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Singular Perturbation Analysis of the Neutron Transport Equation

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Summary I for ANS Conference(u)

The attached paper on singular perturbation analysis in neutron transport is for submission at the next ANS conference.

Singular Perturbation Analysis of the Neutron Transport Equation

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A singular perturbation technique is applied to the one-speed, one-dimensional neutron transport equation with isotropic scattering. Our technique extends previous singular perturbation applications[1,2] to higher-order and reduces the transport problem to a series of diffusion theory problems in the interior medium and a series of analytically solvable transport problems in the boundary layers. Asymptotic matching links the two solutions, yielding boundary conditions and a composite expansion valid throughout the media. Our formulation generates an accurate correction for the material interface condition used in global diffusion theory calculations.

Our analysis[3] selects the small ordering parameter ε as the ratio of the neutron mean free path to the physical width of the system. We assume a power series expansion of the angular flux:

$$\Psi(x,\mu;\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j \Psi^j(x,\mu) = \sum_{j=0}^{\infty} \varepsilon^j \sum_{k=0}^{j} \Psi^{jk}(x)\, P_k(\mu), \tag{1}$$

where x is the dimensionless spatial variable and the $P_k(\mu)$'s are Legendre polynomials. Since c, the number of secondary neutrons per collision, relates characteristic lengths in the system, an expansion in ε is assumed:

$$c(\varepsilon) = 1 + \sum_{j=2}^{\infty} \varepsilon^j c_j \quad . \tag{2}$$

Similarly, we allow a small isotropic external source that is expanded:

$$\varepsilon S(x;\varepsilon) = \sum_{j=2}^{\infty} \varepsilon^j Q_j(x) \quad . \tag{3}$$

These expansions are substituted into the transport equation and terms of each order in ε are equated to derive the equations in Table I. These equations are decoupled since the lower-order equations do not depend on the higher-order.

Our analysis finds that the c_j and Q_j expansion coefficients are not uniquely determined as just diffusion coefficients in diffusion theory are not unique. Although the c_j are not unique, $(c-1)$ is required to be $O(\varepsilon^2)$ and the dispersion law, $c\nu_0 \tanh^{-1} 1/\nu_0 = 1$, must be satisfied. Here $\nu_0(\varepsilon)$ is the transport discrete eigenvalue and also has a power series expansion since it relates characteristic lengths.

In fixed source problems one choice for the c_j is $c_2 = (c-1) / \varepsilon^2$ so that $c_j = 0$ for $j \geq 3$, which gives the common form for the interior diffusion equations. Alternatively, we could choose $c_2 = -1 / (3\varepsilon^2 \nu_0^2)$ so that the c_j vanish for j odd and are functions of c_2 for j even. This gives the transport-corrected form of the interior diffusion equations with asymptotic diffusion lengths of $\varepsilon\nu_0$. Our source expansion comes from reasoning that the lowest-order flux expansion, Ψ^{00}, can model the exact flux in an infinite medium with slowly varying source, if we choose $Q_2(x)(c-1) = c_2 \varepsilon S(x;\varepsilon)$.

In criticality problems, $c(\varepsilon)$ becomes an eigenvalue and the c_j depend on the boundary conditions rather than material properties as in fixed source problems. For example, in a bare slab that is $1/\varepsilon$ mfp thick, the eigenvalue is calculated as:

$$c(\varepsilon) = 1 + \sum_{j=2}^{\infty} \varepsilon^j c_j = 1 + \tfrac{1}{3}\pi^2\varepsilon^2 - \tfrac{4}{3}\gamma_1\pi^2\varepsilon^3 - \left(\tfrac{4}{45}\pi^4 - 4\gamma_1^2\pi^2\right)\varepsilon^4 + O(\varepsilon^5) \tag{4}$$

from the interior equations and vacuum boundary conditions of Table I. A similar perturbation expression was derived previously[4], but we solve through $O(\varepsilon^2)$, without simplifying the boundary analysis.

