$Q_{1n}(b_2, b_3) = Y_0(\alpha_n b_3)\, J_1(\alpha_n b_2) - J_0(\alpha_n b_3)\, Y_1(\alpha_n b_2)\,, \tag{V.4}$$

$$M_{0n}(b_3, b_4) = Y_1(\alpha_n b_4)\, J_0(\alpha_n b_3) - J_1(\alpha_n b_4)\, Y_0(\alpha_n b_3)\,, \tag{V.5}$$

$$M_{1n}(b_3, b_4) = Y_1(\alpha_n b_4)\, J_1(\alpha_n b_3) - J_1(\alpha_n b_4)\, Y_1(\alpha_n b_3)\,, \tag{V.6}$$

$$S_{0n}(b_3, b_4) = Y_0(\alpha_n b_4)\, J_0(\alpha_n b_3) - J_0(\alpha_n b_4)\, Y_0(\alpha_n b_3)\,, \tag{V.7}$$

$$S_{1n}(b_3, b_4) = Y_0(\alpha_n b_4)\, J_1(\alpha_n b_3) - J_0(\alpha_n b_4)\, Y_1(\alpha_n b_3)\,, \tag{V.8}$$

with the properties

$$L_{0n}(b_2, b_2) = W_n(b_2)\,, \tag{V.9}$$

$$M_{0n}(b_4, b_4) = W_n(b_4)\,, \tag{V.10}$$

where $W_n(b_2)$ and $W_n(b_4)$ are the Wronskians

$$W_n(x) = Y_1(\alpha_n x)\, J_0(\alpha_n x) - J_1(\alpha_n x)\, Y_0(\alpha_n x) = -\frac{2}{\pi \alpha_n x}\,. \tag{V.11}$$

In terms of the above functions we have the following expressions for the functions $W_0(\alpha_n x)$, $W_1(\alpha_n x)$, $Z_0(\alpha_n x)$, and $Z_1(\alpha_n x)$:

$$W_0(\alpha_n b_2) = \frac{1}{\Delta_n Y_1(\alpha_n b_1)}\, U_0(\alpha_n b_2)\, W_n(b_2)\,, \tag{V.12}$$

$$W_0(\alpha_n b_3) = -\frac{1}{\Delta_n Y_1(\alpha_n b_1)}\left[U_0(\alpha_n b_2)\, Q_{1n}(b_2, b_3) - \frac{K_1}{K_2}\, U_1(\alpha_n b_2)\, Q_{0n}(b_2, b_3)\right], \tag{V.13}$$

$$W_1(\alpha_n b_2) = \frac{1}{\Delta_n\, Y_1(\alpha_n b_1)}\, \frac{K_1}{K_2}\, U_1(\alpha_n b_2)\, W_n(b_2)\,, \tag{V.14}$$

$$W_1(\alpha_n b_3) = \frac{-1}{\Delta_n\, Y_1(\alpha_n b_1)} \left[U_0(\alpha_n b_2)\, L_{1n}(b_2,\, b_3) - \frac{K_1}{K_2}\, U_1(b_2)\, L_{0n}(b_2,\, b_3) \right] , \tag{V.15}$$

$$Z_0(\alpha_n b_4) = \alpha_n b_4\, W_n(b_4) \,, \tag{V.16}$$

$$Z_0(\alpha_n b_3) = \alpha_n b_4\, T_{0n}(b_3,\, b_4) - B_0\, S_{0n}(b_3,\, b_4) \,, \tag{V.17}$$

$$Z_1(\alpha_n b_4) = B_0\, W_n(b_4) \,, \tag{V.18}$$

$$Z_1(\alpha_n b_3) = \alpha_n b_4\, T_{1n}(b_3,\, b_4) - B_0\, S_{1n}(b_3,\, b_4) \,. \tag{V.19}$$

