PREPARED FOR THE U.S. DEPARTMENT OF ENERGY,
UNDER CONTRACT DE-AC02-76-CHO-3073

PPPL-3258
UC-420, 427

PPPL-3258

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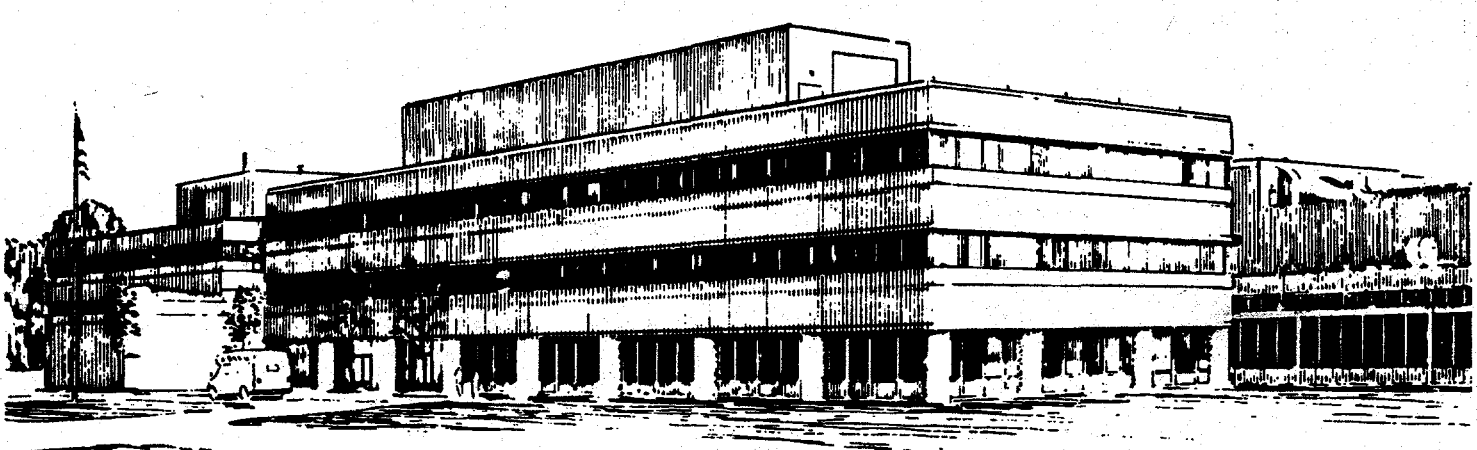
August 1997

PPPL PRINCETON PLASMA PHYSICS LABORATORY



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Statistically-Averaged Rate Equations for Intense Nonneutral Beam Propagation Through a Periodic Solenoidal Focusing Field Based on the Nonlinear Vlasov-Maxwell Equations

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Abstract

This paper presents a detailed formulation and analysis of the rate equations for statistically-averaged quantities for an intense nonneutral beam propagating through a periodic solenoidal focusing field $\mathbf{B}^{sol}(\mathbf{x}) = B_z(s)\hat{\mathbf{e}}_z - (1/2)B_z'(s)r\hat{\mathbf{e}}_r$, where $B_z(s+S) = B_z(s)$, s is the axial coordinate, and $S = const.$ is the axial periodicity length. The analysis is based on the nonlinear Vlasov-Maxwell equations in the electrostatic approximation, assuming a thin beam with characteristic beam radius $r_b \ll S$, negligibly small axial momentum spread about the directed value $p_z = \gamma_b m\beta_b c$, and $\nu/\gamma_b = N_b Z_i^2 e^2/\gamma_b mc^2 \ll 1$. Here, ν is Budker's parameter, $\gamma_b mc^2$ is characteristic kinematic energy of a beam particle, $N_b = \int dXdYdX'dY'F_b$ is the number of beam particles per unit axial length, and $F_b(X,Y,X',Y',s)$ is the distribution function of the beam particles in the transverse phase space (X,Y,X',Y') appropriate to the Larmor frame. The global rate equation is derived for the self-consistent nonlinear evolution of the statistical average $\langle\chi\rangle = N_b^{-1}\int dXdYdX'dY'\chi F_b$, where $\chi(X,Y,X',Y',s)$ is a general phase function. The results are applied to investigate the nonlinear evolution of the generalized entropy, mean canonical angular momentum $\langle P_\theta\rangle$, center-of-mass motion for $\langle X\rangle$ and $\langle Y\rangle$, mean kinetic energy $(1/2)\langle X'^2+Y'^2\rangle$, mean-square beam radius $\langle X^2+Y^2\rangle$, and coupled rate equations for the unnormalized transverse emittance $\epsilon(s)$ and root-mean-square beam radius $R_b(s) = \langle X^2+Y^2\rangle^{1/2}$. Global energy balance is discussed, and the coupled rate equations for $\epsilon(s)$ and $R_b(s)$ are examined for the class of axisymmetric beam distributions F_b with fixed shape density profile $n_b(R,s) = [N_b/\pi r_b^2(s)]f(R/r_b(s))$. Here, $R = (X^2+Y^2)^{1/2}$ is the radial distance from the beam axis, $r_b(s)$ is the outer radius of the beam envelope, and the density shape function $f(R/r_b)$ is allowed to have general functional form. Most importantly, it is found that $d\epsilon(s)/ds = 0$ for general shape function $f(R/r_b)$, and the envelope equation for the outer beam radius $r_b(s)$ is similar to the envelope equation for a Kapchinskij-Vladimirskij beam distribution, appropriately modified by a geometric factor g to reflect the shape of the function $f(R/r_b)$.

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I. INTRODUCTION

Periodic focusing accelerators [1–4] have a wide range of applications ranging from basic scientific research, to applications [5–8] such as heavy ion fusion, tritium production, and nuclear waste treatment. There is growing interest in developing an improved understanding of the nonlinear dynamics, stability, and transport properties of intense nonneutral beams propagating through a periodic focusing field [8], both with respect to identifying operating regimes for quiescent beam propagation with negligible effects of collective instabilities [8–18], and with respect to minimizing or eliminating halo production [19–22]. Particularly useful in describing intense beam propagation in periodic focusing transport systems are kinetic models [9–11,17,18,23–26] based on the nonlinear Vlasov-Maxwell equations [1], which incorporate the self-consistent evolution of the distribution of beam particles F_b and the interaction of the beam particles with the electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$.

This paper presents a detailed formulation and analysis of the rate equations for statistically-averaged quantities for an intense nonneutral beam propagating through a periodic solenoidal focusing field $\mathbf{B}^{sol}(\mathbf{x}) = B_z(s)\hat{\mathbf{e}}_z - (1/2)B_z'(s)r\hat{\mathbf{e}}_r$, where $B_z(s+S) = B_z(s)$, s is the axial coordinate, and $S = const.$ is the axial periodicity length. The analysis is based on the nonlinear Vlasov-Maxwell equations in the electrostatic approximation [1,26]. It assumes a thin beam with characteristic beam radius $r_b \ll S$, negligibly small axial momentum spread about the directed value $p_z = \gamma_b m\beta_b c$, where $\gamma_b mc^2$ is the characteristic kinematic energy of a beam particle, and $\nu/\gamma_b \ll 1$, where $\nu = N_b Z_i^2 e^2/mc^2$ is Budker's parameter. Here, $Z_i e$ is the particle charge, $N_b = \int dXdYn_b$ is the number of beam particles per unit axial length, and $F_b(X,Y,X',Y',s)$ is the distribution function of the beam particles in the transverse phase space (X,Y,X',Y') appropriate to the Larmor frame [26]. Particularly useful in experimental applications and in numerical simulation models, such as the nonlinear δf-scheme [18], is an understanding of the self-consistent nonlinear evolution of various statistical averages [1,27,28], $\langle\chi\rangle = N_b^{-1}\int dXdYdX'dY'\chi F_b$, where χ is a phase function defined on the four-dimensional phase space (X,Y,X',Y'). Such models for the evolution of statistically averaged quantities have been developed and applied by Sacherer [27] for the case of an elliptical cross-section beam propagating through a periodic quadrupole lattice, by Lee and Cooper [28] for an axisymmetric beam propagating through a solenoidal focusing field, and by Struckmeier and Hofmann [17] for beam propagation through general periodic focusing systems.

The organization of this paper is the following. The theoretical model and assumptions are summarized in Sec. II. In Sec. III, the global rate equation is derived for general phase function $\chi(X, Y, X', Y', s)$, and the results are applied to investigate the nonlinear evolution of generalized entropy, mean canonical angular momentum $\langle P_\theta \rangle$, center-of-mass motion for $\langle X \rangle$ and $\langle Y \rangle$, mean kinetic energy $(1/2)\langle X'^2 + Y'^2 \rangle$, mean-square beam radius $\langle X^2 + Y^2 \rangle$, and coupled rate equations for the unnormalized transverse emittance $\epsilon(s)$ and root-mean-square beam radius $R_b(s) = \langle X^2 + Y^2 \rangle^{1/2}$. Here, $\epsilon(s)$ is defined by $(1/4)\epsilon^2(s) = \langle X'^2 + Y'^2 \rangle \langle X^2 + Y^2 \rangle - \langle XX' + YY' \rangle^2$. The rate equations obtained in Sec. III are derived from the fully nonlinear Vlasov-Poisson equations allowing for azimuthal asymmetries $(\partial/\partial\theta \neq 0)$, and are valid no matter how complex the nonlinear evolution of the system. In Sec. IV, following a discussion of global energy balance, and the rate equations for the special case where F_b corresponds to the Kapchinskij-Vladimirskij (KV) distribution [23], we examine the coupled rate equations for the unnormalized beam emittance $\epsilon(s)$ and rms beam radius $R_b(s)$ for the class of axisymmetric beam distributions F_b with fixed-shape density profile $n_b(R, s) = [N_b/\pi r_b^2(s)] f(R/r_b(s))$ [28]. Here, $R = (X^2 + Y^2)^{1/2}$ is the radial distance from the beam axis, $r_b(s)$ is the outer radius of the beam envelope, and the density shape function $f(R/r_b)$ is allowed to have general functional form. Most importantly, it is found that $d\epsilon(s)/ds = 0$, corresponding to emittance conservation for general density shape function $f(R/r_b(s))$, and that the envelope equation for the outer beam radius $r_b(s)$ is similar to the envelope equation [28] for a KV beam distribution [23], appropriately modified by a geometric factor g to reflect the shape of the function $f(R/r_b)$. This is similar to the result obtained by Lee and Cooper [28] for the case of axisymmetric beam propagation through a solenoidal focusing field and general density shape function $f(r/r_b)$.

II. THEORETICAL MODEL AND ASSUMPTIONS

We consider a thin, intense nonneutral beam with characteristic radius r_b and axial velocity $V_b = \beta_b c$ propagating in the z-direction through the periodic solenoidal focusing field [1]

$$\mathbf{B}^{sol}(\mathbf{x}) = B_z(s)\hat{\mathbf{e}}_z - \frac{1}{2} r B_z'(s)\hat{\mathbf{e}}_r. \tag{1}$$

Here, s is the axial coordinate, $r = (x^2 + y^2)^{1/2}$ is the radial distance from the beam axis, $B_z(s+S) = B_z(s)$ is the axial magnetic field with fundamental periodicity length $S = const.$, 'prime' denotes derivative with respect to s, $r_b \ll S$ is assumed, consistent with the thin-beam approximation, and $\gamma_b mc^2$ is the characteristic energy of a beam particle, where $\gamma_b = (1-\beta_b^2)^{-1/2}$. Consistent with the thin-beam approximation, the transverse momentum of a beam particle and the axial momentum spread are assumed to be small in comparison with the directed axial momentum $\gamma_b m \beta_b c$, where m is the rest mass, and c is the speed of light *in vacuo*. In addition, it is assumed that

$$\frac{\nu}{\gamma_b} = \frac{Z_i^2 e^2 N_b}{\gamma_b mc^2} \ll 1, \tag{2}$$

where ν is Budker's parameter, $Z_i e$ is the particle charge, $N_b = \int dxdy n_b$ is the number of particles per unit axial length, and $n_b(x,y,s)$ is the particle density. Equation (2) assures that the self-field intensity is sufficiently weak that $|Z_i e\phi^s/\gamma_b mc^2| \ll 1$, where ϕ^s is the electrostatic potential due to the beam space charge. However, the present analysis does permit the potential energy $Z_i e\phi^s$ to be comparable in magnitude with the transverse kinetic energy $(p_x^2 + p_y^2)/2\gamma_b m$ of a beam particle.

