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Fundamental Properties of Perturbation-Theoretical Integral Representations. III*

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Asymptotic behavior and subtraction problem of the perturbation-theoretical integral representation (PTIR) are investigated in detail. Six theorems are rigorously proved in this connection. It is shown that a function represented by an unsubtracted PTIR may asymptotically increase in particular directions. The relation between the asymptotic behavior and the subtraction number is clarified for the subtracted PTIR. As a by-product one obtains a consistent definition of a finite part of the integral involving $x^{-1}\theta(x)$.

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DISCLAIMER

I. INTRODUCTION

Analyticity and uniqueness properties of perturbation-theoretical integral representations (PTIR) were investigated in detail in our previous papers.[1)2)] But asymptotic property of PTIR was studied only for some subseries of perturbation expansion,[2)] and no general discussions are made on this subject as yet. Recently, PTIR has been applied to the investigation of the high-energy behavior of the scattering amplitude in the Bethe-Salpeter formalism[3)], and we have found that the <u>unsubtracted</u> PTIR can describe the Regge behavior and more general high-energy behaviors. This is an interesting feature of PTIR, which is not known in the conventional dispersion theory. It will, therefore, be desirable to explore the asymptotic behavior and subtraction problem of PTIR.

For simplicity, we consider the two-variable PTIR only:

$$f(s,t) = \int_0^1 dz \int_0^\infty d\alpha \, \frac{\rho(z,\alpha)}{\alpha - zs-(1-z)t} \, . \tag{1.1}$$

Here $\int_0^1 dz \int_0^\infty d\alpha$ should be understood as $\int_{-\infty}^\infty dz \int_{-\infty}^\infty d\alpha$ with an integrand having a support $\{0 \le z \le 1, \ \alpha \ge 0\}$. If (1.1) is not convergent at $\alpha = \infty$, it is necessary to make subtractions. From the knowledge of the conventional dispersion theory, one might make the following subtraction:

$$\frac{f(s,t) - f(s,t_0)}{t - t_0} = \int_0^1 dz \int_0^\infty d\alpha \frac{\tilde{\rho}(z,\alpha)}{\alpha - zs - (1-z)t}, \qquad (1.2)$$

with $t_0 < 0$. But if one applies this subtraction procedure to a function for which (1.1) converges, then one finds that $\tilde{\rho}(z,\alpha)$ is generally completely different from $\rho(z,\alpha)$. Indeed, an elementary calculation shows

$$\tilde{\rho}(z,\alpha) = \int_0^z dx \int_0^\infty d\beta \; x\rho(x,\beta) \; \delta'(x(\alpha-t_0)-z(\beta-t_0)). \qquad (1.3)$$

Since the invariance of the weight function is important in the subtraction procedure, (1.2) should not be accepted as a good one. Hence, we shall merely subtract a constant term $f(s_0,t_0)$ instead of a one-variable function. In general, we have the following subtracted PTIR[1)]:

$$f(s,t) = P(s,t) + \int_0^1 dz \int_0^\infty d\alpha \left[\frac{z(s-s_0)+(1-z)(t-t_0)}{\alpha - zs_0 - (1-z)t_0}\right]^N \frac{\rho(z,\alpha)}{\alpha - zs - (1-z)t}, \qquad (1.4)$$

where $P(s,t)$ is an (N-1)th order polynomial of s and t, (s_0,t_0) being a fixed point in the analyticity domain of PTIR, i.e., $(s_0,t_0) \in D_{st}$. Then the following questions will arise. Is (1.4) general enough to represent any function holomorphic in D_{st} and bounded by a polynomial of $|s|$ and $|t|$ at infinity? If the subtracted PTIR is possible, how can we determine the subtraction number N? To answer these questions was the motivation of the present work.

In the next section, we give six theorems which can be proved rigorously. The proof of each theorem is given in the Appendix of the same number. In Section 3, we discuss the singularity in z of the weight function in the unsubtracted PTIR. In Section 4, we consider the subtracted PTIR, and answer the questions of last paragraph with a reasonable degree of confidence. The final section is devoted to another topic concerning a new consistent definition of the finite part of $x^{-1}\theta(x)$, which is a by-product of the consideration in Section 4. In Appendix VII, we derive various formulas given in concrete examples of the text.

Throughout this paper, n, m, N stand for zero or positive integers, and A, B, C, M, R denote big positive numbers, whereas ϵ, δ, γ, σ, η, etc. usually represent small positive numbers. While s, t, $\hat{s}$, $\hat{t}$, $\tilde{s}$, $\tilde{t}$ are complex variables, s', t', s'', t'' are used only as real variables.

II. THEOREMS

The first aim of this section is to investigate the asymptotic property of PTIR under certain assumptions. First, we consider (1.1). The convergence of the integral in the right-hand side is, of course, implicitly assumed in (1.1). Since it is too difficult to deal with the general case, we shall assume that $\rho(z,\alpha)$ decreases at least like $\alpha^{-\delta}(\delta > 0)$ at infinity, and similarly that $\rho(z,\alpha)$ has a finite number of singularities in z of order $z^{-1+\sigma}$ $(\sigma > 0)$ at worst. For finite values of α, $\rho(z,\alpha)$ will be a distribution of α, which may be dependent on z as is suggested by the following simple example.

<u>Example 1.</u>

$$f(s,t) = (a-s)^{-1}(b-t)^{-1}, \quad (a \geq 0,\ b \geq 0). \tag{2.1}$$

Its weight function is, of course, given by

$$\rho(z,\alpha) = \delta'(\alpha-za-(1-z)b). \tag{2.2}$$

It will be natural to expect in a naive sense that if the above conditions on $\rho(z,\alpha)$ are satisfied then the function f(s,t) given by (1.1) vanishes at infinity in D_{st}. But if we want to <u>prove</u> this statement, we must express it in a mathematically well-defined manner. Care must be taken for the definition of the asymptotic region because, for example, no matter how large $|s|$ may be, the function (2.1) does not become small if t approaches to b. Namely, in general, we should avoid to consider the asymptotic behavior in a neighborhood of an unbounded singularity. Now, we obtain the following theorem.

THEOREM I. Let $f(s,t)$ be an analytic function which can be represented as (1.1), where the weight function $\rho(z,\alpha)$ has the following properties. There exists a function of z,

$$H(z) \equiv \prod_{i=1}^{m} |z-z_i|^{1-\sigma}, \quad (0 < \sigma < 1,\ 0 \le z_i \le 1), \tag{2.3}$$

such that

$$\hat{\rho}(z,\alpha) \equiv H(z)\ \rho(z,\alpha) \tag{2.4}$$

satisfies the following conditions.

(i) There exists an integer n such that

$$\hat{\rho}(z,\alpha) = (\partial/\partial\alpha)^n\ \varphi(z,\alpha), \tag{2.5}$$

where $\varphi(z,\alpha)$ is a continuous function of z and α.

(ii) There exists a positive number R (z-independent) such that when $\alpha > R$ $\hat{\rho}(z,\alpha)$ is a function of α and satisfies the following inequalities:

$$\text{a)} \quad |\hat{\rho}(z,\alpha)| < A\alpha^{-\delta}, \tag{2.6}$$

$$\text{b)} \quad |\hat{\rho}(z,\alpha+\Delta\alpha) - \hat{\rho}(z,\alpha)| < B|\Delta\alpha|^{\mu} \quad \text{for } |\Delta\alpha| \le \varkappa, \tag{2.7}$$

where δ, μ, $\varkappa$, A, B are positive constants.

Then, f(s,t) has the following properties:

(A) For any <u>closed</u> subset K of D_{st}, we can always find positive numbers γ, C, M such that

$$|f(s,t)| < C(|s|+|t|)^{-\gamma} \tag{2.8}$$

whenever $|s|+|t| > M$ and $(s,t) \in K$.

(B) We can always find a positive number M such that if for any z satisfying $0 \le z \le 1$ s and t satisfy the inequality

$$|zs + (1-z)t| > M \tag{2.9}$$

and if $(s,t) \in D_{st}$, then (2.8) holds.

In the above theorem, the condition (2.7) is called the Hölder condition or the Lipschitz condition of order μ. This assumption is usually necessary if one wants to discuss bounds of a singular integral. The main result in the theorem is, of course, the property (A). The property (B) is a special consequence of the assumption (2.6). For instance, the following example does not have the property (B).

<u>Example 2.</u>

$$f(s,t) = (-t)^{-1} \sum_{n=0}^{\infty} (n^2-s)^{-1} , \tag{2.10}$$

whose weight function is

$$\rho(z,\alpha) = \sum_{n=0}^{\infty} \delta'(\alpha-zn^2) . \tag{2.11}$$

Theorem I can easily be generalized to the case of the subtracted PTIR.

THEOREM II. If $f(s,t)$ is represented as (1.4) instead of (1.1), and if (2.6) is replaced by the condition

(i) a′) $$|\hat{\rho}(z,\alpha)| < A\,\alpha^{N-\delta}\ , \tag{2.12}$$

then one has

$$|f(s,t)| < C(|s|+|t|)^{N-\gamma} \tag{2.13}$$

instead of (2.8).

From Example 1, we can expect that the unboundedness of $f(s,t)$ at the boundary of D_{st} is generally caused by singularities in α of $\rho(z,\alpha)$. Indeed, this is true, namely, we have the following theorem.

THEOREM III. If $f(s,t)$ is represented as (1.4), and if $\hat{\rho}(z,\alpha)$ defined by (2.4) is a continuous function satisfying (2.12) and (2.7) for any $\alpha \geq 0$, then we can always find positive numbers γ and C such that

$$|f(s,t)| < C(1 + |s|+|t|)^{N-\gamma} \tag{2.14}$$

in the whole D_{st}.

In the above theorem it should be remarked that the Hölder condition (2.7) is required also for $\alpha = 0$ and $\Delta\alpha < 0$ with the convention

$$\rho(z,\beta) = 0 \quad \text{for } \beta < 0\ . \tag{2.15}$$

Now, our next task is to consider the inverse problem of Theorem I. Namely, we want to find PTIR for a given function $f(s,t)$ holomorphic in D_{st} and bounded at infinity. This problem was discussed already twice,[1)2)] but the proof of the theorem was still incomplete.[4)]

As was noticed previously,[2)] it is not necessary to assume that $f(s,t)$ be holomorphic in the whole D_{st}. This is because we have the following theorem, which is essentially due to Glaser.[5)]

THEOREM IV Let $f(s,t)$ be holomorphic in domains D_+ and D_- separately, and both boundary values on E coincide with each other, where

$$D_+ \equiv \{s,t;\ \mathrm{Im}\, s > 0,\ \mathrm{Im}\, t > 0\},$$

$$D_- \equiv \{s,t;\ \mathrm{Im}\, s < 0,\ \mathrm{Im}\, t < 0\}, \qquad (2.16)$$

$$E \equiv \{s,t;\ \mathrm{Im}\, s = \mathrm{Im}\, t = 0,\ \mathrm{Re}\, s < 0,\ \mathrm{Re}\, t < 0\}.$$

If $|f(s,t)|$ is bounded by a polynomial of $|s|$ and $|t|$ in any closed subset of D_+ and D_- , then $f(s,t)$ is holomorphic in D_{st}.