The above relations allow us to write the secular equation (III.26) and the integrals I_{pp} and I_{pn} in a manner such that the constants K_1, K_2, K_3, and B_0 are explicitly shown. We obtain

$$\alpha_n b_4\, U_0(\alpha_n b_2) \left(L_{1n} M_{0n} \frac{K_2}{K_1} - Q_{1n} M_{1n} \frac{K_3}{K_1} \right) + \alpha_n b_4\, U_1(\alpha_n b_2) \left(Q_{0n} M_{1n} \frac{K_3}{K_2} - L_{0n} M_{0n} \right)$$

$$+ B_0 \left[U_0(\alpha_n b_2) \left(Q_{1n} S_{1n} \frac{K_3}{K_1} - L_{1n} S_{0n} \frac{K_2}{K_1} \right) + U_1(\alpha_n b_2) \left(L_{0n} S_{0n} - Q_{0n}\, S_{1n} \frac{K_3}{K_2} \right) \right] = 0 \tag{V.20}$$

for the secular equation, and

$$I_{1pp} = \frac{1}{2\Delta_p^2}\, G_{0pp}^{(1)} \,, \tag{V.21}$$

with

$$G_{0pp}^{(1)} = \{ b_2^2 [U_0^2(\alpha_p b_2) + U_1^2(\alpha_p b_2)] - b_1^2\, U_0^2(\alpha_p b_1) \} \Delta_p^2 \,; \tag{V.22}$$

$$I_{2pp} = \frac{1}{2\Delta_p^2\, Y_1^2(\alpha_p b_1)} \left[G_{0pp}^{(2)} + G_{2pp}^{(2)} \left(\frac{K_1}{K_2} \right)^2 - 2G_{1pp}^{(2)} \frac{K_1}{K_2} \right] , \tag{V.23}$$

with

$$G_{0pp}^{(2)} = U_0^2(\alpha_p b_2)\, [b_3^2 (Q_{1p}^2 + L_{1p}^2) - b_2^2\, W_p^2(b_2)] \,, \tag{V.24}$$

$$G_{1pp}^{(2)} = U_0(\alpha_p b_2)\, U_1(\alpha_p b_1)\, b_3^2 (Q_{1p} Q_{0p} + L_{0p} L_{1p}) \,, \tag{V.25}$$

$$G_{2pp}^{(2)} = U_1^2(\alpha_p b_2)\, [b_3^2 (Q_{0p}^2 + L_{0p}^2) - b_2^2\, W_p^2(b_2)] \,; \tag{V.26}$$

$$I_{3pp} = \frac{1}{2}\, [(\alpha_p b_4)^2\, G_{0pp}^{(3)} + G_{2pp}^{(3)} B_0^2 + 2\, \alpha_p b_4\, B_0 G_{1pp}^{(3)}] \,, \tag{V.27}$$

with

$$G_{0pp}^{(3)} = b_4^2\, W_p^2(b_4) - b_3^2\, (T_{0p}^2 + T_{1p}^2) \,, \tag{V.28}$$

$$G^{(3)}_{1pp} = b_3^2\,(M_{0p}S_{0p} + M_{1p}S_{1p})\,, \tag{V.29}$$

$$G^{(3)}_{2pp} = b_4^2\,W_p^2(b_4) - b_3^2(S_{0p}^2 + S_{1p}^2)\,; \tag{V.30}$$

while for $p \neq n$:

$$I_{1pn} = \alpha_n b_2\,U_0(\alpha_p b_2)\,U_1(\alpha_n b_2) - \alpha_p b_2\,U_0(\alpha_n b_2)\,U_1(\alpha_p b_2) = G^{(1)}_{0pn}\,\frac{1}{\Delta_p\Delta_n}\,, \tag{V.31}$$

with

$$G^{(1)}_{0pn} = \Delta_p\Delta_n[\alpha_n b_2\,U_0(\alpha_p b_2)\,U_1(\alpha_n b_2) - \alpha_p b_2\,U_0(\alpha_n b_2)\,U_1(\alpha_p b_2)]\,; \tag{V.32}$$

$$I_{2pn} = \frac{1}{\Delta_n\Delta_p\,Y_1(\alpha_p b_1)\,Y_1(\alpha_n b_1)}\left[G^{(2)}_{0pn} + G^{(2)}_{2pn}\left(\frac{K_1}{K_2}\right)^2 - G^{(2)}_{1pn}\,\frac{K_1}{K_2}\right], \tag{V.33}$$