The present analysis is carried out in the electrostatic approximation, where the self-electric field produced by the beam space charge is $\mathbf{E}^s = -\nabla\phi^s$, and the electrostatic potential $\phi^s(x,y,s)$ is determined self-consistently from Poisson's equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi^s = -4\pi Z_i e n_b. \tag{3}$$

In Eq. (3), $n_b(x,y,s)$ is the particle density, and we have approximated $\nabla^2 \simeq \nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in the thin-beam approximation with $r_b \ll S$. In addition, the axial beam current $Z_i e n_b V_{zb}$, where $V_{zb}(x,y,s)$ is the average axial velocity, produces a transverse self-magnetic field $\mathbf{B}^s = \nabla \times A_z^s \hat{\mathbf{e}}_z$, where $A_z^s(x,y,s)$ is determined self-consistently from $\nabla_\perp^2 A_z = -4\pi Z_i e n_b V_{zb}$. In circumstances where the average axial velocity is approximately uniform over the beam cross section with $V_{zb} \simeq V_b = \beta_b c = const.$, which we assume to be the case, a comparison with Eq. (3) shows that the self-field potentials, $\phi^s(x,y,s)$ and $A_z^s(x,y,s)$, are related by the familiar expression [1]

$$A_z^s = \beta_b \phi^s. \tag{4}$$

Therefore, to summarize, in the thin-beam approximation the beam particles interact with the electric and magnetic fields, $\mathbf{E}^s$ and $\mathbf{B}$, described by

$$\mathbf{E}^s = -\nabla\phi^s(x,y,s),$$

$$\mathbf{B} = \mathbf{B}^{sol} + \mathbf{B}^s = B_z(s)\hat{\mathbf{e}}_z - \frac{1}{2} r B_z'(z)\hat{\mathbf{e}}_r + \beta_b \nabla\phi^s(x,y,s) \times \hat{\mathbf{e}}_z. \tag{5}$$

Here, $\mathbf{B}^{sol}(\mathbf{x})$ is the periodic solenoidal field defined in Eq. (1), and the electrostatic potential $\phi^s(x,y,s)$ is determined in terms of the particle density $n_b(x,y,s)$ from Poisson's equation (3).

In the present analysis, we make use of a kinetic approach based on the nonlinear Vlasov-Poisson equations [1,26] to describe the dynamics of the beam particles and their interaction with the field configuration in Eq. (5). In this regard, it is convenient to introduce the normalized Larmor frequency $\Omega_L(s)$ and the normalized electrostatic potential $\psi(x,y,s)$ defined by

$$\Omega_L(s) = -\sqrt{\kappa_z(s)} = -\frac{Z_i e B_z(s)}{2\gamma_b m \beta_b c^2},$$

$$\psi(x,y,s) = \frac{Z_i e}{\gamma_b^3 m \beta_b^2 c^2}\phi^s(x,y,s). \tag{6}$$

It is also convenient to transform to a frame of reference rotating about the beam axis at the local Larmor frequency $\Omega_L(s)$. We introduce the accumulated phase of rotation from s_0 to s defined by $\theta_L(s) = -\int_{s_0}^{s} ds\sqrt{\kappa_z(s)}$, where $d\theta_L/ds = \Omega_L$. Then the transverse orbits, $X(s)$ and $Y(s)$, in the rotating frame, are related to the transverse orbits, $x(s)$ and $y(s)$, in the laboratory frame by

$$
\begin{aligned}
X &= x\cos\theta_L(s) + y\sin\theta_L(s), \\
Y &= -x\sin\theta_L(s) + y\cos\theta_L(s).
\end{aligned} \tag{7}
$$

Finally, it is assumed that the beam particles have negligibly small spread in axial momentum about the average value $\gamma_b m\beta_b c$. Then, in the transverse phase space variables (X, Y, X', Y') appropriate to the Larmor frame, it can be shown that the distribution function $F_b(X, Y, X', Y', s)$ evolves according to the nonlinear Vlasov equation [26]

$$\frac{\partial F_b}{\partial s} + X'\frac{\partial F_b}{\partial X} + Y'\frac{\partial F_b}{\partial Y} - \left(\kappa_z(s)X + \frac{\partial\psi}{\partial X}\right)\frac{\partial F_b}{\partial X'} - \left(\kappa_z(s)Y + \frac{\partial\psi}{\partial Y}\right)\frac{\partial F_b}{\partial Y'} = 0, \tag{8}$$

where the normalized potential $\psi(X, Y, s)$ is determined self-consistently from Poisson's equation

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right)\psi = -\frac{2\pi K}{N_b}\int dX'dY'F_b. \tag{9}$$

Here, $n_b(X, Y, s) = \int dX'dY'F_b$ is the particle density, $N_b = \int dXdYn_b$ is the number of particles per unit axial length, and we have introduced the self-field perveance K defined by [1,26]

$$K = \frac{2N_b Z_i^2 e^2}{\gamma_b^3 m\beta_b^2 c^2}, \tag{10}$$

which is a (dimensionless) measure of the self-field intensity. Note in Eq. (8) that $X'$and Y' correspond to normalized velocity variables in the $X-Y$ plane (i.e., X' denotes dX/ds and Y' denotes dY/ds), and the coefficients of $\partial F_b/\partial X'$ and $\partial F_b/\partial Y'$ correspond to the particle accelerations in the X- and Y-directions, respectively.

The Vlasov-Poisson equations (8) and (9) constitute the basic dynamical equations used in the present analysis. They describe, in the Larmor frame, the nonlinear evolution of the charged particle beam as it propagates through the periodic solenoidal field $\kappa_z(s+S) = \kappa_z(s)$. In particular, Eq. (8) describes the incompressible evolution of the distribution function $F_b(X, Y, X', Y', s)$ in the four-dimensional phase space (X, Y, X', Y'), and Eq. (9) determines self-consistently the normalized potential $\psi(X, Y, s)$ in terms of the particle density $n_b(X, Y, s) = \int dX'dY' F_b$. In subsequent sections, we make use of Eqs. (8) and (9) to investigate the evolution of various global (statistically averaged) quantities of physical interest. In this regard, when carrying out averages of the Vlasov equation (8) over the phase space (X, Y, X', Y'), we assume that a perfectly conducting cylindrical wall is located at radius $r = R = (X^2 + Y^2)^{1/2} = r_w$, and impose the boundary condition

$$\left[\frac{1}{R}\frac{\partial}{\partial\theta}\psi(R, \theta, s)\right]_{R=r_w} = 0, \tag{11}$$

which corresponds to zero tangential electric field at the conducting wall. Here, (R, θ) correspond to cylindrical coordinates in the Larmor frame defined by $X = R\cos\theta$ and $Y = R\sin\theta$. In addition, it is assumed that the distribution function $F_b(X, Y, X', Y', s)$ satisfies

$$F_b = 0, \quad \text{for } X' \to \pm\infty \text{ or } Y' \to \pm\infty, \tag{12}$$

and that there are no beam particles beyond some radius r_0, i.e.,

$$F_b = 0, \quad \text{for } (X^2 + Y^2)^{1/2} \geq r_0, \quad \text{where } r_0 < r_w. \tag{13}$$

Note that Eq. (13) implies that the beam density n_b is zero in the vacuum region $r_0 \leq R \leq r_w$.

As a final point in concluding this section, it should be noted that the characteristics of the nonlinear Vlasov equation (8) correspond to the single-particle equations of motion, e.g., $X'(s) = dX(s)/ds$ and $dX'(s)/ds = -\kappa_z(s)X - \partial\psi/\partial X$ for the X-motion, and similar equations for the Y-motion. Indeed, these equations of motion can be derived from the Hamiltonian $H_\perp(X, Y, X', Y', s)$ defined by

$$H_\perp = \frac{1}{2}(X'^2 + Y'^2) + \frac{1}{2}\kappa_z(s)(X^2 + Y^2) + \psi(X, Y, s). \tag{14}$$

Because $\kappa_z(s)$ is s-dependent for a periodic focusing lattice, it is clear from Eq. (14) that $H_\perp$ is *not* a single-particle constant of the motion. Therefore, it is not expected that total energy (kinetic energy plus potential energy plus self-field energy) will be globally conserved by the nonlinear Vlasov-Poisson equations (8) and (9).

III. GLOBAL RATE EQUATIONS AND CONSERVATION RELATIONS

We now make use of the Vlasov-Poisson equations (8) and (9), together with the boundary conditions in Eqs. (11)–(13), to derive rate equations and conservation relations that describe the nonlinear dynamics of the beam and its interaction with the field configuration in Eq. (5). In this regard, the statistical average of a phase function $\chi(X,Y,X',Y',s)$ over the four-dimensional phase space (X,Y,X',Y') is denoted by $\langle\chi\rangle$ and is defined in the usual manner by [26–28]

$$\langle\chi\rangle = \frac{1}{N_b}\int dXdYdX'dY'\chi F_b. \qquad (15)$$

Here, $N_b = \int dXdYn_b = \int dXdYdX'dY'F_b$ is the number of particles per unit axial length. The phase-space integral in Eq. (15) can also be expressed in cylindrical coordinates as $\int dXdYdX'dY'\cdots = \int_0^{2\pi}d\theta\int_0^{r_w}dRR\int_{-\infty}^{\infty}dX'\int_{-\infty}^{\infty}dY'\cdots$. The most basic conservation relation evident from Eq. (8) corresponds to the conservation of the total number of particles per unit axial length. Operating on Eq. (8) with $\int dXdYdX'dY'\cdots$, integrating by parts with respect to X, Y, X' and Y', and making use of Eqs. (12) and (13), readily gives

$$\frac{d}{ds}N_b = \int dXdYdX'dY'\frac{\partial F_b}{\partial s} = 0. \qquad (16)$$

Equation (16) is simply a statement that $N_b = const.$, no matter how complicated the nonlinear evolution of the system.

For general phase function $\chi(X,Y,X',Y',s)$, it follows from the definition of statistical average in Eq. (15) that

$$\frac{d}{ds}\langle\chi\rangle = \langle\frac{\partial\chi}{\partial s}\rangle + \int dXdYdX'dY'\chi\frac{\partial F_b}{\partial s}. \qquad (17)$$

Multiplying the Vlasov equation (8) by χ, operating with $N_b^{-1}\int dXdYdX'dY'\ldots$, integrating by parts with respect to X, Y, X' and Y', and making use of Eqs. (12) and (13), the final term in Eq. (17) can be simplified. This gives

$$\frac{d}{ds}\langle\chi\rangle = \left\langle\frac{\partial\chi}{\partial s} + X'\frac{\partial\chi}{\partial X} + Y'\frac{\partial\chi}{\partial Y} - \left(\kappa_z(s)X + \frac{\partial\psi}{\partial X}\right)\frac{\partial\chi}{\partial X'} - \left(\kappa_z(s)Y + \frac{\partial\psi}{\partial Y}\right)\frac{\partial\chi}{\partial Y'}\right\rangle. \qquad (18)$$

The general rate equation (18) can be used to evaluate $(d/ds)\langle\chi\rangle$ for a wide variety of choices of phase function χ of physical interest.