For completeness, we write here the explicit shape of the domain D_{st}.[1)2)] Let

$$D^*[s,t] = \{s,t;\ \mathrm{Im}\, s > 0\ ,\ \mathrm{Im}\, t < 0\ ,\ \mathrm{Im}\, st^* \geq 0\}\ ,$$

$$D^*[s] = \{s,t;\ \mathrm{Im}\, s = 0\ ,\ \mathrm{Re}\, s \geq 0\}\ . \qquad (2.17)$$

Then D_{st} is the complement of

$$D^*[s,t] \cup D^*[t,s] \cup D^*[s] \cup D^*[t] . \tag{2.18}$$

Hence, of course, D_{st} includes D_+, D_-, and E. D_{st} is the envelope of holomorphy of $D_+ \cup D_- \cup E$.

Now, our main theorem is as follows.

THEOREM V Let f(s,t) be holomorphic in domains D_+ and D_- separately, and both boundary values on E coincide with each other. Moreover, assume that f(s,t) satisfies the following boundedness conditions.

(i) For any closed subset K of D_+ and D_-, there exist positive numbers δ, A, M such that

$$|f(s,t)| < A\left(|s|+|t|\right)^{-\delta} \tag{2.19}$$

whenever $|s|+|t| > M$ and $(s,t) \in K$.

(ii) For the same K, one has

$$|(\partial/\partial t)f(s,t)| < B(|s|+|t|)^{-\gamma} \sum_{i=1}^{m} |z_i s+(1-z_i)t|^{-1} \tag{2.20}$$

whenever $|s| + |t| > M$ in K, where $\gamma > 0$, $B > 0$ and $0 \leq z_i < 1$ $(i = 1, 2, \ldots , m)$.

Then f(s,t) can be represented as (1.1), where $\rho(z,\alpha)$ is defined for every z except for $z_1, z_2, \ldots , z_m$, and 1.

For a fixed z, $\rho(z,\alpha)$ is a distribution of α, which is given by

$$\rho(z,\alpha) = (2\pi i)^{-1} \lim_{\varepsilon\to 0+} [\psi(z,\alpha+i\varepsilon)-\psi(z,\alpha-i\varepsilon)]. \tag{2.21}$$

Here $\psi(z,w)$ is a holomorphic function of w except for $w \geq 0$, and we have for $w < 0$

$$\psi(z,w) = (1-z)^{-1} \int_0^\infty ds' \frac{\partial}{\partial w} f_s\left(s', \frac{w-zs'}{1-z}\right), \tag{2.22}$$

where $f_s(s',t)$ is the absorptive part of f(s,t), i.e.,

$$f_s(s',t) \equiv (2\pi i)^{-1} \lim_{\varepsilon\to 0+} [f(s'+i\varepsilon,t)-f(s'-i\varepsilon,t)]. \tag{2.23}$$

In the above theorem, the boundedness condition (i) is essentially equivalent to the result (A) of Theorem I. We may conjecture that the condition (i) will be enough to give the essential results of Theorem V because we know no counterexample to this statement, but it seems to be technically extremely difficult to eliminate a condition on a partial derivative of f(s,t).[6)]

One might suppose that the right-hand side of (2.20) may be replaced by $B(|s|+|t|)^{-\gamma} |t|^{-1}$, but this bound is not general enough to admit a simple example $(-s)^{-\frac{1}{2}}(-s-t)^{-\frac{1}{2}}$. It should be remarked that z_i cannot be equal to unity in (2.20). We have, of course, some examples which satisfy (i) but not (ii).

Example 3.

$$f(s,t) = (-s)^{-\frac{1}{4}}(-t)^{-\frac{1}{4}} \exp[-(-t)^{\frac{1}{2}}], \quad (\mathrm{Re}(-t)^{\frac{1}{2}} \geq 0). \qquad (2.24)$$

This function does not satisfy the condition (ii), but it still has the representation (1.1) with a weight function satisfying all the conditions of Theorem I.

Since the conditions of Theorem V are imposed only on (s,t) belonging to K, $\rho(z,\alpha)$ is not necessarily bounded by $\alpha^{-\sigma}$ $(\sigma > 0)$. For instance, Example 2 satisfies all the conditions of Theorem V. It seems to be very difficult to prove a fairly general theorem which gives the boundedness of $\rho(z,\alpha)$ at $\alpha = \infty$. The following theorem is too restrictive to be practical.

THEOREM VI. If the condition (ii) of Theorem V is replaced by the stronger condition that

(ii') $$|(\partial/\partial t) f(s,t)| < B(1+|s|+|t|)^{-1-\gamma} \qquad (2.25)$$

in the whole D_+ and D_-, then we have

$$|\rho(z,\alpha)| < C(1+\alpha)^{-\sigma} \qquad (2.26)$$

for $z \neq 1$, where $0 < \sigma < \gamma$ and C is a big positive number.

Both Theorems V and VI concern the unsubtracted PTIR. The extension to the subtracted PTIR is by no means trivial. The reason why it is difficult will be clarified in Section 4.

III. SINGULARITIES IN z

In the preceding section, we have assumed that singularities in z of $\rho(z,\alpha)$ are integrable ones in the usual sense. But this restriction is too stringent for practical applications, and we should admit for $\rho(z,\alpha)$ to include a distribution of z independent of α. For example, if f(s,t) is independent of s, $\rho(z,\alpha)$ is necessarily proportional to $\delta(z)$.

When $\rho(z,\alpha)$ contains such a distribution of z, the asymptotic behavior (2.8) is no longer assured. The following examples will be instructive.

<u>Example 4.</u> If

$$\rho(z,\alpha) = \delta(\alpha-a)\ \delta^{(n)}(z-z_0) \tag{3.1}$$

with $a \geq 0$ and $0 \leq z_0 \leq 1$, then

$$f(s,t) = \frac{n!\,(t-s)^n}{[a-z_0 s-(1-z_0)t]^{n+1}} . \tag{3.2}$$

<u>Example 5.</u> The weight function

$$\rho(z,\alpha) = \alpha^{-\sigma}\ \delta^{(n)}(z-z_0) \tag{3.3}$$

with $0 < \mathrm{Re}\ \sigma < 1$ and $0 \leq z_0 \leq 1$ corresponds to

$$f(s,t) = \Gamma(1-\sigma)\ \Gamma(n+\sigma)\ \frac{(t-s)^n}{[-z_0 s-(1-z_0)t]^{n+\sigma}} \tag{3.4}$$

From the above examples we see that if $\rho(z,\alpha)$ contains an (n+1)th order singularity at $z = z_0$, then $|f(s,t)|$ can increase as $(|s|+|t|)^n$ <u>only when</u> one goes to infinity in the direction in which $|z_0 s + (1-z_0)t|$ remains small. The purpose of this section is to show that the above statement is generally true, but no claim of rigor is made for the reasoning in this section.

In what follows it is very important to consider the following distribution introduced by Schwartz[7]:

$$\begin{aligned} Y_\lambda(x) &= [\Gamma(\lambda)]^{-1} \,\mathrm{Pf}\, x^{\lambda-1}\, \theta(x) \qquad \text{for } \lambda \neq 0, -1, -2, \ldots, \\ &= \delta^{(n)}(x) \qquad \text{for } \lambda = -n. \end{aligned} \tag{3.5}$$

If $\varphi(x)$ is a test function, $\int Y_\lambda(x)\,\varphi(x)\,dx$ is an entire function of a complex parameter λ. $Y_\lambda(x)$ can be understood as the discontinuity of an analytic function $\Gamma(1-\lambda)(-x)^{\lambda-1}$. Thus a δ-function and its derivatives can be regarded as special cases of a power of x.

For simplicity, we consider the singularity at $z = 0$. In this case we expect a special asymptotic behavior of $f(s,t)$ when $s \to \infty$ but $t/s \to 0$. In a neighborhood of $z = 0$ the weight function may be written as

$$\rho(z,\alpha) \simeq F(z)\,\varpi(\alpha) \qquad \text{at } z \approx 0, \tag{3.6}$$

where the symbol $\simeq$ means that the ratio of both sides tends to unity. Then the behavior for $s \to \infty$ but $t/s \to 0$ is determined by

$$I \equiv \int_0^1 dz \, \frac{F(z)}{\beta - zs} \, , \tag{3.7}$$

where $\beta \equiv \alpha - t$. For practical applications, the following case is important and seems to be sufficiently general:

$$F(z) = c \, \mathrm{Pf} \left[z^{\lambda-1} (\log 1/z)^{\nu} (\log \log 1/z)^{\xi} \cdots \right]. \tag{3.8}$$

For simplicity, we consider the case

$$F(z) = \mathrm{Pf} \left[z^{\lambda-1} (\log 1/z)^{\nu} \right] , \tag{3.9}$$

since the extension to the general case is straightforward. Putting $s = -r\, e^{i\theta}$, $(r > 0,\ |\theta| < \pi)$ and $z = y/r$, we have

$$I = r^{-\lambda} \int_0^r dy \, \frac{\mathrm{Pf} \left[y^{\lambda-1} (\log r - \log y)^{\nu} \right]}{\beta + y e^{i\theta}} \, . \tag{3.10}$$

As is easily seen by a binomial expansion, the leading term of the numerator is $(\log r)^{\nu}$ because the singularity of $\log y$ at $y = 0$ does not lead to infinity for $r \to \infty$. Thus one gets

$$I \simeq r^{-\lambda} (\log r)^{\nu} \, J \tag{3.11}$$

with

$$J \equiv \int_0^{\infty} dy \, \frac{\mathrm{Pf} \left[y^{\lambda-1} \right]}{\beta + y e^{i\theta}} = \frac{\pi}{\sin \pi\lambda} \, \beta^{\lambda-1} \, e^{-i\lambda\theta} \, , \tag{3.12}$$

When the integration of (3.12) is carried out, we first assume $0 < \mathrm{Re}\,\lambda < 1$, and then analytically continue the result with respect to λ. Hence, (3.12) is true even for $\mathrm{Re}\,\lambda \leq 0$. Thus we have

$$I \simeq (\pi/\sin \pi\lambda)\ \beta^{\lambda-1}\ (-s)^{-\lambda}\ [\log(-s)]^{\nu}\ . \tag{3.13}$$

Putting $\lambda = -\mu$, ($\mathrm{Re}\,\mu \geq 0$), we obtain the following asymptotic behavior of f(s,t):

$$f(s,t) \simeq \int_0^\infty d\alpha\ \frac{\pi}{\sin \pi(\mu+1)} \cdot \frac{\varphi(\alpha)}{(\alpha-t)^{\mu+1}}\ (-s)^{\mu}\ [\log(-s)]^{\nu}. \tag{3.14}$$

Now, in the Bethe-Salpeter approach to the high-energy behavior of the scattering amplitude,[3)] the momentum transfer in the crossed channel was treated as a parameter, hence μ and ν may be functions of the momentum transfer. Thus we were able to describe the Regge and Regge-cut behaviors by PTIR. However, if we consider the S-matrix theory and, hence, t is identified with the momentum transfer, μ and ν in (3.14) cannot be functions of the momentum transfer. In this case, μ and ν should be considered as functions of α. But, unfortunately, we cannot carry out the integration over α if μ and ν are dependent on α.