with

$$G^{(2)}_{0pn} = [\alpha_n\,U_0(\alpha_p b_2)\,U_0(\alpha_n b_2)\,Q_{1p}L_{1n} - \alpha_p\,U_0(\alpha_n b_2)\,U_0(\alpha_p b_2)\,Q_{1n}L_{1p}]b_3\,, \tag{V.34}$$

$$\begin{aligned} G^{(2)}_{1pn} = {} & \alpha_n b_3\,[U_0(\alpha_p b_2)\,U_1(\alpha_n b_2)\,Q_{1p}L_{0n} + U_0(\alpha_n b_2)\,U_1(\alpha_p b_2)\,Q_{0p}L_{1n}] \\ & - \alpha_p b_3\,[U_0(\alpha_n b_2)\,U_1(\alpha_p b_2)\,Q_{1n}L_{0p} + U_0(\alpha_p b_2)\,U_1(\alpha_n b_2)\,Q_{0n}L_p] \\ & - b_2[\alpha_p\,U_0(\alpha_n b_2)\,U_1(\alpha_p b_2) - \alpha_n\,U_0(\alpha_p b_2)\,U_1(\alpha_n b_2)]\,W_p(b_2)\,W_n(b_2)\,, \end{aligned} \tag{V.35}$$

$$G^{(2)}_{2pn} = b_3(\alpha_n Q_{0p}L_{0n} - \alpha_p Q_{0n}L_{0p})\,U_1(\alpha_p b_2)\,U_1(\alpha_n b_2)\,; \tag{V.36}$$

$$I_{3pn} = G^{(3)}_{0pn} + G^{(3)}_{2pn}B_0^2 + G^{(3)}_{1pn}B_0\,, \tag{V.37}$$

with

$$G^{(3)}_{0pn} = \alpha_n b_3(\alpha_p b_4)^2\,M_{0n}M_{1p} - \alpha_p b_3(\alpha_n b_4)^2\,M_{0p}M_{1n}\,, \tag{V.38}$$

$$G^{(3)}_{2pn} = (\alpha_p S_{0n}S_{1p} - \alpha_n S_{0p}S_{1n})b_3\,, \tag{V.39}$$

$$G^{(3)}_{1pn} = \alpha_n\alpha_p b_4 b_3(M_{0p}S_{1n} - M_{0n}S_{1p}) + (\alpha_n^2 S_{0p}M_{1n} - \alpha_p^2 S_{0n}M_{1p})b_3 b_4\,. \tag{V.40}$$

Next we express the weighted time constants τ_{11}, τ_{22}, τ_{12}, and τ_{21} in terms of the above relations. We now have

$$\tau_{pp} = \frac{1}{\alpha_p^2\kappa_1}\,\frac{\Lambda^0_{pp} - 2\Lambda^{(1)}_{pp}(K_1/K_2) + \Lambda^{(2)}_{pp}(K_1/K_2)^2}{\Omega^0_{pp} + 2\Omega^{(1)}_{pp}(K_1/K_2) + \Omega^2_{pp}(K_1/K_2)^2} \qquad (p = 1, 2)\,, \tag{V.41}$$

$$\tau_{pn} = 2\left(\frac{W_p}{W_n} \cdot \frac{Y_1(\alpha_p b_1)}{Y_1(\alpha_n b_1)} \cdot \frac{V_p}{V_n}\right)\left(\frac{1}{\alpha_p^2(\alpha_n^2 - \alpha_p^2)\kappa_1}\right)\left(\frac{\Lambda_{pn}^{(0)} - \Lambda_{pn}^{(1)}(K_1/K_2) + \Lambda_{pn}^{(2)}(K_1/K_2)^2}{\Lambda_{pp}^{(1)} + \Lambda_{pp}^{(1)}(K_1/K_2) + \Lambda_{pp}^{(3)}(K_1/K_2)^2}\right)(p \neq n,\ 1,\ 2)\ , \tag{V.42}$$

where the following matrix elements have been introduced:

$$\xi_{0p} = U_0(\alpha_p b_2)\, Q_{1p}\ , \tag{V.43}$$

$$\xi_{1p} = U_1(\alpha_p b_2)\, Q_{0p}\ , \tag{V.44}$$

$$\left.\begin{aligned} D_{pp}(B_0) &= (\alpha_p b_4)^2 G_{0pp}^{(3)} + G_{2pp}^{(3)} B_0^2 + 2\alpha_p b_4 G_{1pp}^{(3)} B_0 \\ D_{pn}(B_0) &= G_{0pn}^{(3)} + G_{2pn}^{(3)} B_0^2 + G_{1pn}^{(3)} B_0 \end{aligned}\right\}, \tag{V.45}$$