Entropy Conservation: It is important to note that the total derivative operation on χ within the angular brackets on the right-hand side of Eq. (18) is identical to the total derivative operation on F_b in the nonlinear Vlasov equation (8). Furthermore, for smooth, differentiable $G(F_b)$, it follows from Eq. (8) that

$$\left\{\frac{\partial}{\partial s} + X'\frac{\partial}{\partial X} + Y'\frac{\partial}{\partial Y} - \left(\kappa_z(s)X + \frac{\partial\psi}{\partial X}\right)\frac{\partial}{\partial X'} - \left(\kappa_z(s)Y + \frac{\partial\psi}{\partial Y}\right)\frac{\partial}{\partial Y'}\right\} G(F_b) = 0. \quad (19)$$

Making use of Eq. (18), or operating directly on Eq. (19) with $\int dXdYdX'dY'\cdots$ readily gives

$$\frac{d}{ds}\int dXdYdX'dY'G(F_b) = 0. \qquad (20)$$

That is, any smooth, differentiable function $G(F_b)$ integrated over the four-dimensional phase space (X, Y, X', Y') is a globally conserved quantity. The case $G(F_b) = F_b$ corresponds simply to $dN_b/ds = 0$ in Eq. (16). Many other choices of $G(F_b)$, such as F_b^2, $-F_b\ell nF_b$, etc., are also globally conserved quantities. For example, using the standard definition of entropy S it follows from Eq. (20) that

$$\frac{d}{ds}S = -\frac{d}{ds}\int dXdYdX'dY'F_b\ell nF_b = 0. \qquad (21)$$

Equations (20) and (21) are expected results because the nonlinear collective processes contained in the Vlasov-Poisson equations (8) and (9) are known to conserve both entropy [Eq. (21)] and generalized entropy [Eq. (20)] no matter how complicated the evolution of the system. Nonetheless, Eq. (20) represents a powerful global constraint on the nonlinear dynamics of the beam.

Conservation of Canonical Angular Momentum: In the normalized Larmor-frame variables used in the present analysis, the canonical angular momentum P_θ of an individual particle is defined by [26]

$$P_\theta = XY' - YX'. \qquad (22)$$

Substituting $\chi = XY' - YX'$ into Eq. (18) readily gives

$$\frac{d}{ds}\langle XY' - YX' \rangle = \left\langle Y\frac{\partial\psi}{\partial X} - X\frac{\partial\psi}{\partial Y}\right\rangle = -\left\langle \frac{\partial\psi}{\partial\theta}\right\rangle. \tag{23}$$

Here, use has been made of $\partial/\partial X = \cos\theta(\partial/\partial R) - R^{-1}\sin\theta(\partial/\partial\theta)$ and $\partial/\partial Y = \sin\theta(\partial/\partial R) + R^{-1}\cos\theta(\partial/\partial\theta)$ in cylindrical coordinates (R,θ), where $X = R\cos\theta$ and $Y = R\sin\theta$. To simplify the right-hand side of Eq. (23) it is convenient to express Poisson's equation (9) in cylindrical coordinates, i.e.,

$$\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2} = -\frac{2\pi K}{N_b}n_b, \tag{24}$$

where $n_b(R,\theta,s) = \int dX'dY'F_b$ is the density of beam particles. From Eqs. (15) and (24) it follows that

$$\left\langle \frac{\partial\psi}{\partial\theta}\right\rangle = \frac{1}{N_b}\int_0^{2\pi} d\theta \int_0^{r_w} dRR\frac{\partial\psi}{\partial\theta}n_b$$

$$= -\frac{1}{2\pi K}\int_0^{2\pi} d\theta \int_0^{r_w} dRR\frac{\partial\psi}{\partial\theta}\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}\right)$$

$$= -\frac{1}{2\pi K}\int_0^{2\pi} d\theta \int_0^{r_w} dRR\left(-\frac{\partial\psi}{\partial R}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial\psi}{\partial\theta}\frac{\partial}{\partial\theta}\frac{\partial\psi}{\partial\theta}\right). \tag{25}$$

In Eq. (25), we have integrated by parts with respect to R and made use of $[\partial\psi/\partial\theta]_{R=r_w} = 0$ [see Eq. (11)]. Next, we integrate by parts with respect to θ on the right-hand side of Eq. (25), and make use of the fact that $\partial\psi/\partial R$ and $\partial\psi/\partial\theta$ are periodic functions of θ with azimuthal period 2π. This readily gives $\langle\partial\psi/\partial\theta\rangle = 0$, and Eq. (23) reduces to

$$\frac{d}{ds}\langle P_\theta\rangle = \frac{d}{ds}\langle XY' - YX'\rangle = 0. \tag{26}$$

Equation (26) corresponds to global conservation of canonical angular momentum, also a very powerful constraint on the nonlinear evolution of the system. Note that the boundary condition in Eq. (11), corresponding to a perfectly conducting wall located at radius $R = r_w$, has played an important role in assuring that $\langle\partial\psi/\partial\theta\rangle = 0$ and hence that Eq. (26) is satisfied. Note also that it has *not* been assumed that $\partial F_b/\partial\theta$ and $\partial\psi/\partial\theta$ are zero (azimuthally symmetric beam) in deriving the conservation relation in Eq. (26). That is, the beam can (in principle) develop large-amplitude azimuthal distortions, and the global conservation of canonical angular momentum in Eq. (26) will remain a valid conservation constraint.

Center-of-Mass Motion: Substituting $\chi = X$ in Eq. (18) readily gives

$$\frac{d}{ds}\langle X\rangle = \langle X'\rangle, \tag{27}$$

and similarly, for $\chi = X'$, we obtain from Eq. (18)

$$\frac{d}{ds}\langle X'\rangle = -\kappa_z(s)\langle X\rangle - \left\langle\frac{\partial\psi}{\partial X}\right\rangle. \tag{28}$$

Substituting Eq. (27) into Eq. (28) then gives

$$\frac{d^2}{ds^2}\langle X\rangle + \kappa_z(s)\langle X\rangle = -\left\langle\frac{\partial\psi}{\partial X}\right\rangle. \tag{29}$$

Equations identical in form to Eqs. (27) and (28) can be obtained for the average Y-motion, i.e., $(d/ds)\langle Y\rangle = \langle Y'\rangle$ and $(d/ds)\langle Y'\rangle = -\kappa_z\langle Y\rangle - \langle\partial\psi/\partial Y\rangle$, which can be combined to give

$$\frac{d^2}{ds^2}\langle Y\rangle + \kappa_z(s)\langle Y\rangle = -\left\langle\frac{\partial\psi}{\partial Y}\right\rangle. \tag{30}$$

Equations (29) and (30) describe the evolution of the beam centroid $(\langle X\rangle, \langle Y\rangle)$ in response to the periodic focusing magnetic field $\kappa_z(s+S) = \kappa_z(s)$ and the average self-field components $-\langle\partial\psi/\partial X\rangle$ and $-\langle\partial\psi/\partial Y\rangle$. Evidently, if the beam is initially centered (at $s = s_0$, say) with $\langle X\rangle_{s=s_0} = 0 = \langle Y\rangle_{s=s_0}$ and $\langle X'\rangle_{s=s_0} = 0 = \langle Y'\rangle_{s=s_0}$, then it follows from Eqs. (27)–(30) that $\langle X\rangle = 0 = \langle Y\rangle$ at subsequent values of s.

Rate equations for the evolution of other average quantities are also readily obtained. For example, substituting $\chi = X^n$, where n is an integer, into Eq. (18) gives

$$\frac{d}{ds}\langle X^n\rangle = n\langle X'X^{n-1}\rangle. \tag{31}$$

Similarly, it follows from Eq. (18) that

$$\frac{d}{ds}\langle X'^m\rangle = -\kappa_z(s)m\langle XX'^{m-1}\rangle - m\left\langle X'^{m-1}\frac{\partial\psi}{\partial X}\right\rangle, \tag{32}$$

and

$$\frac{d}{ds}\langle X^n X'^m\rangle = n\langle X^{n-1}X'^{m+1}\rangle - \kappa_z(s)m\langle X^{n+1}X'^{m-1}\rangle - m\left\langle X'^{m-1}X^n\frac{\partial\psi}{\partial X}\right\rangle. \tag{33}$$

Analogous equations for the evolution of $\langle Y^n\rangle$, $\langle Y'^m\rangle$, $\langle Y^nY'^m\rangle$, $\langle X^nY^m\rangle$, etc. are also readily obtained but will not be presented here.

Two quantities of particular physical interest are the mean kinetic energy $(1/2)\langle X'^2 + Y'^2\rangle$ and the mean-square beam radius $\langle X^2+Y^2\rangle$. We now make use of the general rate equation (18) to derive equations for the evolution of $(1/2)\langle X'^2+Y'^2\rangle$ and $\langle X^2+Y^2\rangle$.

<u>Rate Equation for Mean Kinetic Energy $(1/2)\langle X'^2+Y'^2\rangle$:</u> Substituting $\chi = (1/2)(X'^2+Y'^2)$ into Eq. (18), we obtain

$$\frac{d}{ds}\frac{1}{2}\langle X'^2+Y'^2\rangle = -\kappa_z(s)\langle XX'+YY'\rangle - \left\langle X'\frac{\partial\psi}{\partial X}+Y'\frac{\partial\psi}{\partial Y}\right\rangle. \tag{34}$$

It also readily follows from Eq. (18) that $(d/ds)\langle X^2+Y^2\rangle = 2\langle XX'+YY'\rangle$, so that Eq. (34) can be expressed in the equivalent form

$$\frac{d}{ds}\frac{1}{2}\langle X'^2+Y'^2\rangle = -\kappa_z(s)\frac{d}{ds}\frac{1}{2}\langle X^2+Y^2\rangle - \left\langle X'\frac{\partial\psi}{\partial X}+Y'\frac{\partial\psi}{\partial Y}\right\rangle. \tag{35}$$

The first term on the right-hand side of Eq. (35) is related to the change in potential energy associated with the particle motion in the periodic solenoidal focusing field, whereas the second term on the right-hand side is related to the change in self-field energy.