The special asymptotic behavior (3.14) is, of course, due to the choice of the special direction $t/s \to 0$. Our next task is to see that if the asymptotic behavior is considered in any <u>other</u> direction then $f(s,t)$ satisfies (2.8), provided that $\rho(z,\alpha)$ satisfies all the conditions of Theorem 1 except for $z = 0$ and

$$z^{n+1-\delta}\,\rho(z,\alpha) = 0 \quad \text{for } z = 0,\ (\delta > 0). \tag{3.15}$$

First, we shall show an inequality

$$\left| \int_0^1 dz \int_0^\infty d\alpha \, \frac{\rho(z,\alpha)}{\alpha - zs - (1-z)t} \left(\frac{\alpha - t}{\alpha + a} \right)^n \right| < A\left(|s| + |t|\right)^{n-\gamma} \tag{3.16}$$

for $|s|+|t| > M$ with $0 < \gamma < \delta$ and $a > 0$. Without loss of generality, a smaller value than unity can be taken as the upper limit of the integral. Then the integrand can be rewritten as

$$\frac{1}{(1-z)^n} \cdot \frac{\rho(z,\alpha)}{(\alpha+a)^n} \left[P\left(z(\alpha - s), (1-z)(\alpha - t)\right) + \frac{(-z)^n (\alpha - s)^n}{\alpha - zs - (1-z)t} \right], \tag{3.17}$$

where $P(x,y)$ is a certain $(n-1)$th order polynomial of x and y. The integral coming out from this polynomial part is, of course, convergent in the sense of a distribution, and gives an $(n-1)$th order polynomial of s and t. As for the second term of (3.17), because of the assumption (3.15) the weight functions

$$(-1)^k \, (1-z)^{-n} \, {}_nC_k \, \alpha^k (\alpha+a)^{-n} \, z^n \, \rho(z,\alpha), \qquad (k = 0, 1, \ldots, n), \tag{3.18}$$

satisfy all the conditions of Theorem 1. Thus we have established (3.16).

If we consider a special case

$$\rho(z,\alpha) = F(z)\ \delta(\alpha - \alpha_0), \qquad (\alpha_0 \geq 0), \tag{3.19}$$

(3.16) leads to

$$\left| \int_0^1 dz \int_0^\infty d\alpha \frac{\rho(z,\alpha)}{\alpha - zs - (1-z)t} \right| < A' \frac{(|s|+|t|)^{n-\gamma}}{|t|^n} \tag{3.20}$$

for $|s|+|t| > M$ and $|t| > R$ with $R \geq 2\alpha_0$. Next, we consider the case in which $\rho(z,\alpha)$ is an integrable function of α and

$$\rho(z,\alpha) = 0 \qquad \text{for} \quad \alpha > R. \tag{3.21}$$

Such a function can be written as

$$\rho(z,\alpha) = \lim_{N\to\infty} N^{-1} \sum_{k=0}^{N} \rho(z,\alpha)\,\delta(\alpha - N^{-1}kR), \tag{3.22}$$

just as done in the definition of the Riemann integral. Using (3.20) with (3.19) and taking $M > 4R$, we obtain (3.20) for the present $\rho(z,\alpha)$. Thus the expected result has been established to a certain extent. As is indicated by Example 5, the contribution from $\alpha > R$ in the general case will not be important. Summarizing the above investigation, we have the following statement.

CONJECTURE I. Let $f(s,t)$ be a function holomorphic in D_{st}. If for any closed subset K of D_+ and D_-, whenever $|s|+|t| > M$, the inequality

$$|f(s,t)| < A(|s|+|t|)^{-\delta} \left[1 + \sum_{i=1}^{m} \left(\frac{|s| + |t|}{|z_i s+(1-z_i)t|} \right)^{\lambda_i} \right] \tag{3.23}$$

with $\delta > 0$, $0 \leq z_i \leq 1$, and $\lambda_i \geq 0$ $(i = 1,2,...,m)$ holds, then $f(s,t)$ can be represented as (1.1), and in a neighborhood of $z = z_i$ $\rho(z,\alpha)$ is defined as a distribution of z and the order of the singularity does not exceed

$$|z - z_i|^{-\lambda_i-1+\delta} \tag{3.24}$$

apart from logarithmic factors. Conversely, if in (1.1) $\rho(z,\alpha)$ has singularities at $z = z_i$ $(i = 1,2,...,m)$ of order (3.24) and satisfies all the conditions of Theorem I in all other points, then $f(s,t)$ satisfies the inequality (3.23).

IV. SUBTRACTED PTIR

The purpose of this section is to extend the conjecture made in Section 3 to the case of the subtracted PTIR (1.4). Since the general mathematical consideration is prohibitively difficult, we shall deduce the general conclusion from the previous results for the unsubtracted PTIR and some concrete examples of the subtracted PTIR.

Example 6.

$$f(s,t) = \int_0^1 dz \int_0^\infty d\alpha \, \frac{[zs+(1-z)t]^N \alpha^\lambda \delta^{(n)}(z)}{\alpha^N[\alpha-zs-(1-z)t]} . \tag{4.1}$$

This example is a generalization of Example 5. The integral is convergent for

$$N > \mathrm{Re}\,\lambda > N-1,$$

and

$$n \geq N > \mathrm{Re}\,\lambda > -1, \tag{4.2}$$

and then

$$f(s,t) = \Gamma(\lambda+1)\,\Gamma(n-\lambda)\,(t-s)^n(-t)^{\lambda-n} . \tag{4.3}$$

Example 7.

$$f(s,t) = (-s)^\mu \, (-t)^\nu . \tag{4.4}$$

Its weight function is given by

$$\rho(z,\alpha) = Y_{-\mu}(z)\, Y_{-\nu}(1-z)\, Y_{\mu+\nu+1}(\alpha), \tag{4.5}$$

where $Y_\lambda(x)$ is defined by (3.5).

It will be natural to inquire whether or not the existence of distributional singularities in z (in the sense of Section 3) can be predicted by the asymptotic behavior of f(s,t). In Example 6, the special asymptotic behavior of order $|s|^n$ in the direction $t/s \to 0$ is certainly owing to the (n+1)th order singularity at z = 0. But consider Example 7 with $\mu > 0$ and $\nu > 0$. The $(\mu+1)$th order singularity at z = 0 gives the asymptotic behavior of order $|s|^\mu$ in the direction $t/s \to 0$. However, the asymptotic behavior in the general direction

$$[z_0 s+(1-z_0)t]/s \to 0, \qquad (0 < z_0 < 1) \tag{4.6}$$

is of order $|s|^{\mu+\nu}$, which is stronger than $|s|^\mu$. Therefore, if we consider a function

$$f(s,t) = (-s)^\mu(-t)^\nu+(-s)^{\mu+\nu}, \tag{4.7}$$

it exhibits no special asymptotic behavior in the direction $t/s \to 0$. Thus the answer to the above question is negative. Namely, we cannot say anything about the non-existence of distributional singularities in z of $\rho(z,\alpha)$ from the asymptotic behavior of f(s,t) in the subtracted PTIR. This means that the introduction of distributions of z is quite natural and inevitable in the subtracted PTIR, and this is the reason why it is difficult to extend the proof of Theorem V to the case of the subtracted PTIR.

Thus, we arrive at the following conjecture.

CONJECTURE II. Let $f(s,t)$ be a function holomorphic in D_{st}. If for any closed subset K of D_+ and D_-, whenever $|s|+|t| > M$, the inequality

$$|f(s,t)| < A\left(|s|+|t|\right)^{\lambda} \left[1 + \sum_{i=1}^{m} \left(\frac{|s| + |t|}{|z_i s+(1-z_i)t|}\right)^{\lambda_i}\right] \tag{4.8}$$

with $\lambda < N$, $0 \le z_i \le 1$, and $\lambda_i > 0$ $(i = 1,2,\ldots,m)$ holds, then $f(s,t)$ can be represented as (1.4), and the singularities in z of $\rho(z,\alpha)$ of more than $(\lambda+1)$th order can be located only at $z = z_i$ with the order of at most $|z-z_i|^{-\lambda-\lambda_i-1}$. The second statement of Conjecture I is likewise extended.

We can now answer the questions in Section 1. The answer to the first question is "yes". The subtracted PTIR will be general enough to represent any reasonable function holomorphic in D_{st} if $\rho(z,\alpha)$ is a distribution of not only α but also z. The answer to the second question will be as follows. If for any (non-zero and non-negative) complex number k one always has

$$|f(s,ks)| < A\ |s|^{\lambda} \tag{4.9}$$

when $|s|> M$, where (s,ks) belongs to a closed subset K of D_+ and D_-, then N is determined as the minimal non-negative integer greater than λ. Therefore, the number of subtractions in PTIR is usually less than (sometimes equal to) that in the double dispersion representation.

The weight function $\rho(z,\alpha)$ sometimes contains a new distribution which is not well known so far. The following example will be such an interesting one.

Example 8. If we operate $\partial^2/\partial\mu\partial\nu$ on the function of Example 7, we see that a function

$$f(s,t) = (-s)^{\mu} \log(-s)(-t)^{\nu} \log(-t) \tag{4.10}$$

has a weight function

$$\begin{aligned}\rho(z,\alpha) = {} & Y_{-\mu}(z)\; Y_{-\nu}(1-z)\; Y_{\mu+\nu+1}(\alpha)\{[\psi(-\mu)-\log z \\ & +\log\alpha - \psi(\mu+\nu+1)][\psi(-\nu)-\log(1-z)+\log\alpha \\ & -\psi(\mu+\nu+1)]-\psi'(\mu+\nu+1)\}.\end{aligned} \tag{4.11}$$

Here $\psi(x)$ stands for the polygamma function, i.e.,

$$\psi(x) \equiv \Gamma'(x)/\Gamma(x) = -\gamma + \sum_{n=0}^{\infty} \left[(n+1)^{-1} - (x+n)^{-1} \right]. \tag{4.12}$$

where γ is Euler's constant. We are interested in the limit $\mu \to 0$ and $\nu \to 0$, namely, we want to find the weight function of

$$f(s,t) = \log(-s)\log(-t). \tag{4.13}$$

From (4.11), noticing

$$\lim_{\mu\to 0} Y_{-\mu}(z)\ \psi(-\mu) = -z^{-1} + \delta(z) \lim_{\mu\to 0} [\Gamma(-\mu)+\psi(-\mu)]$$

$$= -z^{-1} - 2\gamma \cdot \delta(z), \tag{4.14}$$

We have

$$\rho(z,\alpha) = z^{-1} + \delta(z)\ (\log z + \gamma - \log \alpha)$$

$$+ (1-z)^{-1} + \delta(1-z)\ [\log(1-z) + \gamma - \log \alpha]. \tag{4.15}$$

Unfortunately, (4.15) is not well defined at $z = 0$ and $z = 1$. But (4.15) suggests that the sum $z^{-1} + \delta(z)\log z$ will give a meaningful result. Indeed, by a direct calculation we can show

$$\log(-s)\log(-t) = \int_0^1 dz \int_0^\infty d\alpha\ \frac{[zs+(1-z)t+1]\rho(z,\alpha)}{(\alpha+1)[\alpha-zs-(1-z)t]}, \tag{4.16}$$

with

$$\rho(z,\alpha) = \mathrm{Pf}[z^{-1}\theta(z)]+\mathrm{Pf}[(1-z)^{-1}\theta(1-z)]$$

$$-[\delta(z)+\delta(1-z)]\log \alpha. \tag{4.17}$$

Here, the distribution $\mathrm{Pf}[x^{-1}\theta(x)]$ is defined by

$$\int dx\ \mathrm{Pf}[x^{-1}\theta(x)]\varphi(x) \equiv \lim_{\epsilon\to 0+} \int dx\ [x^{-1}\theta(x-\epsilon)+\delta(x-\epsilon)\log x]\varphi(x), \tag{4.18}$$

where $\varphi(x)$ is a test function. Comparison of (4.15) with (4.17) leads to the identification

$$\mathrm{Pf}[z^{-1}\theta(z)] = z^{-1} + \delta(z)(\log z + \gamma), \quad (z \geq 0). \tag{4.19}$$

V. FINITE PART OF $x^{-n}\theta(x)$

In (4.18) we have defined a distribution $\mathrm{Pf}[x^{-1}\theta(x)]$. A similar distribution was introduced by Schwartz in his famous book.[7] But his definition of the finite part of $x^{-1}\theta(x)$ is simply to discard the logarithmically divergent part. As was noticed by himself, his definition has a serious difficulty, namely, it is not invariant under the transformation of the integration variable. For instance, according to his prescription, one has

$$\int_0^1 dx\ \mathrm{Pf}\left[x^{-1}\theta(x)\right]_{\text{Schwartz}} = 0. \tag{5.1}$$

But if one puts $x = 2y$, then one obtains

$$\int_0^{1/2} dy\ \mathrm{Pf}\left[y^{-1}\theta(y)\right]_{\text{Schwartz}} = -\log 2\,. \tag{5.2}$$

Thus the value of the integral changes. This is quite unsatisfactory, and his distribution cannot be used for practical calculations. On the other hand, our definition of $\mathrm{Pf}\left[x^{-1}\theta(x)\right]$ is free from this difficulty as is easily checked. Therefore, it will be desirable to investigate the properties of our distribution.

