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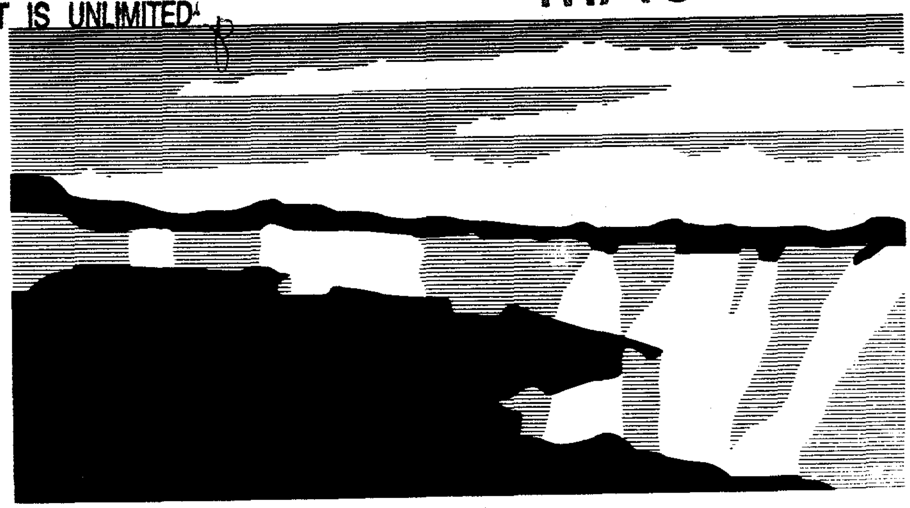

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THE PLANAR GREEN'S FUNCTION

IN AN INFINITE MULTIPLYING MEDIUM

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INTRODUCTION

Throughout the history of neutron transport theory, the study of simplified problems that have analytical or semi-analytical solutions has been a foundation for more complicated analyses. Analytical transport results are often used as benchmarks or in pedagogical settings. Benchmark problems in infinite homogeneous media have been studied continually, beginning with the monograph by Case, DeHoffmann, and Placzek.[1] A fundamental problem considered in this work is the Green's function in an infinite medium. The Green's function problem considers an infinite planar source emitting neutral particles in the single direction μ'. Recently, this Green's function has been used to obtain solutions for finite media.[2-4] These solutions, which hinge on accurate and fast evaluation of the infinite medium Green's function, use Fourier and Laplace transform inversion techniques for the evaluation.[5,6] Until now, only absorbing media have been considered, and applications were therefore limited to non-multiplying media. In an effort to relax this limitation, the infinite medium Green's function is numerically evaluated for an infinite multiplying medium using the double-sided Laplace transform inversion. Of course, no steady-state mathematical solution exists for an infinite multiplying medium with a source present; however, the non-physical solution in an infinite medium can be used in finite media problems where the solution is physically realizable.

THE INFINITE MEDIUM GREEN'S FUNCTION

Much effort has been dedicated to evaluating the infinite medium Green's function. In plane geometry, the Green's function, $G(x-x',\mu|\mu')$, is essentially the angular flux at (x,μ) resulting from a planar source of neutral particles at x' emitting in the direction μ'. The transport equation for the Green's function with $x' = 0$ (no loss of generality) is given by

$$\left[\mu \frac{\partial}{\partial x} + 1\right] G(x,\mu|\mu') = \frac{c}{2}\int_{-1}^{1} d\mu'' \, G(x,\mu''|\mu') + \delta(x)\,\delta(\mu - \mu') \quad , \tag{1}$$

where c is the number of secondary particles per collision, and the boundary condition requires a bounded Green's function at infinity. A Fourier transform is applied to Eq. (1) and the uncollided portion is inverted analytically, leaving for the infinite medium Green's function

$$G(x,\mu|\mu') = \frac{e^{-|x/\mu'|}}{|\mu'|}\,\delta(\mu - \mu')\,\theta\!\left(\frac{x}{\mu'}\right) + G_c(x,\mu|\mu') \quad . \tag{2}$$