To simplify the self-field contribution on the right-hand side of Eq. (35), we make use of the generalized rate equation (18) with $\chi = \psi(X,Y,s)$ and the definition of statistical average in Eq. (15) to write

$$\begin{aligned}\left\langle X'\frac{\partial\psi}{\partial X}+Y'\frac{\partial\psi}{\partial Y}\right\rangle &= \frac{d}{ds}\langle\psi\rangle - \left\langle\frac{\partial\psi}{\partial s}\right\rangle\\ &= \frac{1}{N_b}\int dXdYdX'dY'\psi\frac{\partial F_b}{\partial s}\\ &= \frac{1}{N_b}\int dXdY\psi\frac{\partial}{\partial s}\int dX'dY'F_b.\end{aligned} \tag{36}$$

The beam density $n_b(X,Y,s) = \int dX'dY'F_b$ can be eliminated in Eq. (36) in terms of $\nabla_\perp^2\psi = (\partial^2/\partial X^2 + \partial^2/\partial Y^2)\psi$ by means of Poisson's equation (9). This gives, after some integration by parts,

$$\left\langle X'\frac{\partial\psi}{\partial X} + Y'\frac{\partial\psi}{\partial Y}\right\rangle = -\frac{1}{2\pi K}\int dXdY\psi\frac{\partial}{\partial s}\nabla_\perp^2\psi = \frac{1}{4\pi K}\frac{\partial}{\partial s}\int dXdY|\nabla_\perp\psi|^2. \quad (37)$$

In simplifying Eq. (37), we have expressed $\int dXdY\cdots = \int_0^{2\pi} d\theta \int_0^{r_w} dRR\cdots$ and integrated by parts with respect to R and θ, enforcing the boundary conditions that $[\partial\psi/\partial\theta]_{R=r_w} = 0$ and $[\psi]_{R=r_w} = const.$ at the conducting wall. (For convenience, the reader may wish to assume that the conducting wall is grounded, with $[\psi]_{R=r_w} = 0$.) In the normalized units used here,

$$\mathcal{E}_F(s) \equiv \frac{1}{4\pi K}\int dXdY|\nabla_\perp\psi|^2, \quad (38)$$

will be recognized as the self-field energy of the beam particles.

Substituting Eqs. (37) and (38) into Eq. (35), the rate equation (35) for the change in mean kinetic energy can be expressed as

$$\frac{d}{ds}\frac{1}{2}\langle X'^2 + Y'^2\rangle = -\kappa_z(s)\frac{d}{ds}\frac{1}{2}\langle X^2 + Y^2\rangle - \frac{d}{ds}\mathcal{E}_F(s), \quad (39)$$

or equivalently,

$$\frac{d}{ds}\left\{\frac{1}{2}\langle X'^2 + Y'^2\rangle + \frac{1}{2}\kappa_z(s)\langle X^2 + Y^2\rangle + \mathcal{E}_F(s)\right\} = \frac{1}{2}\frac{d\kappa_z(s)}{ds}\langle X^2 + Y^2\rangle. \quad (40)$$

The left-hand side of Eq. (40) will be recognized as the rate of change of mean total energy of the system. Indeed making use of the definition of $H_\perp$ in Eq. (14), and recognizing that the field energy can also be expressed as $\mathcal{E}_F(s) = \langle\psi\rangle$ [see Eqs. (15), (24), and (38)], Eq. (40) can be expressed in the equivalent form $(d/ds)\langle H_\perp\rangle = (1/2)(d\kappa_z/ds)\langle X^2 + Y^2\rangle$. It is clear from Eq. (40) that the mean total energy of the system is conserved whenever $d\kappa_z(s)/ds = 0$. For example, this is true for the case of a uniform solenoidal focusing field with $\kappa_z(s) = \kappa_{z0} = const.$ (for all s). It is also true for a periodic step-function lattice in the localized regions of s-space where $\kappa_z = 0$ or $\kappa_z = \kappa_{z0} = const.$

As a final point in concluding this subsection, it should be emphasized that rate equations in Eq. (35) and Eq. (39) [or Eq. (40)] are fully equivalent.

<u>**Rate Equation for Mean-Square Beam Radius $\langle X^2 + Y^2\rangle$:**</u> Substituting $\chi = X^2 + Y^2$ into the generalized rate equation (18) gives

$$\frac{d}{ds}\langle X^2+Y^2\rangle = 2\langle XX'+YY'\rangle. \tag{41}$$

Furthermore, substituting $\chi = XX' + YY'$ into Eq. (18) and carrying out the required derivative operations, we obtain

$$\begin{aligned}\frac{d}{ds}\langle XX'+YY'\rangle &= \frac{d^2}{ds^2}\frac{1}{2}\langle X^2+Y^2\rangle \\ &= \langle X'^2+Y'^2\rangle - \kappa_z(s)\langle X^2+Y^2\rangle - \left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle.\end{aligned} \tag{42}$$

Equation (42) and Eq. (35) [or Eq. (39)] represent coupled rate equations for the evolution of $\langle X^2+Y^2\rangle$ and $\langle X'^2+Y'^2\rangle$.

For future reference, it is useful to simplify the final term on the right-hand side of Eq. (42). Making use of the representation $\langle X\partial\psi/\partial X + Y\partial\psi/\partial Y\rangle = \langle R\partial\psi(R,\theta,s)/\partial R\rangle$ in cylindrical coordinates, and the definition of statistical average in Eq. (15), we obtain

$$\begin{aligned}\left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle &= \frac{1}{N_b}\int_0^{2\pi} d\theta \int_0^{r_w} dRRR\frac{\partial\psi}{\partial R}n_b \\ &= -\frac{1}{2\pi K}\int_0^{2\pi} d\theta \int_0^{r_w} dRRR\frac{\partial\psi}{\partial R}\left[\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\theta^2}\right] \\ &= -\frac{1}{2\pi K}\int_0^{2\pi} d\theta \int_0^{r_w} dR\frac{1}{2}\left[\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right)^2 - \frac{\partial}{\partial R}\left(\frac{\partial\psi}{\partial\theta}\right)^2\right].\end{aligned} \tag{43}$$

Here, use has been made of Poisson's equation (24) to eliminate $n_b(R,\theta,s)$ in favor of $\nabla_\perp^2\psi$ in cylindrical coordinates, and the term proportional to $\partial^2\psi/\partial\theta^2$ has been integrated by parts with respect to θ, making use of $\psi(R,\theta+2\pi,s) = \psi(R,\theta,s)$. Integrating by parts with respect to R in Eq. (43), and making use of $[\partial\psi/\partial\theta]_{R=r_w} = 0$ [Eq. (11)], we obtain

$$\left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle = -\frac{1}{4\pi K}\int_0^{2\pi} d\theta \left[R\frac{\partial\psi}{\partial R}\right]^2_{R=r_w}. \tag{44}$$

It is convenient to express $\psi(R,\theta,s) = \bar{\psi}(R,s) + \delta\psi(R,\theta,s)$, where $\bar{\psi} \equiv (2\pi)^{-1}\int_0^{2\pi} d\theta\psi$ and $\int_0^{2\pi} d\theta\delta\psi = 0$. If ψ is azimuthally symmetric ($\partial\psi/\partial\theta = 0$), then $\psi = \bar{\psi}$ and $\delta\psi = 0$. In general, however, there may be (instability-induced) perturbations in the system, in which case $\delta\psi(R,\theta,s)$ is non-zero. Therefore, in the general case, Eq. (44) can be expressed as

$$\left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle = -\frac{1}{4\pi K}\int_0^{2\pi} d\theta \left\{\left[R\frac{\partial\bar{\psi}}{\partial R}\right]^2_{R=r_w} + \left[R\frac{\partial\delta\psi}{\partial R}\right]^2_{R=r_w}\right\}. \tag{45}$$

The first term on the right-hand side of Eq. (45) is associated with the dc space charge of the beam and can be evaluated in closed form (see below). The second term on the right-hand side of Eq. (45) involves the perturbed charge density and is typically much smaller than the first term. To evaluate $[R\partial\bar{\psi}/\partial R]_{R=r_w}$ in Eq. (45), we operate on Poisson's equation (24) with $\int_0^{2\pi} d\theta \int_0^{r_w} dRR\ldots$ and make use of $\int_0^{2\pi} d\theta \int_0^{r_w} dRRn_b(R,\theta,s) = N_b$. This readily gives

$$\left[R\frac{\partial}{\partial R}\bar{\psi}\right]_{R=r_w} = -K, \tag{46}$$

where K is the self-field perveance defined in Eq. (10). Substituting Eq. (46) into Eq. (45) then gives

$$\left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle = -\frac{1}{2}K(1+\Delta), \tag{47}$$

where Δ is defined by

$$\Delta \equiv \frac{1}{K^2}\int_0^{2\pi}\frac{d\theta}{2\pi}\left[R\frac{\partial}{\partial R}\delta\psi\right]^2_{R=r_w}. \tag{48}$$

Substituting Eq. (47) into Eq. (42), it is found that

$$\frac{d^2}{ds^2}\frac{1}{2}\langle X^2+Y^2\rangle + \kappa_z(s)\langle X^2+Y^2\rangle - \frac{1}{2}K(1+\Delta) = \langle X'^2+Y'^2\rangle. \tag{49}$$

Equation (49) describes the evolution of $\langle X^2+Y^2\rangle$ and is fully equivalent to Eq. (42). Here, the nonlinear evolution of the mean-square radius [Eq. (49) or Eq. (42)] is coupled to the evolution of the mean kinetic energy $(1/2)\langle X'^2+Y'^2\rangle$ [Eq. (39) or Eq. (35)]. Particularly important in Eq. (49) is that the self-field perveance $K = const.$, no matter how complicated the nonlinear evolution of the system. Furthermore, the term proportional to Δ in Eq. (49) is either zero ($\Delta = 0$ for an azimuthally symmetric beam), or small ($\Delta \ll 1$) in many regimes of practical interest. Equation (49) [or, equivalently, Eq. (42)] is an exact consequence of the fully nonlinear Vlasov-Maxwell equations (8) and (9).

<u>**Coupled Rate Equations for rms Emittance $\epsilon(s)$ and rms Beam Radius $R_b(s)$:**</u> For future reference, it is convenient to rewrite the coupled rate equations for the mean kinetic energy $(1/2)\langle X'^2+Y'^2\rangle$ in Eq. (35) [or Eq. (39)] and the mean-square beam radius

$\langle X^2 + Y^2 \rangle$ in Eq. (42) [or Eq. (49)] in terms of the root-mean square beam radius $R_b(s)$ defined by [27,28]

$$R_b(s) = \langle X^2 + Y^2 \rangle^{1/2}, \tag{50}$$

and the unnormalized beam emittance $\epsilon(s)$ defined by [1,26–28]

$$\begin{aligned}\epsilon^2(s) &= 4[\langle X'^2 + Y'^2 \rangle \langle X^2 + Y^2 \rangle - \langle XX' + YY' \rangle^2] \\ &= 4\left[\langle X'^2 + Y'^2 \rangle R_b^2 - \left(\frac{1}{2}\frac{d}{ds}R_b^2\right)^2\right].\end{aligned} \tag{51}$$

In Eq. (51), use has been made of $R_b^2 = \langle X^2 + Y^2 \rangle$ and $\langle XX' + YY' \rangle = (d/ds)(1/2)\langle X^2 + Y^2 \rangle = (d/ds)(R_b^2/2)$. Substituting Eq. (51) into Eq. (42) and eliminating $\langle X'^2 + Y'^2 \rangle$ in favor of $\epsilon^2(s)$, some straightforward algebra shows that Eq. (42) can be expressed as

$$\frac{d^2}{ds^2}R_b(s) + \kappa_z(s)R_b(s) = \frac{\epsilon^2(s)/4}{R_b^3(s)} - \frac{1}{R_b(s)}\left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle. \tag{52}$$

Equation (52) is fully equivalent to Eq. (42). If we further express $\langle X\partial\psi/\partial X + Y\partial\psi/\partial Y \rangle$ in terms of the self-field perveance by means of Eq. (47), then Eq. (52) for the rms beam radius $R_b(s)$ can also be expressed as

$$\frac{d^2}{ds^2}R_b(s) + \left[\kappa_z(s) - \frac{(K/2)(1+\Delta)}{R_b^2(s)}\right]R_b(s) = \frac{\epsilon^2(s)/4}{R_b^3(s)}. \tag{53}$$

Equation (53) is fully equivalent to Eqs. (52) and (42), and has been derived from the *fully nonlinear Vlasov-Poisson equations* (8) and (9) for *general* beam distribution function $F_b(X, Y, X', Y', s)$ and self-consistent normalized potential $\psi(X, Y, s)$ defined in Eq. (6). In deriving Eq. (53), no a priori assumption has been made that $\partial F_b/\partial\theta = 0$ or $\partial\psi/\partial\theta = 0$. The striking feature of Eq. (53) is that it is similar in form to the analogous equation for $R_b(s)$ derived for the restrictive assumption of a Kapchinskij-Vladimirskij beam distribution [23] F_b^{KV} [which has *uniform* beam density $n_b = const.$ for $0 \leq R < r_b(s) \equiv \sqrt{2}R_b(s)$, and $n_b = 0$ for $R > r_b(s)$], in which case $\epsilon(s) = const.$, and $\Delta = 0$ because $(\partial/\partial\theta)F_b^{KV} = 0$ and $\partial\psi/\partial\theta = 0$.