We define

$$\mathrm{Pf}\left[x^{-n}\theta(x)\right] \equiv \frac{(-1)^{n-1}}{(n-1)!} \quad \frac{d^n}{dx^n} \left[\theta(x-\varepsilon)\log x\right] \tag{5.3}$$

for any positive integer n, where $\epsilon \to 0+$ should be taken after the integration over x is carried out. It is easy to see that

$$\mathrm{Pf}\; x^{-n} = \mathrm{Pf}\left[x^{-n}\theta(x)\right] + (-1)^n\, \mathrm{Pf}\left[(-x)^{-n}\theta(-x)\right] , \tag{5.4}$$

where $\mathrm{Pf}\; x^{-1}$ is identical with Cauchy's principal part and $\mathrm{Pf}\; x^{-n}$ can be derived from it by successive differentiations (with an appropriate coefficient). Thus (5.3) is a natural generalization of $\mathrm{Pf}\; x^{-n}$.

Let $\varphi(x)$ be a test function, which is, of course, an infinitely differentiable function. We can easily calculate the integral

$$F[\varphi] \equiv \int_0^a dx\; \mathrm{Pf}\left[x^{-n}\theta(x)\right]\; \varphi(x), \tag{5.5}$$

according to the definition (5.3). We obtain

$$F[\varphi] = [(n-1)!]^{-1} \left\{ - \sum_{j=0}^{n-2} (n-j-2)!\; a^{-n+j+1}\; \varphi^{(j)}(a) + \varphi^{(n-1)}(a) \log a - \int_0^a dx\; \varphi^{(n)}(x)\; \log x \right\} . \tag{5.6}$$

Especially, for $n = 1$ we have

$$\int_0^a dx \,\mathrm{Pf}\left[x^{-1}\theta(x)\right] \varphi(x) = \varphi(a)\log a - \int_0^a dx\, \varphi'(x)\log x. \tag{5.7}$$

According to (5.3), if one puts $x = ky$, $(k > 0)$, then one obtains

$$\mathrm{Pf}\left[x^{-n}\theta(x)\right] = k^{-n}\,\mathrm{Pf}\left[y^{-n}\theta(y)\right] + (-1)^{n-1}[(n-1)!]^{-1}k^{-n}\log k\;\delta^{(n-1)}(y). \tag{5.8}$$

The appearance of an additional term guarantees the invariance under the transformation of the integration variable.

Finally, carrying out the differentiation in (5.3), we have

$$\mathrm{Pf}\left[x^{-n}\theta(x)\right] = x^{-n}\theta(x-\varepsilon) + \sum_{j=1}^{n-1}(-1)^j n(n-j)^{-1}(j!)^{-1}x^{-n+j}\delta^{(j-1)}(x-\varepsilon)$$

$$+(-1)^{n-1}[(n-1)!]^{-1}\;\delta^{(n-1)}(x-\varepsilon)\log x. \tag{5.9}$$

Hence, we have the multiplication law

$$x^{\mu}\,\mathrm{Pf}\left[x^{-n}\theta(x)\right] = x^{-n+\mu}\,\theta(x) \tag{5.10}$$

<u>only for</u> $\mu > n - 1$, but

$$x^{\mu}\,\mathrm{Pf}\left[x^{-n}\theta(x)\right] \neq \mathrm{Pf}\left[x^{-n+\mu}\,\theta(x)\right] \tag{5.11}$$

for $0 < \mu \leq n - 1$. For instance,

$$\int_0^1 dx \; Pf \left[x^{-1} \theta(x) \right] = 0, \tag{5.12}$$

but

$$\int_0^1 dx \cdot x \; Pf \left[x^{-2} \theta(x) \right] = -1. \tag{5.13}$$

The above definition may be useful for practical calculations. We can now consistently calculate the finite part of a logarithmically divergent integral. It might be particularly useful for the calculation of an infrared-divergent transition amplitude.

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APPENDIX I. PROOF OF THEOREM I.

We shall first prove the following lemma. The method is an extension of that of Källén[8] and Frye and Warnock.[9]

Lemma 1 Let

$$F(w) \equiv \int_0^\infty d\alpha \; \frac{\rho(\alpha)}{\alpha - w} , \tag{A1.1}$$

where $\rho(\alpha)$ has the following properties.

(i) There exists a continuous function $\varphi(\alpha)$ such that

$$\rho(\alpha) = \varphi^{(n)}(\alpha) . \tag{A1.2}$$

(ii) There exists a positive number R such that for $\alpha > R$ $\rho(\alpha)$ is a function of α satisfying the following conditions

a) $$|\rho(\alpha)| < A\,\alpha^{-\delta}, \tag{A1.3}$$

b) $$|\rho(\alpha+\Delta\alpha) - \rho(\alpha)| < B\,|\Delta\alpha|^{\mu} \quad \text{for } |\Delta\alpha| \leq \varkappa, \tag{A1.4}$$

where δ, μ, $\varkappa$, A, B are positive constants.

Then we can always find positive numbers δ', C, M such that

$$|F(w)| < C|w|^{-\delta'} \tag{A1.5}$$

whenever $|w| > M$ except for the positive real axis.

<u>Proof:</u> From (A1.3) we have for $\alpha > R$

$$|\rho(\alpha+\Delta\alpha) - \rho(\alpha)| < 2A\,\alpha^{-\delta}, \tag{A1.6}$$

provided that $\alpha >> |\Delta\alpha|$. Hence (A1.4) and (A1.6) yield

$$|\rho(\alpha+\Delta\alpha) - \rho(\alpha)| < (2A\,\alpha^{-\delta})^{\nu}(B\cdot|\Delta\alpha|^{\mu})^{1-\nu} \tag{A1.7}$$

with $0 < \nu < 1$. We can, therefore, take

b') $$|\rho(\alpha+\Delta\alpha) - \rho(\alpha)| < B'\,\alpha^{-\delta'}|\Delta\alpha|^{\mu'} \quad \text{for } |\Delta\alpha| \le \varkappa \tag{A1.8}$$

with $0 < \delta' < \delta$ and $\mu' > 0$ instead of (A1.4) without loss of generality.

Let $r \equiv |w| > 2R$ and $\varkappa < R$. We divide the integral (A1.1) into five parts:

$$\int_0^\infty = \int_0^{r/2} + \int_{r/2}^{r-\varkappa} + \int_{r-\varkappa}^{r+\varkappa} + \int_{r+\varkappa}^{2r} + \int_{2r}^{\infty}. \tag{A1.9}$$

1°) We may assume $n \ge 1$ without loss of generality.

$$\left|\int_0^{r/2}\right| = \left|\int_0^{r/2} d\alpha\, \frac{\varphi^{(n)}(\alpha)}{\alpha - w}\right|$$

$$\le \sum_{j=1}^{n} (j-1)!\left|\frac{\varphi^{(n-j)}(r/2)}{(r/2 - w)^j}\right| + n!\left|\int_0^{r/2} d\alpha\, \frac{\varphi(\alpha)}{(\alpha-w)^{n+1}}\right|. \tag{A1.10}$$

From (A1.3), for $\alpha > R$ there is a j-th order polynomial $P_j(\alpha)$, $(n \geq j \geq 1)$, such that

$$\left|\varphi^{(n-j)}(\alpha)\right| < P_j(\alpha)\alpha^{-\delta} . \tag{A1.11}$$

Hence,

$$\left|\frac{\varphi^{(n-j)}(r/2)}{(r/2 - w)^j}\right| < \frac{P_j(r/2)(r/2)^{-\delta}}{(r/2)^j} = O(r^{-\delta}), \tag{A1.12}$$

$$\left|\int_0^{r/2} d\alpha \frac{\varphi(\alpha)}{(\alpha-w)^{n+1}}\right| \leq \frac{\max\limits_{0 \leq \alpha \leq r/2} \varphi(\alpha)}{(r/2)^n} = O(r^{-\delta}). \tag{A1.13}$$

2°)

$$\left|\int_{r/2}^{r-\varkappa}\right| \leq \int_{r/2}^{r-\varkappa} d\alpha \frac{|\rho(\alpha)|}{|\alpha-w|} \leq A \int_{r/2}^{r-\varkappa} d\alpha \frac{\alpha^{-\delta}}{r-\alpha}$$

$$\leq A(r/2)^{-\delta} \log\left(r/2\varkappa\right) = o(r^{-\delta'}). \tag{A1.14}$$

3°)

$$\int_{r-\varkappa}^{r+\varkappa} = \int_{r-\varkappa}^{r+\varkappa} d\alpha \frac{\rho(\alpha)-\rho(r)}{\alpha - w} + \rho(r) \int_{r-\varkappa}^{r+\varkappa} \frac{d\alpha}{\alpha-w} .$$

(A1.15)

Because of (Al.8) we have

$$\left| \int_{r-\varkappa}^{r+\varkappa} d\alpha \; \frac{\rho(\alpha)-\rho(r)}{\alpha-w} \right| \le B' \int_{r-\varkappa}^{r+\varkappa} d\alpha \; \frac{\alpha^{-\delta'}|\alpha-r|^{\mu'}}{|\alpha-r|}$$

$$\le 2B'(r-\varkappa)^{-\delta'} \int_0^{\varkappa} \beta^{-1+\mu'} d\beta$$

$$= 2B'\mu'^{-1} \varkappa^{\mu'}(r-\varkappa)^{-\delta'} = O(r^{-\delta'}). \qquad \text{(Al.16)}$$

$$\left| \rho(r) \int_{r-\varkappa}^{r+\varkappa} \frac{d\alpha}{\alpha-w} \right| \le A\, r^{-\delta} \left| \log \frac{r+\varkappa-w}{r-\varkappa-w} \right| = o(r^{-\delta'}). \qquad \text{(Al.17)}$$

4°)

$$\left| \int_{r+\varkappa}^{2r} \right| \le A \int_{r+\varkappa}^{2r} d\alpha \; \frac{\alpha^{-\delta}}{\alpha-r} \le A(r+\varkappa)^{-\delta}\log(r/\varkappa) = o(r^{-\delta'}). \qquad \text{(Al.18)}$$

5°)

$$\left| \int_{2r}^{\infty} \right| \le A \int_{2r}^{\infty} d\alpha \; \frac{\alpha^{-\delta}}{\alpha-r} \le A \int_{R}^{\infty} d\alpha \; \frac{\alpha^{-\delta}}{(\alpha/2)^{1-\delta'}\; r^{\delta'}} = O(r^{-\delta'}) \qquad \text{(Al.19)}$$

Thus we obtain (Al.5). Q.E.D.