The collided Green's function can be reformulated as[3]

$$G_c(x,\mu|\mu') = \frac{1}{\mu'-\mu}\frac{c}{2}\left[h(x,\mu') - h(x,\mu)\right] \quad , \tag{3}$$

with

$$h(x,\mu) = \mathrm{sgn}(\mu)\, e^{-|x/\mu|}\, \theta\left(\frac{x}{\mu}\right) + \tilde{h}(x,\mu) \ , \tag{4a}$$

$$\tilde{h}(x,\mu) = \frac{\mu}{2\pi} \int_{-\infty}^{\infty} dk\, \frac{e^{ikx}}{1 + ik\mu} \frac{cL(k)}{1 - cL(k)} \ , \tag{4b}$$

where $L(k) = \tan^{-1}k/k$. Decomposing the Green's function into two components of the same function $h(x,\mu)$ has the advantage of allowing inversion of the individual terms for each μ and μ', instead of inverting for the Green's function at every μ,μ' combination. Given the form of Eq. (4b), the most straightforward means of evaluating the integral is to convert the infinite integral into semi-infinite integrals, convert these into infinite series of finite integrals, and evaluate the series.[3] A further simplification is obtained by examination of Eq. (4a), which yields the useful properly $h(x,\mu) = -h(-x,-\mu)$. Thus, two integrals must be evaluated to obtain full knowledge of the Green's function: $\tilde{h}(x,\mu)$, and $\tilde{h}(x,-\mu)$.

A very useful alternative for evaluating Eq. (4b) comes by converting the integral along the real k axis to an integral along the imaginary k axis. This is accomplished by making the substitution $s = ik$, which results in

$$\tilde{h}(x,\mu) = \frac{\mu}{2\pi i} \int_{-i\infty}^{i\infty} ds\, \frac{e^{sx}}{1 + \mu s} \frac{cL(s)}{1 - cL(s)} \ . \tag{5}$$

The Fourier transform inversion has been converted to a Laplace transform inversion, or more specifically, a double-sided Laplace transform inversion, where the direct function is *not* limited to $f(t) = 0$ for $t < 0$.[7]

THE DOUBLE-SIDED LAPLACE TRANSFORM INVERSION

From the standard formula for the Laplace transform inversion, the double-sided inversion is given as

$$f(t) = \frac{e^{\gamma t}}{2\pi} \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \; \bar{F}(\gamma + i\omega) \quad , \tag{6a}$$

with γ being the distance from the Bromwich contour to the imaginary axis. Separating the image function into real and imaginary parts and expanding the exponential into sines and cosines yields [for $f(t)$ real]

$$f(t) = \frac{e^{\gamma t}}{\pi} \int_{0}^{\infty} d\omega \left[\cos(\omega t) \operatorname{Re} \bar{F}(\gamma + i\omega) - \sin(\omega t) \operatorname{Im} \bar{F}(\gamma + i\omega)\right] \; . \tag{6b}$$

As usual, this integral may be recast as an infinite series of finite integrals and evaluated using the Euler-Knopp transformation to accelerate the convergence of the series.

For $c < 1$, examination of the image function in Eq. (5) yields the following information: there is a branch cut resulting from the inverse tangent at $s = -1$ and extending to $s = -\infty$; there is a pole at $-\kappa_0$, where $-1 \le -\kappa_0 < 0$, which is the zero of the infinite medium dispersion relation

$$1 - c \frac{\tan^{-1} is}{is} = 0 \; ;$$

there is a simple pole at $-1/\mu$. Using the double-sided Laplace transform formulation, the Bromwich contour may be evaluated *anywhere* between $-1 < \operatorname{Re} s \le 0$. If the contour is evaluated between the pole and the imaginary axis, the individual contour integrals combine such that