The rate equations (35) and (42) can be used to describe the nonlinear evolution of the unnormalized rms emittance $\epsilon(s)$ defined in Eq. (51). From Eq. (51), we obtain

$$\frac{d}{ds}\frac{1}{4}\epsilon^2(s) = \langle X^2+Y^2\rangle \frac{d}{ds}\langle X'^2+Y'^2\rangle$$
$$+\langle X'^2+Y'^2\rangle \frac{d}{ds}\langle X^2+Y^2\rangle - 2\langle XX'+YY'\rangle \frac{d}{ds}\langle XX'+YY'\rangle. \qquad (54)$$

Substituting Eqs. (35) and (42) into Eq. (54) and rearranging terms, it can be shown that Eq. (54) reduces to

$$\frac{d}{ds}\frac{1}{8}\epsilon^2(s) = \frac{1}{2}\frac{d}{ds}\langle X^2+Y^2\rangle \left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle - \langle X^2+Y^2\rangle \left\langle X'\frac{\partial\psi}{\partial X} + Y'\frac{\partial\psi}{\partial Y}\right\rangle, \qquad (55)$$

or equivalently,

$$\frac{d}{ds}\frac{1}{8}\epsilon^2(s) = R_b^2\left\{\frac{1}{R_b}\frac{dR_b}{ds}\left\langle X\frac{\partial\psi}{\partial X} + Y\frac{\partial\psi}{\partial Y}\right\rangle - \left\langle X'\frac{\partial\psi}{\partial X} + Y'\frac{\partial\psi}{\partial Y}\right\rangle\right\}. \qquad (56)$$

If we make use of Eqs. (37), (38), and (47), then Eq. (56) can also be expressed as

$$\frac{d}{ds}\frac{1}{8}\epsilon^2(s) = R_b^2\left\{-\frac{1}{R_b}\frac{dR_b}{ds}\frac{1}{2}K(1+\Delta) - \frac{d}{ds}\mathcal{E}_F\right\}, \qquad (57)$$

where $\mathcal{E}_F(s)$ is the self-field energy defined in Eq. (38).

To summarize, the rate equation (52) [or Eq. (53)] for the rms beam radius $R_b(s)$ together with the rate equation (56) [or Eq. (57)] for the unnormalized beam emittance $\epsilon(s)$ are fully equivalent to the original rate equations for the mean-square radius $\langle X^2+Y^2\rangle$ in Eq. (42) and mean kinetic energy $(1/2)\langle X'^2+Y'^2\rangle$ in Eq. (35). These equations have been derived from the fully nonlinear Vlasov-Poisson equations (8) and (9) and are valid no matter how complex the nonlinear evolution of the system. Which of these representations is best to use depends in practice on the particular application under consideration. In the remainder of this paper, we will use primarily (but not exclusively) the rate equations for $R_b(s)$ and $\epsilon(s)$.

IV. APPLICATION OF GLOBAL RATE EQUATIONS

The global rate equations and conservation relations derived in Sec. III have been obtained from the nonlinear Vlasov-Poisson equations (8) and (9) and are applicable over a wide range of system parameters consistent with the assumptions enumerated in Sec. II. As such, the results obtained in Sec. III place very powerful global constraints on the nonlinear evolution of the system and can be used to benchmark numerical simulation codes as well as obtain valuable insights regarding the nonlinear beam dynamics and collective processes affecting its propagation. In this section, we make use of the global rate equations derived in Sec. III to investigate several aspects of beam propagation in regimes of practical interest. These include the energy rate equation for matched beam propagation (Sec. IV.A), and the coupled rate equations for the rms beam radius $R_b(s)$ and unnormalized beam emittance $\epsilon(s)$ for azimuthally symmetric beam propagation ($\partial/\partial\theta = 0$), corresponding to the Kapchinskij-Vladimirskij distribution [23] (Sec. IV.B), and beam distributions with fixed-shape density profile (Sec. IV.C).

A. Global Energy Balance

The global energy balance equation can be expressed in the three equivalent forms given in Eqs. (35), (39) and (40). For present purposes, we make use of the form given in Eq. (40), where $\mathcal{E}_F(s)$ is the normalized self-field energy defined in Eq. (38), and the coupling coefficient $\kappa_z(s+S) = \kappa_z(s)$ for the solenoidal field is assumed to have fundamental periodicity length S. The energy balance equation (40) is valid no matter how complex the nonlinear evolution of the system. For example, if $\mathcal{E}_F(s)$ exhibits growth and saturates nonlinearly due to a collective instability, then the mean kinetic energy $(1/2)\langle X'^2 + Y'^2\rangle$ and the mean-square beam radius $\langle X^2 + Y^2\rangle$ must adjust self-consistently according to Eq. (40). Evidently, in the special case of a uniform focusing field with

$$\kappa_z(s) = \kappa_{z0} = const., \tag{58}$$

the total energy is conserved according to Eq. (40). In this case $d\kappa_z(s)/ds = 0$, and Eq. (40) reduces to

$$\frac{1}{2}\langle X'^2 + Y'^2\rangle + \frac{1}{2}\kappa_{z0}\langle X^2 + Y^2\rangle + \mathcal{E}_F(s) = const., \tag{59}$$

which is a statement of energy conservation between mean kinetic energy, potential energy, and self-field energy. When $\kappa_z(s) = \kappa_{z0} = const.$, other rate equations, such as Eq. (53) for the rms beam radius $R_b(s)$ also undergo corresponding simplifications.

For a periodic focusing field $\kappa_z(s+S) = \kappa_z(s)$, examination of Eq. (40) shows that the total mean energy $U(s)$ defined by

$$U(s) = \frac{1}{2}\langle X'^2 + Y'^2\rangle + \frac{1}{2}\kappa_z(s)\langle X^2 + Y^2\rangle + \mathcal{E}_F(s) \tag{60}$$

also varies periodically for quiescent (instability-free) propagation of a *matched beam* with $R_b(s+S) = R_b(s)$, where $R_b^2(s) = \langle X^2 + Y^2\rangle$ is the mean-square beam radius. For present purposes, we further assume that the functional form of $\kappa_z(s)$ and the rms beam radius $R_b(s)$ have *half-period symmetry* of the form

$$\begin{aligned} \kappa_z(S/2 - s) &= \kappa_z(S/2 + s), \\ R_b(S/2 - s) &= R_b(S/2 + s), \end{aligned} \tag{61}$$

for $0 \leq s \leq S/2$, and all subsequent lattice intervals. A case in point is illustrated in Fig. 1 for a periodic step-function lattice where

$$\kappa_z(s) = \begin{cases} \kappa_{z0} = const., & 0 \leq s < (\eta/2)S, \\ 0, & (\eta/2)S < s < (1-\eta/2)S, \\ \kappa_{z0} = const., & (1-\eta/2)S < s \leq S, \end{cases} \tag{62}$$

over the fundamental interval $0 \leq s \leq S$. Of course the functional form of $\kappa_z(s)$ in Eq. (62) and Fig. 1 repeats for each subsequent lattice interval because $\kappa_z(s+S) = \kappa_z(s)$. We integrate the energy balance equation (40) over one lattice period by operating on Eq. (40) with $\int_s^{s+S} ds \cdots$. Because $d\kappa_z/ds$ has odd-function symmetry over a half-lattice interval, whereas $R_b^2(s) = \langle X^2 + Y^2 \rangle$ has even-function symmetry over a half-lattice interval, it follows from Eq. (61) that

$$\int_s^{s+S} ds \frac{d\kappa_z}{ds} \langle X^2 + Y^2 \rangle = 0. \tag{63}$$

Therefore, Eq. (40) gives $\int_s^{s+S} ds dU(s)/ds = 0$, or equivalently

$$U(s+S) = U(s), \tag{64}$$

which corresponds to a *periodic* variation of the total energy $U(s)$ with fundamental periodicity length S.

B. Rate Equations for Kapchinskij-Vladimirskij Beam Distribution

We now specialize to the case of azimuthally symmetric beam propagation ($\partial F_b/\partial\theta = 0 = \partial\psi/\partial\theta$) through a periodic solenoidal focusing field $\kappa_z(s+S) = \kappa_z(s)$, placing particular emphasis on the rate equations for the rms beam radius $R_b(s)$ [Eqs. (52) or (53)] and the unnormalized beam emittance $\epsilon(s)$ [Eqs. (55) or (57)]. As a first example, we consider the widely-studied case of a Kapchinskij-Vladimirskij beam distribution [23] F_b^{KV}, which generates self-consistently the step-function density profile [1,23,26,28]

$$n_b(R,s) = \begin{cases} \dfrac{N_b}{2\pi R_b^2(s)}, & 0 \le R < r_b(s) = \sqrt{2}R_b(s), \\ 0, & \sqrt{2}R_b(s) = r_b(s) < R \le r_w. \end{cases} \tag{65}$$

Here, $n_b(R,s) = \int dX'dY'F_b$ is the beam density, $r_b(s) \equiv \sqrt{2}R_b(s)$ is the outer radius of the beam envelope, and $N_b = 2\pi \int_0^{r_b(s)} dRRn_b(R,s)$ is the number of particles per unit axial length. Note from Eq. (65) that the mean-square beam radius is $\langle X^2 + Y^2 \rangle \equiv N_b^{-1} 2\pi \int_0^{r_b} dRRR^2 n_b(R,s) = R_b^2(s)$, as expected. Because $\partial\psi/\partial\theta = 0$, it follows that the coefficient $\Delta = 0$ [see Eq. (48)] in the rate equation (53) for the rms beam radius $R_b(s)$. Substituting $\Delta = 0$ and $R_b(s) = r_b(s)/\sqrt{2}$ into Eq. (53), we obtain

$$\frac{d^2}{ds^2} r_b(s) + \left[\kappa_z(s) - \frac{K}{r_b^2(s)} \right] r_b(s) = \frac{\epsilon^2(s)}{r_b^3(s)}. \tag{66}$$

Equation (66) is the familiar envelope equation [1,28] for the outer radius $r_b(s)$ of a Kapchinskij-Vladimirskij beam, derived as a particular application of the general rate equations developed in Sec. III.