Now, we apply Lemma 1 to

$$F(z,w) \equiv \int_0^{\infty} d\alpha \; \frac{\hat{\rho}(z,\alpha)}{\alpha-w} . \qquad \text{(Al.20)}$$

Then there exist positive numbers δ', C_1, M_1 such that

$$|F(z,w)| < C_1 \; |w|^{-\delta'} \tag{A1.21}$$

whenever $|w| > M_1$ except for the positive real axis.

Let

$$K_w \equiv \{w;\; w = zs+(1-z)t, \quad 0 \le z \le 1, \quad (s,t) \in K\}. \tag{A1.22}$$

Then K_w is a closed set which does not intersect $\{w \ge 0\}$. Since the intersection of K_w and $\{|w| \le M_1\}$ is compact and $F(z,w)$ is holomorphic there, $F(z,w)$ is bounded there, i.e.,

$$|F(z,w)| < C_2. \tag{A1.23}$$

Thus

$$|F(z,w)| < C_0(M_1 + |w|)^{-\delta'} \quad \text{in } K_w, \tag{A1.24}$$

where $C_0 = \max(2^{\delta'} C_1, M_1^{\delta'} C_2)$. By definition, we have

$$f(s,t) = \int_0^1 dz \left[H(z)\right]^{-1} F(z,\; zs+(1-z)t). \tag{A1.25}$$

Hence,

$$|f(s,t)| < C_0 \int_0^1 dz \left[H(z)\right]^{-1} \{M_1+|zs+(1-z)t|\}^{-\delta'} \tag{A1.26}$$

for $(s,t) \in K$. We divide the integration range $[0,1]$ into $I[s,t]$ in which the inequality

$$|zs + (1-z)t| < \tfrac{1}{2} M^{\frac{1}{2}}\left(|s|+|t|\right)^{\frac{1}{2}} \tag{A1.27}$$

holds and the remaining part. Then

$$|f(s,t)| < C_0 M_1^{-\delta'} \int_{I[s,t]} dz \quad [H(z)]^{-1}$$

$$+ C_0 \int_0^1 dz\, [H(z)]^{-1}\left\{M_1 + \tfrac{1}{2} M^{\frac{1}{2}}\left(|s|+|t|\right)^{\frac{1}{2}}\right\}^{-\delta'}. \tag{A1.28}$$

Since the z integral is convergent, the second term evidently behaves like $O\left(\left(|s|+|t|\right)^{-\delta'/2}\right)$. Therefore, the problem is to estimate the first term. For this purpose, we use the following lemma.

<u>Lemma 2.</u> Let s and t be complex, $M > 0$, and

$$I[s,t] \equiv \left\{z;\ 0 \le z \le 1,\ |zs+(1-z)t| < \tfrac{1}{2}M^{\frac{1}{2}}\left(|s|+|t|\right)^{\frac{1}{2}}\right\}. \tag{A1.29}$$

We denote the Lebesgue measure of a set S by $\mu(S)$. Then

$$\mu(I[s,t]) < 4M^{\frac{1}{2}}\left(|s|+|t|\right)^{-\frac{1}{2}} \quad \text{for } |s|+|t| > 4M. \tag{A1.30}$$

<u>Proof:</u> Putting $\xi = 2z - 1$, $u = s+t$, and $v = s-t$, we have

$$zs + (1-z)t = \tfrac{1}{2}(u + \xi v),$$

$$4M \le |s|+|t| \le |u|+|v|. \tag{A1.31}$$

Let

$$J[s,t] \equiv \left\{\xi;\ |\xi| \le 1,\quad |u+\xi v| < M^{\frac{1}{2}}\left(|u|+|v|\right)^{\frac{1}{2}}\right\} \quad . \tag{A1.32}$$

Then it is evident that

$$2\ \mu(I[s,t]) \le \mu(J[s,t]). \tag{A1.33}$$

The inequality

$$|u + \xi v| < M^{\frac{1}{2}}\left(|u|+|v|\right)^{\frac{1}{2}} \tag{A1.34}$$

can be rewritten as

$$|v|^2\ \xi^2 + 2\ \mathrm{Re}(uv^*)\ \xi + |u|^2 - M\left(|u|+|v|\right) < 0. \tag{A1.35}$$

The discriminant D of this quadratic form is

$$D = M\left(|u|+|v|\right)\ |v|^2 - (\mathrm{Im}\ uv^*)^2. \tag{A1.36}$$

For $D \ge 0$

$$\mu(J[s,t]) \le 2D^{\frac{1}{2}}|v|^{-2}, \tag{A1.37}$$

and for $D < 0$

$$\mu(J[s,t]) = 0. \tag{A1.38}$$

Therefore, in general, we have

$$\mu(J[s,t]) \le 2M^{\frac{1}{2}}\ \left(|u|+|v|\right)^{\frac{1}{2}}\ |v|^{-1}. \tag{A1.39}$$

On the other hand, (Al.34) leads to

$$|u| - |\xi|\cdot|v| < M^{\frac{1}{2}}\left(|u| + |v|\right)^{\frac{1}{2}}. \tag{Al.40}$$

Since $|\xi| \leq 1$, we have

$$|v| \geq |\xi| \cdot |v| > |u| - M^{\frac{1}{2}}\left(|u| + |v|\right)^{\frac{1}{2}}. \tag{Al.41}$$

Adding $|v|$ to (Al.41) and dividing it by two, we obtain

$$|v| > \tfrac{1}{2}\left(|u|+|v|\right)^{\frac{1}{2}} \left[\left(|u|+|v|\right)^{\frac{1}{2}} - M^{\frac{1}{2}}\right] . \tag{Al.42}$$

Substitution of (Al.42) in (Al.39) yields

$$\mu(J[s,t]) < 4M^{\frac{1}{2}} \left[\left(|u|+|v|\right)^{\frac{1}{2}} - M^{\frac{1}{2}}\right]^{-1} . \tag{Al.43}$$

Hence, (Al.33) together with (Al.31) leads to

$$\mu(I[s,t]) < 2M^{\frac{1}{2}} \left[\left(|s|+|t|\right)^{\frac{1}{2}} - M^{\frac{1}{2}}\right]^{-1}$$

$$\leq 4M^{\frac{1}{2}} \left(|s|+|t|\right)^{-\frac{1}{2}}. \tag{Al.44}$$

Thus, we have proved (Al.30). Q.E.D.

Now, it is obvious that

$$[H(z)]^{-1} < a\left(1 + \sum_{i=1}^{m} |z-z_i|^{-1+\sigma}\right) \tag{Al.45}$$

if a is sufficiently large. Hence, putting $\eta \equiv \mu(I[s,t])$, we have

$$\int_{I[s,t]} dz\, [H(z)]^{-1} < a\eta + a \left[\sum_{i=1}^{m} \int_{z_i-\eta}^{z_i+\eta} dz \quad |z-z_i|^{-1+\sigma} + \eta \cdot \eta^{-1+\sigma} \right]$$

$$= O(\eta^{\sigma}). \qquad \text{(A1.46)}$$

Lemma 2 tells us that

$$\eta < 4M^{\frac{1}{2}}\left(|s|+|t|\right)^{-\frac{1}{2}} \quad . \qquad \text{(A1.47)}$$

Thus the first term of (A1.28) behaves like $(|s|+|t|)^{-\sigma/2}$ at most. Then putting

$$\gamma \equiv \min(\delta'/2,\ \sigma/2), \qquad \text{(A1.48)}$$

we obtain the statement (A) of Theorem I. The statement (B) is likewise obtained by applying (A1.21) to (A1.25) directly ($M_1 = M$, $C_0 = 2^{\delta'} C_1$).

APPENDIX II. PROOF OF THEOREM II

Lemma 1 of Appendix I together with

$$\rho(\alpha) = \frac{\hat{\rho}(z,\alpha)}{[\alpha - zs_0 - (1-z)t_0]^N} \tag{A2.1}$$

leads to

$$\left| \int_0^\infty d\alpha \, \frac{\hat{\rho}(z,\alpha)}{[\alpha - zs_0 - (1-z)t_0]^N [\alpha - zs - (1-z)t]} \right| < C_0 \left[M_1 + |zs + (1-z)t| \right]^{-\delta'} . \tag{A2.2}$$

Hence,

$$|f(s,t)| \le |P(s,t)| + C_0 \sum_{j=0}^{N} {}_N C_j \, |s-s_0|^j \, |t-t_0|^{N-j} \, \varphi_j(s,t) \tag{A2.3}$$

with

$$\varphi_j(s,t) \equiv \int_0^1 dz \cdot z^j (1-z)^{N-j} [H(z)]^{-1} [M_1 + |zs + (1-z)t|]^{-\delta'} . \tag{A2.4}$$

Since $|z^j (1-z)^{N-j}| \le 1$, we see

$$\varphi_j(s,t) = O\left((|s| + |t|)^{-\gamma} \right), \tag{A2.5}$$

according to the proof of Theorem I. Thus

$$|f(s,t)| = O\left((|s| + |t|)^{N-\gamma} \right). \tag{A2.6}$$

APPENDIX III. PROOF OF THEOREM III.

If we can prove the following lemma, then the rest of the proof is equivalent to that of Theorem II.

Lemma 3. If for any $\alpha \geq 0$

$$|\rho(\alpha)| < A\,\alpha^{-\delta}, \tag{A3.1}$$

$$|\rho(\alpha+\Delta\alpha) - \rho(\alpha)| < B|\Delta\alpha|^{\mu} \quad \text{for } |\Delta\alpha| \leq \lambda, \tag{A3.2}$$

where $1 > \delta > 0$, $\mu > 0$, and $\lambda > 0$, then we can always find positive numbers δ' and C such that

$$\left| \int_0^\infty d\alpha \, \frac{\rho(\alpha)}{\alpha - w} \right| < C\left(1+|w|\right)^{-\delta'} \tag{A3.3}$$

for any non-zero and non-positive w.

Proof: Let $\varkappa = \lambda/3$. In the case $r \equiv |w| \geq 2\varkappa$, we have[10)]

$$\left| \int_0^\infty \right| < C' r^{-\delta'} \tag{A3.4}$$

immediately from the proof of Lemma 1 of Appendix I. In the case $r < 2\varkappa$, (A3.2) yields

$$|\rho(\alpha)| < B\alpha^{\mu} \qquad \text{for } 0 \leq \alpha \leq 3\varkappa. \tag{A3.5}$$

because of the convention (2.15). Hence,

$$\left| \int_0^{3\varkappa} \right| \le B \int_0^{3\varkappa} d\alpha \, |\alpha - r|^{-1+\mu} + \left| \rho(r) \int_0^{3\varkappa} \frac{d\alpha}{\alpha - w} \right|$$

$$< B\mu^{-1} \left[r^{\mu} + (3\varkappa - r)^{\mu} \right] + Br^{\mu} \left| \log \frac{3\varkappa - w}{-w} \right|$$

$$< \text{const.} \tag{A3.6}$$

$$\left| \int_{3\varkappa}^{\infty} \right| < A \int_{3\varkappa}^{\infty} d\alpha \, \frac{\alpha^{-\delta}}{\alpha - 2\varkappa} = \text{const.} \tag{A3.7}$$

Thus we obtain

$$\left| \int_0^{\infty} \right| \le C(1+r)^{-\delta'} \tag{A3.8}$$

for any w except for $w \ge 0$. Q.E.D.