$$\int_{-i\infty}^{i\infty} ds \, (\bullet) + \int_{C_\infty} + \int_{C_1} + \int_{C_{\mu+}} + \int_{C_{\mu-}} + \int_{\Gamma+} + \int_{\Gamma-} = 2\pi i \;\; \operatorname{Res}_{-\kappa_0} \; ; \tag{7a}$$

however, if the Bromwich contour is evaluated between the pole and the branch cut at $s =$ -1, there are no poles inside the closed contour:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} ds\,(\bullet) + \int_{C_\infty} + \int_{C_1} + \int_{C_{\mu+}} + \int_{C_{\mu-}} + \int_{\Gamma+} + \int_{\Gamma-} = 0 \ . \tag{7b}$$

Combining Eqs. (7a) and (7b), it is easily seen that the original Bromwich contour ($\gamma = 0$) is the modified Bromwich contour plus the contribution of the pole:

$$\int_{-i\infty}^{i\infty} ds\,(\bullet) = \int_{\gamma-i\infty}^{\gamma+i\infty} ds\,(\bullet) + 2\pi i \ \mathrm{Res}_{-\kappa_0} \ . \tag{7c}$$

The residue may be analytically calculated from the image function. The numerical result of calculating the inversion using the left or right hand side of Eq. (7c) is identical. This distinction may seem unnecessary; however, when the case for $c \geq 1$ is considered, it becomes essential.

When $c < 1$, there are actually 2 poles at $s = \pm\kappa_0$. As c approaches 1 the poles converge at the origin and then split again along the imaginary axis for $c > 1$. Now the standard Bromwich contour for the double-sided Laplace transform inversion, which lies along the imaginary axis, crosses over two poles. Shifting the contour as shown in Fig. 1 and as discussed above allows evaluation of the integral through the shifted contour plus contributions from poles and contours around the poles as appropriate:

$$\int_{-i\infty}^{i\infty} ds\,(\bullet) = \int_{\gamma-i\infty}^{\gamma+i\infty} ds\,(\bullet) - \left(\int_{C_{k+}} + \int_{C_{k-}}\right) + 2\pi i \sum \mathrm{Residues} \ . \tag{8}$$

This shifting of contours allows evaluation of the infinite medium Green's function for a multiplying medium.

Using the Bromwich contour shift depicted in Fig. 1, the only contributions from the poles at $\pm ik_0$ come from the semi-circular contours around the poles. In the figure, the poles are not enclosed by the contour; therefore, the residues lie outside the contour and need not be evaluated. Using the standard limiting processes for contour integration for these contours, the expression for $\tilde{h}(x,\mu)$ becomes

$$\tilde{h}(x,\mu) = \frac{\mu}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} ds\, \frac{e^{sx}}{1+\mu s}\, \frac{cL(s)}{1-cL(s)} - \frac{\mu k_0(1+k_0^2)}{1-c+k_0^2}\, \frac{\sin(k_0 x) - k_0\mu\cos(k_0 x)}{1 + k_0^2\mu^2} \quad . \tag{9}$$

Note that these expressions are real (it can be shown that the contour integrals around the branch cut combine to a real term). In Ref. 1, when discussing the solution to the isotropic point source solution in an infinite multiplying medium, the authors note that different solutions are obtained by using different contours. In this case, when the semi-circular contours are included in the closed contour (and residues where appropriate) Eq. (9) is obtained regardless of the path chosen. That is, both semi-circles can be taken to the left, right, or any combination thereof and the result is Eq. (9). However, if one of the poles is ignored, the result is effectively taking only one of the two pole contributions and an $e^{\pm ik_0 x}$ term is obtained without a conjugate term to make the final result real. This produces an incoming or outgoing "wave" in the infinite medium, where the previous analysis produced a "standing wave." Similar analyses as those above are undertaken to treat the cases where $x = 0$ and $\mu = \mu'$.