$$\left.\begin{aligned} &\Gamma_{pp}^{(0)} = \xi_{0p}^2 D_{pp}\ ;\ \Gamma_{pp}^{(1)} = \xi_{0p}\xi_{1p} D_{pp}\ ;\ \Gamma_{pp}^{(2)} = \xi_{1p}^2 D_{pp} \\ &\Gamma_{pn}^{(0)} = \xi_{0p}\xi_{0n} D_{pn}\ ;\ \Gamma_{pn}^{(1)} = (\xi_{0p}\xi_{1n} + \xi_{1p}\xi_{0n}) D_{pn}\ ;\ \Gamma_{pn}^{(2)} = \xi_{1p}\xi_{1n} D_{pn} \end{aligned}\right\}, \tag{V.46}$$

$$\Lambda_{pp}^{(0)} = V_p[G_{0pp}^{(1)} + (\rho_2 c_2/\rho_1 c_1) G_{0pp}^{(2)}] + (\rho_3 c_3/\rho_1 c_1)\Gamma_{pp}^{(0)} \tag{V.47}$$

$$\Lambda_{pp}^{(1)} = V_p(\rho_2 c_2/\rho_1 c_1) G_{1pp}^{(2)} + (\rho_3 c_3/\rho_1 c_1)\Gamma_{pp}^{(1)} \tag{V.48}$$

$$\Lambda_{pp}^{(2)} = V_p(\rho_2 c_2/\rho_1 c_1) G_{2pp}^{(2)} + (\rho_3 c_3/\rho_1 c_1)\Gamma_{pp}^{(2)} \tag{V.49}$$

$$\Omega_{pp}^{(0)} = V_p[G_{0pp}^{(1)} - 2G_{1pp}^{(2)} + (K_2/K_1) G_{0pp}^{(2)}] + (K_3/K_1)\Gamma_{pp}^{(0)} \tag{V.50}$$

$$\Omega_{pp}^{(1)} = 1/2[V_p G_{2pp}^{(2)} - 2(K_3/K_1)\Gamma_{pp}^{(1)}] \tag{V.51}$$

$$\Omega_{pp}^{(2)} = (K_3/K_1)\Gamma_{pp}^{(2)} \tag{V.52}$$

$$\Lambda_{pn}^{(0)} = V_p V_n\, [G_{0pn}^{(1)} + (\rho_2 c_2/\rho_1 c_1) G_{0pn}^{(2)}] + (\rho_3 c_3/\rho_1 c_1)\Gamma_{pn}^{(0)} \tag{V.53}$$

$$\Lambda_{pn}^{(1)} = V_p V_n(\rho_2 c_2/\rho_1 c_1) G_{1pn}^{(2)} + (\rho_3 c_3/\rho_1 c_1)\Gamma_{pn}^{(1)} \tag{V.54}$$

$$\Lambda_{pn}^{(2)} = V_p V_n(\rho_2 c_2/\rho_1 c_1) G_{2pn}^{(2)} + (\rho_3 c_3/\rho_1 c_1)\Gamma_{pn}^{(2)} \tag{V.55}$$

where

$$\Lambda_{np}^{(i)} = -\Lambda_{pn}^{(i)} \qquad (i = 0,\ 1,\ 2)\ . \tag{V.56}$$

$$V_n = \alpha_n b_4 M_{0n} - B_0 S_{0n} \tag{V.57}$$

$$V_p = \alpha_p b_4 M_{0p} - B_0 S_{0p} \tag{V.58}$$

Introduction of the relations (V.41) and (V.42) into Eqs. (IV.11) and (IV.12) yields

$$\left(\frac{1}{s_1}+\frac{1}{s_2}\right)\kappa_1=\frac{1}{\alpha_1^2\alpha_2^2}\,\frac{C_0+2C_1(K_1/K_2)+C_2\,(K_1/K_2)^2-2C_3(K_1/K_2)^3+C_4(K_1/K_2)^4}{O_0+2O_1(K_1/K_2)+O_2\,(K_1/K_2)^2+2O_3(K_1/K_2)^3+O_4(K_1/K_2)^4}, \tag{V.59}$$