We now turn to the evolution of the unnormalized beam emittance $\epsilon(s)$ described by Eq. (55), or equivalently, Eqs. (56) and (57). For the step-function density profile in Eq. (65), Poisson's equation (24) is readily integrated to give

$$\psi = \begin{cases} -\frac{1}{2}K\frac{R^2}{r_b^2(s)}, & 0 \le R < r_b(s), \\ \\ -\frac{1}{2}K\left(1 + 2\ell n\frac{R}{r_b(s)}\right), & r_b(s) < R \le r_w. \end{cases} \tag{67}$$

Here, we have taken $\psi(R=0,s)=0$ without loss of generality. Making use of Eq. (67) and $R^2 = X^2 + Y^2$ to evaluate $\partial\psi/\partial X$ and $\partial\psi/\partial Y$ in the region where the beam density n_b is non-zero in (65), it follows from Eq. (55) that

$$\frac{d}{ds}\frac{1}{8}\epsilon^2(s) = -\frac{K}{r_b^2(s)}\left\{\frac{1}{2}\frac{d}{ds}\langle X^2+Y^2\rangle\langle X^2+Y^2\rangle - \langle X^2+Y^2\rangle\langle XX'+YY'\rangle\right\} = 0, \tag{68}$$

where use has been made of $(d/ds)\langle X^2+Y^2\rangle = 2\langle XX'+YY'\rangle$. Therefore, as expected for a Kapchinskij-Vladimirskij beam distribution, the unnormalized beam emittance defined in Eq. (51) is an exactly conserved quantity with $\epsilon(s) = const.$ (independent of s). This conclusion also follows from the representation of the rate equation for $\epsilon(s)$ in Eq. (57), where $\mathcal{E}_F(s)$ is the self-field energy defined in Eq. (38). Making use of Eqs. (38) and (67), we obtain

$$\mathcal{E}_F(s) = \frac{1}{2K}\int_0^{r_w} dRR\left(\frac{\partial\psi}{\partial R}\right)^2 = \frac{1}{2}K\left(\frac{1}{4} + \ell n\frac{r_w}{r_b(s)}\right). \tag{69}$$

Therefore, from Eq. (69), $(d/ds)\mathcal{E}_F(s) = -(K/2)r_b^{-1}(s)(d/ds)r_b(s)$, and the rate equation (57) also gives $(d/ds)\epsilon^2(s) = 0$ when $\Delta = 0$. This calculation also clearly shows that if a KV beam distribution develops an (instability-induced, say) asymmetry such that $\Delta \neq 0$, then the unnormalized beam emittance is no longer a conserved quantity according to Eq. (57).

For an azimuthally symmetric KV beam distribution, the constancy of $\epsilon(s)$ permit a determination of the s-dependence of the mean kinetic energy $(1/2)\langle X'^2 + Y'^2\rangle$. Setting $\epsilon^2(s) = \epsilon_0^2 = const.$ in Eq. (51), and solving for the kinetic energy gives

$$\frac{1}{2}\langle X'^2 + Y'^2\rangle = \frac{\epsilon_0^2}{4r_b^2(s)} + \frac{1}{4}\left(\frac{dr_b}{ds}\right)^2 , \tag{70}$$

where use has been made of $2R_b^2(s) = r_b^2(s)$.

To summarize, for a KV beam distribution propagating through a periodic solenoidal focusing field $\kappa_z(s+S) = \kappa_z(s)$, the outer radius $r_b(s)$ of the beam envelope evolves according to the nonlinear envelope equation (66) with $\epsilon^2(s) = \epsilon_0^2 = const.$ Closed expressions for the field energy $\mathcal{E}_F(s)$ and mean kinetic energy $(1/2)\langle X'^2 + Y'^2\rangle$ are given in terms of $r_b(s)$ and other system parameters by Eqs. (69) and (70). For a matched beam with $r_b(s+S) = r_b(s)$, note from Eqs. (69) and (70) that $\mathcal{E}_F(s)$ and $(1/2)\langle X'^2 + Y'^2\rangle$ are also periodic functions of s with fundamental periodicity length S.

C. Rate Equations for Beam Distributions with Fixed-Shaped Density Profile

The assumption of a KV beam distribution F_b^{KV} and corresponding step-function density profile for $n_b(R, s)$ in Eq. (65) is very restrictive. In this section, we carry out an important generalization of the analysis in Sec. IV.B to the class of azimuthally symmetric beam distributions $(\partial F_b/\partial\theta = 0 = \partial\psi/\partial\theta)$ with density profile $n_b = \int dX'dY'F_b$ of the form [28]

$$n_b(R,s) = \begin{cases} \dfrac{N_b}{\pi r_b^2(s)} f\left(\dfrac{R}{r_b(s)}\right), & 0 \le R < r_b(s), \\ 0, & r_b(s) < R \le r_w. \end{cases} \tag{71}$$

Here, $f(R/r_b(s))$ is a smooth, but otherwise unspecified function satisfying $f \ge 0$, and $r_b(s)$ is the outer radial envelope of the beam. We refer to $f(R/r_b)$ as the *density shape function*, and the class of profiles in Eq. (71) as *fixed-shape profiles* because the only dependence on s in Eq. (71) is through the factor $r_b^{-2}(s)$, and the shape-function $f(R/r_b(s))$. Because $N_b \equiv 2\pi \int_0^{r_b} dRRn_b(R,s)$, the normalization on f in Eq. (71) is

$$\int_0^1 dXXf(X) = 1/2. \tag{72}$$

Furthermore, the mean-square beam radius $R_b^2(s) = \langle X^2 + Y^2 \rangle$ for the class of profiles in Eq. (71) is given by $R_b^2(s) = N_b^{-1} 2\pi \int_0^{r_b(s)} dRRR^2 n_b(R, s)$, which reduces to

$$R_b^2(s) = gr_b^2(s), \tag{73}$$

where $g = const.$ is the geometric factor defined by

$$g = 2\int_0^1 dXXX^2 f(X). \tag{74}$$

A simple example for the choice of $f(R/r_b(s))$ in Eq. (71) is the step-function density profile in Eq. (65) corresponding to a KV distribution of beam particles. In this case $f(X) = 1$ for $0 \leq X < 1$, and $f(X) = 0$ for $X > 1$, and the geometric factor in Eq. (74) is $g = 1/2$, which corresponds to $R_b^2(s) = r_b^2(s)/2$. A second example is the parabolic density profile

$$n_b(R, s) = \frac{2N_b}{\pi r_b^2(s)} \left(1 - \frac{R^2}{r_b^2(s)}\right), \tag{75}$$

for $0 \leq R < r_b(s)$, and $n_b(R, s) = 0$ for $r_b(s) < R \leq r_w$. In this case, the density shape function is $f(X) = 2(1 - X^2)$ for $0 \leq X < 1$, and $f(X) = 0$ for $X > 1$. The corresponding geometric factor g and mean-square beam radius calculated from Eq. (74) are $g = 1/3$ and $R_b^2(s) = r_b^2(s)/3$. Clearly, many other examples are possible (Table 1).

We now turn to an examination of the rate equations for the rms beam radius $R_b(s)$ [Eqs. (52) or (53)] and unnormalized beam emittance $\epsilon(s)$ [Eqs. (55), (56) or (57)] for the class of beam density profiles in Eq. (71). Because $\Delta = 0$ for the azimuthally symmetric case considered here [see Eq. 48)], the rate equation (53) for the rms beam radius $R_b(s) = \sqrt{g} r_b(s)$ readily gives

$$\frac{d^2}{ds^2} r_b(s) + \left[\kappa_z(s) - \frac{K/2g}{r_b^2(s)}\right] r_b(s) = \frac{\epsilon^2(s)/4g^2}{r_b^3(s)}. \tag{76}$$

Equation (76) describes the nonlinear evolution of the outer beam envelope $r_b(s)$ for the general class of density profiles in Eq. (71). Apart from the geometric factor g, Eq. (76) is identical in form to the envelope equation (66) derived for a KV beam distribution. Indeed, for a KV beam, $g = 1/2$ and Eq. (76) reduces exactly to Eq. (66), as expected.

With regard to the unnormalized beam emittance $\epsilon(s)$, we make use of the rate equation in the form given in Eq. (57). Setting $\Delta = 0$ and $R_b^2(s) = gr_b^2(s)$, where g is defined in Eq. (74) for general $f(X)$, we obtain

$$\frac{d}{ds}\frac{1}{8}\epsilon^2(s) = gr_b^2(s)\left\{-\frac{1}{r_b}\frac{dr_b}{ds}\frac{1}{2}K - \frac{d}{ds}\mathcal{E}_F\right\}, \tag{77}$$

where $\mathcal{E}_F(s)$ is the field energy defined by (for $\partial\psi/\partial\theta = 0$)

$$\mathcal{E}_F(s) = \frac{1}{2K}\int_0^{r_w} dRR\left(\frac{\partial\psi}{\partial R}\right)^2. \tag{78}$$

Substituting Eq. (71) into Poisson's equation (24), we obtain the equation for $\psi(R,s)$, i.e.,

$$\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial\psi}{\partial R} = -\frac{2K}{r_b^2(s)}f\left(\frac{R}{r_b(s)}\right). \tag{79}$$

Equation (79) can be formally integrated to determine $\partial\psi/\partial R$, which is required in Eq. (78). This gives

$$\frac{\partial\psi}{\partial R} = \begin{cases} -K\dfrac{2}{R}\displaystyle\int_0^{R/r_b(s)} dXXf(X), & 0 \le R < r_b(s), \\ \\ -K\dfrac{1}{R}, & r_b(s) < R \le r_w. \end{cases} \tag{80}$$

Note from Eq. (80) that $\partial\psi/\partial R$ is continuous at $R = r_b(s)$ because of the normalization condition $\int_0^1 dXXf(X) = 1/2$. Substituting Eq. (80) into the expression for $\mathcal{E}_F(s)$ in Eq. (78), and carrying out the integration over R gives

$$\mathcal{E}_F(s) = \frac{1}{2}K\left\{4\int_0^1 \frac{dX}{X}\left(\int_0^X dXXf(X)\right)^2 + \ell n\frac{r_w}{r_b(s)}\right\}. \tag{81}$$

It is important to note in Eq. (81) that the first term on the right-hand side is constant (independent of s) for general choice of density shape function $f(R/r_b(s))$. Therefore, the only s-variation of $\mathcal{E}_F(s)$ occurs through the logarithmic term in Eq. (81), which gives

$$\frac{d}{ds}\mathcal{E}_F(s) = -\frac{1}{2}\frac{K}{r_b(s)}\frac{d}{ds}r_b(s). \tag{82}$$

Substituting Eq. (82) into Eq. (77) then gives the important result

$$\frac{d}{ds}\epsilon^2(s) = 0, \tag{83}$$

corresponding to conservation of beam emittance, with $\epsilon(s) = \epsilon_0 = const.$

Equations (76) and (83) are similar to the results first obtained by Lee and Cooper [28] for the case of azimuthally symmetric beam propagation through a solenoidal focusing field, making the assumption that the density profile has the fixed profile shape in Eq. (66). Equations (76) and (83) indeed constitute powerful results for axisymmetric beam propagation. While emittance conservation is a well-known result for a KV beam, the fact that $\epsilon(s) = const.$ for general profile shape function $f(R/r_b(s))$ has several important implications. First, because $\epsilon(s) = \epsilon_0 = const.$ in the envelope equation (76) for the outer beam radius $r_b(s)$, Eq. (76) can be solved numerically for $r_b(s)$ for a broad range of lattice functions $\kappa_z(s+S) = \kappa_z(s)$, system parameters κ and ϵ_0, and values of the geometric parameter g, which depends on the choice of shape function $f(R/r_b(s))$. Second, once the outer beam radius $r_b(s)$ is determined from Eq. (76), the self-consistent evolution of $\mathcal{E}_F(s)$ can be determined from Eq. (81). Finally, similar to the result obtained for a KV beam in Sec. IV.B [see Eq. (70)], the definition of emittance in Eq. (51) can be used to determine the evolution of the mean kinetic energy $(1/2)\langle X'^2 + Y'^2\rangle$. We readily obtain, for $\epsilon(s) = \epsilon_0 = const.$,

$$\frac{1}{2}\langle X'^2 + Y'^2\rangle = \frac{\epsilon_0^2}{8gr_b^2(s)} + \frac{1}{2}g\left(\frac{dr_b}{ds}\right)^2. \tag{84}$$

For the particular choice of a KV beam distribution, where the geometric factor is $g = 1/2$, Eq. (84) reduces to Eq. (70), as expected.