APPENDIX IV. PROOF OF THEOREM IV

By multiplying $f(s,t)$ by $(st)^{-N-\delta}$, $(\delta > 0)$, we can assume

$$|f(s,t)| < A(|s|+|t|)^{-\delta} \quad \text{for } |s|+|t| > M \tag{A4.1}$$

without loss of generality.

According to the edge-of-the-wedge theorem[11), $f(s,t)$ is a single analytic function holomorphic in $D_+ \cup D_- \cup \mathcal{N}(E)$, where $\mathcal{N}(E)$ stands for a complex neighborhood of E. For $(\hat{s}, \hat{t}) \in D_+$ we consider

$$F(\hat{s}, \hat{t}, \xi) \equiv (2\pi i)^{-1} \int_{-\infty}^{\infty} d\xi' \frac{f(\xi'\hat{s} - \epsilon, \xi'\hat{t} - \epsilon)}{\xi' - \xi}, \tag{A4.2}$$

where $\operatorname{Im} \xi > 0$, and ϵ is an infinitesimal positive constant. The right-hand side of (A4.2) is well defined because of the analyticity and the boundedness (A4.1) of $f(s,t)$.

If we take particular points $\arg \hat{s} = \arg \hat{t}$, then we can close the ξ' contour of (A4.2) by adding a large semi-circle because $(\xi'\hat{s} - \epsilon, \xi'\hat{t} - \epsilon) \in D_+ \cup D_- \cup \mathcal{N}(E)$, and we obtain

$$F(\hat{s}, \hat{t}, \xi) = f(\xi\hat{s} - \epsilon, \xi\hat{t} - \epsilon). \tag{A4.3}$$

Because of the uniqueness of analytic continuation, (A4.3) tells us that $F(\hat{s}, \hat{t}, \xi)$ is the analytic extension of $f(\xi\hat{s} - \epsilon, \xi\hat{t} - \epsilon)$ to the topological product of D_+ and $\{\mathrm{Im}\ \xi > 0\}$. Thus $f(s,t)$ is holomorphic in

$$D' \equiv \{s,t;\ s = \xi\hat{s},\ t = \xi\hat{t},\ (\hat{s},\hat{t}) \in D_+,\ \mathrm{Im}\ \xi > 0\}. \qquad (A4.4)$$

We will show $D' = D_{st}$ in the following.

For simplicity, we write

$$\theta \equiv \arg s, \quad \varphi \equiv \arg t,$$
$$\hat{\theta} \equiv \arg \hat{s}, \quad \hat{\varphi} \equiv \arg \hat{t}, \quad \psi \equiv \arg \xi, \qquad (A4.5)$$

then

$$\theta = \hat{\theta} + \psi, \quad \varphi = \hat{\varphi} + \psi, \qquad (A4.6)$$

and

$$0 < \hat{\theta} < \pi,\ 0 < \hat{\varphi} < \pi,\ 0 < \psi < \pi. \qquad (A4.7)$$

Since D_{st} is explicitly given as the complement of (2.18), we compare D' with it in the following.

1°) $D' \supset D_+$ and $D' \supset D_-$ are evident (the choices of ψ are $\psi \approx 0$ and $\psi \approx \pi$, respectively).

2°) When $\mathrm{Im}\ s > 0$ and $\mathrm{Im}\ t < 0$, the points belonging to D_{st} are characterized by $\mathrm{Im}\ st^* < 0$. This condition can be rewritten as

$$0 < \theta < \pi < \varphi < 2\pi, \quad 0 < \varphi - \theta < \pi. \qquad (A4.8)$$

On the other hand, if we choose ξ as

$$\varphi - \pi < \psi < \theta, \quad |\xi| = 1, \tag{A4.9}$$

then we have

$$\hat{s} = |s|\, e^{i(\theta-\psi)}, \quad \hat{t} = |t|\, e^{i(\varphi-\psi)}, \tag{A4.10}$$

hence $(\hat{s}, \hat{t}) \in D_+$. Thus the points of D_{st} belong to D'. Conversely, if Im st* ≥ 0, i.e., $\varphi - \theta \geq \pi$, which in turn implies $\hat{\varphi} - \hat{\theta} \geq \pi$. This contradicts (A4.7). Thus both domains in this portion coincide with each other.

3°) When Im s < 0 and Im t > 0, the problem is reduced to the above case by interchanging s and t.

4°) When Im s = 0, the points of D_{st} is characterized by Re s < 0 with arg t ≠ 0. As for D', (A4.6) and (A4.7) imply arg s ≠ 0 and arg t ≠ 0, hence Im s = 0 gives Re s < 0.

5°) The case Im t = 0 is similar to the above.

APPENDIX V. PROOF OF THEOREM V.

We consider a point $(s,t) \in D_{st}$ such that

$$\text{Im}\, s > \epsilon, \quad \text{Im}\, t > \epsilon\,, \tag{A5.1}$$

where $\epsilon > 0$. Cauchy's theorem leads to

$$f(s,t) = (2\pi i)^{-2} \int_C \frac{d\tilde{s}}{\tilde{s}-s} \int_C \frac{d\tilde{t}}{\tilde{t}-t}\, f(\tilde{s},\tilde{t})\,, \tag{A5.2}$$

where the closed contour C is indicated in Fig. 1. Let R be the radius of the semi-circle of C. As $R \to \infty$, the contribution from the semi-circle vanishes because of the condition (i). Hence,

$$f(s,t) = (2\pi i)^{-2} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{d\tilde{s}}{\tilde{s}-s} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} \frac{d\tilde{t}}{\tilde{t}-t}\, f(\tilde{s},\tilde{t})$$

$$= (2\pi i)^{-2} \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} dt' \, \frac{f(s'+i\epsilon, t'+i\epsilon)}{(s'-s+i\epsilon)(t'-t+i\epsilon)}$$

$$= (2\pi i)^{-2} \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} dt' \; f(s'+i\epsilon, t'+i\epsilon)$$

$$\cdot \int_0^1 dz \, [zs'+(1-z)t'-zs-(1-z)t+i\epsilon]^{-2}. \tag{A5.3}$$

We want to interchange the order of the s' and t' integrations and the z integration. But this is not trivial because the denominator of the integrand may not necessarily be large when $|s'|$ and $|t'|$ are large.

<u>Lemma 4.</u> Let

$$I[R,s',t'] \equiv \{z;\ 0 \le z \le 1,\ \max(|s'|,|t'|) > R > 1,$$

$$|zs'+(1-z)t'| < |s't'|^{\frac{1}{2}-\sigma}\}, \tag{A5.4}$$

where s' and t' are real and $0 < \sigma < \frac{1}{2}$. Then its Lebesgue measure $\mu(I[R,s',t'])$ <u>uniformly</u> tends to zero as $R \to \infty$.

<u>Proof:</u> We consider two cases $s't' \ge 0$ and $s't' < 0$ separately.

1°) The case $s't' \ge 0$. We may assume $s' \ge 0$ and $t' \ge 0$ without loss of generality. The main inequality in (A5.4) becomes

$$zs' + (1-z)t' < (s't')^{\frac{1}{2}-\sigma} . \tag{A5.5}$$

When $s' = t'$, (A5.5) becomes $1 < s' < s'^{1-2\sigma}$, which is impossible. When $s' > t'$, the points z belonging to $I[R,s',t']$ satisfy

$$0 \le z < \frac{(s't')^{\frac{1}{2}-\sigma}-t'}{s'-t'} , \tag{A5.6}$$

namely,

$$\mu\big(I[R,s',t']\big) \le \frac{(s't')^{\frac{1}{2}-\sigma}-t'}{s'-t'} . \tag{A5.7}$$

If $t' \le 1$, the right-hand side of (A5.7) is $O(R^{-\frac{1}{2}-\sigma})$.
If $t' > 1$,

$$\mu\big(I[R,s',t']\big) < \frac{t'^{\frac{1}{2}-\sigma}(s'^{\frac{1}{2}}-t'^{\frac{1}{2}})}{s'-t'} = \frac{t'^{\frac{1}{2}-\sigma}}{s'^{\frac{1}{2}}+t'^{\frac{1}{2}}}$$

$$< s'^{-\sigma} < R^{-\sigma} . \tag{A5.8}$$

When $s' < t'$, by interchanging (s',z) and $(t',1-z)$ the problem is reduced to the above.

2°) The case $s't' < 0$. We may assume $s' > 0$ and $t' < 0$ without loss of generality. Let $t'' \equiv -t' > 0$. The main inequality in (A5.4) becomes

$$\pm\,[zs'-(1-z)t''] < (s't'')^{\frac{1}{2}-\sigma} . \tag{A5.9}$$

If $zs'-(1-z)t'' \ge 0$, the points z belonging to $I[R,s',t']$ satisfy

$$\frac{t''}{s'+t''} \le z < \frac{(s't'')^{\frac{1}{2}-\sigma}+t''}{s'+t''} . \tag{A5.10}$$

If $zs'-(1-z)t'' < 0$, we have only to interchange (s',z) and $(t',1-z)$. Hence

$$\mu(I[R,s',t']) \le 2\cdot\frac{(s't'')^{\frac{1}{2}-\sigma}}{s'+t''} \le 2R^{-2\sigma}. \tag{A5.11}$$

Thus we have

$$\mu\left(I[R,s',t']\right) = O(R^{-\sigma}) \tag{A5.12}$$

for any case. Q.E.D.

Now, denoting the interval [0,1] by I, we can rewrite (A5.3) as

$$\int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} dt' \int_{0}^{1} dz = \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} dt' \left[\int_{I-I[R,s',t']} dz + \int_{I[R,s',t']} dz\right]. \tag{A5.13}$$

In the first term of the right-hand side, s' and t' satisfy either $\{|s'| \leq R,\ |t'| \leq R\}$ or

$$|zs'+(1-z)t'| \geq |s't'|^{\frac{1}{2}-\sigma}. \tag{A5.14}$$

Therefore, if we choose σ so as to satisfy $\frac{1}{4}\delta > \sigma > 0$, the order of the s' and t' integrations and the z integration can be interchanged on account of the condition (i). The second term tends to zero as $R \to \infty$ because of Lemma 4. Thus

$$f(s,t) = \int_{0}^{1} dz\ \psi_{+}(z,\ zs+(1-z)t) \tag{A5.15}$$

with

$$\psi_{+}(z,w) \equiv (2\pi i)^{-2} \int_{-\infty}^{\infty} ds' \int_{-\infty}^{\infty} dt' \frac{f(s'+i\epsilon,t'+i\epsilon)}{[zs'+(1-z)t'-w+i\epsilon]^2}, \quad (\mathrm{Im}\ w > \epsilon). \tag{A5.16}$$

For w fixed, $\psi_+(z,w)$ is a function of z defined almost everywhere in $0 \le z \le 1$. Since the contribution from $z = 1$ is infinitesimal, we always assume $z \neq 1$ hereafter.

