

# Singular Perturbation Analysis of the Neutron Transport Equation

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## **Summary I for ANS Conference<sub>(u)</sub>**

The attached paper on singular perturbation analysis in neutron transport  
is for submission at the next ANS conference.

# Singular Perturbation Analysis of the Neutron Transport Equation

by

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A singular perturbation technique is applied to the one-speed, one-dimensional neutron transport equation with isotropic scattering. Our technique extends previous singular perturbation applications<sup>1,2</sup> to higher-order and reduces the transport problem to a series of diffusion theory problems in the interior medium and a series of analytically solvable transport problems in the boundary layers. Asymptotic matching links the two solutions, yielding boundary conditions and a composite expansion valid throughout the media. Our formulation generates an accurate correction for the material interface condition used in global diffusion theory calculations.

Our analysis<sup>3</sup> selects the small ordering parameter  $\epsilon$  as the ratio of the neutron mean free path to the physical width of the system. We assume a power series expansion of the angular flux:

$$\Psi(x, \mu; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j \Psi^j(x, \mu) = \sum_{j=0}^{\infty} \epsilon^j \sum_{k=0}^j \Psi^{jk}(x) P_k(\mu), \quad (1)$$

where  $x$  is the dimensionless spatial variable and the  $P_k(\mu)$ 's are Legendre polynomials. Since  $c$ , the number of secondary neutrons per collision, relates characteristic lengths in the system, an expansion in  $\epsilon$  is assumed:

$$c(\epsilon) = 1 + \sum_{j=2}^{\infty} \epsilon^j c_j. \quad (2)$$

Similarly, we allow a small isotropic external source that is expanded:

$$\epsilon S(x; \epsilon) = \sum_{j=2}^{\infty} \epsilon^j Q_j(x). \quad (3)$$

These expansions are substituted into the transport equation and terms of each order in  $\epsilon$  are equated to derive the equations in Table I. These equations are decoupled since the lower-order equations do not depend on the higher-order.

Our analysis finds that the  $c_j$  and  $Q_j$  expansion coefficients are not uniquely determined as just diffusion coefficients in diffusion theory are not unique. Although the  $c_j$  are not unique,  $(c-1)$  is required to be  $O(\epsilon^2)$  and the dispersion law,  $c v_0 \tanh^{-1} 1/v_0 = 1$ , must be satisfied. Here  $v_0(\epsilon)$  is the transport discrete eigenvalue and also has a power series expansion since it relates characteristic lengths.

In fixed source problems one choice for the  $c_j$  is  $c_2 = (c-1) / \epsilon^2$  so that  $c_j = 0$  for  $j \geq 3$ , which gives the common form for the interior diffusion equations. Alternatively, we could choose  $c_2 = -1 / (3\epsilon^2 v_0^2)$  so that the  $c_j$  vanish for  $j$  odd and are functions of  $c_2$  for  $j$  even. This gives the transport-corrected form of the interior diffusion equations with asymptotic diffusion lengths of  $\epsilon v_0$ . Our source expansion comes from reasoning that the lowest-order flux expansion,  $\Psi^{00}$ , can model the exact flux in an infinite medium with slowly varying source, if we choose  $Q_2(x)(c-1) = c_2 \epsilon S(x; \epsilon)$ .

In criticality problems,  $c(\epsilon)$  becomes an eigenvalue and the  $c_j$  depend on the boundary conditions rather than material properties as in fixed source problems. For example, in a bare slab that is  $1/\epsilon$  mfp thick, the eigenvalue is calculated as:

$$c(\epsilon) = 1 + \sum_{j=2}^{\infty} \epsilon^j c_j = 1 + \frac{1}{3} \pi^2 \epsilon^2 - \frac{4}{3} \gamma_1 \pi^2 \epsilon^3 - \left( \frac{4}{45} \pi^4 - 4\gamma_1^2 \pi^2 \right) \epsilon^4 + O(\epsilon^5) \quad (4)$$

from the interior equations and vacuum boundary conditions of Table I. A similar perturbation expression was derived previously<sup>4</sup>, but we solve through  $O(\epsilon^2)$ , without simplifying the boundary analysis.

Our boundary layer analysis formulates a series of inhomogeneous transport problems in purely scattering media that can be solved by various methods of the singular eigenfunction technique<sup>5</sup>. Although the analysis is complicated, it is performed only once for each boundary condition, and the final transient component of the scalar flux is expressed as a

series of exponential integrals rather than in the singular eigenfunctions. For the interface problem the additional  $O(\epsilon^2)$  transient component is:

$$\omega(y) = \pm \Delta \Psi_{xx}^{00}(x_0) \sum_{j=0}^{\infty} \Gamma_j E_{2j+2}(\pm y) / 6 , \quad y \geq 0 \quad (5)$$

where  $y = (x - x_0)/\epsilon$ ,  $x = x_0$  at the boundary, and the  $\Gamma_j$  are constants from the singular eigenfunction analysis. The boundary layer analysis also gives boundary conditions for the interior solutions listed in Table I. For material interfaces the lower-order conditions require continuity of the flux and its derivative, while the  $O(\epsilon^2)$  conditions require a discontinuous interior flux that is matched by the Eq. (5) transient flux. For vacuum boundaries the  $O(1)$  condition requires the flux to vanish at  $x_0$ , while the  $O(\epsilon)$  condition effectively adds an extrapolated endpoint.

Our perturbation techniques yield decoupled interior diffusion equations and boundary layer solutions involving exponential integrals. In analytic and numeric tests detailed in a companion summary<sup>6</sup>, the error in calculated scalar flux, relative to transport results, is reduced to about half that of conventional diffusion theory. Our analysis also provides a consistent method for deriving and comparing various diffusion theory approximations.

TABLE I  
Interior Perturbation Equations and Boundary Conditions

Order	Interior Diffusion Equations for the Common or Transport-Corrected Forms	Boundary Conditions	
		Material Interface at $x = x_0$	Left/Right Vacuum at $x = x_0$
$O(1)$	$-\Psi_{xx}^{00}/3 - c_2 \Psi^{00}(x) = Q_2(x)$	$\Delta \Psi^{00} = \Delta \Psi_x^{00} = 0$	$\Psi^{00} = 0$
$O(\epsilon)$	$-\Psi_{xx}^{10}/3 - c_2 \Psi^{10}(x) = 0$	$\Delta \Psi^{10} = \Delta \Psi_x^{10} = 0$	$\Psi^{10} = \pm \gamma_1 \Psi_x^{00}$
$O(\epsilon^2)$	$-\Psi_{xx}^{20}/3 - c_2 \Psi^{20}(x) = -\frac{4}{15} Q_{2xx}$ + $\left\{ c_4 + \frac{4}{5} c_2^2 \right\} [\Psi^{00}(x) + Q_2(x)/c_2]$	$\Delta \Psi^{20} + \frac{4}{15} \Delta \Psi_{xx}^{00} = 0$ $\Delta \Psi_x^{20} + \frac{4}{15} \Delta \Psi_{xxx}^{00} = 0$	$\Psi^{20} = \pm \gamma_1 \Psi_x^{10}$ - .2190 $\Psi_{xx}^{00}$

Note:  $\Delta \Psi$  is the change across the interface  $\Delta \Psi = \Psi(x_0+) - \Psi(x_0-)$  and  $\gamma_1 = .7104$

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