Table 1 shows a tabulation of values of the geometric factor $g = 2\int_0^1 dXXX^2 f(X)$ defined in Eq. (74) for several choices of the profile shape-function $f(X)$. Here, $f(X)$ is normalized according to $\int_0^1 dXXf(X) = 1/2$ [Eq. (72)], and the rms beam radius $R_b(s)$ is related to the outer beam radius $r_b(s)$ by $R_b^2(s) = gr_b^2(s)$ [Eqs. (73) and (74)]. Note from Table 1 that $g = 1/2$ for a KV beam distribution, which has the step-function density profile in Eq. (65), whereas $g < 1/2$ when the density profile is peaked on axis and decreases monotonically to zero at $r = r_b(s)(X = 1)$. On the other hand, if the density profile is strongly peaked off axis, then $g > 1/2$. Indeed, for an infinitesimally thin annulus centered at $r = r_b(s)$, which corresponds to $f(X) = (1/2)\delta(X-1)$ in Table 1, we obtain the geometric factor $g = 1$.

We conclude this section with a brief discussion of properties of the envelope equation (76) for general geometric factor g. Defining

$$K_g \equiv \frac{1}{2g}K,$$

$$\epsilon_g \equiv \frac{1}{2g}\epsilon, \tag{85}$$

it follows that Eq. (76) can be expressed in the equivalent form

$$\frac{d^2}{ds^2}r_b(s) + \left[\kappa_z(s) - \frac{K_g}{r_b^2(s)}\right] r_b(s) = \frac{\epsilon_g^2}{r_b^3(s)}. \tag{86}$$

Equation (86) is identical in form to the envelope equation (66) for a KV beam provided we make the replacements $K \to K_g$ and $\epsilon \to \epsilon_g$ in Eq. (66). Therefore, many of the results obtained in analytical and numerical studies of the envelope equation (66) can be applied directly to Eq. (86) provided we make the replacements implied by Eq. (85). This includes, for example, the existence of self-field-induced nonlinear resonances and chaotic behavior [29] exhibited by the beam envelope in some parameter regimes where there is a mismatch between the beam and the periodic focusing field.

For a periodic focusing lattice with $\kappa_z(s+S) = \kappa_z(s)$, Eq. (86) generally supports nonlinear periodic solutions with $r_b(s+S) = r_b(s)$, corresponding to a *matched-beam* solution in which the period of oscillation of the beam envelope $r_b(s)$ is the *same* as the period of the focusing field $\kappa_z(s)$. In the special case of a uniform focusing field where $\kappa_z(s) = \bar{\kappa}_z = const.$ (independent of s), Eq. (86) also supports a *smooth-beam* solution in which $r_b(s) = r_{bs} = const.$ Setting $d^2r_{bs}/ds^2 = 0$ in Eq. (86) and solving for r_{bs}, we find

$$r_{bs}^2 = \frac{K_g}{2\bar{\kappa}_z} + \left[\left(\frac{K_g}{2\bar{\kappa}_z}\right)^2 + \frac{\epsilon_g^2}{\bar{\kappa}_z}\right]^{1/2} = \frac{1}{2g}\left\{\frac{K}{2\bar{\kappa}_z} + \left[\left(\frac{K}{2\bar{\kappa}_z}\right)^2 + \frac{\epsilon^2}{\bar{\kappa}_z}\right]^{1/2}\right\}. \tag{87}$$

For specified values of beam current (K), field strength ($\bar{\kappa}_z$) and emittance (ϵ), we note from Eq. (87) that the equilibrium beam radius r_{bs} is smaller when the density profile is strongly peaked off axis ($1/2 < g < 1$) than when it is peaked on axis ($g < 1/2$). Moreover, from Eq. (87) the beam radius r_{bs} generally scales as $g^{-1/2}$.

We now examine the envelope equation (86) for a periodic focusing field $\kappa_z(s+S) = \kappa_z(s)$ and matched-beam solutions $r_b(s+S) = r_b(s)$. Inspection of Eqs. (85) and (86) shows that it is useful to introduce the dimensionless quantities (denoted by a 'hat') defined by

$$
\begin{aligned}
\hat{K} &= \frac{K_g S}{\epsilon_g} = \frac{KS}{\epsilon}, \\
\hat{s} &= \frac{s}{S}, \\
\hat{\kappa}_z(\hat{s}) &= \kappa_z(s/S)S^2, \\
\hat{r}_b(\hat{s}) &= \frac{r_b(s/S)}{\sqrt{\epsilon_g S}} = \sqrt{\frac{2g}{\epsilon S}}\, r_b(s/S).
\end{aligned}
\tag{88}
$$

Because $\hat{K} = KS/\epsilon$, we note from Eq. (88) that $\hat{K}$ is a dimensionless measure of beam current, which is proportional to K. Substituting Eq. (88) into Eq. (86), the envelope equation can be expressed in the equivalent form

$$
\frac{d^2}{d\hat{s}^2}\hat{r}_b(\hat{s}) + \left[\hat{\kappa}_z(\hat{s}) - \frac{\hat{K}}{\hat{r}_b^2(\hat{s})}\right]\hat{r}_b(\hat{s}) = \frac{1}{\hat{r}_b^3(\hat{s})}, \tag{89}
$$

where $\hat{\kappa}_z(\hat{s}+1) = \hat{\kappa}_z(\hat{s})$ for a periodic focusing lattice. In solving Eq. (89) numerically for $\hat{r}_b(\hat{s}+1) = \hat{r}_b(\hat{s})$, it is necessary to specify the functional form of $\hat{\kappa}_z(\hat{s})$ and the value of $\hat{K} = KS/\epsilon$.

Typical numerical solutions to Eq. (89) over the interval $0 \leq s/S \leq 1$ are illustrated in Fig. 2 for the choice of step-function lattice in Eq. (62). Here, we assume lattice strength $\kappa_{z0}S^2 = 6.25$ and filling factor $\eta = 0.25$. The two cases shown in Fig. 2 correspond to $\hat{K} = KS/\epsilon = 1$ (low beam current) and $KS/\epsilon = 10$ (high beam current). As expected, the normalized beam radius $\hat{r}_b(\hat{s})$ increases as $\hat{K}$ is increased because of repulsive space-charge effects.

An important quantity in accelerator physics is the so-called *phase advance* (or '*tune*') σ defined by [1]

$$\sigma \equiv \epsilon_g \int_0^S \frac{ds}{r_b^2(s)} = \int_0^1 \frac{d\hat{s}}{\hat{r}_b^2(\hat{s})}. \tag{90}$$

Note that the phase advance σ becomes increasingly depressed as the normalized beam current $\hat{K} = KS/\epsilon$ is increased (larger beam radius $\hat{r}_b$). This is illustrated in Fig. 3 for the choice of lattice parameters $\kappa_{z0}S^2 = 6.25$ and $\eta = 0.25$, where the phase advance σ calculated numerically from Eqs. (89) and (90) is plotted versus KS/ϵ. Here, the vacuum phase advance is $\sigma_v = \lim_{K\to 0}\sigma = 74.69°$ for the choice of lattice parameters in Fig. 3.

V. CONCLUSIONS

In this paper, we have presented a detailed formulation and analysis of the rate equations for statistically-averaged quantities for an intense nonneutral beam propagating through a periodic solenoidal focusing field $\mathbf{B}^{sol}(\mathbf{x})$ described by Eq. (1). The analysis was based on the nonlinear Vlasov-Maxwell equations in the electrostatic approximation, assuming a thin beam with characteristic beam radius $r_b \ll S$, negligibly small axial momentum spread about the directed value $p_z = \gamma_b m \beta_b c$, and $\nu/\gamma_b = N_b Z_i^2 e^2/\gamma_b mc^2 \ll 1$, where ν is Budker's parameter. Following a discussion of the theoretical model and assumptions (Sec. II), the global rate equation was derived (Sec. III) which describes the self-consistent nonlinear evolution of the statistical average $\langle\chi\rangle = N_b^{-1}\int dXdYdX'dY'\chi F_b$, where χ is a general phase function defined on the transverse four-dimensional phase space (X, Y, X', Y'). The results were then applied to investigate the evolution of the generalized entropy, mean canonical angular momentum $\langle P_\theta\rangle$, center-of-mass motion for $\langle X\rangle$ and $\langle Y\rangle$, mean kinetic energy $(1/2)\langle X'^2 + Y'^2\rangle$, mean-square beam radius $\langle X^2 + Y^2\rangle$, and coupled rate equations for the unnormalized transverse emittance $\epsilon(s)$and rms beam radius $R_b(s) = \langle X^2 + Y^2\rangle^{1/2}$. The rate equations obtained in Sec. III are derived from the fully nonlinear Vlasov-Poisson equations allowing for azimuthal asymmetries $(\partial/\partial\theta \neq 0)$, and are valid no matter how complex the nonlinear evolution of the system. Following a discussion of global energy balance (Sec. IV), and the rate equations for the special case where F_b corresponds to the Kapchinskij-Vladimirskij (KV) distribution with step-function radial density profile, we examined the coupled rate equations for the unnormalized beam emittance $\epsilon(s)$ and rms beam radius $R_b(s)$ for the class of axisymmetric beam distributions F_b with fixed-shape density profile $n_b(R, s) = [N_b/\pi r_b^2(s)]f(R/r_b(s))$. Here, $r_b(s)$ is the outer radius of the beam envelope, and the density shape function $f(R/r_b)$ is allowed to have general functional form. Most importantly, it was found that $d\epsilon(s)/ds = 0$, corresponding to emittance conservation for general density shape function $f(R/r_b)$, and that the envelope equation (76) for the outer beam radius $r_b(s)$ is similar to the envelope equation [1,28] for a KV beam distribution [23], approximately modified by the geometric factor g to reflect the shape of the function $f(R/r_b)$. This is similar to the result obtained by Lee and Cooper [28] for the case of axisymmetric beam propagation through a solenoidal focusing field and general density shape function $f(r/r_b)$. Future work will include a determination of axisymmetric distributions F_b that self-consistently generate different functional forms for the density shape function $f(R/r_b)$.

ACKNOWLEDGMENTS

This research was supported by the Department of Energy, and in part by the Office of Naval Research.

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FIGURES

FIG. 1. Periodic step-function lattice in Eq. (62). Here, η is the so-called filling factor.

FIG. 2. Plots of $\hat{r}_b(\hat{s})$ versus $\hat{s} = s/S$ obtained numerically from Eq. (89) for the choice of step-function lattice in Eq. (62). Here, $\kappa_{z0}S^2 = 6.25$ and $\eta = 1/4$, and the two cases correspond to $KS/\epsilon = 1$ (dashed curve) and $KS/\epsilon = 10$ (solid curve).

FIG. 3. Plot of phase advance σ defined in Eq. (90) versus KS/ϵ obtained numerically from (89) for the choice of step-function lattice in Eq. (62) with $\kappa_{z0}S^2 = 6.25$ and $\eta = 1/4$.

TABLES

TABLE I. Table of values of the constant A and geometric factor $g = 2\int_0^1 dX X X^2 f(X)$ for several choices of the density shape function $f(X)$. Here, $f(X) = 0$ for $X > 1$, and the constant A is chosen so that $\int_0^1 dX X f(X) = 1/2$ [Eq. (72)].