We can carry out one of integrations in (A5.16) as follows.

$$\psi_+(z,w) = (2\pi i)^{-2} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tilde{s} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tilde{t} \, \frac{f(\tilde{s},\tilde{t})}{[z\tilde{s}+(1-z)\tilde{t}-w]^2}$$

$$= (2\pi i)^{-2} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tilde{s} \int_{C} d\tilde{t} \, \frac{f(\tilde{s},\tilde{t})}{[z\tilde{s}+(1-z)\tilde{t}-w]^2} . \qquad \text{(A5.17)}$$

Cauchy's theorem leads to

$$\psi_+(z,w) = (2\pi i)^{-1}(1-z)^{-1} \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tilde{s} \, \frac{\partial}{\partial w} f\left(\tilde{s}, \frac{w-z\tilde{s}}{1-z}\right) . \qquad \text{(A5.18)}$$

Because of the condition (ii), the integral (A5.18) is convergent if $z \neq z_i$. Thus $\psi_+(z,w)$ is well defined except for $z = z_1, z_2, \ldots, z_m, 1$.

Our next task is to investigate the analyticity of $\psi_+(z,w)$ in w for z fixed. It is evident from (A5.16) that $\psi_+(z,w)$ is holomorphic in $\mathrm{Im}\, w > \epsilon$. Next, we consider the analytic continuation to

$$\{w;\ \epsilon \ge \mathrm{Im}\, w \ge 0,\ \mathrm{Re}\, w < 0\}. \qquad \text{(A5.19)}$$

For this purpose, we investigate the analyticity of $f\left(\tilde{s}, \frac{w-z\tilde{s}}{1-z}\right)$ in $\tilde{s}$ when w is fixed in the second quadrant. This can be easily done by using (2.18). The result is illustrated in Fig.2 in case of $z \neq 0$. The shaded areas stand for singularity regions, which are defined by

$$z^{-1}\,\mathrm{Im}\,w \leq \mathrm{Im}\,\tilde{s} \leq (\mathrm{Im}\,w/\,\mathrm{Re}\,w)\;\mathrm{Re}\,\tilde{s}, \tag{A5.20}$$

and

$$0 \geq \mathrm{Im}\,\tilde{s} \geq (\mathrm{Im}\,w/\,\mathrm{Re}\,w)\;\mathrm{Re}\,\tilde{s}. \tag{A5.21}$$

In case of $z = 0$, there is no singularity in the upper half-plane. Thus we can analytically continue $\psi_+(z,w)$ to (A5.19) by deforming the $\tilde{s}$ contour of (A5.18). For Im w = 0, the $\tilde{s}$ becomes like Fig. 3. The singularity regions now become two cuts shown in Fig. 3. (The change of the limit $-\infty+i\epsilon$ into $-\infty-i\epsilon$ causes no trouble because of the condition (ii) and continuity.)

In the next step, we fix w on the negative real axis. Then we can further deform the $\tilde{s}$ contour into the lower half-plane. Since the contribution from a large semi-circle vanishes because of the condition (ii)[12), we finally obtain

$$\psi_+(z,w) = (2\pi i)^{-1}(1-z)^{-1} \int_{C'} d\tilde{s}\; \frac{\partial}{\partial w}\, f\left(\tilde{s}, \frac{w-z\tilde{s}}{1-z}\right), \tag{A5.22}$$

where the contour C' is shown in Fig. 4.

All the above procedure can be done quite analogously for a point $(s,t) \in D_-$ such that

$$\text{Im } s < -\epsilon \ , \qquad \text{Im } t < -\epsilon . \tag{A5.23}$$

Then we obtain

$$\psi_-(z,w) \equiv (2\pi i)^{-1}(1-z)^{-1} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d\tilde{s} \ \frac{\partial}{\partial w} \ f\left(\tilde{s}, \ \frac{w-z\tilde{s}}{1-z}\right) . \tag{A5.24}$$

For w on the negative real axis, we have

$$\psi_-(z,w) = (2\pi i)^{-1}(1-z)^{-1} \int_{C'} d\tilde{s} \ \frac{\partial}{\partial w} f\left(\tilde{s}, \frac{w-z\tilde{s}}{1-z}\right) . \tag{A5.25}$$

Therefore, we see

$$\psi_+(z,w) = \psi_-(z,w) \tag{A5.26}$$

on the negative real axis. This means that $\psi_+(z,w)$ and $\psi_-(z,w)$ define an analytic function $\psi(z,w)$ which is holomorphic except for the ϵ-neighborhood of the positive real axis.

Finally, we investigate the asymptotic behavior of $\psi(z,w)$. For this purpose, we again apply the condition (ii) to (A5.18). In the present case, since $|w|$ is large, it is necessary to investigate the behavior of the integrand much more closely. Since the intersection of K and the disc $|s|+|t| \leq M$ is compact, we have

$$|(\partial/\partial t)f(s,t)| < B_0. \tag{A5.27}$$

Therefore, the condition (ii) can be rewritten as

$$|(\partial/\partial t)f(s,t)| < B'\left(M+|s|+|t|\right)^{-\gamma} \sum_{i=1}^{m} \left(\epsilon_i+|z_i s+(1-z_i)t|\right)^{-1} \tag{A5.28}$$

in the whole K, where

$$B' \equiv \max(2^{1+\gamma}\, B,\ M^{\gamma}\ \epsilon_1 B_0),$$

$$\epsilon_i \equiv \min_{(s,t)\in K} |z_i s+(1-z_i)t| > 0. \tag{A5.29}$$

For Im $w > \epsilon$, we use (A5.28).

$$\left| \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tilde{s}\ \frac{\partial}{\partial w} f\left(\tilde{s},\ \frac{w-z\tilde{s}}{1-z}\right) \right|$$

$$\leq \int_{-\infty}^{\infty} ds' \left| \frac{\partial}{\partial w}\ f\left(s'+i\epsilon,\ \frac{w-zs'-iz\epsilon}{1-z}\right) \right|$$

$$< (1-z)^{-1}\ B' \sum_{i=1}^{m} I_i\ , \tag{A5.30}$$

where

$$I_i \equiv \int_{-\infty}^{\infty} ds'\ (M+|s'|+|v-ks'|)^{-\gamma}\ (\epsilon_i+|(1-z_i)v-k's'|)^{-1} \tag{A5.31}$$

with

$$v \equiv (1-z)^{-1}(w-iz\epsilon), \qquad (\mathrm{Im}\, v > 0),$$

$$k \equiv z(1-z)^{-1} \geq 0,$$

$$k' \equiv (z-z_i)(1-z)^{-1} \leq (1-z_i)k. \qquad (A5.32)$$

The assumption $z \neq z_i$ implies $k' \neq 0$. Writing $\mathrm{Re}\, v \equiv v'$ and $\mathrm{Im}\, v \equiv v'' > 0$, we have

$$I_i < 2^\gamma \int_{-\infty}^{\infty} ds' \left(M+|s'|+|v'-ks'|+|v''|\right)^{-\gamma}\left(\epsilon_i+|(1-z_i)v'-k's'|\right)^{-1} . \qquad (A5.33)$$

The transformation $u = k's'-(1-z_i)v'$ leads to

$$I_i < \frac{2^\gamma}{|k'|} \int_{-\infty}^{\infty} du \left(M+ \frac{|u+v'_i|}{|k'|} + \frac{|ku+k_i''v_i'|}{|k'|} + |v''|\right)^{-\gamma}\left(\epsilon_i+|u|\right)^{-1}, \qquad (A5.34)$$

with $v_i' \equiv (1-z_i)v'$ and $k_i'' \equiv k-k'(1-z_i)^{-1} \geq 0$ ($z_i \neq 1$ by assumption). We make use of the following inequality, which can be easily proved: If $a \geq b \geq 0$, one has

$$|X+Y| + |aX+bY| \geq c\left(|X|+|Y|\right) \qquad (A5.35)$$

for any real values of X and Y, where

$$c \equiv \min\left(1, \frac{a-b}{2}, \frac{a-b}{2a}\right). \qquad (A5.36)$$

Applying (A5.35) to the first factor of the integrand of (A5.34), we obtain

$$I_i < 2^\gamma \ |k'|^{-1} \int_{-\infty}^{\infty} du \ [M+h(|u|+|v_i'|)+|v''|]^{-\gamma} \ (\epsilon_i+|u|)^{-1}, \tag{A5.37}$$

where $h > 0$ because of (A5.36) with $k' \neq 0$. Let $h' \equiv \min(1, h(1-z_i))$. Then choosing σ such that $0 < \sigma < \gamma$, we have

$$\begin{aligned} I_i &< 2^\gamma |k'|^{-1} \int_{-\infty}^{\infty} du (\epsilon_i + |u|)^{-1} \ (M+h|u|)^{-\gamma+\sigma} \ (M+h'|v|)^{-\sigma} \\ &= O(|v|^{-\sigma}) \\ &= O(|w|^{-\sigma}) \end{aligned} \tag{A5.38}$$

Thus it has been established that

$$\psi(z,w) = O(|w|^{-\sigma}) \tag{A5.39}$$

for $\text{Im}\ w > \epsilon$. The same is true also for $\text{Im}\ w < -\epsilon$. Hence, Lindelöf's asymptotic theorem tells us that (A5.39) is true also in

$$\{w; \ |\text{Im}\ w| \leq \epsilon, \ \text{Re}\ w < 0\}. \tag{A5.40}$$

Now, the analyticity and the boundedness (A5.39) of $\psi(z,w)$ yield

$$\psi(z,w) = (2\pi i)^{-1} \int_{C'} d\tilde{w} \ \frac{\psi(z,\tilde{w})}{\tilde{w} - w} . \tag{A5.41}$$

Taking the <u>improper</u> limit $\epsilon \to 0+$ in (A5.22) and in (A5.41), we obtain

$$\psi(z,w) = (1-z)^{-1} \int_0^\infty ds' \frac{\partial}{\partial w} f_s\left(s', \frac{w-zs'}{1-z}\right) \qquad \text{(A5.42)}$$

for $w < 0$, and

$$\psi(z,w) = \int_0^\infty d\alpha \frac{\rho(z,\alpha)}{\alpha - w} \qquad \text{(A5.43)}$$

for $w \not\geq 0$. To interchange the order of $\epsilon \to 0+$ and an integration is not made in the usual sense, but it <u>defines</u> a distribution. Therefore, the asymptotic behavior (A5.39) is not necessarily inherited by $\rho(z,\alpha)$. Theorem V has now been established by (A5.15) with $\psi_+ = \psi$, (A5.43), and (A5.42).

APPENDIX VI. PROOF OF THEOREM VI.

The proof is the same with that of Theorem V except for the asymptotic behavior of $\psi(z,w)$. In the present case, instead of (A5.30), we have

$$\left| \int_{-\infty+i\epsilon}^{+\infty+i\epsilon} d\tilde{s} \; \frac{\partial}{\partial w} f\left(\tilde{s}, \frac{w-z\tilde{s}}{1-z}\right) \right| \leq B \int_{-\infty}^{\infty} ds' \left(1+|s'|+ \frac{|w-zs'-iz\epsilon|}{1-z} \right)^{-1-\gamma}$$

$$\leq 2^{1+\gamma} B \int_{-\infty}^{\infty} ds' \; \left(1+|s'|+|v'-ks'|+|v''|\right)^{-1-\gamma} \qquad \text{(A6.1)}$$

in the whole D_+, where $v = v'+iv''$ and k are given in (A5.32). Since the last expression of (A6.1) is nothing but a special case of the right-hand side of (A5.33), we obtain

$$|\psi_+(z,w)| \leq C'(1+|w|)^{-\sigma}. \qquad \text{(A6.2)}$$

The same is true also for $\psi_-(z,w)$ with $w \in D_-$. In the present case, (A6.2) holds regardless to ϵ, hence we can interchange the order of $\epsilon \to 0+$ and the integration in the ordinary sense. Thus we obtain (2.26).