Our boundary layer analysis formulates a series of inhomogeneous transport problems in purely scattering media that can be solved by various methods of the singular eigenfunction technique[5]. Although the analysis is complicated, it is performed only once for each boundary condition, and the final transient component of the scalar flux is expressed as a series of exponential integrals rather than in the singular eigenfunctions. For the interface problem the additional $O(\varepsilon^2)$ transient component is:

$$\omega(y) = \pm \Delta\Psi^{00}_{xx}(x_0) \sum_{j=0}^{\infty} \Gamma_j E_{2j+2}(\pm y) / 6 \quad , \qquad y \gtrless 0 \tag{5}$$

where $y = (x - x_0)/\varepsilon$, $x = x_0$ at the boundary, and the Γ_j are constants from the singular eigenfunction analysis. The boundary layer analysis also gives boundary conditions for the interior solutions listed in Table I. For material interfaces the lower-order conditions require continuity of the flux and its derivative, while the $O(\varepsilon^2)$ conditions require a discontinuous interior flux that is matched by the Eq. (5) transient flux. For vacuum boundaries the $O(1)$ condition requires the flux to vanish at x_0, while the $O(\varepsilon)$ condition effectively adds an extrapolated endpoint.

Our perturbation techniques yield decoupled interior diffusion equations and boundary layer solutions involving exponential integrals. In analytic and numeric tests detailed in a companion summary[6], the error in calculated scalar flux, relative to transport results, is reduced to about half that of conventional diffusion theory. Our analysis also provides a consistent method for deriving and comparing various diffusion theory approximations.

TABLE I
Interior Perturbation Equations and Boundary Conditions

Order	Interior Diffusion Equations for the Common or Transport-Corrected Forms	Boundary Conditions: Material Interface at $x = x_0$	Boundary Conditions: Left/Right Vacuum at $x = x_0$
$O(1)$	$-\Psi^{00}_{xx}/3 - c_2\Psi^{00}(x) = Q_2(x)$	$\Delta\Psi^{00} = \Delta\Psi^{00}_x = 0$	$\Psi^{00} = 0$
$O(\varepsilon)$	$-\Psi^{10}_{xx}/3 - c_2\Psi^{10}(x) = 0$	$\Delta\Psi^{10} = \Delta\Psi^{10}_x = 0$	$\Psi^{10} = \pm\gamma_1 \Psi^{00}_x$
$O(\varepsilon^2)$	$-\Psi^{20}_{xx}/3 - c_2\Psi^{20}(x) = -\frac{4}{15} Q_{2xx} + \left\{ c_4 + \frac{4}{5}c_2^2 \right\} [\Psi^{00}(x) + Q_2(x)/c_2]$	$\Delta\Psi^{20} + \frac{4}{15}\Delta\Psi^{00}_{xx} = 0$; $\Delta\Psi^{20}_x + \frac{4}{15}\Delta\Psi^{00}_{xxx} = 0$	$\Psi^{20} = \pm\gamma_1 \Psi^{10}_x - .2190\,\Psi^{00}_{xx}$

Note: $\Delta\Psi$ is the change across the interface $\Delta\Psi = \Psi(x_0+) - \Psi(x_0-)$ and $\gamma_1 = .7104$

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2 E. W. LARSEN, "Diffusion Theory as an Asymptotic Limit of Transport Theory for Nearly Critical Systems with Small Mean Free Paths," *Annals of Nuclear Energy*, **7** 249, (1980).

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4 V. C. BOFFI, V. G. MOLINARI, and G. SPIGA, "Asymptotic Solutions in Neutron Transport Theory," *Nucl. Sci. Eng.*, **62**, 332 (1977).

5 K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley, Reading, Massachusetts (1967).

6 D. C. LOSEY and J. C. LEE, "Singular Perturbation Applications to Neutron Transport", submitted for presentation at the present conference.