$f(X)$ for $0 \leq X < 1$	A	$g \equiv 2\int_0^1 dX X X^2 f(X)$
A	1	1/2
$A(1-X^2)$	2	1/3
$A(1-X^2)^2$	3	1/4
$A(1-X^2)^n$, $n>0$	$n+1$	$(n+2)^{-1}$
$A\cos\left(\frac{\pi}{2}X\right)$	$\frac{1}{2}\frac{(\pi/2)^2}{(\pi/2-1)} = 2.161$	$\frac{\pi/2}{(\pi/2-1)} - 6\left(\frac{2}{\pi}\right)^2 = 0.320$
$AX^2(1-X^2)$	6	1/2
$A\delta(X-1)$	1/2	1

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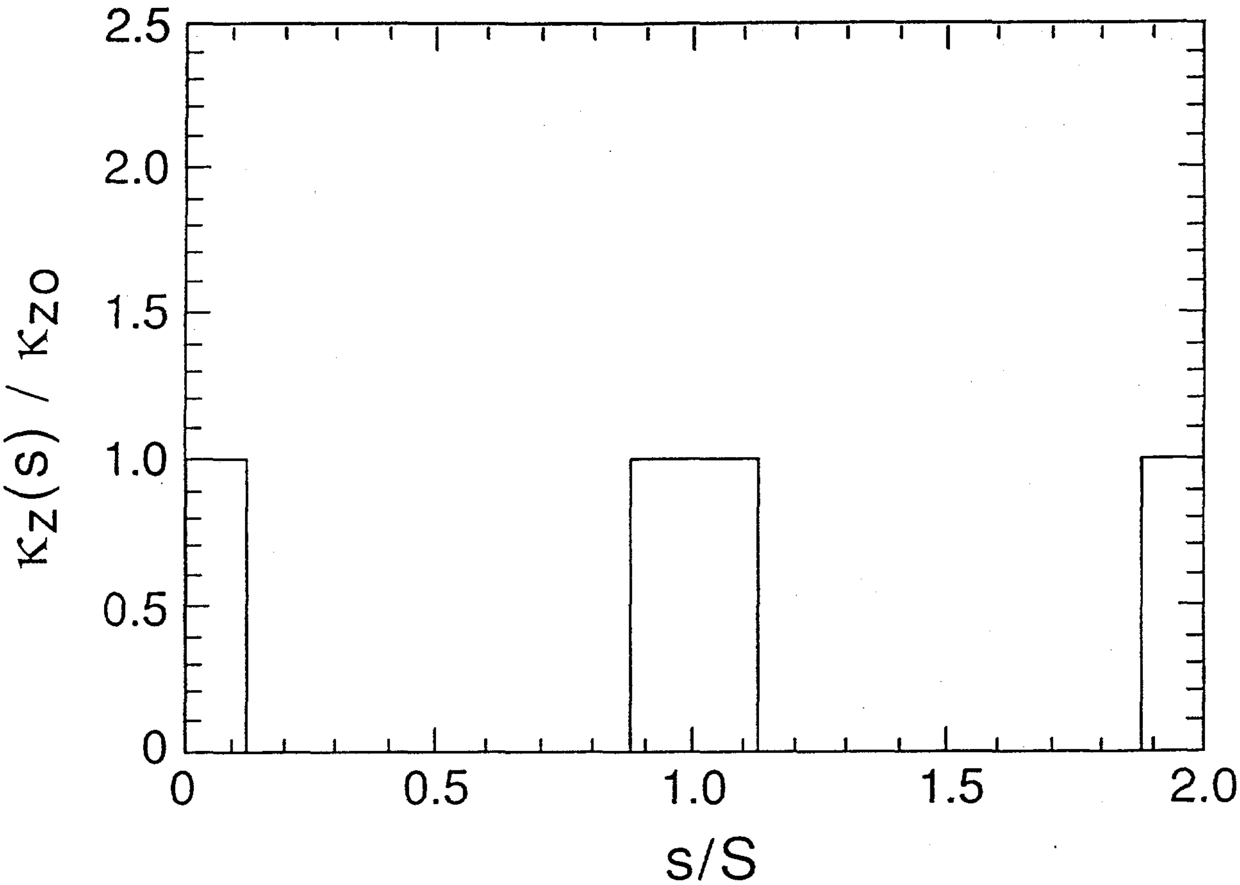

Fig. 1

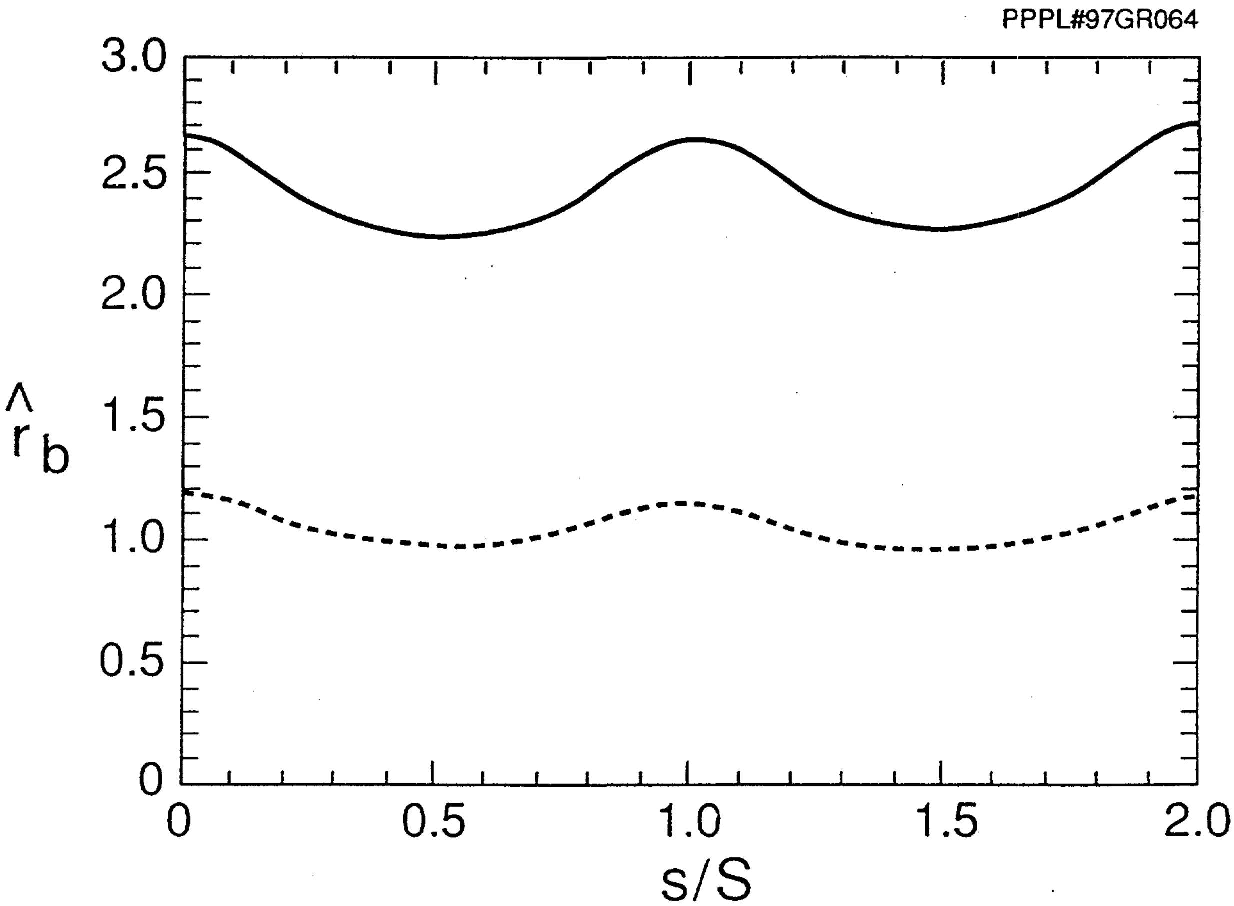

Fig. 2

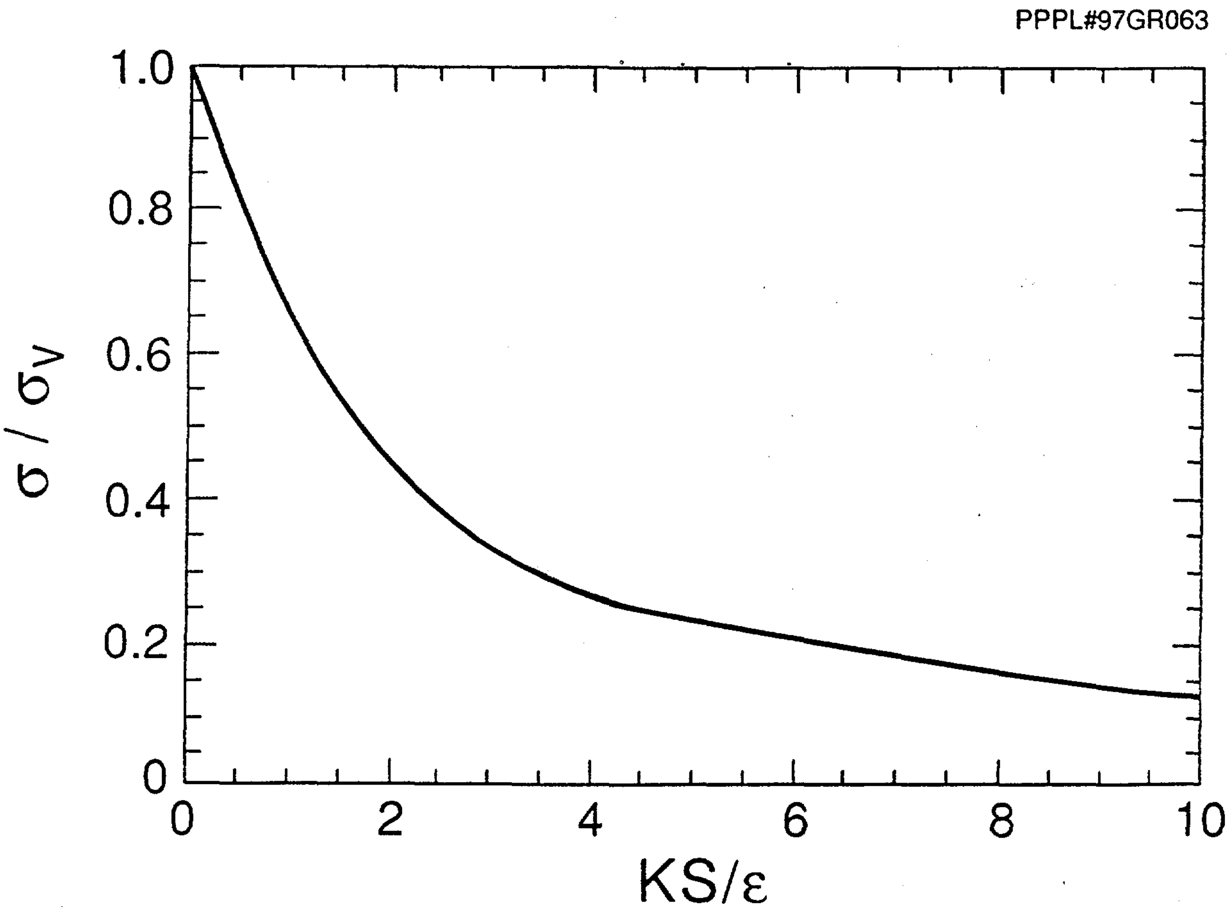

Fig. 3

External Distribution in Addition to UC-420

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