APPENDIX VII. DERIVATION OF FORMULAS IN EXAMPLES

Examples 1 and 2. Trivial.

Example 3. The weight function can be easily calculated by using the representation of $(-s)^{-\frac{1}{4}}\ (-t)^{-\frac{1}{4}}$ (see Example 7) and

$$\exp[-(-t)^{\frac{1}{2}}] = \pi^{-1} \int_0^\infty d\alpha \ \frac{\sin \alpha^{\frac{1}{2}}}{\alpha - t} \ . \tag{A7.1}$$

The result is

$$\rho(z,\alpha) = \tfrac{1}{2}\, \pi^{-\frac{1}{2}}\ [\Gamma(\tfrac{1}{4})]^{-2}\ z^{-\frac{1}{2}} \int_z^1 dx \cdot x^{\frac{1}{4}}(1-x)^{-\frac{3}{4}}(x-z)^{-\frac{3}{2}} J_0\left([x\alpha/(x-z)]^{\frac{1}{2}}\right), \tag{A7.2}$$

whose singularities are located at $z = 0$ (order $z^{-\frac{3}{4}}$) and at $\alpha = 0$ (order $\alpha^{-\frac{1}{2}}$) only, and (A7.2) behaves like $O(\alpha^{-\frac{1}{4}})$ as $\alpha \to \infty$.

Example 4. Trivial.

Example 5. See the next.

Example 6. When $N > \mathrm{Re}\ \lambda > N-1$, we have

$$\int_0^1 dz \int_0^\infty d\alpha \ \frac{[zs+(1-z)t]^N\ \alpha^\lambda\, \delta(z-z_0)}{\alpha^N[\alpha - zs-(1-z)t]} = \Gamma(\lambda+1)\Gamma(-\lambda)[-z_0 s-(1-z_0)t]^\lambda . \tag{A7.3}$$

Differentiating (A7.3) by z_0 n times, we obtain (4.3).

If n ≥ N, one has an identity

$$\int dz \frac{[zs+(1-z)t]^N}{\alpha-zs-(1-z)t} \delta^{(n)}(z) = \int dz \frac{\alpha^N \delta^{(n)}(z)}{\alpha-zs-(1-z)t} . \tag{A7.4}$$

Hence, for n ≥ N > Re λ > -1,

$$f(s,t) = n!\ (t-s)^n \int_0^\infty d\alpha \frac{\alpha^\lambda}{(\alpha-t)^{n+1}}$$

$$= \Gamma(\lambda+1)\Gamma(n-\lambda)(t-s)^n(-t)^{\lambda-n}. \tag{A7.5}$$

<u>Example 7.</u> First, we assume $0 > \text{Re}\,\mu > -\frac{1}{2}$ and $0 > \text{Re}\,\nu > -\frac{1}{2}$. Then we can use (2.21) with (2.22).

$$\psi(z,w) = (1-z)^{-1} \int_0^\infty ds' \frac{\partial}{\partial w}\left[\frac{s'^\mu}{\Gamma(\mu+1)\Gamma(-\mu)}\left(-\frac{w-zs'}{1-z}\right)^\nu\right]$$

$$= \left[\Gamma(\mu+1)\Gamma(-\mu)\right]^{-1}(-\nu)z^{-\mu-1}(1-z)^{-\nu-1}\int_0^\infty dx \cdot x^\mu(x-w)^{\nu-1}$$

$$= [\Gamma(-\mu)\Gamma(-\nu)]^{-1}\Gamma(-\mu-\nu)z^{-\mu-1}(1-z)^{-\nu-1}(-w)^{\mu+\nu}. \tag{A7.6}$$

Then we obtain (4.5). For the general case, we analytically continue (4.5) with respect to μ and ν after making subtractions. By this way we get the correct result because of the invariance property of the weight function in the subtraction procedure (1.4).

<u>Example 8.</u> We want to prove (4.16). We denote the right-hand side of (4.16) by f(s,t). Then

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t} f(s,t) = \int_0^1 dz \int_0^\infty d\alpha \frac{2}{[\alpha - zs - (1-z)t]^3} = \frac{1}{st}. \qquad \text{(A7.7)}$$

Since f(s,t) is a symmetric function of s and t, (A7.7) leads to

$$f(s,t) = \log(-s)\log(-t) + \varphi(s) + \varphi(t), \qquad \text{(A7.8)}$$

where φ is an unknown function. On the other hand, because of (5.12), we obtain

$$f(s,s) = -2 \int_0^\infty d\alpha \frac{(s+1)\log\alpha}{(1+\alpha)(\alpha - s)} = [\log(-s)]^2, \qquad \text{(A7.9)}$$

which implies $\varphi(s) \equiv 0$.

REFERENCES AND FOOTNOTES

1) N. Nakanishi, Phys. Rev. 127, 1380 (1962).

2) N. Nakanishi, J. Math. Phys. 4, 1385 (1963).

3) N. Nakanishi, Phys. Rev. 133, B214 (1964); Phys. Rev. to be published.

4) The proof given in Ref. 2 is incomplete in the following respects. (i) The integral (2.11) of Ref. 2 is not well defined because the contours necessarily pass through some zero points of the denominator of the integrand. (ii) $(\partial/\partial t)f(s,t)$ does not necessarily behave like $O\left(|t|^{-1-\delta}\right)$ at infinity. (iii) The analytic continuation of $g(w,z)$ to the lower half-plane of w is not good because then some part of the s' contour belongs to the singularity region.

5) J. Bros and V. Glaser, preprint.
A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento 23, 954 (1962). The proof given in the latter paper is incomplete because the ξ' contour crosses the singularity region of the integrand at $\xi' = 0$ and on the large semi-circle. To avoid this difficulty, it is essential to use the edge-of-the-wedge theorem as is done in our proof (see Appendix IV). The equivalence of D_{st} and the envelope of holomorphy of $D_+ \cup D_- \cup E$ was first pointed out in Ref. 2.

6) It is generally impossible to deduce $f'(x) = O(x^{-1-\delta})$, $(x > 0,\ \delta > 0)$, from $f(x) = O(x^{-\delta})$ even if $f(x)$ is holomorphic on the positive real axis, because, for example,

$f(x) = x^{-\delta}\sin x$. The author is much indebted to Dr. Pincus and Dr. Marr for valuable discussions on the asymptotic behavior of a derivative.

7) L. Schwartz, Théorie des Distributions (Herman and C^{ie}, Paris, 1950), Chapter II. The symbol Pf denotes Hadamard's finite part, which means to discard the divergent part of the integral in a consistent manner.

8) G. Källén, Dan. Mat. Fys. Medd. 27, No. 12 (1953), Appendix.

9) G. Frye and R. L. Warnock, Phys. Rev. 130, 478 (1963), Appendix A.

10) Instead of the part 1°) of the proof of Lemma 1, we now have simply

$$\left| \int_0^{r/2} \right| < A(1-\delta)^{-1}\,(r/2)^{-\delta},$$

assuming $\delta < 1$.

11) H. J. Bremermann, R. Oehme, and J. G. Taylor, Phys. Rev. 109, 2178 (1958); F. J. Dyson, Phys. Rev. 110, 579 (1958).

12) The inequality (2.20) holds also for an arbitrary closed subset of D_{st}. This can be proved as follows. From (A4.2) together with (A4.3) we have

$$|(\partial/\partial t)f(s,t)| \leq (2\pi)^{-1}\int_{-\infty}^{\infty} d\xi' \left| \frac{(\partial/\partial\hat{t})f(\xi'\hat{s}-\epsilon,\ \xi'\hat{t}-\epsilon)}{\xi'-\xi} \right| ,$$

where $s = \xi\hat{s}-\epsilon$, $t = \xi\hat{t}-\epsilon$, and $|\xi| = 1$. Substituting (2.20) in the integrand and taking $\epsilon \to 0+$, we easily obtain the desired result.

FIGURE CAPTIONS

Fig. 1 The contour C on the $\tilde{s}$ or $\tilde{t}$ plane.

Fig. 2 The singularity regions of $f\left(\tilde{s}, \frac{w-z\tilde{s}}{1-z}\right)$ on the $\tilde{s}$ plane when w lies in the second quadrant.

Fig. 3 The deformed $\tilde{s}$ contour when w lies on the negative real axis.

Fig. 4 The contour C' on the $\tilde{s}$ plane.

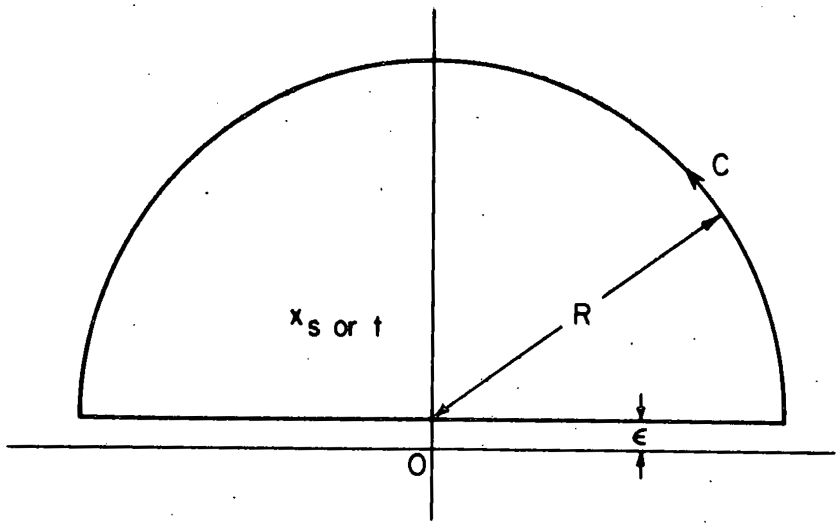

FIGURE 1

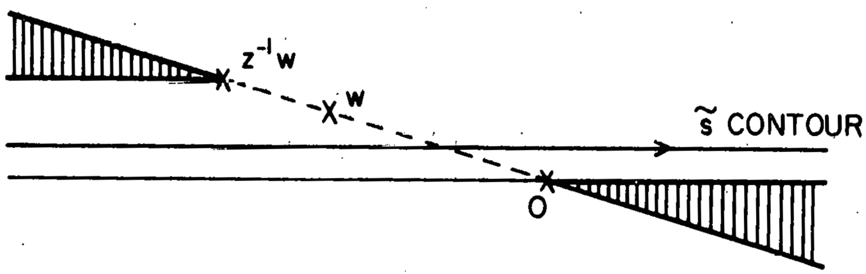

FIGURE 2

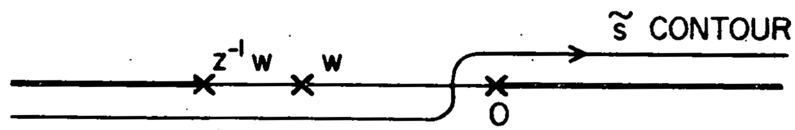

FIGURE 3

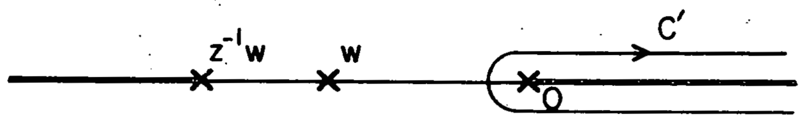

FIGURE 4