NUMERICAL DEMONSTRATION

With this analysis, we now have the means of calculating the infinite medium Green's function for any homogeneous medium. Fig. 2 shows the scalar Green's function as a function of distance from the source ($x' = 0$, $\mu' = 1$). Note that for $c < 1$ the Green's function appears as expected, and that as c approaches 1 the Green's function increases without bound. At $c = 1$ the Green's function is negative and asymptotically linear, and as c progresses above 1 oscillations becomes apparent. The Green's function method may now be applied to problems which contain multiplying media, such as the critical slab problem or heterogeneous subcritical media with inhomogeneous source terms.

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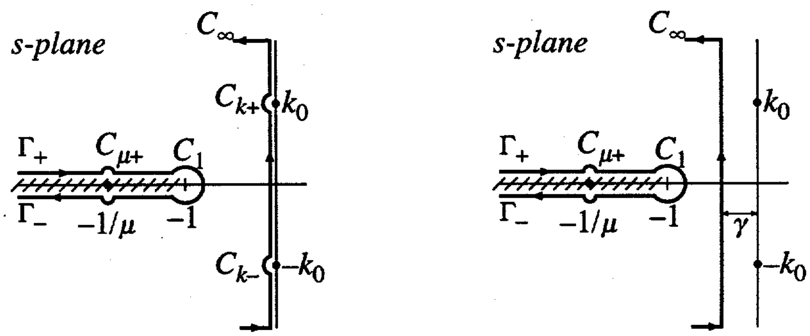

Fig. 1. $\tilde{h}(x,\mu)$ Laplace transform inversion contours for $c > 1$.

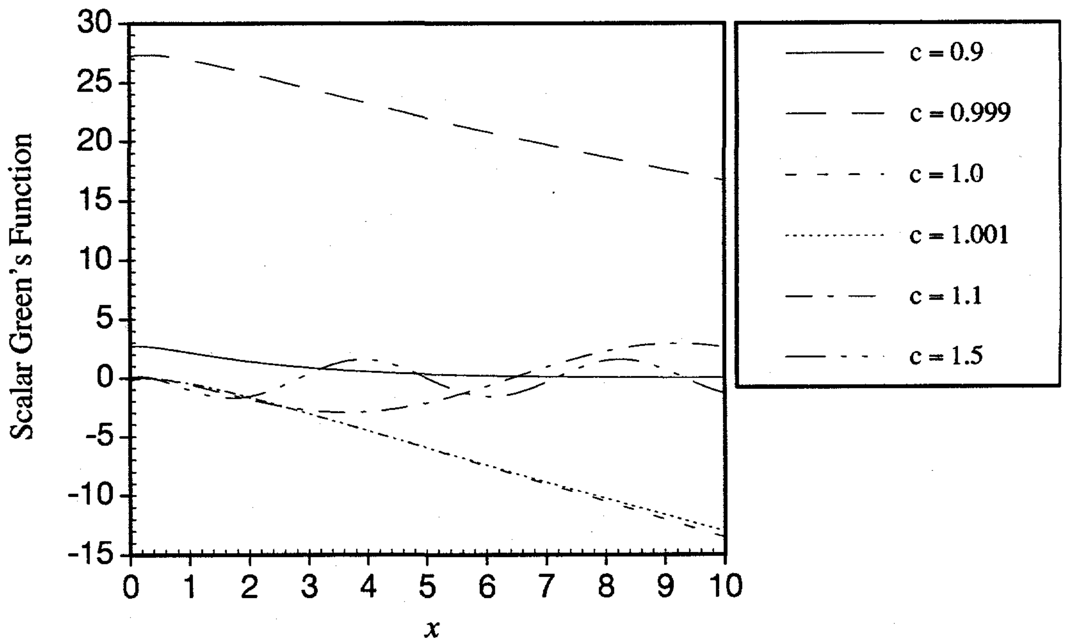

Fig. 2. The scalar Green's function as a function of distance from the source for source parameters $x' = 0$, $\mu' = 1$.