$$\frac{\kappa_1^2\alpha_1^2\alpha_2^2}{s_1s_2}=\frac{E_0-2E_1(K_1/K_2)+E_2(K_1/K_2)^2-2E_3(K_1/K_2)^3+E_4(K_1/K_2)^4}{O_0+2O_1(K_1/K_2)+O_2(K_1/K_2)^2+2O_3(K_1/K_2)^3+O_4(K_1/K_2)^4}, \tag{V.60}$$

with

$$C_0=\alpha_2^2\Lambda_{11}^{(0)}\Omega_{22}^{(0)}+\alpha_1^2\Lambda_{22}^{(0)}\Omega_{11}^{(0)}\,, \tag{V.61}$$

$$C_1=\alpha_2^2(\Lambda_{11}^{(0)}\Omega_{22}^{(1)}-\Lambda_{11}^{(1)}\Omega_{22}^{(0)})+\alpha_1^2(\Lambda_{22}^{(0)}\Omega_{11}^{(1)}-\Lambda_{22}^{(1)}\Omega_{11}^{(0)})\,, \tag{V.62}$$

$$C_2=\alpha_2^2(\Lambda_{11}^{(0)}\Omega_{22}^{(2)}-4\Lambda_{11}^{(1)}\Omega_{22}^{(1)}+\Lambda_{11}^{(2)}\Omega_{22}^{(0)})+\alpha_1^2(\Lambda_{22}^{(0)}\Omega_{11}^{(2)}-4\Lambda_{22}^{(1)}\Omega_{11}^{(1)}+\Lambda_{22}^{(2)}\Omega_{11}^{(0)})\,, \tag{V.63}$$

$$C_3=\alpha_2^2(\Lambda_{11}^{(1)}\Omega_{22}^{(2)}-\Lambda_{11}^{(2)}\Omega_{22}^{(1)})+\alpha_1^2(\Lambda_{22}^{(1)}\Omega_{11}^{2}-\Lambda_{22}^{(2)}\Omega_{11}^{(1)})\,, \tag{V.64}$$

$$C_4=\alpha_2^2\Lambda_{11}^{(2)}\Omega_{22}^{(2)}+\alpha_1^2\Lambda_{22}^{(2)}\Omega_{11}^{(2)}\,, \tag{V.65}$$

$$O_0=\Omega_{11}^{(0)}\Omega_{22}^{(0)}\,, \tag{V.66}$$

$$O_1=\Omega_{22}^{(1)}\Omega_{11}^{(0)}+\Omega_{11}^{(1)}\Omega_{22}^{(0)}\,, \tag{V.67}$$

$$O_2=\Omega_{11}^{(0)}\Omega_{22}^{(2)}+4\Omega_{11}^{(1)}\Omega_{22}^{(1)}+\Omega_{11}^{(2)}\Omega_{22}^{(0)}\,, \tag{V.68}$$

$$O_3=\Omega_{11}^{(1)}\Omega_{22}^{(2)}+\Omega_{11}^{(2)}\Omega_{22}^{(1)}\,, \tag{V.69}$$

$$O_4=\Omega_{11}^{(2)}\Omega_{22}^{(2)}\,, \tag{V.70}$$

$$E_0=\Lambda_{11}^{(0)}\Lambda_{22}^{(0)}-\frac{4}{(\alpha_2^2-\alpha_1^2)^2}\Lambda_{12}^{(0)2}\,, \tag{V.71}$$

$$E_1=\Lambda_{11}^{(0)}\Lambda_{22}^{(1)}+\Lambda_{11}^{(1)}\Lambda_{22}^{(0)}-\frac{4}{\left(\alpha_2^2-\alpha_1^2\right)^2}\Lambda_{12}^{(0)}\Lambda_{12}^{(1)}\,, \tag{V.72}$$

$$E_2=\Lambda_{11}^{0}\Lambda_{22}^{(2)}+4\Lambda_{11}^{(1)}\Lambda_{22}^{(1)}+\Lambda_{11}^{(2)}\Lambda_{22}^{(0)}-\frac{4}{\left(\alpha_2^2-\alpha_1^2\right)^2}(2\Lambda_{12}^{(0)}\Lambda_{12}^{(2)}+\Lambda_{12}^{(1)^2}) \tag{V.73}$$

$$E_3=\Lambda_{11}^{(1)}\Lambda_{22}^{(2)}+\Lambda_{11}^{(2)}\Lambda_{22}^{(1)}-\frac{4}{\left(\alpha_2^2-\alpha_1^2\right)^2}\Lambda_{12}^{(2)}\Lambda_{12}^{(1)}\,, \tag{V.74}$$

$$E_4 = \Lambda_{11}^{(2)}\Lambda_{22}^{(2)} - \frac{4}{(\alpha_2^2 - \alpha_1^2)^2}\Lambda_{12}^{(2)2} . \tag{V.75}$$

The environmental conditions represented via the Biot number B_0 can be eliminated by the use of the secular equation (V.23). We then obtain

$$B_0 = \alpha_n b_4 \frac{U_0(\alpha_n b_2)[(K_2/K_1)L_{1n}T_{0n} - (K_3/K_1)Q_{1n}T_{1n}] + U_1(\alpha_n b_2)\,[(K_3/K_2)Q_{0n}T_{1n} - L_{0n}T_{0n}]}{U_0(\alpha_n b_2)[(K_2/K_1)L_{1n}s_{0n} - (K_3/K_1)Q_{1n}s_{1n}] + U_1(\alpha_n b_2)\,[(K_3/K_2)Q_{0n}s_{1n} - L_{0n}s_{0n}]} . \tag{V.76}$$

Because Eq. (V.76) has to be satisfied by the two space eigenvalues α_1 and α_2, one obtains

$$\left(\frac{\alpha_2}{\alpha_1}\right) \frac{U_0(\alpha_2 b_2)[(K_2/K_1)L_{12}T_{02} - (K_3/K_1)Q_{12}T_{12}] + U_1(\alpha_2 b_2)[(K_3/K_2)Q_{02}T_{12} - L_{02}T_{02}]}{U_0(\alpha_2 b_2)[(K_2/K_1)L_{12}s_{02} - (K_3/K_1)Q_{12}s_{12}] + U_1(\alpha_2 b_2)](K_3/K_2)Q_{02}s_{12} - L_{02}s_{02}]} \tag{V.77}$$

$$- \frac{U_0(\alpha_1 b_2)[(K_2/K_1)L_{11}T_{01} - (K_3/K_1)Q_{11}T_{11}] + U_1(\alpha_1 b_2)[K_3/K_2)Q_{01}T_{11} - L_{01}T_{01}]}{U_0(\alpha_1 b_2)[(K_2/K_1)L_{11}s_{01} - (K_3/K_1)Q_{11}s_{11}] + U_1(\alpha_1 b_2)[(K_3/K_2)Q_{01}s_{11} - L_{01}s_{01}]} = 0 .$$

Finally, introduction of (V.76), evaluated for either value α_1 or α_2, into Eqs. (V.59) and (V.60) yields, together with the relation (V.77), three equations to determine α_1, α_2, and κ_1.

The numerical problem associated with the above procedure has been coded for the IBM 360/91 computer with satisfactory results. Details of the numerical techniques utilized will be presented in a forthcoming paper.

VI. CONCLUSIONS

We have developed a general theory of time-dependent heat diffusion phenomena which shows the feasibility of measuring the thermal diffusivity with a single temperature measurement, eliminating the necessity of measuring the power input. When the effects of the enclosure of the sample (one-region problem) can be neglected, a very simple relation for the thermal diffusivity of the sample has been obtained, Eq. (II.26). For the general case the value of the thermal diffusivity has to be obtained by iteration of three transcendental equations. The input (parameters) needed for the calculation are the two lowest time eigenvalues observed in the temperature-time curves and the material properties of the regions surrounding the sample.

The method devised in this work may become an important tool for the measurement of small samples, when it is only possible to measure the central temperature accurately.

APPENDIX A

Orthogonality Conditions and the Normalization Integral

Calling $X_0(\alpha_n r)$ any of the functions U_0, W_0, Z_0, one has from the Bessel differential equation:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)X_0(\alpha_n r) + \alpha_n^2\, X_0(\alpha_n r) = 0\ , \tag{A.1}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}\right)X_0(\alpha_p r) + \alpha_p^2\, X_0(\alpha_p r) = 0\ . \tag{A.2}$$

From the above equations one obtains

$$I_{ipn} = (\alpha_n^2 - \alpha_p^2)\int_{x_1}^{x_2} r\, dr\, X_0(\alpha_p r)\, X_0(\alpha_n r) = \left[X_0(\alpha_n r)\, r\frac{d}{dr}X_0(\alpha_p r)\right]_{x_1}^{x_2} - \left[X_0(\alpha_p r)\, r\frac{d}{dr}X_0(\alpha_n r)\right]_{x_1}^{x_2}. \tag{A.3}$$

We then write Eq. (A.3) for the three regions involved in the problem, that is,

$$I_{1pn} = (\alpha_n^2 - \alpha_p^2)\int_{b_1}^{b_2} r\, dr\, U_0(\alpha_p r)\, U_0(\alpha_n r) = b_2[\alpha_n\, U_0(\alpha_p b_2)\, U_1(\alpha_n b_2) - \alpha_p U_0(\alpha_n b_2)\, U_1(\alpha_p b_2)]\ , \tag{A.4}$$

$$I_{2pn} = (\alpha_n^2 - \alpha_p^2)\int_{b_2}^{b_3} r\, dr\, W_0(\alpha_p r)\, W_0(\alpha_n r) = b_3\,[\alpha_n\, W_0(\alpha_p b_3)\ W_1(\alpha_n b_3) - \alpha_p\, W_0(\alpha_n b_3)\, W_1(\alpha_p b_3)] - b_2[\alpha_n\, W_0(\alpha_p b_2)\, W_1(\alpha_n b_2) - \alpha_p\, W_0(\alpha_n b_2)\, W_1(\alpha_p b_2)]\ , \tag{A.5}$$

$$I_{3pn} = -\,b_3[\alpha_n\, Z_0(\alpha_p b_3)\, Z_1(\alpha_n b_3) - \alpha_p\, Z_0(\alpha_n b_3)\, Z_1(\alpha_p b_3)]\ , \tag{A.6}$$

where we applied the boundary conditions (III.31) and (III.36).

Multiplying Eq. (A.4) by $[Y_1(\alpha_p b_1)\, Y_1(\alpha_n b_1)]^{-1}$, Eq. (A.5) by K_2/K_1, and Eq. (A.6) by $(K_3/K_1)\,(\eta_{3p}/\beta_{2p})\ (\eta_{3n}/\beta_{2n})$ and utilizing the boundary conditions (III.32) up to (III.35), we obtain the orthogonality condition (III.37).

To obtain the normalization integral we follow a slightly different method. We can rewrite the Bessel differential equation in the form

$$\alpha_p^2 r^2\, \frac{d}{dr}X_0^2(\alpha_p r) + \frac{d}{dr}\left[r\frac{d}{dr}X_0(\alpha_p r)\right]^2 = 0\ , \tag{A.7}$$

from which, upon integration between the limits x_1 and x_2, we obtain

$$I_{ipp} = \frac{1}{2}[x_2^2\, X_0^2(\alpha_p x_2) - x_1^2\, X_0^2(\alpha_p x_1) + x_2^2\, X_1^2(\alpha_p x_2) - x_1^2\, X_1^2(\alpha_p x_1)]\ . \tag{A.8}$$

For each one of the regions, we have

$$I_{1pp} = \int_{b_1}^{b_2} r\,dr\,U_0^2(\alpha_p r) = \frac{1}{2}[b_2^2\,U_0^2(\alpha_p b_2) - b_1^2\,U_0^2(\alpha_p b_1) + b_2^2\,U_1(\alpha_p b_2) - b_1^2\,U_1(\alpha_p b_1)]\ , \quad (A.9)$$

$$I_{2pp} = \int_{b_2}^{b_3} r\,dr\,W_0^2(\alpha_p r) = \frac{1}{2}[b_3^2\,W_0^2(\alpha_p b_3) - b_2^2\,W_0^2(\alpha_p b_2) + b_3^2\,W_1^2(\alpha_p b_3) - b_2^2\,W_1^2(\alpha_p b_2)]\ , \quad (A.10)$$

$$I_{3pp} = \int_{b_3}^{b_4} r\,dr\,Z_0^2(\alpha_p r) = \frac{1}{2}[b_4^2\,Z_0^2(\alpha_p b_4) - b_3^2\,Z_0(\alpha_p b_3) + b_4^2\,Z_1^2(\alpha_p r) - b_3^2 Z_1^2\,(\alpha_p b_3)]\ . \quad (A.11)$$

Multiply Eq. (A.9) by $[Y_1(\alpha_p b_1)]^{-2}$, Eq. (A.10) by K_2/K_1 and Eq. (A.11) by $(\eta_{3p}/\beta_{2p})^2 \times (K_3/K_1)$, sum the results, and apply the boundary conditions (III.31) up to (III.35) to obtain N_{pp} as given by Eq. (III.45). Also,

$$U_1(\alpha_n r) = Y_1(\alpha_n b_1) J_1(\alpha_n r) - J_1(\alpha_n b_1) Y_1(\alpha_n r) \quad (A.12)$$

$$W_1(\alpha_n r) = \eta_{1n} J_1(\alpha_n r) - \eta_{02n} Y_1(\alpha_n r) \quad (A.13)$$

$$Z_1(\alpha_n r) = \beta_{2n} J_1(\alpha_n r) - \beta_{1n} Y_1(\alpha_n r) \quad (A.14)$$

APPENDIX B

Expansion of the U_0 and U_1 Functions for Thin Samples

When the ratio δ/b_2 is smaller than unity ($\delta = b_2 - b_1$), one can expand the Bessel functions $J_0(x)$ and $Y_0(x)$ in the Taylor series

$$X_n(\alpha_p b_2 - \alpha_p \delta) = X_n(\alpha_n b_2) - \frac{1}{2}(\alpha_p \delta)[X_{n-1}(\alpha_p b_2) - X_{n+1}(\alpha_p b_2)]$$
$$+ \frac{1}{2!2^2}(\alpha_p \delta)^2[X_{n-2}(\alpha_p b_2) - 2X_n(\alpha_p b_2) + X_{n+2}(\alpha_p b_2)]$$
$$- \frac{1}{2^3 3!}(\alpha_p \delta)^3[X_{n-3}(\alpha_p b_2) - 3X_{n-1}(\alpha_p b_2) + 3X_{n+1}(\alpha_p b_2) - X_{n+3}(\alpha_p b_2)]\ \ (\alpha_p \delta < 1)\ . \quad (B.1)$$

Introduction of the above expansion into the functions $U_0(\alpha_p b_2)$ and $U_1(\alpha_p b_2)$, together with the relations

$$W_p(b_2) = Y_n(\alpha_p b_2)\,J_{n-1}(\alpha_p b_2) - J_n(\alpha_p b_2)\,Y_{n-1}(\alpha_p b_2) = -\frac{2}{\pi\alpha_p b_2}\ , \quad (B.2)$$

$$J_1(\alpha_p b_2)\,Y_2(\alpha_p b_2) - Y_1(\alpha_p b_2)\,J_2(\alpha_p b_2) = -\frac{2}{\pi\alpha_p b_2}\ , \quad (B.3)$$

$$J_1(\alpha_p b_2)\,Y_3(\alpha_p b_2) - Y_1(\alpha_p b_2)\,J_3(\alpha_p b_2) = -\frac{8}{\pi(\alpha_p b_2)^2}, \tag{B.4}$$

$$Y_1(\alpha_p b_2)\,J_4(\alpha_p b_2) - J_1(\alpha_p b_2)\,Y_4(\alpha_p b_2) = \frac{48}{\pi(\alpha_p b_2)^3} - \frac{2}{\pi(\alpha_p b_2)}, \tag{B.5}$$

$$J_0(\alpha_p b_2)\,Y_4(\alpha_p b_2) - Y_0(\alpha_p b_2)\,J_4(\alpha_p b_2) = -\frac{96}{\pi(\alpha_p b_2)^4} + \frac{16}{\pi(\alpha_p b_2)^2}, \tag{B.6}$$

yields the results (II.21) and (II.22).

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